

Article

# Asymptotically Flat Boundary Conditions for the $U(1)^3$ Model for Euclidean Quantum Gravity

Sepideh Bakhoda <sup>1,2,\*</sup> , Hossein Shojaie <sup>1</sup>  and Thomas Thiemann <sup>2,\*</sup><sup>1</sup> Department of Physics, Shahid Beheshti University, Tehran 1983969411, Iran; h-shojaie@sbu.ac.ir<sup>2</sup> Institute for Quantum Gravity, FAU Erlangen—Nürnberg, Staudtstraße 7, 91058 Erlangen, Germany

\* Correspondence: sepideh.bakhoda@gravity.fau.de (S.B.); thomas.thiemann@fau.de (T.T.)

**Abstract:** A generally covariant  $U(1)^3$  gauge theory describing the  $G_N \rightarrow 0$  limit of Euclidean general relativity is an interesting test laboratory for general relativity, specially because the algebra of the Hamiltonian and diffeomorphism constraints of this limit is isomorphic to the algebra of the corresponding constraints in general relativity. In the present work, we study boundary conditions and asymptotic symmetries of the  $U(1)^3$  model and show that while asymptotic spacetime translations admit well-defined generators, boosts and rotations do not. Comparing with Euclidean general relativity, one finds that the non-Abelian part of the  $SU(2)$  Gauss constraint, which is absent in the  $U(1)^3$  model, plays a crucial role in obtaining boost and rotation generators.

**Keywords:** asymptotically flat boundary conditions; classical and quantum gravity;  $U(1)^3$  model; asymptotic charges



**Citation:** Bakhoda, S.; Shojaie, H.; Thiemann, T. Asymptotically Flat Boundary Conditions for the  $U(1)^3$  Model for Euclidean Quantum Gravity. *Universe* **2021**, *7*, 68. <https://doi.org/10.3390/universe7030068>

Academic Editor: Astrid Eichhorn

Received: 13 February 2021

Accepted: 10 March 2021

Published: 13 March 2021

**Publisher's Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



**Copyright:** © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

## 1. Introduction

In the framework of the Ashtekar variables in terms of which General Relativity (GR) is formulated as a  $SU(2)$  gauge theory [1–3], attempts to find an operator corresponding to the Hamiltonian constraint of the Lorentzian vacuum canonical GR led to the result [4] that the Lorentzian Hamiltonian constraint can be written in terms of the Euclidean Hamiltonian constraint and the volume operator. Since the latter is under much control in the context of Loop Quantum Gravity (LQG) [5–8], an essential step towards quantising the Lorentzian Hamiltonian constraint is the quantisation of the Euclidean Hamiltonian constraint. An important consistency check for a successful quantisation of the Hamiltonian constraint is the anomaly free implementation of the hypersurface deformation algebra [9]. In its current form, the algebra of Hamiltonian and spatial diffeomorphism constraints does close, but with wrong structure “constants”, and in that sense it suffers from an anomaly.

In order to improve the situation, we are motivated to study structurally similar but simpler theories and see what lessons can be learned from the outcome. The  $G_N \rightarrow 0$  limit of Euclidean gravity introduced by Smolin [10] is one of these models which is described by a  $U(1)^3$  gauge theory. This model contains three Gauss constraints, three spatial diffeomorphism constraints and a Hamiltonian constraint whose constraint algebra for the Hamiltonian and diffeomorphism constraints is isomorphic to that of general relativity. This property, in addition to the Abelian nature of its gauge group, make the  $U(1)^3$  theory an interesting toy model to scrutinise, and there has been interesting recent work on it [11–15].

There exist two different approaches to work out the quantum theory for constrained theories. The first one, known as Dirac quantisation [16], quantises the entire kinematical phase space producing a kinematical Hilbert space. Then, physical states are those which are annihilated by all constraints operators acting on the kinematical Hilbert space. Therefore, the physical sector of the theory in this approach is constructed in quantum theory. In the second approach, called “reduced phase space quantisation”, one solves the constraints at the classical level and obtains a physical phase space whose variables are

called observables, which are gauge invariant quantities, and then quantises the physical phase space, yielding the physical Hilbert space. For the  $U(1)^3$  model, there has been much recent progress [11–15] along the Dirac quantisation approach, and the analysis of the reduced phase space track was begun in [17,18]. In that work and also the present one, for simplicity we restrict to the spatial topology of  $\mathbb{R}^3$  with the asymptotically flat boundary conditions. As asymptotically flat spacetimes are of great importance in GR, this paper is devoted to investigating their properties in the  $U(1)^3$  model. The results of the present paper were used in [17,18], which was in fact the main motivation for the present study.

To achieve asymptotic symmetry generators [19–29], we seek for boundary terms to the constraints that produce well-defined phase space functions and Poisson brackets, while lapse function and shift vector obey decay behaviours corresponding to asymptotic symmetry transformations. When we are working in the context of Lorenzian or Euclidean GR, one expects these well-defined functions to generate the Poincaré or ISO(4) group, respectively, depending on the signature. Regarding  $U(1)^3$  theory, we therefore examine to what extent we can recover ISO(4) transformations. In fact, the question is whether there are well-defined generators for all generators of ISO(4) in this model or not, and what the main reason for answering yes/no to this question is.

The architecture of this paper is as follows:

In Section 2, we briefly review the background material needed for the subsequent analysis. In Section 2.1, first we express the constraints of GR in terms of both ADM and  $SU(2)$  variables. Then observing that they are not well-defined and functionally differentiable, we revisit the results and reasoning, presented in [30] and in [31,32], to improve the constraints using suitable boundary terms to well-defined generators of Poincaré and ISO(4) group for Lorentzian and Euclidean GR respectively. In Section 2.2, the  $U(1)^3$  model will be concisely introduced with a focus on the boundary conditions (fall-off behaviour) of the canonical fields and the constraints which are functions of those.

In Section 3, we try to make the constraints well-defined. To do this, we follow the usual approach and first extract boundary term variations which a priori violate differentiability of the constraints and then, if possible, subtract corresponding boundary terms from the constraints themselves, rendering them functionally differentiable. As usual, we also require the constraints to be integrable. It is well known that there is a delicate interplay between the correspondingly allowed fall-off behaviour of the test functions smearing the constraints and whether the associated constraint functional including the boundary term generates local gauge transformations or global symmetries. Surprisingly, we show that in contrast to the  $SU(2)$  model, in the  $U(1)^3$  model, there are no well-defined generators for (Euclidean) boosts and rotations. On the other hand, spacetime translations are still allowed asymptotic symmetries.

In Section 4, we compare the results of Sections 2.1 and 3 and exhibit the reason for why (Euclidean) boost and rotation generators in the  $U(1)^3$  model cannot be defined.

In the last section we conclude with a brief summary.

## 2. Background

### 2.1. Review of Asymptotically Flat Boundary Conditions for the $SU(2)$ Case

In general, consistent boundary conditions are supposed to provide a well-defined symplectic structure as well as finite (integrable) and differentiable constraints. Both features generically require to include boundary terms into the constraints which are vanishing when the constraint generates gauge and non-vanishing when it generates symmetries, depending on the fall-off behaviour of the test functions of the constraint functional. In the ADM formulation of asymptotically flat spacetimes, it is assumed that on spatial slices asymptotic spheres are equipped with asymptotically cartesian coordinates  $x^a$  at spatial infinity—i.e.,  $r \rightarrow \infty$ , where  $r^2 = x^a x_a$ . Taking this as the starting point and

seeking appropriate boundary conditions, one sees that on any hypersurface, the fall-off behaviours of the spatial metric  $q_{ab}$  and its conjugate momentum  $\pi^{ab}$  have to be:

$$\begin{aligned} q_{ab} &= \delta_{ab} + \frac{h_{ab}}{r} + \mathcal{O}(r^{-2}), \\ \pi^{ab} &= \frac{p^{ab}}{r^2} + \mathcal{O}(r^{-3}), \end{aligned} \tag{1}$$

where  $h_{ab}$  and  $p^{ab}$  are smooth tensor fields on the asymptotic 2-sphere. In (1), the first condition follows directly from the form of the spacetime metric for the asymptotically flat case, while the second one is a consequence of demanding a non-vanishing ADM momentum. In order to eliminate the logarithmic singularity existing in the symplectic structure, the leading terms in (1) need to admit additional certain parity conditions, such as:

$$h_{ab}\left(-\frac{x}{r}\right) = h_{ab}\left(\frac{x}{r}\right), \quad p^{ab}\left(-\frac{x}{r}\right) = -p^{ab}\left(\frac{x}{r}\right). \tag{2}$$

Indeed, the integral of  $p^{ab}\dot{h}_{ab}$  over the sphere, which is the coefficient of the singularity, vanishes owing to (2). The parity conditions also give rise to the finite and integrable Poincaré or ISO(4) charges.

Furthermore, aiming to retain the boundary conditions (1) invariant under the hypersurface deformations:

$$\begin{aligned} \delta q_{ab} &= \frac{-2sN}{\sqrt{q}}\left(\pi_{ab} - \frac{1}{2}\pi q_{ab}\right) + \mathcal{L}_{\vec{N}}q_{ab}, \\ \delta \pi^{ab} &= -N\sqrt{q}({}^{(3)}R^{ab} - \frac{1}{2}q^{ab}({}^{(3)}R) - \frac{sN}{2\sqrt{q}}(\pi_{cd}\pi^{cd} - \frac{1}{2}\pi^2)q^{ab} \\ &\quad + \frac{2sN}{\sqrt{q}}(\pi^{ac}\pi_c{}^b - \frac{1}{2}\pi^{ab}\pi) + \sqrt{q}(D^a D^b N - q^{ab}D_a D^b N) + \mathcal{L}_{\vec{N}}\pi^{ab}, \end{aligned} \tag{3}$$

One is required to restrict the fall-off behaviours of the lapse function,  $N$ , and the shift vector,  $N^a$ . In (3),  $q := \det(q_{ab})$ ,  ${}^{(3)}R_{ab}$  is the Ricci tensor of the spatial hypersurface;  $D_a$  is the torsion free metric compatible connection with respect to  $q_{ab}$ ; and  $s$  denotes the signature of the spacetime metric—i.e.,  $s = +1$  and  $s = -1$  for Euclidean and Lorentzian spacetimes, respectively. It turns out that the most general fall-off behaviours of lapse and shift which also give rise to the generators of the asymptotic Poincaré and ISO(4) groups are:

$$\begin{aligned} N &= \beta_a x^a + \alpha + S + \mathcal{O}(r^{-1}), \\ N^a &= \beta^a{}_b x^b + \alpha^a + S^a + \mathcal{O}(r^{-1}), \end{aligned} \tag{4}$$

where  $\beta_a$  and  $\beta_{ab} (= -\beta_{ba})$  are arbitrary constants representing (Euclidean) boosts and rotations. Here, if  $v^a$  denotes the velocity, then the boost parameter is  $\beta^a = \frac{v^a}{\sqrt{1+s v^2}} =: \gamma v^a$ , which satisfies the identity  $\gamma^2 + s\gamma^2 v^2 = 1$ . This identity indicates that for a Euclidean boost, when  $s = 1$  the sine and cosine appear in the transformation matrix, which says that the Euclidean boost is nothing but a rotation in the  $x^0, \vec{x}$  plane. In turn, the arbitrary function  $\alpha$  and arbitrary vector  $\alpha^a$  represent time and spatial translations, respectively, and  $S, S^a$ , which are odd functions on the asymptotic  $S^2$ , correspond to the so called supertranslations.

On the other hand, in vacuum GR the Hamiltonian and diffeomorphism constraints are:

$$\begin{aligned} H[N] &:= \int d^3x N \left( \frac{-s}{\sqrt{q}} \left[ (q_{ac}q_{bd} - \frac{1}{2}q_{ab}q_{cd})\pi^{ab}\pi^{cd} \right] - \sqrt{q}({}^{(3)}R) \right), \\ H_a[N^a] &:= -2 \int d^3x N^a D_b \pi_a{}^b, \end{aligned} \tag{5}$$

These are the generators of gauge transformations, and they have to be finite and functionally differentiable so that their Poisson bracket with any function on the phase space can be computed. With regard to (1) and (4), it is easy to check that the constraints (5) are neither finite nor differentiable. To remedy this situation, a surface integral should be subtracted from the constraint functionals. More in detail, with appropriate boundary conditions the result of the ill-defined contribution to the variation in the constraint should be the variation in a surface term, and thus one can define new expressions for the constraints by subtracting the boundary terms so that they become functionally differentiable. Finally, one examines the convergence (integrability) of the improved expressions. Having carried out this procedure, the authors of [30] obtained the following well-defined constraints:

$$\begin{aligned}
 J[N] &:= H[N] + 2 \oint dS_d \sqrt{q} q^{a[b} q^{c]d} [N \partial_b q_{ca} - \partial_b N (q_{ca} - \delta_{ca})], \\
 J_a[N^a] &:= H_a[N^a] + 2 \oint dS_a N_b \pi^{ab},
 \end{aligned}
 \tag{6}$$

where  $\oint$  is the integration over the asymptotic 2-sphere.

This analysis in terms of the ADM variables language can be translated to the Ashtekar variables  $(A_a^i, E_i^a)$ , where the connection,  $A_a^i$ , is an  $su(2)$ -valued one form and its momentum conjugate,  $E_i^a$ , is a densitised 3-Bein. However, achieving this is challenging, since there is an internal  $su(2)$  frame whose asymptotic behaviour has to be determined.

Accordingly, the boundary conditions (1) and (2) in terms of the Ashtekar variables can be written as:

$$\begin{aligned}
 E_i^a &= \delta_i^a + \frac{f_i^a}{r} + \mathcal{O}(r^{-2}), \\
 A_a^i &= \frac{g_a^i}{r^2} + \mathcal{O}(r^{-3}),
 \end{aligned}
 \tag{7}$$

where:

$$\delta_i^a = \begin{cases} 1 & \text{if } (a, i) = (x, 1), (y, 2), (z, 3) \\ 0 & \text{otherwise.} \end{cases}$$

Additionally,  $f_i^a$  and  $g_a^i$  are tensor fields defined on the asymptotic 2-sphere with the following definite parity conditions:

$$f_i^a\left(-\frac{x}{r}\right) = f_i^a\left(\frac{x}{r}\right), \quad g_a^i\left(-\frac{x}{r}\right) = -g_a^i\left(\frac{x}{r}\right).
 \tag{8}$$

By the decay conditions (7) and (8), it is assured that the symplectic structure is well-defined.

In Euclidean GR, the constraint surface in the  $(A, E)$ -phase space is defined by the vanishing of the following functionals called Gauss, Hamiltonian and diffeomorphism constraints, respectively

$$\begin{aligned}
 G_i[\Lambda^i] &= \int d^3x \Lambda^i \left( \partial_a E_i^a + \epsilon_{ijk} A_a^j E_k^a \right), \\
 H[N] &= \int d^3x N \epsilon_{ijk} F_{ab}^i E_j^a E_k^b, \\
 H_a[N^a] &= \int d^3x N^a \left( F_{ab}^j E_j^b - A_a^j G_j \right).
 \end{aligned}
 \tag{9}$$

where:

$$F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + \epsilon^i{}_{jk} A_a^j A_b^k,
 \tag{10}$$

is the curvature 2-form of  $A$ ,  $\Lambda^i$  is the Lagrange multiplier associated with  $G_i$  and  $N$  is the densitised lapse function with weight  $-1$ . It is desired to attain a well-defined form of these functionals with the smearing functions including ISO(4) generators (4). To do this,

first one has to ascertain an appropriate decay behaviour for  $\Lambda^i$ . Since the leading term of  $G_i$  is  $\mathcal{O}(r^{-2})$  odd, the convergence of  $G_i[\Lambda^i]$  requires the following fall-off condition:

$$\Lambda^i = \frac{\lambda^i}{r} + \mathcal{O}(r^{-2}), \tag{11}$$

where  $\lambda^i$  are even functions defined on the asymptotic 2-sphere. It is straightforward to verify that (11) also ensures the differentiability of  $G_i[\Lambda^i]$ . Even after subtracting the surface integral, destroying the differentiability of the Hamiltonian and diffeomorphism constraints (9), it turns out that they are convergent only for translations and not for boosts and rotations. This situation should be cured in such a way that (1) the generators stay functionally differentiable and (2) the well-defined generator for translations which is already available can be recovered up to a pure gauge. As shown in [31,32], ultimately the well-defined forms of the symmetry generators are:

$$\begin{aligned} J[N] &= H[N] - \oint dS_a N \epsilon_{ijk} A_b^i E_j^a E_k^b - G_i[\Lambda_B^i] + \oint dS_a E_a^i \bar{\Lambda}_B^i, \\ J_a[N^a] &= H_a[N^a] - \oint dS_a N^a A_b^i E_i^b - G_i[\Lambda_R^i] + \oint dS_a E_i^a \bar{\Lambda}_R^i, \end{aligned} \tag{12}$$

where  $\Lambda_R^i = \Lambda^i + \bar{\Lambda}_R^i = \Lambda^i - \frac{1}{2} \epsilon_{ijk} \delta_a^j \delta_k^b \beta_b^a$  and  $\Lambda_B^i = \Lambda^i + \bar{\Lambda}_B^i = \Lambda^i + \delta_i^a \beta_a$ . The second term appearing in either expressions in (12) is the surface term subtracted to make the original functionals (9) differentiable. The third term is subtracted to get rid of the source of divergence for boosts and rotations but puts them again in the status of non-differentiability, which is modified by adding the last term. As expected, the volume terms added to the constraints are proportional to the Gauss constraint and thus do not change the translation generator on the constraint surface of the Gauss constraint.

### 2.2. Review of $U(1)^3$ Model for Euclidean Quantum Gravity

In [10], Smolin introduced the weak field limit of the Euclidean gravity  $G_N \rightarrow 0$ , where  $G_N$  is the Newtonian gravitational constant, by expanding the phase space variables  $(A, E)$  as:

$$\begin{aligned} E &= E_0 + G_N E_1 + G_N^2 E_2 + \dots, \\ A &= A_0 + G_N A_1 + G_N^2 A_2 + \dots, \end{aligned} \tag{13}$$

at the level of the action. The resulting theory is not to be confused with standard perturbation theory. More precisely, consider the Hamiltonian for Euclidean gravity:

$$\mathcal{H}[E, A] = \frac{1}{G_N} \int d^3x \left( N^a H_a + NH + \Lambda^i G_i \right). \tag{14}$$

Rescaling the dimensionful quantities in (14) by  $G_N$ , namely the connection  $A_a^i \rightarrow G_N A_a^i$  and the Lagrange multiplier  $\Lambda^i \rightarrow G_N \Lambda^i$ , the Gauss constraint of (9) and (10) become

$$\begin{aligned} G_i &= \mathcal{D}_a E_i^a = \partial_a E_i^a + \epsilon_{ij}{}^k G_N A_a^j E_k^a, \\ F_{ab}^i &= \partial_a A_b^i - \partial_b A_a^i + \epsilon^i{}_{jk} G_N A_a^j A_b^k, \end{aligned} \tag{15}$$

respectively. From (15), it is obvious that the internal gauge symmetry is still  $SU(2)$ . However, in the limit  $G_N \rightarrow 0$ , the second term, which causes the self-interaction of the connection, is switched off. The Poisson bracket of a pair of Gauss constraints then commutes, as one has:

$$\{G_i[\Lambda^i], G_j[\Delta^j]\} \propto G_N, \tag{16}$$

and the symmetry group contracts from  $SU(2)$  to three independent Abelian internal gauge symmetry  $U(1)$  copies—namely  $U(1)^3$ , each of which corresponds to one of the

gauge fields  $A^i$  ( $i = 1, 2, 3$ ). Consequently, the constraints remain first class and have the following simpler forms:

$$\begin{aligned}
 C_j[\Lambda^j] &= \int d^3x \Lambda^j \partial_a E_j^a, \\
 C_a[N^a] &= \int d^3x N^a \left( F_{ab}^j E_j^b - A_a^j \partial_b E_j^b \right), \\
 C[N] &= \int d^3x N F_{ab}^j E_k^a E_l^b \epsilon^{ijkl},
 \end{aligned}
 \tag{17}$$

where  $C_j$ ,  $C_a$  and  $C$  are the Gauss, diffeomorphism and Hamiltonian constraints for the  $U(1)^3$  model respectively and  $F_{ab}^j = \partial_a A_b^j - \partial_b A_a^j$  is the corresponding curvature. The Hamiltonian then reads as:

$$\mathcal{H}[E, A] = \frac{1}{G_N} \int d^3x \left( N^a C_a + NC + \Lambda^i C_i \right),
 \tag{18}$$

and the only non-vanishing Poisson brackets of the pair of the constraints in the algebra will be:

$$\begin{aligned}
 \{C_a[N^a], C_b[M^b]\} &= C_a[\mathcal{L}_{\vec{N}} M^a], \\
 \{C_a[N^a], C[N]\} &= C[\mathcal{L}_{\vec{N}} N], \\
 \{C[N], C[M]\} &= C_a[E^{ia} E_i^b (ND_b M - MD_b N)].
 \end{aligned}
 \tag{19}$$

The algebra, except for the vanishing Poisson bracket of a pair of Gauss constraints,  $C_i$ 's, is isomorphic to the algebra of GR, as can be easily seen.

It transpires that the  $U(1)^3$  contraction of Euclidean  $SU(2)$  GR retains much of the essential structure of Euclidean GR while being technically simpler, and therefore provides an ideal testing ground for Euclidean GR and even for Lorentzian GR, as explained in the introduction. For exciting recent work on the Dirac quantisation approach of this theory, see [11–15].

### 3. Generators of Asymptotic Symmetries for $U(1)^3$ Model

As the  $U(1)^3$  model is a test laboratory for Euclidean GR, it is of interest to know whether the boundary conditions and asymptotic symmetries of these two theories are identical. The question to be answered in this section is whether the  $ISO(4)$  group can be considered as the asymptotic symmetries of the  $U(1)^3$  model. In other words, is there a way to construct well-defined functionals from the constraints (17) while the lapse and the shift include the  $ISO(4)$  generators? Although the model being pursued is not Euclidean GR and there is no physical reason for why the  $ISO(4)$  group is the asymptotic symmetry, since the two theories are structurally very similar as far as their constraint algebras are concerned, it is natural to investigate to what extent the model admits (a subgroup of) the  $ISO(4)$  group as an asymptotic symmetry group. In what follows, we examine the constraints (17) and try to make them well-defined.

#### 3.1. Gauss Constraint

The action of the Gauss constraint on the phase space variables is:

$$\begin{aligned}
 \delta_\Lambda A_a^j &= \{C_i[\Lambda^i], A_a^j\} = -\partial_a \Lambda^j, \\
 \delta_\Lambda E_j^a &= \{C_i[\Lambda^i], E_j^a\} = 0.
 \end{aligned}
 \tag{20}$$

Thus, one sees that:

$$\begin{aligned}
 \delta C_j[\Lambda^j] &= \int d^3x \Lambda^j \partial_a \delta E_j^a = \oint dS_a \Lambda^j \delta E_j^a - \int d^3x (\partial_a \Lambda^j) \delta E_j^a = - \int d^3x (\partial_a \Lambda^j) \delta E_j^a \\
 &= \int d^3x \left[ (\delta_\Lambda A_a^j) \delta E_j^a - (\delta_\Lambda E_j^a) \delta A_a^j \right],
 \end{aligned}
 \tag{21}$$



is functionally differentiable. Here, the surface term has been dropped because  $\delta E_j^a = \mathcal{O}(r^{-1})$  is even and  $\Lambda^j = \mathcal{O}(r^{-1})$  is even. As  $\partial_a E_j^a = \mathcal{O}(r^{-2})$  is odd, the integrand of  $C_j[\Lambda^j]$  is  $\mathcal{O}(r^{-3})$  odd and hence the constraint is also finite.

### 3.2. Scalar Constraint

It is straightforward to see that:

$$\begin{aligned} \delta_N A_c^i(x) &= \{C[N], A_c^i(x)\} = -2N\epsilon_{ijk} F_{ac}^j E_k^a, \\ \delta_N E_i^c(x) &= \{C[N], E_i^c(x)\} = 2\epsilon_{ikl} \partial_b (NE_k^c E_l^b), \end{aligned} \tag{22}$$

Thus, the variation in this constraint is:

$$\begin{aligned} \delta C[N] &= \int d^3x \epsilon_{jkl} N \left( \delta F_{ab}^j E_k^a E_l^b + 2F_{ab}^j E_l^b \delta E_k^a \right) \\ &= \int d^3x \epsilon_{jkl} N \left( (\partial_a \delta A_b^j - \partial_b \delta A_a^j) E_k^a E_l^b + 2F_{ab}^j E_l^b \delta E_k^a \right) \\ &= 2 \int d^3x \epsilon_{jkl} \left( \partial_a (NE_k^a E_l^b \delta A_b^j) - \delta A_b^j \partial_a (NE_k^a E_l^b) + NF_{ab}^j E_l^b \delta E_k^a \right) \\ &= \int d^3x 2\epsilon_{jkl} \left( NF_{ab}^j E_l^b \delta E_k^a - \delta A_b^j \partial_a (NE_k^a E_l^b) \right) + 2 \int dS_a (NE_k^a E_l^b \delta A_b^j) \\ &= \int d^3x \left( \delta A_b^j (\delta_N E_j^b) - (\delta_N A_a^k) \delta E_k^a \right) + 2\delta \oint dS_a \epsilon_{jkl} (NE_k^a E_l^b A_b^j). \end{aligned} \tag{23}$$

We pulled the variation out of the surface integral in (23) because the correction terms are  $\mathcal{O}(r^{-1})$  even for a translation and  $\mathcal{O}(1)$  odd for a boost. Now, we define the new generator as:

$$C'[N] := C[N] - 2 \oint dS_a N \epsilon_{jkl} A_b^j E_k^a E_l^b, \tag{24}$$

which is functionally differentiable. At this step, one is supposed to check whether it is finite.

$$\begin{aligned} C'[N] &= \int d^3x \epsilon_{jkl} \left( F_{ab}^j E_k^a E_l^b N - 2\partial_a (A_b^j E_k^a E_l^b N) \right) \\ &= 2 \int d^3x \epsilon_{jkl} \left( \frac{1}{2} F_{ab}^j E_k^a E_l^b N - E_k^a E_l^b N \partial_a A_b^j - A_b^j E_l^b N \partial_a E_k^a - A_b^j E_k^a N \partial_a E_l^b - A_b^j E_k^a E_l^b \partial_a N \right) \\ &= -2 \int d^3x \epsilon_{jkl} \left( A_b^j E_l^b N \partial_a E_k^a + A_b^j E_k^a N \partial_a E_l^b + A_b^j E_k^a E_l^b \partial_a N \right). \end{aligned} \tag{25}$$

Here, terms of the form  $AEN\partial E$  are  $\mathcal{O}(r^{-4})$  even for a translation and  $\mathcal{O}(r^{-3})$  odd for a boost. Thus, they are convergent. The last term which is of the form  $AEE\partial N$  vanishes for a translation but is divergent for a boost. Therefore,  $C'[N]$  is well-defined for a translation and the source of its divergence for a boost is:

$$\begin{aligned} -2 \int d^3x \epsilon_{jkl} (A_b^j E_k^a E_l^b \beta_a) &= - \int d^3x \frac{1}{r^2} (\beta_a \epsilon_{jkl} g_b^j \delta_k^a \delta_l^b) - \int d^3x \beta_a \epsilon_{jkl} A_b^j \delta_k^a E_l^b + \text{finite} \\ &= - \int d^3x \beta_a \epsilon_{jkl} A_b^j \delta_k^a E_l^b + \text{finite}, \end{aligned} \tag{26}$$

where, in going from the first line to the second one, we used the parity of  $g_b^j$  to drop the linear singularity. Thus, we are left with the logarithmic singularity (26), which is non-vanishing for a boost and also not a linear combination of constraints. Consequently, time translations have a well-defined generator (24), but boosts do not! A more detailed argument for this happens in the Abelian  $U(1)^3$  theory but not in the non-Abelian  $SU(2)$  theory is given in Section 4.

### 3.3. Vector Constraint

The vector constraint acts on the canonical variables as follows:

$$\begin{aligned} \delta_{\vec{N}} A_c^i(x) &= \{C_a[N^a], A_c^i(x)\} = -\mathcal{L}_{\vec{N}} A_c^i, \\ \delta_{\vec{N}} E_i^c(x) &= \{C_a[N^a], E_i^c(x)\} = -\mathcal{L}_{\vec{N}} E_i^c. \end{aligned} \tag{27}$$

Hence, the variation in the constraint is:

$$\begin{aligned} \delta C_a[N^a] &= \int d^3x N^a \left( \delta F_{ab}^j E_j^b + F_{ab}^j \delta E_j^b - \delta A_a^j \partial_b E_j^b - A_a^j \partial_b \delta E_j^b \right) \\ &= \int d^3x N^a \left( \partial_a \delta A_b^j E_j^b - \partial_b \delta A_a^j E_j^b + \partial_a A_b^j \delta E_j^b - \partial_b A_a^j \delta E_j^b - \delta A_a^j \partial_b E_j^b - A_a^j \partial_b \delta E_j^b \right) \\ &= \int d^3x \left( \delta A_a^j \partial_b (N^a E_j^b) - \delta A_b^j \partial_a (N^a E_j^b) + N^a \partial_a A_b^j \delta E_j^b - N^a \partial_b A_a^j \delta E_j^b \right. \\ &\quad \left. - N^a \delta A_a^j \partial_b E_j^b + \partial_b (N^a A_a^j) \delta E_j^b \right) + \oint dS_a (N^a E_j^b \delta A_b^j - N^b E_j^a \delta A_b^j - N^b A_b^j \delta E_j^a) \\ &= \int d^3x \left( \delta A_a^j \left[ E_j^b \partial_b N^a - \partial_b (N^b E_j^a) \right] + \delta E_j^b \left[ N^a \partial_a A_b^j + A_a^j \partial_b N^a \right] \right) \\ &\quad + \oint dS_a (N^a E_j^b \delta A_b^j - N^b E_j^a \delta A_b^j) \\ &= \int d^3x \left( \delta A_a^j \left[ -\mathcal{L}_{\vec{N}} E_j^a \right] + \delta E_j^b \left[ \mathcal{L}_{\vec{N}} A_b^j \right] \right) + \oint dS_a (N^a E_j^b \delta A_b^j - N^b E_j^a \delta A_b^j) \\ &= \int d^3x \left( \delta A_a^j \left[ \delta_{\vec{N}} E_j^a \right] - \delta E_j^b \left[ \delta_{\vec{N}} A_b^j \right] \right) + \delta \oint dS_a (N^a E_j^b A_b^j - N^b E_j^a A_b^j). \end{aligned} \tag{28}$$

Here, the third term of the surface integral in the fourth line is  $\mathcal{O}(1)$  odd for a rotation and  $\mathcal{O}(r^{-1})$  even for a translation and so can be dropped. Furthermore, in the last step one can pull the variation out of the surface integral, since the correction terms are  $\mathcal{O}(r^{-1})$  even for a translation and  $\mathcal{O}(1)$  odd for a rotation.

So the new generator should be defined as:

$$C'_a[N^a] := C_a[N^a] - \oint dS_a (N^a E_j^b - N^b E_j^a) A_b^j, \tag{29}$$

which is functionally differentiable. To check its finiteness, we rewrite (29) as a volume integral:

$$\begin{aligned} C'_a[N^a] &= \int d^3x \left[ N^a F_{ab}^j E_j^b - N^a A_a^j \partial_b E_j^b - \partial_a (N^a E_j^b A_b^j - N^b E_j^a A_b^j) \right] \\ &= \int d^3x \left[ N^a F_{ab}^j E_j^b - N^a A_a^j \partial_b E_j^b - N^a E_j^b F_{ab}^j - \partial_a (N^a E_j^b - N^b E_j^a) A_b^j \right] \\ &= - \int d^3x \left[ N^a A_a^j \partial_b E_j^b + \partial_a (N^a E_j^b - N^b E_j^a) A_b^j \right] \\ &= - \int d^3x A_b^j \left[ E_j^b \partial_a N^a + N^a \partial_a E_j^b - E_j^a \partial_a N^b \right] = - \int d^3x A_b^j \mathcal{L}_{\vec{N}} E_j^b. \end{aligned} \tag{30}$$

In the last line of (30), the term  $AN\partial E$  is  $\mathcal{O}(r^{-4})$  even for a translation and  $\mathcal{O}(r^{-3})$  odd for a rotation, which means it is convergent. The term  $A_b^j E_j^b \partial_a N^a$  vanishes for both translation and rotation because  $\alpha^a$  is a constant and  $\beta_b^a$  is antisymmetric. On the other hand, the other term of the form  $AE\partial N$  is  $\mathcal{O}(r^{-2})$  odd for a rotation and vanishes for a translation. Thus,  $C'_a[N^a]$  is well-defined for a translation and the source of its divergence for a rotation is:



$$\begin{aligned}
 \int d^3x A_b^j E_j^a \beta_a^b &= \int d^3x \beta_a^b A_b^j \left( \delta_j^a + \frac{f_j^a}{r} + \dots \right) \\
 &= \int d^3x \beta_a^b A_b^j \delta_j^a + \int d^3x \beta_a^b A_b^j \frac{f_j^a}{r} + \text{finite} \\
 &= \int d^3x \beta_a^b A_b^j \delta_j^a + \text{finite},
 \end{aligned}
 \tag{31}$$

where the second integral in the second line is convergent since its integrand is  $\mathcal{O}(r^{-3})$  odd. Accordingly, the obstruction term (31) vanishes for spatial translations but not for rotations and it is also not a linear combination of constraints. Hence, spatial translations have a well-defined generator (29), but rotations do not!

#### 4. Comparison with the SU(2) Case

In this section, we scrutinise the situation and exhibit what exactly causes the difference between the  $U(1)^3$  and  $SU(2)$  theories—i.e., why the former does not admit generators for boosts and rotations but the latter does. First, we split  $F_{ab}^i$  and  $G_i$  into its Abelian and non-Abelian parts—i.e.,  $F_{ab}^i = F_{ab}^{i+} + F_{ab}^{i-}$  and  $G_i = G_i^+ + G_i^-$  where  $F_{ab}^{i+} = \partial_a A_b^i - \partial_b A_a^i$ ,  $F_{ab}^{i-} = \epsilon_{ijk} A_a^j A_b^k$ ,  $G_i^+ = \partial_a E_i^a$  and  $G_i^- = \epsilon_{ijk} A_a^j E_k^a$ . Accordingly, the Hamiltonian and diffeomorphism constraints also split into two terms corresponding to the plus and minus pieces of  $F_{ab}^i$  and  $G_i$ , respectively—namely  $H[N] = H^+[N] + H^-[N]$  and  $H_a[N^a] = H_a^+[N^a] + H_a^-[N^a]$ , where:

$$H^+[N] = \int d^3x N \epsilon_{jkl} F_{ab}^{j+} E_k^a E_l^b, \quad H^-[N] = \int d^3x N \epsilon_{jkl} F_{ab}^{j-} E_k^a E_l^b \tag{32}$$

$$H_a^+[N^a] = \int d^3x N^a (F_{ab}^{j+} E_j^b - A_a^j G_j^+), \quad H_a^-[N^a] = \int d^3x N^a (F_{ab}^{j-} E_j^b - A_a^j G_j^-) = 0. \tag{33}$$

Due to the boundary conditions,  $F_{ab}^{i-} = \mathcal{O}(r^{-4})$  is even and  $G_i^- = \mathcal{O}(r^{-2})$  is odd. Hence, the integrand of  $H^-[N]$  is  $\mathcal{O}(r^{-4})$  even for a translation and  $\mathcal{O}(r^{-3})$  odd for a boost. Therefore, the minus parts of these constraints are already finite. We show that  $H^-[N]$  is also functionally differentiable. Its action on the canonical variables is:

$$\begin{aligned}
 \delta_{N^-} A_c^l &:= \{H^-[N], A_c^l(x)\} = -2N \epsilon_{ilk} \epsilon_{imn} A_c^m A_b^n E_k^b, \\
 \delta_{N^-} E_l^c &:= \{H^-[N], E_l^c(x)\} = 2N \epsilon_{ijk} \epsilon_{iln} E_j^c E_k^b A_b^n.
 \end{aligned}
 \tag{34}$$

Using (34), one observes:

$$\begin{aligned}
 \delta H^-[N] &= \int d^3x N \epsilon_{jkl} (\delta F_{ab}^{j-} E_k^a E_l^b + F_{ab}^{j-} \delta E_k^a E_l^b + F_{ab}^{j-} E_k^a \delta E_l^b) \\
 &= \int d^3x (\delta A_c^l (2N \epsilon_{ijk} \epsilon_{iln} E_j^c E_k^b A_b^n) + \delta E_l^c (2N \epsilon_{ilk} \epsilon_{imn} A_c^m A_b^n E_k^b)) \\
 &= \int d^3x (\delta A_c^l (\delta_{N^-} E_l^c) - \delta E_l^c (\delta_{N^-} A_c^l)),
 \end{aligned}
 \tag{35}$$

which means  $H^-[N]$  is differentiable. Consequently, what needs to be modified are  $H^+[N] = C[N]$  and  $H_a^+[N^a] = C_a[N^a]$ , thus all failure to be well-defined is rooted in the  $U(1)^3$  part of the Hamiltonian and diffeomorphism constraints of the  $SU(2)$  theory. Thus, as far as finding the source of divergence and non-differentiability is concerned, calculations are the same in both theories. This brings us back to (26) and (31) for boosts and rotations, respectively.

For a boost, the source of divergence can be expressed as:

$$\begin{aligned}
 - \int d^3x \beta_a \epsilon_{jkl} A_b^j \delta_k^a E_l^b &= \int d^3x (\beta_a \delta_k^a) G_k^- = \int d^3x \bar{\Lambda}_B^k (G_k - G_k^+) \\
 &= G_k[\bar{\Lambda}_B^k] - \int d^3x \partial_b (\bar{\Lambda}_B^k E_k^b) \\
 &= G_k[\bar{\Lambda}_B^k] - \oint dS_b \bar{\Lambda}_B^k E_k^b,
 \end{aligned}
 \tag{36}$$

where we used  $\partial_b \bar{\Lambda}_B^k = 0$ ; see (12). As expected, at the end the volume term in (36) is proportional to  $G_k$ . Therefore, both terms can be subtracted from the Hamiltonian constraint, resulting in  $J[N]$  in (12), which is the final well-defined generator. Thus, we have extracted the source of the subtlety: in the  $U(1)^3$  case, the absence of the non-Abelian piece  $G_k^-$  in the Gauss constraint is responsible for excluding a well-defined boost generator.

We proceed to (31) and write:

$$\begin{aligned}
 \int d^3x A_b^j \beta_a^b \delta_j^a &= \int d^3x A_b^j (\epsilon_{ijk} \delta_k^b \bar{\Lambda}_R^i) \\
 &= \int d^3x A_b^j (\epsilon_{ijk} E_k^b \bar{\Lambda}_R^i) - \int d^3x A_b^j (\epsilon_{ijk} \frac{f_k^b}{r} \bar{\Lambda}_R^i) + \text{finite} \\
 &= \int d^3x \bar{\Lambda}_R^i G_i^- + \text{finite} \\
 &= \int d^3x \bar{\Lambda}_R^i (G_i - G_i^+) + \text{finite} \\
 &= G_i[\bar{\Lambda}_R^i] - \oint dS_a \bar{\Lambda}_R^i E_i^a + \text{finite},
 \end{aligned}
 \tag{37}$$

where  $\partial_a \bar{\Lambda}_R^i = 0$  has been used and the second integral in the second line is dropped, since it is  $\mathcal{O}(r^{-3})$  odd. Again, the volume term is proportional to the Gauss constraint, as desired. It is straightforward to investigate that  $J_a[N^a]$  in (12), which subtracts (37), is the well-defined generator for the spatial translations and rotations. Hence again, just like in (36), the presence of  $G_i^-$  (which is zero in the  $U(1)^3$  case) plays a crucial role in obtaining the rotation generator.

One can rephrase the above technical reasoning in intuitive terms: The  $SU(2)$  Gauss constraint generates rotations on the internal tangent space associated with the internal indices  $j, k, l, \dots$  while asymptotic rotations act on the spatial tangent space corresponding to the indices  $a, b, c, \dots$ . Due to the boundary conditions  $E_j^a \propto \delta_j^a$ , these tangent spaces get identified in leading order, so it is not surprising that one can “undo” an unwanted asymptotic rotation by an internal one. This cannot work in the  $U(1)^3$  case, because the Gauss constraint does not generate internal rotations.

### 5. Conclusions

Due to the fact that the  $G_N \rightarrow 0$  limit of Euclidean general relativity is an interesting toy model for Lorentzian GR, this paper is devoted to studying its boundary conditions, yielding a well-defined symplectic structure and finite and integrable charges associated with the asymptotic symmetries. We have demonstrated that in the  $U(1)^3$  model, the boundary terms appearing in the constraint volume term variation are variations in boundary terms and therefore all constraints can be improved to differentiable functionals. However, these functionals are not finite for boosts and rotations and we have shown that the reason for this is precisely the lack of the non-Abelian term, which is  $\epsilon_{ijk} A_a^j E_k^a$  in the Gauss constraint of this model as compared to that of general relativity.

Although the study of surface charges and asymptotic symmetries is interesting in its own right for any classical field theory, our main motivation is the quantum theory of that model. In the Abelian case, the reduced phase space approach turns out to be technically simple enough even without reference matter [33] because the constraints are linear polynomials in one of the canonical variables (the connection), which thus can give

us new insights into quantum theory from a different angle than the Dirac quantisation approach does. To compute the corresponding physical Hamiltonian [17,18], generating the time evolution of gauge invariant observables requires a careful treatment of boundary conditions, as one has to invert partial differential operators (Green functions).

**Author Contributions:** Conceptualization, S.B., H.S. and T.T.; methodology, S.B., H.S. and T.T.; validation, S.B., H.S. and T.T.; formal analysis, S.B., H.S. and T.T.; investigation, S.B., H.S. and T.T.; writing—original draft preparation, S.B., H.S. and T.T.; writing—review and editing, S.B., H.S. and T.T. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Data sharing not applicable.

**Acknowledgments:** S.B. thanks the Ministry of Science, Research and Technology of Iran and FAU Erlangen-Nürnberg for financial support.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

- Ashtekar, A. New Hamiltonian formulation of general relativity. *Phys. Rev. D* **1987**, *36*, 1587–1603. [[CrossRef](#)]
- Ashtekar, A. New Variables for Classical and Quantum Gravity. *Phys. Rev. Lett.* **1986**, *57*, 2244–2247. [[CrossRef](#)] [[PubMed](#)]
- Barbero, J. Real Ashtekar variables for Lorentzian signature space-times. *Phys. Rev. D* **1995**, *51*, 5507–5510. [[CrossRef](#)] [[PubMed](#)]
- Thiemann, T. Quantum Spin Dynamics (QSD). *Class. Quant. Grav.* **1998**, *15*, 839–873. [[CrossRef](#)]
- Ashtekar, A.; Pullin, J. (Eds.) *Loop Quantum Gravity—The First 30 Years*; World Scientific: Singapore, 2017.
- Pullin, J.; Gambini, R. *A First Course in Loop Quantum Gravity*; Oxford University Press: Oxford, UK, 2011.
- Rovelli, C. *Quantum Gravity*; Cambridge University Press: Cambridge, UK, 2008.
- Thiemann, T. *Modern Canonical Quantum General Relativity*; Cambridge University Press: Cambridge, UK, 2007.
- Hojman, S.A.; Kuchar, K.; Teitelboim, C. Geometrodynamics Regained. *Ann. Phys.* **1976**, *96*, 88–135. [[CrossRef](#)]
- Smolin, L. The  $G_{Newton} \rightarrow 0$  limit of Euclidean quantum gravity. *Class. Quant. Grav.* **1992**, *9*, 883–894. [[CrossRef](#)]
- Tomlin, C.; Varadarajan, M. Towards an Anomaly-Free Quantum Dynamics for a Weak Coupling Limit of Euclidean Gravity. *Phys. Rev. D* **2013**, *87*, 044039. [[CrossRef](#)]
- Laddha, A. Hamiltonian Constraint in Euclidean LQG revisited: First hints of off-shell Closure. *arXiv* **2014**, arXiv:1401.0931.
- Henderson, A.; Laddha, A.; Tomlin, C. Constraint algebra in loop quantum gravity reloaded. II. Toy model of an Abelian gauge theory: Spatial diffeomorphisms. *Phys. Rev. D* **2013**, *88*, 044029. [[CrossRef](#)]
- Henderson, A.; Laddha, A.; Tomlin, C. Constraint algebra in loop quantum gravity reloaded. I. Toy model of a  $U(1)^3$  gauge theory. *Phys. Rev. D* **2013**, *88*, 044028. [[CrossRef](#)]
- Varadarajan, M. On quantum propagation in Smolin’s weak coupling limit of 4d Euclidean Gravity. *Phys. Rev. D* **2019**, *100*, 066018. [[CrossRef](#)]
- Dirac, P.A.M. Lectures on quantum mechanics. *J. Math. Phys.* **1960**, *1*, 434.
- Bakhoda, S.; Thiemann, T. Covariant Origin of the  $U(1)^3$  model for Euclidean Quantum Gravity. *arXiv* **2020**, arXiv:2011.00031.
- Bakhoda, S.; Thiemann, T. Reduced Phase Space Approach to the  $U(1)^3$  model for Euclidean Quantum Gravity. *arXiv* **2020**, arXiv:2010.16351.
- Regge, T.; Teitelboim, C. Role of Surface Integrals in the Hamiltonian Formulation of General Relativity. *Ann. Phys.* **1974**, *88*, 286. [[CrossRef](#)]
- Sachs, R. Asymptotic symmetries in gravitational theory. *Phys. Rev.* **1962**, *128*, 2851. [[CrossRef](#)]
- Newman, E.T.; Unti, T.W. Behavior of Asymptotically Flat Empty Spaces. *J. Math. Phys.* **1962**, *3*, 891. [[CrossRef](#)]
- Penrose, R. Asymptotic properties of fields and space-times. *Phys. Rev. Lett.* **1963**, *10*, 66. [[CrossRef](#)]
- Penrose, R. Zero rest mass fields including gravitation: Asymptotic behavior. *Proc. R. Soc. Lond. A* **1965**, *284*, 159.
- Barnich, G.; Troessaert, C. Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited. *Phys. Rev. Lett.* **2010**, *105*, 111103. [[CrossRef](#)]
- Campiglia, M.; Laddha, A. Asymptotic symmetries and subleading soft graviton theorem. *Phys. Rev. D* **2014**, *90*, 124028. [[CrossRef](#)]
- Campiglia, M.; Laddha, A. New symmetries for the Gravitational S-matrix. *JHEP* **2015**, *1504*, 076. [[CrossRef](#)]
- Corvino, J.; Schoen, R.M. On the asymptotics for the vacuum Einstein constraint equations. *J. Differ. Geom.* **2006**, *73*, 185. [[CrossRef](#)]
- Huang, L.H. Solutions of special asymptotics to the Einstein constraint equations. *Class. Quant. Grav.* **2010**, *27*, 245002. [[CrossRef](#)]
- Henneaux, M.; Troessaert, C. BMS Group at Spatial Infinity: the Hamiltonian (ADM) approach. *JHEP* **2018**, *3*, 147. [[CrossRef](#)]

- 
30. Beig, R.; óMurchadha, N. The Poincaré group as the symmetry group of canonical general relativity. *Ann. Phys.* **1987**, *174*, 463. [[CrossRef](#)]
  31. Thiemann, T. Generalized boundary conditions for general relativity for the asymptotically flat case in terms of Ashtekar's variables. *Class. Quant. Grav.* **1995**, *12*, 181. [[CrossRef](#)]
  32. Campiglia, M. Note on the phase space of asymptotically flat gravity in Ashtekar-Barbero variables. *Class. Quant. Grav.* **2015**, *32*, 14. [[CrossRef](#)]
  33. Giesel, K.; Thiemann, T. Scalar Material Reference Systems and Loop Quantum Gravity. *Class. Quant. Grav.* **2015**, *32*, 135015. [[CrossRef](#)]