Article

Painlevé–Gullstrand form of the Lense–Thirring Spacetime

Joshua Baines [iD], Thomas Berry [iD] and Alex Simpson [iD] and Matt Visser *[iD]

School of Mathematics and Statistics, Victoria University of Wellington, P.O. Box 600,
Wellington 6140, New Zealand; joshua.baines@sms.vuw.ac.nz (J.B.); thomas.berry@sms.vuw.ac.nz (T.B.);
alex.simpson@sms.vuw.ac.nz (A.S.)
* Correspondence: matt.visser@sms.vuw.ac.nz

Abstract: The standard Lense–Thirring metric is a century-old slow-rotation large-distance *approximation* to the gravitational field outside a rotating massive body, depending only on the total mass and angular momentum of the source. Although it is not an *exact* solution to the vacuum Einstein equations, asymptotically the Lense–Thirring metric approaches the Kerr metric at large distances. Herein we shall discuss a specific *variant* of the standard Lense–Thirring metric, carefully chosen for simplicity, clarity, and various forms of improved mathematical and physical behaviour, (to be more carefully defined in the body of the article). We shall see that this Lense–Thirring variant can be viewed as arising from the linearization of a suitably chosen tetrad representing the Kerr spacetime. In particular, we shall construct an explicit unit-lapse Painlevé–Gullstrand variant of the Lense–Thirring spacetime, one that has flat spatial slices, a very simple and physically intuitive tetrad, and extremely simple curvature tensors. We shall verify that this variant of the Lense–Thirring spacetime is Petrov type I, (so it is *not* algebraically special), but nevertheless possesses some very straightforward timelike geodesics, (the "rain" geodesics). We shall also discuss on-axis and equatorial geodesics, ISCOs (innermost stable circular orbits) and circular photon orbits. Finally, we wrap up by discussing some astrophysically relevant estimates, and analyze what happens if we extrapolate down to small values of *r*; verifying that for sufficiently slow rotation we explicitly recover slowly rotating Schwarzschild geometry. This Lense–Thirring variant can be viewed, in its own right, as a "black hole mimic", of direct interest to the observational astronomy community.

Keywords: general relativity; rotation; Kerr spacetime; Lense–Thirring spacetime

1. Introduction

Only two years after the discovery of the original Schwarzschild solution in 1916 [1], in 1918 Lense and Thirring found an approximate solution to the vacuum Einstein equations at large distances from a stationary isolated body of mass *m* and angular momentum *J* [2]. In suitable coordinates, at asymptotically large distances one has [2–10]:

$$
\begin{aligned}
ds^2 \;=\; & -\left[1 - \frac{2m}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right] dt^2 - \left[\frac{4J\sin^2\theta}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right] d\phi\, dt \\
& + \left[1 + \frac{2m}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right]\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right].
\end{aligned}
\tag{1}
$$

Here the sign conventions are compatible with MTW [5] (33.6), and Hartle [7] (14.22) (trying to explicitly calculate approximate curvature components for this approximate metric is quite slow, and the results are quite horrid). It took another 45 years before Roy Kerr found the corresponding *exact* solution in 1963 [11,12]. Some books explicitly focussed on the Kerr spacetime include [10,13,14].

Nevertheless the Lense–Thirring metric continues to be of interest for two main reasons: (1) Lense–Thirring is *much* easier to work with than the full Kerr solution; and (2) For a real rotating planet or star, generically possessing non-trivial mass multipole moments, the vacuum solution outside the surface is not *exactly* Kerr; it is only asymptotically Kerr [9] (there is no Birkhoff theorem for rotating bodies in 3 + 1 dimensions [15–19]). Consequently, the only region where one should trust the Kerr solution (as applied to a real rotating star or planet) is in the asymptotic regime, where in any case it reduces to the Lense–Thirring metric (note in particular that Chandrasekhar's book [13] does not include any weak-field analysis, and is focussed primarily on the exact Kerr spacetime. See also relevant observational details in references [20–22]).

Below we shall, by suitably adjusting the sub-dominant $\mathcal{O}(r^{-2})$ terms, recast a *variant* of the standard Lense–Thirring metric of Equation (1) into Painlevé–Gullstrand form—in this form of the metric (up to coordinate transformations) one has

$$ds^2 = -dt^2 + \delta_{ij}(dx^i - v^i dt)(dx^j - v^j dt). \tag{2}$$

That is, the constant-t spatial 3-slices of the metric are all flat, and the lapse function is unity ($g^{tt} = -1$). See the early references [23–25], and more recently [26–30] (note that the vector v^i, representing the "flow" of space, is *minus* the shift vector in the ADM formalism). Two of the virtues of putting the metric into Painlevé–Gullstrand form is that mathematically it is particularly easy to work with, and physically it is very easy to interpret—in particular, the analogue spacetimes built from excitations in moving fluids are typically (conformally) of Painlevé–Gullstrand form [31–42], and so give a very concrete and physically intuitive visualization of such spacetimes.

2. Variants on the Theme of the Lense–Thirring Metric

Let us now take the original Lense–Thirring metric (1) and seek to modify and simplify it in various ways, while retaining the good features of the asymptotic large-distance behaviour.

- First, we note that at $J = 0$, for a non-rotating source we do have the Birkhoff theorem, so it makes sense to consider the modified metric

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{2m}{r}\right) dt^2 - \left[\frac{4J \sin^2 \theta}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right] d\phi\, dt \\
&\quad + \frac{dr^2}{1 - 2m/r} + r^2(d\theta^2 + \sin^2 \theta\, d\phi^2).
\end{aligned}
\tag{3}
$$

  This modified metric asymptotically approaches standard Lense–Thirring (1) at large distances, but has the very strong advantage that for $J = 0$ it is an *exact* solution of the vacuum Einstein equations (trying to explicitly calculate approximate curvature components for this approximate metric is slightly faster, and the results are somewhat less horrid).

- Second, "complete the square". Consider the modified metric

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - 2m/r} \\
&\quad + r^2\left(d\theta^2 + \sin^2 \theta \left(d\phi - \left[\frac{2J}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right)\right] dt\right)^2\right).
\end{aligned}
\tag{4}
$$

  This modified metric again asymptotically approaches standard Lense–Thirring (1) at large distances, but now has the *two* very strong advantages that (i) for $J = 0$ it is an exact solution of the vacuum Einstein equations *and* (ii) that the azimuthal dependence is now in partial Painlevé–Gullstrand form: $g_{\phi\phi}(d\phi - v^\phi\, dt)^2 = g_{\phi\phi}(d\phi - \omega dt)^2$. See the early references [23–25], and more recently references [26–30]. We shall soon see

that "completing the square" in the Lense–Thirring metric is equivalent to linearizing a suitably chosen tetrad in the Kerr spacetime (trying to explicitly calculate approximate curvature components for this approximate metric is again slightly faster, and the results are again somewhat less horrid).

- Third, put the r–t plane into standard Painlevé–Gullstrand form [23–30] (we note that the shift function is $v^r = -\sqrt{2m/r}$ for a Schwarzschild black hole). We then have the modified metric

$$
\begin{aligned}
ds^2 \;=\; & -dt^2 + \left(dr + \sqrt{2m/r}\, dt \right)^2 \\
& + r^2 \left(d\theta^2 + \sin^2 \theta \left(d\phi - \left[\frac{2J}{r^3} + \mathcal{O}\!\left(\frac{1}{r^4} \right) \right] dt \right)^2 \right).
\end{aligned}
\tag{5}
$$

This modified metric again asymptotically approaches standard Lense–Thirring (1) at large distances, but has the *three* very strong advantages that (i) for $J = 0$ it is an exact solution of the vacuum Einstein equations, (ii) that the azimuthal dependence is now in partial Painlevé–Gullstrand form, with $g_{\phi\phi}(d\phi - v^\phi\, dt)^2 = g_{\phi\phi}(d\phi - \omega dt)^2$, *and* (iii) that all the spatial dependence is in exact Painlevé–Gullstrand type form, in the sense that the constant-t spatial 3-slices are now flat, and so very easy to interpret (trying to explicitly calculate approximate curvature components for this approximate metric is again slightly faster, and the results are again somewhat less horrid).

- Fourth, drop the $\mathcal{O}(1/r^4)$ terms in the ϕ dependence. That is, consider the specific and fully explicit metric:

$$
ds^2 \;=\; -dt^2 + \left(dr + \sqrt{2m/r}\, dt \right)^2 + r^2 \left(d\theta^2 + \sin^2 \theta \left(d\phi - \frac{2J}{r^3} dt \right)^2 \right).
\tag{6}
$$

By construction, for $J = 0$ this is the Painlevé–Gullstrand version of the Schwarzschild metric [23–28]. By construction, at large distances this asymptotically approaches the "standard" form of Lense–Thirring as given in Equation (1), and so it also asymptotically approaches Kerr. By construction even for $J \neq 0$ this metric is in exactly Painlevé–Gullstrand form (in particular, with flat spatial 3-slices, and as we shall soon see, unit lapse, and easily constructed timelike geodesics). These observations make this specific form (6) of the Lense–Thirring spacetime particularly interesting (both mathematically and physically) and worth further investigation (explicitly calculating curvature components for this specific metric is now very much faster, and as we shall soon see, the results are quite tractable).

We emphasize that the five spacetimes represented by these five metrics, (1)–(3)–(4)–(5)–(6), are all physically different from each other. They may have the same asymptotic limit at large distances, but differ in many crucial technical details. In particular, as we shall soon see, the tetrads, curvature components, and the analysis of geodesics is much easier for the fully explicit Painlevé–Gullstrand form of the metric (6) than it is for any of the (1)–(3)–(4)–(5) variants.

Furthermore both the mathematics and the physics of our Painleve–Gullstrand Lense–Thirring variant (6) is much cleaner than it is for any of the (1)–(3)–(4)–(5) variants. Among other issues, we should emphasize that our Painleve–Gullstrand Lense–Thirring variant (6) can be viewed as a "black hole mimic", so it is of direct astrophysical interest to observational astronomers. The physics point here is that since Lense–Thirring is asymptotically Kerr, it will pass all the usual weak-field observational tests of general relativity; but with potentially different strong-field physics, the deep-field observations could differ. In this regard it is very much like many other black hole mimics currently under theoretical development—as long as one keeps the weak-field physics compatible with standard general relativity one will have a viable candidate for a black hole mimic.

3. Metric Components, Tetrad, and Curvature

We shall now analyze the metric, a particularly natural choice of tetrad (vierbein), the curvature invariants, and the orthonormal tetrad components of the curvature tensors for our Painlevé–Gullstrand variant (6) of the Lense–Thirring spacetime.

3.1. Metric Components

From (6) it is easy to read off the metric components

$$
g_{ab} = \left[
\begin{array}{c|ccc}
-1 + \frac{2m}{r} + \frac{4J^2 \sin^2\theta}{r^4} & \sqrt{\frac{2m}{r}} & 0 & -\frac{2J\sin^2\theta}{r} \\
\hline
\sqrt{\frac{2m}{r}} & 1 & 0 & 0 \\
0 & 0 & r^2 & 0 \\
-\frac{2J\sin^2\theta}{r} & 0 & 0 & r^2\sin^2\theta
\end{array}
\right]_{ab}.
\tag{7}
$$

Thence one can easily verify that for the inverse metric

$$
g^{ab} = \left[
\begin{array}{c|ccc}
-1 & \sqrt{\frac{2m}{r}} & 0 & -\frac{2J}{r^3} \\
\hline
\sqrt{\frac{2m}{r}} & 1 - \frac{2m}{r} & 0 & \sqrt{\frac{2m}{r}}\frac{2J}{r^3} \\
0 & 0 & \frac{1}{r^2} & 0 \\
-\frac{2J}{r^3} & \sqrt{\frac{2m}{r}}\frac{2J}{r^3} & 0 & \frac{1}{r^2\sin^2\theta} - \frac{4J^2}{r^6}
\end{array}
\right]^{ab}.
\tag{8}
$$

Note particularly that $g^{tt} = -1$, so that the lapse function is unity; this fact will be particularly useful when we come to analyzing the geodesics.

3.2. Tetrads

There are good physics reasons (for instance, the ultimate inclusion of fermions by formulating the general-relativistic Dirac equation) for viewing the tetrad formalism as being more fundamental than the metric formalism. Let us see what we can do along these lines.

Denote tetrad labels by an overhat: That is set $\hat{a}, \hat{b} \in \{\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi}\}$. Furthermore set $\eta_{\hat{a}\hat{b}} = \text{diag}(-1, 1, 1, 1)$. To find a suitable covariant tetrad (co-tetrad) $e^{\hat{a}}{}_a$, we wish to find a particular solution of $g_{ab} = \eta_{\hat{a}\hat{b}}\, e^{\hat{a}}{}_a\, e^{\hat{b}}{}_b$. Then working from the line-element (6) an obvious and straightforward choice for the co-tetrad is

$$
\begin{aligned}
e^{\hat{t}}{}_a &= (1;\, 0, 0, 0); & e^{\hat{r}}{}_a &= \left(\sqrt{\frac{2m}{r}};\, 1, 0, 0\right); \\[2mm]
e^{\hat{\theta}}{}_a &= r(0;\, 0, 1, 0); & e^{\hat{\phi}}{}_a &= r\sin\theta\left(-\frac{2J}{r^3};\, 0, 0, 1\right).
\end{aligned}
\tag{9}
$$

This choice of co-tetrad is of course not unique (the underlying metric is unaffected by any arbitrary local Lorentz transformation $L^{\hat{a}}{}_{\hat{b}}$ on the ortho-normal tetrad/co-tetrad indices). However this co-tetrad is particularly well-adapted to the coordinate system used in (6). Once the co-tetrad has been chosen, the contravariant tetrad (usually just called the tetrad) is then uniquely defined by $e_{\hat{a}}{}^a = \eta_{\hat{a}\hat{b}}\, e^{\hat{b}}{}_b\, g^{ba}$.

The tetrad therefore will (as expected) satisfy

$$
\begin{aligned}
\eta^{\hat{a}\hat{b}}\, e_{\hat{a}}{}^a\, e_{\hat{b}}{}^b &= \eta^{\hat{a}\hat{b}}\, (\eta_{\hat{a}\hat{c}}\, e^{\hat{c}}{}_c\, g^{ca})(\eta_{\hat{b}\hat{d}}\, e^{\hat{d}}{}_d\, g^{da}) = \eta_{\hat{c}\hat{d}}\, (e^{\hat{c}}{}_c\, g^{ca})(e^{\hat{d}}{}_d\, g^{da}) \\
&= g_{cd}\, g^{ca}\, g^{db} = g^{ab}.
\end{aligned}
\tag{10}
$$

A brief computation, or comparison with the inverse metric (8), leads to

$$
e_{\hat{t}}{}^{a} = \left(1; -\sqrt{\frac{2m}{r}}, 0, \frac{2J}{r^3}\right); \qquad e_{\hat{r}}{}^{a} = (0; 1, 0, 0);
$$

$$
e_{\hat{\theta}}{}^{a} = \frac{1}{r}\,(0; 0, 1, 0); \qquad e_{\hat{\phi}}{}^{a} = \frac{1}{r\sin\theta}(0; 0, 0, 1). \tag{11}
$$

Note that the last 3 of these tetrad vectors are exactly those that would be expected for flat Euclidean 3-space, and that for this choice of tetrad all of the nontrivial physics is tied up in the timelike vector $e_{\hat{t}}{}^{a}$. For our purposes the tetrad and co-tetrad are most usefully employed in converting tensor coordinate components into an orthonormal basis. Observe that the angular momentum J shows up linearly in this tetrad; and that the tetrad above is manifestly a linear perturbation of the corresponding tetrad for the Painlevé–Gullstrand version of Schwarzschild spacetime—it is in this precise sense that our Painleve–Gullstrand variant of the Lense–Thirring spacetime can be thought of as a linearization, at the level of the tetrad, of the Kerr spacetime. Furthermore, while this tetrad works very nicely for our (6) Painlevé–Gullstrand variant of Lense–Thirring, it will at best be *much messier* for any and all of the (1)–(3)–(4)–(5) variants of Lense–Thirring spacetime.

3.3. Curvature Invariants

While the specific Lense–Thirring spacetime variant we are interested in, that of Equation (6), is not (exactly) Ricci-flat, it is easy to calculate the Ricci scalar and Ricci invariant and verify that they are particularly simple and that asymptotically they are suitably small. We have

$$
R = \frac{18J^2 \sin^2\theta}{r^6}; \tag{12}
$$

and

$$
R_{ab}\,R^{ab} = 3R^2. \tag{13}
$$

Note that all the right things happen as $J \to 0$. Note that all the right things happen as $r \to \infty$. Ultimately, it is the observation that these quantities fall-off very rapidly with distance that justifies the assertion that this is a *useful* "approximate" solution to the vacuum Einstein equations.

A more subtle calculation is to evaluate the Weyl invariant:

$$
\begin{aligned}
C_{abcd}\,C^{abcd} &= \frac{48m^2}{r^6} - \frac{144J^2(2\cos^2\theta + 1)}{r^8} + \frac{864mJ^2 \sin^2\theta}{r^9} + \frac{1728J^4 \sin^4\theta}{r^{12}} \\
&= \frac{48m^2}{r^6} - \frac{144J^2(3 - 2\sin^2\theta)}{r^8} + \frac{48m}{r^3}\,R + \frac{16}{3}\,R^2 \\
&= \frac{48m^2}{r^6} - \frac{432J^2}{r^8} + \frac{16}{r^2}\left(1 + \frac{3m}{r}\right)R + \frac{16}{3}\,R^2.
\end{aligned} \tag{14}
$$

Note that this is exactly what you would expect for Schwarzschild, $48m^2/r^6$, plus a rapid fall-off angular-momentum-dependent term, $\mathcal{O}(J^2/r^8)$. Similarly for the Kretschmann scalar we have

$$
R_{abcd}R^{abcd} = C_{abcd}\,C^{abcd} + \frac{1728J^4 \sin^4\theta}{r^{12}} = C_{abcd}\,C^{abcd} + \frac{17}{3}\,R^2. \tag{15}
$$

3.4. Curvature Tensors

Calculating the Ricci and Einstein tensors is (in the tetrad basis) straightforward.

- Taking $R_{\hat{a}\hat{b}} = e_{\hat{a}}{}^{a}\, e_{\hat{b}}{}^{b}\, R_{ab}$, in terms of the Ricci scalar R we have:

$$
R_{\hat{a}\hat{b}} = R \left[\begin{array}{c|ccc} -1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]. \tag{16}
$$

  Notice the perhaps somewhat unexpected pattern of zeros and minus signs.
- Taking $G_{\hat{a}\hat{b}} = e_{\hat{a}}{}^{a}\, e_{\hat{b}}{}^{b}\, G_{ab}$, in terms of the Ricci scalar R we have:

$$
G_{\hat{a}\hat{b}} = \frac{R}{2} \left[\begin{array}{c|ccc} -1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{array}\right]. \tag{17}
$$

  Notice the perhaps somewhat unexpected pattern of zeros and minus signs.
- Algebraically, this implies that the Ricci and Einstein tensors are type I in the Hawking–Ellis (Segre–Plebanski) classification [43,44].

In contrast, calculating the Weyl and Riemann tensors is somewhat tedious.

- Take $C_{\hat{a}\hat{b}\hat{c}\hat{d}} = e_{\hat{a}}{}^{a}\, e_{\hat{b}}{}^{b}\, e_{\hat{c}}{}^{c}\, e_{\hat{d}}{}^{d}\, C_{abcd}$.
  - The terms quadratic in J are:

$$
\begin{aligned}
C_{\hat{t}\hat{r}\hat{t}\hat{r}} &= -2C_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = -2C_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = 2C_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = 2C_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -C_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} \\
&= -\frac{2m}{r^3} - \frac{12J^2 \sin^2\theta}{r^6} = -\frac{2m}{r^3} - \frac{2}{3}R.
\end{aligned} \tag{18}
$$

  - There are also several terms linear in J:

$$
\begin{aligned}
\frac{1}{2}C_{\hat{t}\hat{r}\hat{\theta}\hat{\phi}} &= C_{\hat{t}\hat{\theta}\hat{r}\hat{\phi}} = -C_{\hat{t}\hat{\phi}\hat{r}\hat{\theta}} = \frac{3J\cos\theta}{r^4}; \\
C_{\hat{t}\hat{r}\hat{r}\hat{\phi}} &= -C_{\hat{t}\hat{\theta}\hat{\theta}\hat{\phi}} = -\frac{3J\sin\theta}{r^4}; \\
C_{\hat{t}\hat{r}\hat{t}\hat{\phi}} &= -C_{\hat{r}\hat{\theta}\hat{\theta}\hat{\phi}} = \frac{3J\sin\theta\sqrt{2m/r}}{r^4}.
\end{aligned} \tag{19}
$$

- Take $R_{\hat{a}\hat{b}\hat{c}\hat{d}} = e_{\hat{a}}{}^{a}\, e_{\hat{b}}{}^{b}\, e_{\hat{c}}{}^{c}\, e_{\hat{d}}{}^{d}\, R_{abcd}$.
  - The terms quadratic in J are:

$$
\begin{aligned}
R_{\hat{t}\hat{r}\hat{t}\hat{r}} &= -\frac{2m}{r^3} - \frac{27J^2 \sin^2\theta}{r^6} = -\frac{2m}{r^3} - \frac{3}{2}R. \\
R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} &= -R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = \frac{m}{r^3} + \frac{9J^2 \sin^2\theta}{r^6} = \frac{m}{r^3} + \frac{1}{2}R.
\end{aligned} \tag{20}
$$

  - There are also several terms linear in J:

$$
\begin{aligned}
R_{\hat{t}\hat{r}\hat{\theta}\hat{\phi}} &= 2R_{\hat{t}\hat{\theta}\hat{r}\hat{\phi}} = -2R_{\hat{t}\hat{\phi}\hat{r}\hat{\theta}} = \frac{6J\cos\theta}{r^4}; \\
R_{\hat{t}\hat{r}\hat{r}\hat{\phi}} &= -R_{\hat{t}\hat{\theta}\hat{\theta}\hat{\phi}} = -\frac{3J\sin\theta}{r^4}; \\
R_{\hat{t}\hat{r}\hat{t}\hat{\phi}} &= -R_{\hat{r}\hat{\theta}\hat{\theta}\hat{\phi}} = \frac{3J\sin\theta\sqrt{2m/r}}{r^4}.
\end{aligned} \tag{21}
$$

> – There are now also several terms completely independent of *J*:

$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = -R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = \frac{1}{2}R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{m}{r^3};$$ (22)

Overall, the tetrad components of the Weyl and Riemann tensors are quite tractable; the coordinate components are considerably more complicated.

4. Petrov Type I

It is straightforward (if somewhat tedious) to check that the Painlevé–Gullstrand version of the Lense–Thirring metric is Petrov type I. (that is, the Lense–Thirirng geometry is *not algebraically special*). To do this, one proceeds by first calculating the mixed Weyl tensor $C^{\hat{a}\hat{b}}{}_{\hat{c}\hat{d}}$. Now since this object is antisymmetric in the individual pairs $[\hat{a}\hat{b}]$ and $[\hat{c}\hat{d}]$ this can effectively be thought of as a real 6×6 matrix according to the scheme $A \longleftrightarrow [\hat{a}\hat{b}]$ and $B \longleftrightarrow [\hat{c}\hat{d}]$ as follows:

$$1 \leftrightarrow [\hat{1}\hat{2}]; \qquad 2 \leftrightarrow [\hat{1}\hat{3}]; \qquad 3 \leftrightarrow [\hat{1}\hat{4}]; \qquad 4 \leftrightarrow [\hat{3}\hat{4}]; \qquad 5 \leftrightarrow [\hat{4}\hat{2}]; \qquad 6 \leftrightarrow [\hat{2}\hat{3}].$$

Note that this 6×6 matrix $C^A{}_B$ is not symmetric, nor should it be symmetric.
It is particularly useful to first define the two real quantities

$$\Xi_1 = -\frac{m}{r^3} - \frac{6J^2\sin^2\theta}{r^6}; \qquad \Xi_2 = \frac{3J\cos\theta}{r^4}.$$ (23)

Further defining $s = \sin(\theta)$ the 6×6 matrix $C^A{}_B$ is:

$$C^A{}_B = \left[\begin{array}{ccc|ccc} -2\,\Xi_1 & 0 & -3\sqrt{\frac{2m}{r}}\frac{Js}{r^4} & -2\,\Xi_2 & -\frac{3Js}{r^4} & 0 \\ 0 & \Xi_1 & 0 & -\frac{3Js}{r^4} & \Xi_2 & 0 \\ -3\sqrt{\frac{2m}{r}}\frac{Js}{r^4} & 0 & \Xi_1 & 0 & 0 & \Xi_2 \\ \hline 2\,\Xi_2 & \frac{3Js}{r^4} & 0 & -2\,\Xi_1 & 0 & -3\sqrt{\frac{2m}{r}}\frac{Js}{r^4} \\ \frac{3Js}{r^4} & -\Xi_2 & 0 & 0 & \Xi_1 & 0 \\ 0 & 0 & -\Xi_2 & -3\sqrt{\frac{2m}{r}}\frac{Js}{r^4} & 0 & \Xi_1 \end{array}\right].$$ (24)

Note that the 6×6 matrix $C^A{}_B$ is traceless, $C^A{}_A = 0$ (as it must be since $C^{ab}{}_{ab} = 0$). This asymmetric matrix nevertheless has the partial symmetry

$$C^A{}_B = \left[\begin{array}{c|c} S_R & S_I \\ \hline -S_I & S_R \end{array}\right].$$ (25)

Here S_R and S_I are themselves symmetric traceless 3×3 matrices.
There are 6 distinct eigenvalues, appearing in complex conjugate pairs. Explicitly, defining the third real quantity

$$\Xi_3 = \left(\frac{3Js}{r^4}\right)^2 - \left(3\sqrt{\frac{2m}{r}}\frac{Js}{r^4}\right)^2 = 9\left(1 - \frac{2m}{r}\right)\frac{J^2\sin^2\theta}{r^8},$$ (26)

the 6 eigenvalues of $C^A{}_B$ are

$$\Xi_1 + i\,\Xi_2; \qquad -\frac{1}{2}(\Xi_1 + i\,\Xi_2) \pm \sqrt{\tfrac{9}{4}(\Xi_1 + i\,\Xi_2)^2 - \Xi_3};$$ (27)

and

$$\Xi_1 - i\,\Xi_2; \qquad -\frac{1}{2}(\Xi_1 - i\,\Xi_2) \pm \sqrt{\tfrac{9}{4}(\Xi_1 - i\,\Xi_2)^2 - \Xi_3}.$$ (28)

Note that the sum over all six eigenvalues yields zero, as it must do since the underlying matrix $C^A{}_B$ is traceless.

The fact that there are (generically) 6 distinct eigenvalues guarantees that the Jordan canonical form of $C^A{}_B$ is trivial, and therefore that the spacetime is of Petrov type I (that is, the Painlevé–Gullstrand from of Lense–Thirring is *not algebraically special*). Relevant discussion can be found on pages 49 and 50 of the "Exact solutions" book by Stephani et al. [45]. Note that those authors prefer to rearrange the 6×6 real matrix $C^A{}_B$ into a pair of 3×3 complex matrices:

$$S_\pm = S_R \pm i S_I = \begin{bmatrix} -2(\Xi_1 \pm i\Xi_2) & \mp\frac{3iJs}{r^4} & -3\sqrt{\frac{2m}{r}}\frac{Js}{r^4} \\ \mp\frac{3iJs}{r^4} & \Xi_1 \pm i\Xi_2 & 0 \\ -3\sqrt{\frac{2m}{r}}\frac{Js}{r^4} & 0 & \Xi_1 \pm i\Xi_2 \end{bmatrix}. \tag{29}$$

This really makes no difference to the physics of the discussion, but does simplify the matrix algebra. It is easy to check that the 3 eigenvalues of S_+ are given by (27), and that the 3 eigenvalues of S_- are given by (28). In view of the fact that $\mathrm{tr}(S_\pm) = 0$, each set of 3 eigenvalues must independently sum to zero, as they explicitly do.

Let us now consider some special cases:

- On the rotation axis we have $\theta \to 0$. So $\Xi_1 \to -\frac{m}{r^3}$, while $\Xi_2 \to \frac{3J}{r^4}$ and $\Xi_3 \to 0$. Then $S_\pm$ are both complex diagonal and the 6 Weyl eigenvalues collapse to

$$\{\lambda\} \to \left\{ -\frac{m}{r^3} \pm \frac{3iJ}{r^4}, \ -\frac{m}{r^3} \pm \frac{3iJ}{r^4}; -2\left(-\frac{m}{r^3} \pm \frac{3iJ}{r^4}\right) \right\}. \tag{30}$$

So on-axis the 6 Weyl eigenvalues *are* degenerate.

- On the equator we have $\theta \to \pi/2$. So $\Xi_1 \to -\frac{m}{r^3} - \frac{6J^2}{r^6}$, while $\Xi_2 \to 0$ and finally we have $\Xi_3 \to 9(1 - \frac{2m}{r})J^2/r^8$. Then $S_+ = S_-$ and the eigenvalues collapse to the twice-repeated degenerate values

$$\{\lambda\} \to \left\{ \Xi_1, -\frac{1}{2}\Xi_1 \pm \sqrt{\frac{9}{4}(\Xi_1)^2 - \Xi_3} \right\}. \tag{31}$$

- Finally note that when $J \to 0$ we have $\Xi_1 \to -\frac{m}{r^3}$, while $\Xi_2 \to 0$ and $\Xi_3 \to 0$. Then $S_\pm$ are both real and diagonal and the 6 Weyl eigenvalues collapse to

$$\{\lambda\} \to \left\{ -\frac{m}{r^3}, -\frac{m}{r^3}, -\frac{m}{r^3}, -\frac{m}{r^3}; \frac{2m}{r^3}, \frac{2m}{r^3} \right\}.$$

This is exactly the repeated eigenvalue structure you would expect for the Schwarzschild spacetime.

That is: While the Weyl eigenvalues are degenerate on-axis, on the equator, and in the non-rotating $J \to 0$ limit, the generic situation is that there are six distinct eigenvalues; our Painlevé–Gullstrand variant of Lense–Thirring is Petrov type I. While, generally speaking, Petrov type I is normally associated with a *lack* of special properties, we shall soon see that the Painlevé–Gullstrand variant of Lense–Thirring still has many very nice features when it comes to the analysis of geodesics.

5. "Rain" Geodesics

For our Painlevé–Gullstrand variant of Lense–Thirring spacetime at least some of the timelike geodesics, the "rain" geodesics corresponding to a test object being dropped from spatial infinity with zero initial velocity and zero angular momentum, are particularly

easy to analyze (these are sometimes called ZAMOs—zero angular momentum observers). Consider the vector field

$$V^a = -g^{ab}\nabla_b t = -g^{ta} = \left(1; -\sqrt{\frac{2m}{r}}, 0, \frac{2J}{r^3}\right). \tag{32}$$

This implies

$$V_a = -\nabla_a t = (-1; 0, 0, 0). \tag{33}$$

Thence $g_{ab}V^a V^b = V^a V_a = -1$, so V^a is a future-pointing timelike vector field with unit norm, a 4-velocity. However, then this vector field has zero 4-acceleration:

$$A_a = V^b\nabla_b V_a = -V^b\nabla_b\nabla_a t = -V^b\nabla_a\nabla_b t = V^b\nabla_a V_b = \frac{1}{2}\nabla_a(V^b V_b) = 0. \tag{34}$$

Thus, the integral curves of V^a are timelike *geodesics*. For this construction to work it is *essential* that the metric be unit-lapse—so while this works nicely for our (6) Painlevé–Gullstrand variant of Lense–Thirring, it will *fail* for any and all of the (1)–(3)–(4)–(5) variants of Lense–Thirring spacetime.

Specifically, the integral curves represented by

$$\frac{dx^a}{d\tau} = \left(\frac{dt}{d\tau}; \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\phi}{d\tau}\right) = \left(1; -\sqrt{2m/r}, 0, 2J/r^3\right) \tag{35}$$

are timelike geodesics. Integrating two of these equations is trivial

$$t(\tau) = \tau; \qquad \theta(\tau) = \theta_\infty; \tag{36}$$

so that the time coordinate t can be identified with the proper time of these particular geodesics, and θ_∞ is the original (and permanent) value of the θ coordinate for these particular geodesics.

Furthermore, algebraically one has

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{m}{r}; \tag{37}$$

so these particular geodesics mimic Newtonian infall from spatial infinity with initial velocity zero.

Finally note that

$$\frac{d\phi}{dr} = \frac{d\phi/d\tau}{dr/d\tau} = -\frac{2J/r^3}{\sqrt{2m/r}} = -\frac{2J}{\sqrt{2m}}\, r^{-5/2}, \tag{38}$$

which is easily integrated to yield

$$\phi(r) = \phi_\infty + \frac{4J}{3\sqrt{2m}}\, r^{-3/2}. \tag{39}$$

Here ϕ_∞ is the initial value of the ϕ coordinate (at $r = \infty$) for these particular geodesics. Note the particularly clean and simple way in which rotation of the source causes these "rain" geodesics to be deflected. These pleasant features are specific to our (6) Painlevé–Gullstrand variant of Lense–Thirring, and *fail* for the (1)–(3)–(4)–(5) variants of Lense–Thirring spacetime.

6. On-Axis Geodesics

The on-axis geodesics are particularly important for both theoretical and observational reasons. Observationally, they are relevant when considering polar "jets" from rotating

astrophysical sources (QSOs and AGNs; quasi-stellar objects and active galactic nuclei). Certainly this holds at large distances well outside the source, and arguably is relevant at medium distances as well. See references [46–51]. Theoretically, they are relevant as simple models and consistency checks on the overall formalism. Working on-axis we have either $\theta \equiv 0$ or $\theta \equiv \pi$, and so $\dot{\theta} = 0$. Working on-axis we can, without loss of generality, also choose $\dot{\phi} = 0$. Then we need only consider the t–r plane, and the specific variant of the Lense–Thirring metric that we are interested in effectively reduces to

$$ds^2 \quad \to \quad -dt^2 + \left(dr + \sqrt{2m/r}\,dt\right)^2. \tag{40}$$

That is, we effectively have

$$g_{ab} \to \begin{bmatrix} -1 + \frac{2m}{r} & \sqrt{\frac{2m}{r}} \\ \sqrt{\frac{2m}{r}} & 1 \end{bmatrix}; \qquad g^{ab} \to \begin{bmatrix} -1 & \sqrt{\frac{2m}{r}} \\ \sqrt{\frac{2m}{r}} & 1 - \frac{2m}{r} \end{bmatrix}. \tag{41}$$

This observation is enough to guarantee that on-axis the geodesics of our specific (6) Painlevé–Gullstrand variant of the Lense–Thirring spacetime are *identical* to those for the Painlevé–Gullstrand version of the Schwarzschild spacetime (For a related discussion, see for instance the discussion by Martel and Poisson in reference [29]). For the on-axis null curves $x^a(t) = (t, r(t))$ we have $g_{ab}\,(dx^a/dt)\,(dx^b/dt) = 0$ implying

$$-1 + \left(\frac{dr}{dt} + \sqrt{2m/r}\right)^2 = 0. \tag{42}$$

That is, for on-axis null curves (as expected for a black hole) we have

$$\frac{dr}{dt} = -\sqrt{\frac{2m}{r}} \pm 1. \tag{43}$$

For on-axis timelike geodesics we parameterize by proper time $x^a(\tau) = (t(\tau), r(\tau))$. Then we have $g_{ab}\,(dx^a/d\tau)\,(dx^b/d\tau) = -1$, implying

$$\left(\frac{dt}{d\tau}\right)^2 \left(-1 + \left(\frac{dr}{dt} + \sqrt{2m/r}\right)^2\right) = -1. \tag{44}$$

From the time translation Killing vector $K^a = (1; 0, 0, 0)^a \to (1, 0)^a$ we construct the conserved quantity:

$$K_a\,(dx^a/d\tau) = k. \tag{45}$$

Thence

$$\left(\frac{dt}{d\tau}\right)\left(\left(-1 + \frac{2m}{r}\right) + \sqrt{\frac{2m}{r}}\frac{dr}{dt}\right) = k. \tag{46}$$

Eliminating $dt/d\tau$ we see

$$k^2\left(-1 + \left(\frac{dr}{dt} + \sqrt{2m/r}\right)^2\right) = -\left(\left(-1 + \frac{2m}{r}\right) + \sqrt{\frac{2m}{r}}\frac{dr}{dt}\right)^2. \tag{47}$$

This is a quadratic for dr/dt, with explicit general solution

$$\frac{dr}{dt} = -\sqrt{\frac{2m}{r}}\,\frac{k^2 - 1 + 2m/r}{k^2 + 2m/r} \pm k\,\frac{\sqrt{k^2 - 1 + 2m/r}}{k^2 + 2m/r}. \tag{48}$$

The limit $k \to \infty$ reproduces the result for on-axis null geodesics given in (43). As $r \to \infty$ one has

$$\lim_{r \to \infty} \left(\frac{dr}{dt} \right) = \pm \sqrt{1 - \frac{1}{k^2}}, \tag{49}$$

which provides a physical interpretation for the parameter k. Indeed

$$k = \frac{1}{\sqrt{1 - \left(\frac{dr}{dt} \right)_\infty^2}} \tag{50}$$

is the asymptotic "gamma factor" of the on-axis geodesic (which may be less than unity, and $\left(\frac{dr}{dt} \right)_\infty$ might formally be imaginary, if the geodesic is bound). As $k \to 1$ the negative root corresponds to the "rain" geodesic falling in from spatial infinity with zero initial velocity, so that $dr/dt = -\sqrt{2m/r}$, while the positive root yields

$$\frac{dr}{dt} = \sqrt{\frac{2m}{r}} \left(\frac{1 - 2m/r}{1 + 2m/r} \right). \tag{51}$$

This represents an outgoing timelike geodesic with $\frac{dr}{dt}$ asymptoting to zero at large distances. Overall, the on-axis geodesics of our variant Lense–Thirring spacetime are quite simple to deal with. Understanding on-axis geodesics boils down to understanding Schwarzschild geodesics.

7. Generic Non-Circular Equatorial Geodesics

For equatorial geodesics we set $\theta = \pi/2$, and consequently $\dot{\theta} = 0$. For generic non-circular equatorial geodesics it proves most efficient to work directly in terms of the conserved Killing quantities associated with the timelike and azimuthal Killing vectors (for circular equatorial geodesics, discussed in the next section, the effective potential proves to be a more useful tool). Working on the equator we need only consider the t–r–ϕ hypersurface, and our specific (6) Painlevé–Gullstrand variant of the Lense–Thirring metric effectively reduces to

$$ds^2 \quad \to \quad -dt^2 + \left(dr + \sqrt{2m/r}\, dt \right)^2 + r^2 \left(d\phi - \frac{2J}{r^3} dt \right)^2. \tag{52}$$

That is, we effectively have

$$g_{ab} \to \left[\begin{array}{c|cc} -1 + \frac{2m}{r} + \frac{4J^2}{r^4} & \sqrt{\frac{2m}{r}} & -\frac{2J}{r} \\ \hline \sqrt{\frac{2m}{r}} & 1 & 0 \\ -\frac{2J}{r} & 0 & r^2 \end{array} \right], \tag{53}$$

and thence

$$g^{ab} \to \left[\begin{array}{c|cc} -1 & \sqrt{\frac{2m}{r}} & -\frac{2J}{r^3} \\ \hline \sqrt{\frac{2m}{r}} & 1 - \frac{2m}{r} & \sqrt{\frac{2m}{r}} \frac{2J}{r^3} \\ -\frac{2J}{r^3} & \sqrt{\frac{2m}{r}} \frac{2J}{r^3} & \frac{1}{r^2} - \frac{4J^2}{r^6} \end{array} \right]. \tag{54}$$

7.1. Equatorial Non-Circular Null Geodesics

For equatorial (non-circular) null geodesics let us parameterize the geodesic curve $x^a(\lambda) = (t(\lambda); x^i(\lambda))$ by some arbitrary affine parameter λ. Then the null condition $g_{ab}\,(dx^a/dt)\,(dx^b/dt) = 0$ implies

$$-1 + \left(\frac{dr}{dt} + \sqrt{2m/r}\right)^2 + r^2 \left(\frac{d\phi}{dt} - \frac{2J}{r^3}\right)^2 = 0. \tag{55}$$

From the time translation and azimuthal Killing vectors, $K^a = (1; 0, 0, 0)^a \to (1; 0, 0)^a$ and $\tilde{K}^a = (0; 0, 0, 1)^a \to (0, 0, 1)^a$, we construct the two conserved quantities:

$$K_a \left(\frac{dx^a}{d\lambda}\right) = k; \qquad \text{and} \qquad \tilde{K}_a \left(\frac{dx^a}{d\lambda}\right) = \tilde{k}. \tag{56}$$

Explicitly these yield

$$\frac{dt}{d\lambda}\left(-1 + \frac{2m}{r} + \frac{4J^2}{r^4} + \sqrt{\frac{2m}{r}}\frac{dr}{dt} - \frac{2J}{r}\frac{d\phi}{dt}\right) = k, \tag{57}$$

and

$$\frac{dt}{d\lambda}\left(-\frac{2J}{r} + r^2\frac{d\phi}{dt}\right) = \tilde{k}. \tag{58}$$

Eliminating $dt/d\lambda$ between these two equations we see

$$\tilde{k}\left(-1 + \frac{2m}{r} + \frac{4J^2}{r^4} + \sqrt{\frac{2m}{r}}\frac{dr}{dt} - \frac{2J}{r}\frac{d\phi}{dt}\right) = kr^2\left(\frac{d\phi}{dt} - \frac{2J}{r^3}\right). \tag{59}$$

This can be solved, either for $d\phi/dt$ or for dr/dt, and then substituted back into the null condition (55) to yield a quadratic, either for dr/dt or for $d\phi/dt$. These quadratics can be solved, exactly, for dr/dt or for $d\phi/dt$, but the explicit results are messy. Recalling that the Lense–Thirring spacetime is at least in its original incarnation a large-distance approximation, it makes sense to peel off the leading terms in an expansion in terms of inverse powers of r.

For dr/dt one then finds

$$\frac{dr}{dt} = -\sqrt{\frac{2m}{r}}\,P(r) \pm \sqrt{Q(r)}, \tag{60}$$

where $P(r)$ and $Q(r)$ are rational polynomials in r that asymptotically satisfy

$$P(r) = 1 - \frac{\tilde{k}^2}{k^2 r^2} + \mathcal{O}(1/r^5); \qquad Q(r) = 1 - \left(1 + \frac{2m}{r}\right)\frac{\tilde{k}^2}{k^2 r^2} + \mathcal{O}(1/r^5). \tag{61}$$

Fully explicit formulae for $P(r)$ and $Q(r)$ can easily be found but are quite messy to write down.

Similarly for $d\phi/dt$ one finds

$$\frac{d\phi}{dt} = \left(\frac{2J}{r^3} - \frac{\tilde{k}}{k r^2}\right)\tilde{P}(r) \pm \sqrt{\frac{2m}{r}}\frac{\tilde{k}}{k r^2}\sqrt{\tilde{Q}(r)}. \tag{62}$$

Here $\tilde{P}(r)$ and $\tilde{Q}(r)$ are rational polynomials in r that asymptotically satisfy

$$\tilde{P}(r) = 1 - \frac{2\tilde{k}(Jk + m\tilde{k})}{k^2 r^3} + \mathcal{O}(r^{-4}); \qquad \tilde{Q}(r) = 1 - \frac{\tilde{k}^2}{k^2 r^2} - \frac{2\tilde{k}(2Jk + m\tilde{k})}{k^2 r^3} + \mathcal{O}(r^{-5}). \tag{63}$$

Fully explicit formulae for $\tilde{P}(r)$ and $\tilde{Q}(r)$ can easily be found but are quite messy to write down. Overall, while equatorial null geodesics are in principle integrable, they are in practice not entirely tractable.

7.2. Equatorial Non-Circular Timelike Geodesics

For equatorial timelike geodesics the basic principles are quite similar. First let us parameterize the curve $x^a(\tau)$ using the proper time parameter. Then the timelike normalization condition $g_{ab}\,(dx^a/d\tau)\,(dx^b/d\tau) = -1$ implies

$$\left(\frac{dt}{d\tau}\right)^2\left(-1+\left(\frac{dr}{dt}+\sqrt{2m/r}\right)^2+r^2\left(\frac{d\phi}{dt}-\frac{2J}{r^3}\right)^2\right) = -1. \tag{64}$$

By considering the time translation and azimuthal Killing vectors, $K^a = (1;0,0,0)^a \rightarrow (1;0,0)^a$ and $\tilde{K}^a = (0;0,0,1)^a \rightarrow (0,0,1)^a$, we construct the two conserved quantities:

$$K_a\left(\frac{dx^a}{d\tau}\right) = k; \qquad \text{and} \qquad \tilde{K}_a\left(\frac{dx^a}{d\tau}\right) = \tilde{k}. \tag{65}$$

Explicitly these yield

$$\frac{dt}{d\tau}\left(-1+\frac{2m}{r}+\frac{4J^2}{r^4}+\sqrt{\frac{2m}{r}}\frac{dr}{dt}-\frac{2J}{r}\frac{d\phi}{dt}\right) = k, \tag{66}$$

and

$$\frac{dt}{d\tau}\left(-\frac{2J}{r}+r^2\frac{d\phi}{dt}\right) = \tilde{k}. \tag{67}$$

Eliminating $dt/d\tau$ between these two equations we see

$$\tilde{k}\left(-1+\frac{2m}{r}+\frac{4J^2}{r^4}+\sqrt{\frac{2m}{r}}\frac{dr}{dt}-\frac{2J}{r}\frac{d\phi}{dt}\right) = kr^2\left(\frac{d\phi}{dt}-\frac{2J}{r^3}\right). \tag{68}$$

Eliminating $dt/d\tau$ between (67) and (64) we see

$$\tilde{k}^2\left(-1+\left(\frac{dr}{dt}+\sqrt{2m/r}\right)^2+r^2\left(\frac{d\phi}{dt}-\frac{2J}{r^3}\right)^2\right) = -\left(-\frac{2J}{r}+r^2\frac{d\phi}{dt}\right)^2. \tag{69}$$

Equation (68) can be solved, either for $d\phi/dt$ or for dr/dt, and then substituted back into the modified timelike normalization condition (69) to yield a quadratic, either for dr/dt or for $d\phi/dt$. As for the null geodesics, it is useful to work perturbatively at large r.

Fot dr/dt one then finds

$$\frac{dr}{dt} = -\sqrt{\frac{2m}{r}}\ P(r)\pm\sqrt{Q(r)}, \tag{70}$$

where $P(r)$ and $Q(r)$ are rational polynomials in r that asymptotically satisfy

$$P(r) = 1 - k^{-2} + \frac{2m}{k^4 r} + \mathcal{O}(1/r^2); \qquad Q(r) = 1 - k^{-2} + \frac{2m(2-k^2)}{k^4 r} + \mathcal{O}(1/r^2). \tag{71}$$

Fully explicit formulae for $P(r)$ and $Q(r)$ can easily be found but are quite messy to write down.

Similarly for $d\phi/dt$ one finds

$$\frac{d\phi}{dt} = \left(\frac{2J}{r^3}-\frac{\tilde{k}}{kr^2}\right)\tilde{P}(r)\pm\sqrt{\frac{2m}{r}}\ \frac{\tilde{k}}{kr^2}\ \sqrt{\tilde{Q}(r)} \tag{72}$$

where $\tilde{P}(r)$ and $\tilde{Q}(r)$ are rational polynomials in r that asymptotically satisfy

$$\tilde{P}(r) = 1 - \frac{2m}{k^2 r} + \mathcal{O}(1/r^2); \qquad \tilde{Q}(r) = 1 - k^{-2} + \frac{2m(2-k^2)}{k^4 r} + \mathcal{O}(1/r^2). \tag{73}$$

Fully explicit formulae for $\tilde{P}(r)$ and $\tilde{Q}(r)$ can easily be found but are quite messy to write down. Overall, while equatorial non-circular timelike geodesics are in principle integrable, they are in practice not entirely tractable.

8. Circular Equatorial Geodesics

For circular equatorial geodesics the use of the effective potential formalism proves to be most efficient. Recall that the line element for our variant of the Lense–Thirring spacetime is:

$$ds^2 = -dt^2 + \left(dr + \sqrt{\frac{2m}{r}}\, dt \right)^2 + r^2 \left(d\theta^2 + \sin^2\theta \left(d\phi - \frac{2J}{r^3} dt \right)^2 \right). \tag{74}$$

Now consider the tangent vector to the worldline of a massive or massless particle, parameterized by some arbitrary affine parameter, λ:

$$g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} = -\left(\frac{dt}{d\lambda}\right)^2 + \left[\left(\frac{dr}{d\lambda}\right) + \sqrt{\frac{2m}{r}}\left(\frac{dt}{d\lambda}\right) \right]^2$$
$$+ r^2 \left\{ \left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta \left[\left(\frac{d\phi}{d\lambda}\right) - \frac{2J}{r^3}\left(\frac{dt}{d\lambda}\right) \right]^2 \right\}. \tag{75}$$

We may, without loss of generality, separate the two physically interesting cases (timelike and null) by defining:

$$\epsilon = \begin{cases} -1 & \text{massive particle, i.e., timelike worldline} \\ 0 & \text{massless particle, i.e., null worldline.} \end{cases} \tag{76}$$

That is, $\frac{ds^2}{d\lambda^2} = \epsilon$. We now consider geodesics on the equatorial plane, that is, we fix $\theta = \frac{\pi}{2}$ (hence, $\frac{d\theta}{d\lambda} = 0$). These geodesics now represent (not yet circular) orbits restricted to the equatorial plane *only*. The timelike/null condition now reads:

$$-\left(\frac{dt}{d\lambda}\right)^2 + \left[\left(\frac{dr}{d\lambda}\right) + \sqrt{\frac{2m}{r}}\left(\frac{dt}{d\lambda}\right) \right]^2 + r^2 \left[\left(\frac{d\phi}{d\lambda}\right) - \frac{2J}{r^3}\left(\frac{dt}{d\lambda}\right) \right]^2 = \epsilon. \tag{77}$$

The Killing symmetries in the t and ϕ–coordinates yield the following expressions for the conserved energy E and angular momentum L per unit mass:

$$E = \left(-1 + \frac{2m}{r} + \frac{4J^2}{r^4}\right)\left(\frac{dt}{d\lambda}\right) + \sqrt{\frac{2m}{r}}\frac{dr}{d\lambda} - \frac{2J}{r}\left(\frac{d\phi}{d\lambda}\right); \tag{78}$$

$$L = r^2\left(\frac{d\phi}{d\lambda}\right) - \frac{2J}{r}\left(\frac{dt}{d\lambda}\right). \tag{79}$$

Treating Equations (77)–(79) as a system of three equations in the three unknowns $\frac{dt}{d\lambda}$, $\frac{dr}{d\lambda}$, and $\frac{d\phi}{d\lambda}$, we can rearrange and solve for $\frac{dr}{d\lambda}$ as a function of the metric parameters, r, E, L, and ϵ *only*. This process yields:

$$\frac{dr}{d\lambda} = \pm\sqrt{\left(E + \frac{2JL}{r^3}\right)^2 - \left(1 - \frac{2m}{r}\right)\left(\frac{L^2}{r^2} - \epsilon\right)}. \tag{80}$$

We can now solve for the effective potential and then use the features of this effective potential to solve for the radial positions of the circular photon orbits and innermost stable circular orbits of our spacetime. The potential is given by:

$$V(r) = E^2 - \left(\frac{dr}{d\lambda}\right)^2 = \left(1 - \frac{2m}{r}\right)\left(\frac{L^2}{r^2} - \epsilon\right) + E^2 - \left(E + \frac{2LJ}{r^3}\right)^2 \tag{81}$$

Notice that in the limit where $J \to 0$, the potential is manifestly that of Schwarzschild. We now consider two separate cases, the massless case where $\epsilon = 0$ and the massive case where $\epsilon = -1$. We shall start our analysis with the massless case.

8.1. Circular Null Orbits

In the massless case where $\epsilon = 0$, our potential reduces to

$$V_0(r) = \left(1 - \frac{2m}{r}\right)\frac{L^2}{r^2} - \frac{4EJL}{r^3} - \frac{4L^2J^2}{r^6} = \frac{L(L(r-2m)r^3 + 4J(JL + Er^3))}{r^6}. \tag{82}$$

The photon ring of our spacetime occurs where $\frac{dV_0(r)}{dr} = 0$. That is, the value of r where the following condition is met:

$$\frac{dV_0(r)}{dr} = -\frac{2L}{r^7}\left(Lr^4 - 3(Lm + 2EJ)r^3 - 12J^2L\right) = 0. \tag{83}$$

This quartic equation has no tractable analytic solution. However, a more tractable semi-analytic solution can be obtained if we solve the two equations $V_0(r) = E^2$ and $\frac{dV_0(r)}{dr} = 0$ simultaneously. That is, we solve the following polynomials simultaneously for r:

$$Lr^4 - 3(Lm + 2EJ)r^3 - 12J^2L = 0\,; \tag{84}$$

and

$$E^2r^6 - L^2r^4 + 2L(Lm + 2EJ)r^3 + 4J^2L^2 = 0. \tag{85}$$

If we eliminate E from these equations, we find

$$r^5 - 6mr^4 + 9mr^3 + 72J^2m - 36rJ^2 = 0. \tag{86}$$

Notice that L has also been eliminated in this process. Now we rearrange:

$$r^3(r - 3m)^2 = 36J^2(1 - 2m/r)r. \tag{87}$$

Thence

$$r = 3m \pm \frac{6J\sqrt{1 - 2m/r}}{r}. \tag{88}$$

This is still exact. To now estimate the value of r corresponding to the location of the photon ring purely in terms of the parameters present in our spacetime, we iterate the lowest-order estimate $r = 3m + \mathcal{O}(J)$ to yield

$$r = 3m \pm \frac{6J\sqrt{1 - 2/3}}{3m} + \mathcal{O}(J^2). \tag{89}$$

Finally

$$r = 3m \pm \frac{2J}{\sqrt{3}\,m} + \mathcal{O}(J^2). \tag{90}$$

Notice that in the limit where $J \to 0$, the photon ring reduces to its known location in Schwarzschild. Furthermore, note that in the Kerr geometry, the photon ring for massless particles occurs at [52]:

$$r_{\text{Kerr}} = 2m \left[1 + \cos \left(\frac{2}{3} \cos^{-1} \left(\pm \frac{J}{m^2} \right) \right) \right]. \tag{91}$$

If we conduct a Taylor series expansion around $J = 0$, we find

$$r_{\text{Kerr}}(J \to 0) = 2m \left(\frac{3}{2} \pm \frac{J}{\sqrt{3}\, m^2} + \mathcal{O}(J^2) \right) = 3m \pm \frac{2J}{\sqrt{3}\, m} + \mathcal{O}(J^2), \tag{92}$$

which is exactly the photon ring location in the Lense–Thirring spacetime. This shows that in the slow-rotation limit, the Kerr solution does reduce to Lense–Thirring as we expect.

In terms of stability of these orbits, we analyse the second derivative of the potential $V_0(r)$:

$$\frac{d^2 V_0(r)}{dr^2} = -\frac{6L}{r^8} \left(28J^2 L + 4(mL + 2EJ)r^3 - Lr^4 \right). \tag{93}$$

However, here we cannot simply eliminate L as we did before. Instead we solve $\frac{dV_0(r)}{dr} = 0$ for L, which gives

$$L = -\frac{6EJr^3}{12J^2 + 3mr^3 - r^4}. \tag{94}$$

Substituting this back into Equation (93) we find

$$\frac{d^2 V_0(r)}{dr^2} = -\frac{72 E^2 J^2 (r^4 + 36J^2)}{r^2 (12J^2 + 3mr^3 - r^4)^2}, \tag{95}$$

which is everywhere negative. Hence all equatorial circular null geodesics in our Painlevé–Gullstrand variant of the Lense–Thirring spacetime are unstable.

8.2. ISCOs (Innermost Stable Circular Orbits)

In the massive case (timelike orbits) where $\epsilon = -1$, our potential reduces to:

$$V_{-1}(r) = \left(1 - \frac{2m}{r} \right) \left(1 + \frac{L^2}{r^2} \right) - \frac{4EJL}{r^3} - \frac{4L^2 J^2}{r^6}. \tag{96}$$

Taking the derivative of this

$$\frac{dV_{-1}(r)}{dr} = \frac{2}{r^7} \left(mr^5 - L(Lr^4 - 3(Lm + 2EJ)r^3 - 12J^2 L). \right) \tag{97}$$

Similarly to the null case, $\frac{dV_{-1}(r)}{dr} = 0$, which is now a quintic, has no analytic solution. However, we can begin to form an analytic solution if we solve both $V_{-1}(r) = E^2$ and $\frac{dV_{-1}(r)}{dr} = 0$ simultaneously. That is, we solve the following polynomials simultaneously for r:

$$mr^5 - L(Lr^4 - 3(Lm + 2EJ)r^3 - 12J^2 L) = 0; \tag{98}$$

$$(E^2 - 1)r^6 + 2mr^5 - L^2 r^4 + 2L(Lm + 2EJ)r^3 + 4J^2 L^2 = 0. \tag{99}$$

If we extract E from the first of these equations, we find

$$E = -\frac{mr^5 - L^2r^4 + 3mL^2r^3 + 12J^2L^2}{6JLr^3}.$$

(100)

Substituting this back into (99) we find the following condition:

$$r^3(3L^2m - L^2r + mr^2)^2 - 36J^2L^2(L^2 + r^2)(r - 2m) = 0.$$

(101)

This condition shows that there exist *many* circular timelike orbits $r(L, J, m)$. Unlike the null case, L is not eliminated, hence we cannot solve for the ISCO location yet. We next find the second derivative of the potential:

$$\frac{d^2V_{-1}(r)}{dr^2} = -\frac{2}{r^8}\left(2mr^5 - 3L^2r^4 + 12L(mL + 2EJ)r^3 + 84J^2L^2\right).$$

(102)

Now substituting our expression for E:

$$\frac{d^2V_{-1}(r)}{dr^2} = -\frac{2}{r^8}\left(L^2(r^4 + 36J^2) - 2mr^5\right).$$

(103)

The condition for an extremal equatorial circular orbit is $\frac{d^2V_{-1}(r)}{dr^2} = 0$, that is:

$$(L^2(r^4 + 36J^2) - 2mr^5) = 0.$$

(104)

Using this condition and our condition for an equatorial circular orbit, Equation (101), we can now eliminate L and hence find

$$mr^6(r - 6m)^2 = 72J^2(r^2 + mr - 10m^2) + 1296J^4(2r - 5m).$$

(105)

This implicitly defines $r(m, J)$ in terms of the mass and angular momentum of the spacetime. Thence, rearranging

$$(r - 6m)^2 = \frac{72J^2r^3(r^2 + mr - 10m^2) + 1296J^4(2r - 5m)}{mr^6}.$$

(106)

Thence, finally

$$r = 6m \pm \frac{6J}{mr^3}\sqrt{2r^3(r^2 + mr - 10m^2) + 36J^2(2r - 5m)}.$$

(107)

This is still exact. However, to estimate the value of $r(m, J)$ corresponding to the location of the ISCO purely in terms of the parameters present in our spacetime, we iterate the zeroth-order estimate $r = 6m + \mathcal{O}(J)$ to yield

$$r = 6m + \frac{4\sqrt{2}}{\sqrt{3}}\frac{J}{m} + \mathcal{O}(J^2).$$

(108)

Notice that in the limit where $J \to 0$, the ISCO reduces to its known location in Schwarzschild. Furthermore, note that in the Kerr geometry, the ISCO for massive particles occurs at [52]:

$$r_{\text{Kerr}} = m\left(3 + Z_2 \pm \sqrt{(3 - Z_1)((3 + Z_1 + 2Z_2)}\right).$$

(109)

Here $x = J/m^2$ and

$$Z_1 = 1 + \sqrt[3]{(1-x^2)}\left(\sqrt[3]{(1+x)} + \sqrt[3]{(1-x)}\right); Z_2 = \sqrt{3x^2 + Z_1^2}. \tag{110}$$

If we conduct a Taylor series expansion around $J = 0$ we find

$$r_{\text{Kerr}}(J \to 0) = 6m + \frac{4\sqrt{2}}{\sqrt{3}}\frac{J}{m} + \mathcal{O}(J^2). \tag{111}$$

which is exactly the ISCO location in the Lense–Thirring spacetime. This shows that in the slow-rotation limit, the Kerr solution does reduce to Lense–Thirring as we expect.

9. Astrophysically Relevant Estimates

Note that in SI units

$$m = \frac{G_N\, m_{physical}}{c^2}; \qquad J = \frac{G_N\, J_{physical}}{c^3}. \tag{112}$$

So dimensionally

$$[m] = [\text{length}]; \qquad [J] = [\text{length}]^2. \tag{113}$$

It is also useful to introduce the quantities $a = J/m$ and $a/m = J/m^2$ so that

$$[a] = [J/m] = [\text{length}]; \qquad [a/m] = [J/m^2] = [\text{dimensionless}]. \tag{114}$$

For uncollapsed objects (stars, planets) we may proceed by *approximating* the source as a constant-density rigidly rotating sphere of radius R_{source}, angular velocity ω, and equatorial velocity $v_{equatorial}$.

In the Newtonian approximation

$$J_{physical} = I\,\omega = \frac{2}{5}m_{physical}\,R_{source}^2\,\omega = \frac{2}{5}m_{physical}\,R_{source}\,v_{equatorial}. \tag{115}$$

Thence in geometrodynamic units we have the approximations

$$J = \frac{2}{5}m\,R_{source}\,\frac{v_{equatorial}}{c}; \qquad a = \frac{J}{m} = \frac{2}{5}R_{source}\,\frac{v_{equatorial}}{c}. \tag{116}$$

Furthermore, (defining $r_{Schwarzschild} = 2m$ in geometrodynamic units),

$$\frac{a}{m} = \frac{J}{m^2} = \frac{4}{5}\frac{R_{source}}{r_{Schwarzschild}}\frac{v_{equatorial}}{c}. \tag{117}$$

Another useful dimensionless parameter is

$$\frac{J}{R_{source}^2} = \frac{1}{5}\frac{r_{Schwarzschild}}{R_{source}}\frac{v_{equatorial}}{c}. \tag{118}$$

Using this discussion, and some quite standard observational results (Table 1), it is possible to *estimate* the parameters m, J, $a = J/m$, $a/m = J/m^2$ and J/R_{source}^2 for various astrophysically interesting objects such as the Earth, Jupiter, Sun, Sagittarius A*, the black hole in M87, and our own Milky Way galaxy. See Table 2 for details.

Table 1. Some observational astrophysical data.

Source	m	R_{source} (m)^2	Rotational Period	References
Earth	$5.97217(13) \times 10^{24}$ kg	6.3781×10^6	1 (sidereal) day	PDG [53]
Jupiter	$318\ m_{earth}$	7.1492×10^7	9.93 h	NASA [54]
Sun	$1.998841(4) \times 10^{30}$ kg	6.857×10^8	22 days	PDG [53], NASA [54]
Sagittarius A*	$4.154(14) \times 10^6\ m_{sun}$	$< 6.7 \times 10^9$	—(indirect)—	GRAVITY [55], Ghez-et-al. [56]
Black hole in M87*	$6.5 \times 10^9\ m_{sun}$	38×10^{12}	—(indirect)—	EHT [57], BlackHoleCam [58]
Milky Way galaxy	$\approx 7 \times 10^{11}\ m_{sun}$	> 25 kpc	$\approx 3 \times 10^8$ yr	Gaia [59], Disk [60], Pattern [61]

Some of the observational data is approximate, incomplete, and indirect. Estimates of the black hole spin in Sgr A* and M87* are best extracted from relatively recent measurements and data fitting, see particularly [55–58,62–64].

To interpret the physical significance of Table 2, first note that Kerr black holes in standard Einstein gravity must satisfy $a/m < 1$, that is $J/m^2 < 1$, in order to avoid development of naked singularities. However no such constraint applies to uncollapsed objects. Observationally, we do seem to have $J/m^2 < 1$ for the object Sagittarius A* and the central object in M87, (which are indeed believed to be Kerr black holes, at least approximately), while $J/m^2 > 1$ for the Earth, Jupiter, Sun, and the Milky Way galaxy.

Table 2. Some astrophysical estimates.

Source	m (m)	J (m)^2	a (m)	J/m^2 (dimensionless)	J/R_{source}^2
Earth	0.004435	0.01755	3.959	892.5	4.315×10^{-16}
Jupiter	1.409	1615	1415	812.9	3.304×10^{-13}
Sun	1477	2.741×10^6	1855	1.256	5.652×10^{-12}
Sagittarius A*	6.5×10^9	1.9×10^{19}	2.9×10^9	≈ 0.44	≈ 0.12
Black hole in M87	3.5×10^{12}	1.1×10^{25}	3.2×10^{12}	≈ 0.90	≈ 0.44
Milky Way galaxy	1.5×10^{15}	2.5×10^{31}	1.7×10^{16}	≈ 11	$\approx 10^{-10}$

The fact that for Sagittarius A* and M87* J/m^2 and J/R_{source}^2 are smaller than unity but rather close to unity indicates that these 2 objects are particularly interesting in terms of being potential black hole mimics. The fact that Sagittarius A* and M87* might (conceivably if somewhat unlikely) be either super-spinars [62] or carry a Taub-NUT charge [63,64] is if anything further justification for considering our "black hole mimic" as a viable alternative to standard Kerr spacetime. As we shall see below, when pushing Lense–Thirring into the regime $a/m > 1$ we still have an event horizon. So for $a/m > 1$ our Lense–Thirring based "black hole mimic" possesses both a good weak-field limit and avoids physically unpleasant naked singularities in the strong field limit.

Secondly, observe that the Lense–Thirring metric should (at least in its original asymptotic form) really only be applied in the region $r > R_{source}$, and for uncollapsed sources we certainly have $J/R_{source}^2 \ll 1$. Even for collapsed sources we still see $J/R_{source}^2 \lesssim 1$. The fact that the dimensionless number $J/R_{source}^2 \ll 1$ for the Earth, Jupiter, Sun, (and even the Milky Way galaxy), is an indication that Lense–Thirring spacetime is a perfectly good approximation for the gravitational field generated by these sources *once one gets beyond the surface of these objects*.

These observations are potentially of interest when studying various black hole mimickers [65–69]. (To include a spherically symmetric halo of dark matter in galactic sources, simply replace $m \rightarrow m(r)$, keeping the spin parameter a fixed, so that $J \rightarrow J(r) = am(r)$. The existence of the gravitationally dominant dark matter halo is really the only good reason for treating spiral galaxies as approximately spherically symmetric).

10. Singularity, Horizon, Ergo-Surface, and the Like

Now recall that the original motivation for considering the Lense–Thirring metric really only makes sense for $r > R_{source}$. In fact the Lense–Thirring metric is likely to be a good approximation to the exterior spacetime geometry only for $J/r^2 \ll 1$, that is $r \gg \sqrt{J}$. However, one can nevertheless ask, (both for pedagogical purposes and with a view to exploring potential black hole mimickers), what happens if we extrapolate our variant of the Lense–Thirring metric down to $r \to 0$, and investigate the possible occurrence of horizons and ergo-surfaces. Indeed, for sufficiently slow rotation, physically one should expect to reproduce the physics of a slowly rotating Schwarzschild black hole, and it is useful to see that this does in fact happen.

Extrapolating our variant of the Lense–Thirring spacetime down to $r = 0$ one sees that there is a point curvature singularity at $r = 0$. Furthermore, note that $\nabla_a r$ becomes timelike for $r < 2m$. That is, $g^{ab} \, \nabla_a r \, \nabla_b r = g^{rr} = 1 - \frac{2m}{r}$, and this changes sign at $r = 2m$. Thence for $r < 2m$ any future-pointing timelike vector must satisfy $V^a \, \nabla_a r < 0$. That is, (in contrast to the Kerr spacetime), there is a *single* horizon at the Schwarzschild radius $r = 2m$, an outer horizon with no accompanying inner horizon. Furthermore the location and presence of this horizon is independent of the value of the spin parameter a, so one never gets a naked singularity. Finally, note that one cannot "stand still" once $g_{tt} < 0$. That is, the time-translation Killing vector becomes spacelike once $g_{ab} \, K^a \, K^b = g_{tt} > 0$ corresponding to

$$1 - \frac{2m}{r} - \frac{4J^2 \sin^2 \theta}{r^4} < 0. \tag{119}$$

That is, there is an ergo-surface located at

$$r_E(\theta)^4 - 2m \, r_E(\theta)^3 - 4J^2 \sin^2 \theta = 0. \tag{120}$$

That is,

$$r_E(\theta) = 2m + \frac{4J^2 \sin^2 \theta}{r_E(\theta)^3}. \tag{121}$$

On axis we have $r_E(\theta = 0) = r_E(\theta = \pi) = 2m$, so that on axis the ergo-surface touches the horizon at $r_H = 2m$. Near the axis, (more precisely for $J \sin^2 \theta / m^2 \ll 1$), the formula for $r_E(\theta)$ can be perturbatively solved to yield

$$r_E(\theta) = 2m \left\{ 1 + \frac{J^2 \sin^2 \theta}{4m^4} - \frac{3J^4 \sin^4 \theta}{16m^8} + \mathcal{O}\left(\frac{J^6 \sin^6 \theta}{m^{12}} \right) \right\}. \tag{122}$$

At the equator we have either

$$r_E(\theta = \pi/2) = 2m \left\{ 1 + \frac{J^2}{4m^4} - \frac{3J^4}{16m^8} + \mathcal{O}\left(\frac{J^6}{m^{12}} \right) \right\}, \tag{123}$$

or

$$r_E(\theta = \pi/2) = \sqrt{2J} \left\{ 1 + \frac{m}{2\sqrt{2J}} + \frac{3m^2}{16J} + \mathcal{O}\left(\frac{m^3}{J^{3/2}} \right) \right\}, \tag{124}$$

depending on whether $J \ll m^2$ or $J \gg m^2$.

Generally we have a quartic to deal with, while there is an exact solution it is so complicated as to be effectively unusable, and the best we can analytically say is to place the simple and tractable lower bounds

$$r_E(\theta) > \max\left\{ 2m, \sqrt{2J \sin \theta} \right\}, \tag{125}$$

and

$$r_E(\theta) > \sqrt[4]{(2m)^4 + 4J^2 \sin^2\theta}. \tag{126}$$

For a tractable upper bound we note

$$r_E(\theta) = 2m + \frac{4J^2 \sin^2\theta}{r_E(\theta)^3} < 2m + \frac{4J^2 \sin^2\theta}{(2m)^3}, \tag{127}$$

whence

$$r_E(\theta) < 2m\left\{1 + \frac{J^2 \sin^2\theta}{4m^4}\right\} < 2m\left\{1 + \frac{J^2}{4m^4}\right\}. \tag{128}$$

Overall, if one does extrapolate our variant of the Lense–Thirring spacetime down to $r = 0$, one finds a point singularity at $r = 0$, a horizon at the Schwarzschild radius, and an ergo-surface at $r_E < 2m\left\{1 + \frac{J^2}{4m^4}\right\}$. While such extrapolation is astrophysically inappropriate for vacuum spacetime surrounding rotating uncollapsed objects in standard general relativity, it may prove interesting for pedagogical reasons, or for exploring additional examples of potential black-hole mimickers.

11. Conclusions

What have we learned form this discussion?

First, the specific variant of the Lense–Thirring spacetime given by the metric

$$ds^2 \;=\; -dt^2 + \left(dr + \sqrt{2m/r}\,dt\right)^2 + r^2\left(d\theta^2 + \sin^2\theta\left(d\phi - \frac{2J}{r^3}dt\right)^2\right) \tag{129}$$

is both mathematically and physically a very tractable and quite reasonable model for the spacetime region exterior to rotating stars and planets. Because this metric is in Painlevé–Gullstrand form, the physical interpretation is particularly transparent. Furthermore, with the slight generalization $m \to m(r)$, with $J \to J(r) = am(r)$, that is,

$$ds^2 \;=\; -dt^2 + \left(dr + \sqrt{2m(r)/r}\,dt\right)^2 + r^2\left(d\theta^2 + \sin^2\theta\left(d\phi - \frac{2J(r)}{r^3}dt\right)^2\right) \tag{130}$$

one can accommodate spherically symmetric dark matter halos, so one has a plausible approximation to the gravitational fields of spiral galaxies. Best of all, this specific Painlevé–Gullstrand variant of the Lense–Thirring spacetime is rather easy to work with.

References

1. Schwarzschild, K. Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte Königlich Preussischen Akademie Wissenschaften* **1916**, *7*, 189.
2. Hans, T.; Josef, L. Über den Einfluss der Eigenrotation der Zentralkörperauf die Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie. *Physikalische Zeitschrift Leipzig Jg* **1918**, *19*, 156–163. English translation by Bahram Mashoon, Friedrich W. Hehl, and Dietmar S. Theiss: "On the influence of the proper rotations of central bodies on the motions of planets and moons in Einstein's theory of gravity". *Gene. Relat. Grav.* **1984**, *16*, 727–741.
3. Herbert, P. On the History of the So-Called Lense–Thirring Effect. Available online: http://philsci-archive.pitt.edu/archive/0000 2681/01/lense.pdf (accessed on 19 April 2021).
4. Ronald, J.A.; Maurice, B.; Menahem, S. *Introduction to General Relativity*, 2nd ed.; It Is Important to Acquire the 1975 Second Edition, the 1965 First Edition Does Not Contain any Discussion of the Kerr Spacetime; McGraw-Hill: New York, NY, USA, 1975.
5. Charles, M.; Kip, T.; John, A.W. *Gravitation*; Freeman: San Francisco, CA, USA, 1973.
6. D'Inverno, R. *Introducing Einstein's Relativity*; Oxford University Press: Oxford, UK, 1992.
7. James, H. *Gravity: An Introduction to Einstein's General Relativity*; Addison Wesley: San Francisco, CA, USA, 2003.
8. Sean, C. *An Introduction to General Relativity: Spacetime and Geometry*; Addison Wesley: San Francisco, CA, USA, 2004.
9. Visser, M. The Kerr spacetime: A brief introduction. *arXiv* **2008**, arXiv:0706.0622.
10. Wiltshire, D.L.; Visser, M.; Scott, S.M. (Eds.) *The Kerr Spacetime: Rotating Black Holes in General Relativity*; Cambridge University Press: Cambridge, UK, 2009.
11. Roy, K. Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.* **1963**, *11*, 237–238.
12. Roy, K. Gravitational collapse and rotation. In *Quasi-Stellar Sources and Gravitational Collapse: Including the Proceedings of the First Texas Symposium on Relativistic Astrophysics, Austin, TX, USA, 16–18 December 1963*; Ivor, R., Alfred, S., Schücking, E.L., Eds.; University of Chicago Press: Chicago, IL, USA, 1965; pp. 99–102.
13. Subrahmanyan, C. *The Mathematical Theory of Black Holes*; Oxford University Press: Oxford, UK, 1992; ISBN 978-0-19-850370-5.
14. O'Neill, B. *The Geometry of Kerr Black Holes*; Dover: New York, NY, USA, 2014; ISBN 978-0-486-4932-8.
15. Garret, B. *Relativity and Modern Physics*; Harvard University Press: Cambridge, MA, USA, 1923.
16. Jørg, T.J. Über die allgemeinen kugelsymmetrischen Lösungen der Einsteinschen, Gravitationsgleichungen im Vakuum. *Ark. Mat. Ast. Fys.* **1921**, *15*, nr.18.
17. Stanley, D.; Joel, F. Schwarzschild and Birkhoff *a la* Weyl. *Am. J. Phys.* **2005**, *73*, 261.
18. Nils, V.J.; Finn, R. On the discovery of Birkhoff's theorem. *Gen. Rel. Grav.* **2006**, *38*, 537–540.
19. Skalala, J. Visser, M. Birkhoff-like theorem for rotating stars in (2+1) dimensions. *arXiv* **2009**, arXiv:0903.2128.
20. Renzetti, G. History of the attempts to measure orbital frame-dragging with artificial satellites. *Cent. Eur. J. Phys.* **2013**, *11*, 531–544. [CrossRef]
21. Iorio, L. Phenomenology of the Lense-Thirring effect in the solar system. *Astrophys. Space Sci.* **2011**, *331*, 351–395. [CrossRef]
22. Bardeen, J.M.; Petterson, J.A. The Lense-Thirring Effect and Accretion Disks around Kerr Black Holes. *Astrophys. J. Lett.* **1975**, *195*, L65. [CrossRef]
23. Paul, P. La mécanique classique et la théorie de la relativité. *C. R. Acad. Sci.* **1921**, *173*, 677–680.
24. Paul, P. La gravitation dans la mécanique de Newton et dans la mécanique d'Einstein. *C. R. Acad. Sci.* **1921**, *173*, 873–886.
25. Gullstrand, A. Allgemeine Lösung des statischen Einkörperproblems in der Einsteinschen Gravitationstheorie. *Arkiv för Matematik Astronomi och Fysik* **1922**, *16*, 1–15.
26. Doran, C. A New form of the Kerr solution. *Phys. Rev. D* **2000**, *61*, 067503. [CrossRef]
27. Hamilton, A.J.; Lisle, J.P. The River model of black holes. *Am. J. Phys.* **2008**, *76*, 519–532. [CrossRef]
28. Natario, J. Painlevé-Gullstrand Coordinates for the Kerr Solution. *Gen. Rel. Grav.* **2009**, *41*, 2579–2586. [CrossRef]
29. Martel, K.; Poisson, E. Regular coordinate systems for Schwarzschild and other spherical space-times. *Am. J. Phys.* **2001**, *69*, 476–480. [CrossRef]
30. Faraoni, V.; Vachon, G. When Painlevé-Gullstrand coordinates fail. *Eur. Phys. J. C* **2020**, *80*, 771. [CrossRef] [PubMed]
31. Visser, M. Acoustic propagation in fluids: An unexpected example of Lorentzian geometry. *arXiv* **1993**, arXiv:Gr-qc/9311028.
32. Visser, M. Acoustic black holes: Horizons, ergospheres, and Hawking radiation. *Class. Quant. Grav.* **1998**, *15*, 1767–1791. [CrossRef]
33. Visser, M. Acoustic black holes. *arXiv* **1995**, arXiv:gr-qc/9901047.
34. Volovik, G. Simulation of Painlevé-Gullstrand black hole in thin He-3-A film. *JETP Lett.* **1999**, *69*, 705–713. [CrossRef]
35. Perez-Bergliaffa, S.E.; Hibberd, K.; Stone, M.; Visser, M. Wave equation for sound in fluids with vorticity. *Phys. D* **2004**, *191*, 121–136. [CrossRef]
36. Visser, M.; Barcelól, C.; Liberati, S. Analog models of and for gravity. *Gen. Rel. Grav.* **2002**, *34*, 1719–1734. [CrossRef]
37. Fischer, U.R.; Visser, M. On the space-time curvature experienced by quasiparticle excitations in the Painlevé–Gullstrand effective geometry. *Ann. Phys.* **2003**, *304*, 22–39. [CrossRef]
38. Novello, M.; Visser, M.; Volovik, G. *Artificial Black Holes*; World Scientific: Singapore, 2002.
39. Barceló, C.; Liberati, S.; Visser, M. Probing semiclassical analog gravity in Bose–Einstein condensates with widely tunable interactions. *Phys. Rev. A* **2003**, *68*, 053613. [CrossRef]

40. Visserm, M.; Weinfurtner, S. Vortex geometry for the equatorial slice of the Kerr black hole. *Class. Quant. Grav.* **2005**, *22*, 2493–2510. [CrossRef]
41. Barceló, C.; Liberati, S.; Visser, M. Analogue gravity. *Living Rev. Rel.* **2005**, *8*, 12. [CrossRef]
42. Visser, M.; Molina-París, C. Acoustic geometry for general relativistic barotropic irrotational fluid flow. *New J. Phys.* **2010**, *12*, 095014. [CrossRef]
43. Martín-Moruno, P.; Visser, M. Generalized Rainich conditions, generalized stress-energy conditions, and the Hawking-Ellis classification. *Class. Quant. Grav.* **2017**, *34*, 225014. [CrossRef]
44. Martín-Moruno, P.; Visser, M. Essential core of the Hawking–Ellis types. *Class. Quant. Grav.* **2018**, *35*, 125003. [CrossRef]
45. Stephani, H.; Kramer, D.; MacCallum, M.; Hoenselaers, C.; Herlt, E. *Exact Solutions of Einstein's Equations*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2003.
46. de Felice, F.; Carlotto, L. Jet dynamics in black hole fields: A collimation mechanism. *Astrophys. J.* **1997**, *418*, 116–126. [CrossRef]
47. Chiconem C.; Mashhoon, B. The Generalized Jacobi equation. *Class. Quant. Grav.* **2002**, *19*, 4231–4248. [CrossRef]
48. Gariel, J.; MacCallum, M.A.H.; Marcilhacy, G.; Santos, N.O. Kerr Geodesics, the Penrose Process and Jet Collimation by a Black Hole. *Astron. Astrophys.* **2010**, *515*, A15. [CrossRef]
49. Gariel, J.; Santos, N.O.; Wang, A. Kerr Geodesics Following the Axis of Symmetry. *Gen. Rel. Grav.* **2016**, *48*, 66. [CrossRef]
50. Gariel, J.; Santos, N.O.; Wang, A. Observable acceleration of jets by a Kerr black hole. *Gen. Rel. Grav.* **2017**, *49*, 43. [CrossRef]
51. Tucker, R.W.; Walton, T.J. On Gravitational Chirality as the Genesis of Astrophysical Jets. *Class. Quant. Grav.* **2017**, *34*, 035005. [CrossRef]
52. Bardeen, J.M.; William, H.; Teukolsky, S.A. Rotating Black Holes: Locally Nonrotating Frames, Energy Extraction, and Scalar Synchrotron Radiation. *Astrophys. J.* **1972**, *178*, 347–370. [CrossRef]
53. Zyla, P.A.; Particle Data Group. The Review of Particle Physics. *Prog. Theor. Exp. Phys.* **2020**, *1*, 083C01. [CrossRef]
54. NASA. Available online: https://solarsystem.nasa.gov/ (accessed on 19 April 2021).
55. Abuter, R.; Amorim, A.; Bauböck, M.; Berger, J.P.; Bonnet, H.; Brandner, W.; Clénet, Y.; Coudé du Foresto, V.; de Zeeuw, P.T. A geometric distance measurement to the Galactic center black hole with 0.3% uncertainty. *Astron. Astrophys.* **2019**, *625*, L10. [CrossRef]
56. Ghez, A.M.; Salim, S.; Weinberg, N.N.; Lu, J.R.; Do, T.; Dunn, J.K.; Matthews, K.; Morris, M.; Yelda, S.; Becklin, E.E.; et al. Measuring Distance and Properties of the Milky Way's Central Supermassive Black Hole with Stellar Orbits. *Astrophys. J.* **2008**, *689*, 1044–1062. [CrossRef]
57. Akiyama, K.; Event Horizon Telescope. First M87 Event Horizon Telescope Results. VI. The Shadow and Mass of the Central Black Hole. *Astrophys. J. Lett.* **2019**, *875*, L6. [CrossRef]
58. Goddi, C.; Falcke, H.; Kramer, M.; Rezzolla, L.; Brinkerink, C.; Bronzwaer, T.; Davelaar, J.R.J.; Deane, R.; Laurentis, M.D.; Desvignes, G.; et al. BlackHoleCam: Fundamental physics of the galactic center. *Int. J. Mod. Phys. D* **2016**, *26*, 1730001. [CrossRef]
59. Necibm L.; Lin, T. Substructure at High Speed II: The Local Escape Velocity and Milky Way Mass with Gaia DR2. *arXiv* **2021**, arXiv:2102.02211.
60. López-Corredoira, M.; Allende Prieto, C.; Garzón, F.; Wang, H.; Liu, C.; Deng, L. Disk stars in the Milky Way detected beyond 25 kpc from its center. *Astron. Astrophys.* **2018**, *618*, L8. [CrossRef]
61. Ortwin, G. Pattern speeds in the Milky Way. *Mem. Della Soc. Astron. Ital. Suppl.* **2011**, *18*, 185.
62. Bambi, C.; Freese, K.; Vagnozzi, S.; Visinelli, L. Testing the rotational nature of the supermassive object M87* from the circularity and size of its first image. *Phys. Rev. D* **2019**, *100*, 044057. [CrossRef]
63. Chakraborty, C.; Bhattacharyya, S. Circular orbits in Kerr-Taub-NUT spacetime and their implications for accreting black holes and naked singularities. *JCAP* **2019**, *5*, 34. [CrossRef]
64. Akiyama, K.; Event Horizon Telescope. First M87 Event Horizon Telescope Results. V. Physical Origin of the Asymmetric Ring. *Astrophys. J. Lett.* **2019**, *875*, L5 [CrossRef]
65. Carballo-Rubio, R.; Di Filippo, F.; Liberati, S.; Visser, M. Phenomenological aspects of black holes beyond general relativity. *Phys. Rev. D* **2018**, *98*, 124009. [CrossRef]
66. Carballo-Rubio, R.; Di Filippo, F.; Liberati, S.; Visser, M. Opening the Pandora's box at the core of black holes. *Class. Quant. Grav.* **2020**, *37*, 145005. [CrossRef]
67. Carballo-Rubio, R.; Di Filippo, F.; Liberati, S.; Visser, M. Geodesically complete black holes. *Phys. Rev. D* **2020**, *101*, 084047. [CrossRef]
68. Carballo-Rubio, R.; Di Filippo, F.; Liberati, S.; Visser, M. On the viability of regular black holes. *JHEP* **2018**, *7*, 23. [CrossRef]
69. Barausse, E.; Berti, E.; Hertog, T.; Hughes, S.A.; Jetzer, P.; Pani, P.; Sotiriou, T.P.; Tamanini, N.; Witek, H.; Yagi, K.; et al. Prospects for Fundamental Physics with LISA. *Gen. Rel. Grav.* **2020**, *52*, 81. [CrossRef]