

Article

Broken Scale Invariance, Gravity Mass, and Dark Energy in Modified Einstein Gravity with Two Measure Finsler like Variables

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Abstract: We study new classes of generic off-diagonal and diagonal cosmological solutions for effective Einstein equations in modified gravity theories (MGTs), with modified dispersion relations (MDRs), and encoding possible violations of (local) Lorentz invariance (LIVs). Such MGTs are constructed for actions and Lagrange densities with two non-Riemannian volume forms (similar to two measure theories (TMTs)) and associated bimetric and/or biconnection geometric structures. For conventional nonholonomic 2 + 2 splitting, we can always describe such models in Finsler-like variables, which is important for elaborating geometric methods of constructing exact and parametric solutions. Examples of such Finsler two-measure formulations of general relativity (GR) and MGTs are considered for Lorentz manifolds and their (co) tangent bundles and abbreviated as FTMT. Generic off-diagonal metrics solving gravitational field equations in FTMTs are determined by generating functions, effective sources and integration constants, and characterized by nonholonomic frame torsion effects. By restricting the class of integration functions, we can extract torsionless and/or diagonal configurations and model emergent cosmological theories with square scalar curvature, R^2 , when the global Weyl-scale symmetry is broken via nonlinear dynamical interactions with nonholonomic constraints. In the physical Einstein–Finsler frame, the constructions involve: (i) nonlinear re-parametrization symmetries of the generating functions and effective sources; (ii) effective potentials for the scalar field with possible two flat regions, which allows for a unified description of locally anisotropic and/or isotropic early universe inflation related to acceleration cosmology and dark energy; (iii) there are “emergent universes” described by off-diagonal and diagonal solutions for certain nonholonomic phases and parametric cosmological evolution resulting in various inflationary phases; (iv) we can reproduce massive gravity effects in two-measure theories. Finally, we study a reconstructing procedure for reproducing off-diagonal FTMT and massive gravity cosmological models as effective Einstein gravity or Einstein–Finsler theories.

Keywords: modified and massive gravity; two measure theories; Einstein and Finsler gravity; off-diagonal cosmological solutions; nonholonomic dynamical Weyl-scale symmetry breaking; (anisotropic) inflation; dark energy; reconstructing procedure



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1. Introduction

Modern cosmology has the very important task of providing a theoretical description of many aspects of the observable universe with exponential expansion (inflation), particle creation, and radiation. We cite books [1–5] on standard cosmology [6–8] and further developments. Then, regarding acceleration cosmology [9,10] and related dark energy and dark matter physics, one a series of works on modified gravity theories, MGTs,

and cosmology [11–16] can be considered. Another direction of research is devoted to nonholonomic and Finsler-like, locally anisotropic cosmological models [17–23]; see [24,25] for an axiomatic approach to Finsler–Lagrange–Hamilton gravity theories. The physics community almost accepted the idea that the Einstein gravity and standard particle physics have to be modified in order to elaborate self-consistent quantum gravity theories and describe existing experimental and observational data in modern cosmology. As a result, a number of MGTs and cosmological scenarios have been elaborated in the last 20 years.

In a series of works [26–33]—see also references therein—a geometric approach (the so-called anholonomic frame deformation method (AFDM)) was developed for the construction of exact and parametric solutions in MGTs, general relativity (GR) and the theory of nonholonomic geometric and classical/quantum information flows. Such solutions use generic off-diagonal metrics (such metrics can not be diagonalized via coordinate transforms in a finite spacetime region; the solutions, in general, have non zero torsion configurations; the Levi-Civita connection can be extracted by imposing additional nonintegrable constraints (in the physics and mathematical literature, two equivalent terms are also used: nonholonomic and/or anholonomic conditions)) and generalized connections when their coefficients depend on all spacetime coordinates via generating and integration functions, for vacuum and non-vacuum configurations. Effective and matter field sources can be considered for possible Killing and non-Killing symmetries, various types of commutative and noncommutative parameters, etc. For Finsler-like modified-gravity theories (FMGTs), the coefficients of geometric and physical objects depend, in general, on (co)fiber velocity (momentum) type coordinates. Following the AFDM, the geometric constructions and variational calculus are preformed with respect to certain classes of (adapted) nonholonomic frames for a formal splitting of spacetime dimension in the form $2(3) + 2 + \dots +$ and a well-defined, geometrically “auxiliary” linear connection which is convenient for performing, for instance, a deformation quantization procedure, or for constructing exact and/or parametric solutions. This allows for the gravitational field equations in MGTs, FMGTs, and GR, and geometric/information flow equations to be decoupled. Such nonholonomic deformations of fundamental geometric objects determined by distortions of nonlinear and linear connection structures were not considered in other approaches with vierbeins (tetrads), $2 + 2$ and/or $3 + 1$ splitting; see standard textbooks on general relativity and exact solutions [34–38]. The methods elaborated by other authors were only successful in the generation of exact solutions with two and three Killing symmetries, but do not provide a geometric/analytic formalism for a general decoupling of gravitational and matter field equations. The surprising result is that such a decoupling is possible for various classes of effective/modified Einstein equations and matter fields, which can be derived for certain physically motivated general assumptions in MGTs.

Let us summarize most of the important ideas and methods developed in References [16,24–33]:

- (a) The (modified) Einstein equations with some effective and/or matter field sources consist of very sophisticated systems of nonlinear partial derivative equations (PDEs). The bulk of most known and important physical applications (of black hole, cosmological and other type solutions) were elaborated for the ansatz of metrics which can be diagonalized by certain frame/coordinate transforms, and when physically important systems of nonlinear PDEs can be reduced to systems of decoupled nonlinear ordinary differential equations (ODEs). In such cases, the generated exact or parametric solutions (i.e., integrals, with possible non-trivial topology, singularities, of different smooth classes, etc.) depend on one space, or time, like the coordinate, being determined by certain imposed symmetries (for instance, spherical/axial ones, which are invariant on some rotations, with Lie algebras symmetries, etc). The integration constants can be found in an explicit form by considering certain symmetry/Cauchy/boundary/asymptotic conditions. In this way, various classes of black/worm hole and isotropic and anisotropic cosmological solutions were constructed;

- (b) The AFDM allows us to decouple and integrate physically important systems of nonlinear PDEs in more general forms than (a) when the integral varieties are parameterized not only by integration constants but also by generating and integration functions subjected to nonholonomic constraints and functional/nonlinear dependence on sources and data for certain classes of “prime metrics and connections”. The resulting “target” off-diagonal metrics and generalized connections depend, in general, on all space–time coordinates. It is important to note that, at the end, we can impose additional nonholonomic constraints and consider “smooth” limits or various type non-trivial topology and/or parametric transitions to Levi–Civita configurations (with zero torsion) and/or diagonal metrics. In this way, we can reproduce well-known black hole/cosmological solutions, which can have deformed horizons (for instance, ellipsoid/toroid symmetries), anisotropic polarized physical constants, for instance, imbedding into nontrivial gravitational vacuum configurations. These new classes of solutions cannot be constructed if we impose particular types of ansatz for diagonalizable metrics, frames of references and/or sources at the beginning, depending on only one spacetime coordinate. This is an important property of nonlinear parametric physical systems subjected to certain nonholonomic constraints. More general solutions with geometric rich structure and various applications for a nonlinear gravitational and matter field dynamics can be found if we succeed in directly solving certain generic nonlinear systems of PDEs which are not transformed into systems of ODEs. Having constructed such general classes of solutions, one might analyze the limits to diagonal configurations and possible perturbative effects. We “loose” the bulk of generic nonlinear solutions with multi-variables if we consider certain “simplified” ansatz for “higher-symmetries”, resulting in ODEs, from the beginning.

Applying the AFDM as explained in paragraph (b), and choosing corresponding types of generating functions and integration functions and constants, it is possible to model various MGTs and accelerating cosmology effects by considering generic off-diagonal interactions and re-parameterizations of generating functions and sources in effective Einstein gravity. In the present paper, we shall elaborate on a unified cosmological scenario for MGTs and GR with nonholonomic off-diagonal interactions when effective Finsler-like variables can be considered for a $2 + 2$ splitting. In such an approach, both inflation and slowly accelerated universe models are reproduced by exact solutions constructed following the AFDM. In general, such solutions are inhomogeneous and have local anisotropy. For a corresponding class of generating and integration functions, and for necessary type of effective sources, we can model effective scalar field potentials with anisotropy and limits to two flat regions. We shall construct and study nonlinear parametric cosmological theories by generalizing the standard models based on Friedman–Lamaître–Robertson–Walker (FLRW), with diagonalizable configurations derived for ODEs. The goal is to address the initial singularity problem and to explain how two periods of exponential expansion with widely different scales can be described via solutions to effective gravitational equations.

A well-known mechanism for generating accelerated expansion as a consequence of vacuum energy can be performed in the context of the scalar field theory paradigm, which is described by an effective potential ${}^{ef}U$ with flat regions. For such “slow roll” configurations of the vacuum field, the kinetic energy terms are small and the resulting energy-momentum tensor is of type $T_{\mu\nu} \simeq g_{\mu\nu} {}^{ef}U$. If the potential ${}^{ef}U$ contains contributions of some modified gravity terms (two measures, massive gravity, etc.), we can analyse the possible effects of such terms in the inflationary phase. However, this is not enough to elaborate a theory of modern cosmology with acceleration and dark energy and dark matter contributions. Theoreticians developed different quintessential, k -essence and “variable gravity” inflation scenarios [39–46] and $f(R)$ modified models, with contributions from massive gravity, Finsler-like theories, bi-metrics and bi-connections and/or generic off-diagonal interactions; see [14,15,17,18,24,25,30,47,48] and references therein.

The solutions with anisotropies and flat regions can be used for speculations on the phase that proceeds the inflation, and may explain both the non-singular origin of universe

and the early universe evolution. This is similar to the concept of “emergent universe”, which was considered with the aim of solving the problem of initial singularity, including the singularity theorems for inflationary cosmology driven by scalar field [49–54]. In our approach, with solutions constructed following the AFDM, the universe does not start as a static Einstein universe but as a parametric effective one, when the scalar field rolls with an almost constant speed for a non-singular configuration with small anisotropies.

Let us briefly explain the origins of and motivations for the present work. The main ideas and methods for constructing generic off-diagonal solutions in MGTs come from Refs. [26–31]. In articles [11–19], various examples were given of instances where the gravitational and matter field equations in MGTs can be re-defined and solved as certain effective/generalized Einstein equations or their Finsler-like modifications. A series of papers [55–59] is devoted to a new class of modified-measure gravity-matter theories containing different terms in the pertinent Lagrangian action, for instance, one with a non-Riemannian integration measure and another with standard Riemannian integration measure. We shall call such models two measure theories (TMTs) of gravity. In a more general case, two non-metric densities [60] are considered. An important feature of such theories is that the constructions are with global Weyl-scale invariance and further dynamic breaking. In particular, the second action term is the standard Riemannian integration measure containing a Weyl-scale symmetry, preserving R^2 , or more general $f(R)$ terms, which, in this work, may encode modifications from massive gravity, bi-metric and bi-connection theories. The latter formalism and geometrization of such TMTs allows for the representation of the corresponding gravitational field equations as certain effective Einstein equations in nonholonomic variables; see various applications in modern cosmology, (super) string/brane theories, non-Abelian confinement, etc. [61,62]. The main goal of this article is to develop the AFDM for generating exact solutions in TMTs formulated in non-holonomic and Einstein–Finsler variables—see also a partner work [63]—and analyze its possible implications in modern cosmology and for dark energy and dark matter physics.

The work is organized as follows. Geometric preliminaries on nonholonomic Lorentz manifolds and relativistic Lagrange–Finsler spaces are provided in Section 2. Then, in Section 3, we formulate a geometric approach to MGT cosmology in the framework of TMT with nonholonomic variables and effective Einstein–Finsler gravity theories. We apply the AFDM for the construction of generic off-diagonal cosmological solutions in various MGTs in Section 4. Cosmological models with locally anisotropic effective scalar potentials and two flat regions are studied in Section 5. We devote Section 6 to the formulation of certain conditions when modified massive gravity can be reproduced as TMTs and effective GR theories, with nonholonomic Finsler-like variables, and speculate on a potential reconstructing procedure for such massive gravity cosmological models. Finally, we provide a discussion and conclusions in Section 7.

2. Nonholonomic Variables and (Modified) Einstein and Lagrange–Finsler Equations

In this section, we outline some necessary results from the geometry of four dimensional (4D) Lorentz manifolds with so-called canonical nonholonomic variables, which can be transformed into Finsler–Lagrange-like variables. The motivation for considering canonical variables is that they can prove certain general decoupling and integration properties of gravitational field equations in MGTs and GR. However, Finsler–Lagrange-like variables and the associated almost-symplectic structures can be used for deformation and other types of quantization procedure in gravity theories. Proofs and details can be found, for instance, in [24,25].

2.1. Geometric Objects and GR and MGTs in Nonholonomic Variables

Let us consider a 4D pseudo-Riemannian manifold V , defined by a metric structure

$$g = g_{\alpha\beta}(u^\gamma) du^\alpha \otimes du^\beta \quad (1)$$

of signature $(+, +, + -)$, with local coordinates $u = \{u^\gamma\}$, where indices $\alpha, \beta, \gamma, \dots$ run values $1, 2, 3$ and (for the time-like coordinate) 4 . The Einstein summation rule on up/low repeating indices is applied if the contrary is not stated. For a corresponding causality structure, the postulates of the special relativity theory, the principle of equivalence, etc, are locally combined (see a review of axiomatic approaches in GR- and Finsler-like modified theories in [17,24,25]); such a curved space–time is called a Lorentz manifold. In this work, we study generalizations of geometric and gravitational and cosmological models when certain nonholonomic (nonintegrable, anholonomic) distributions and related bimeasure structures, and Lagrangians for MGTs, are considered on \mathbf{V} .

On a curved spacetime \mathbf{V} , we can always introduce a nonholonomic $2 + 2$ splitting, which is determined by a non-integrable distribution

$$\mathbf{N} : T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V},$$

where $T\mathbf{V}$ is the tangent bundle of \mathbf{V} , the Whitney sum \oplus defines a conventional splitting into horizontal (h), $h\mathbf{V}$, and vertical (v), $v\mathbf{V}$, subspaces. In local coordinates

$$\mathbf{N} = N_i^a(x^k, y^b) dx^i \otimes \partial/\partial y^a, \tag{2}$$

states a nonlinear connection, N–connection structure. For such a N–connection decomposition, the indices and coordinates split in the form $u = (x, y)$, or $u^\alpha = (x^i, y^a)$, for $x = \{x^i\}$ and $y = \{y^a\}$, with $i, j, k, \dots = 1, 2$ and $a, b, c, \dots = 3, 4$, which is respectively adapted to a nonholonomic $2 + 2$ splitting. The data (\mathbf{V}, \mathbf{N}) define a nonholonomic manifold with a prescribed fibered structure described locally by fiber-like coordinates y^a .

In our works, “boldface” symbols are used to emphasize that certain geometric/physical objects are defined for spaces enabled with a $2 + 2$ splitting determined by an N-connection structure. On pseudo-Riemannian manifolds, introducing an N-connection with a $2 + 2$ splitting is equivalent to the convention that there are used certain subclasses of local (N-adapted) bases $\mathbf{e}_\mu = (\mathbf{e}_i, e_a)$ and their duals $\mathbf{e}^\nu = (e^j, \mathbf{e}^b)$, where

$$\mathbf{e}_i = \frac{\partial}{\partial x^i} - N_i^c \frac{\partial}{\partial y^c}, e_a = \partial_a = \frac{\partial}{\partial y^a} \text{ and } e^j = dx^j, \mathbf{e}^b = dy^b + N_k^b dx^k. \tag{3}$$

Such frames are called nonholonomic because they generally satisfy the relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma,$$

with nontrivial anholonomy coefficients $W_{ia}^b = \partial_a N_i^b, W_{ji}^b = \Omega_{ij}^b = \mathbf{e}_j(N_i^b) - \mathbf{e}_i(N_j^b)$. For zero W –coefficients, we obtain holonomic bases, which allow us to consider coordinate transforms $\mathbf{e}_\alpha \rightarrow \partial_\alpha$ and $\mathbf{e}^\beta \rightarrow du^\beta$.

On any manifold V and its tangent and cotangent bundle, there are also possible general vierbein (tetradic) transformations $e_\alpha = e_\alpha^{\underline{a}}(u) \partial/\partial u^{\underline{a}}$ and $e^\beta = e^\beta_{\underline{b}}(u) du^{\underline{b}}$, where the coordinate indices are underlined in order to distinguish them from arbitrary abstract ones and the matrix $e^\beta_{\underline{b}}$ is inverse to $e_\alpha^{\underline{a}}$ for orthonormalized bases. We do not use boldface symbols for such transformations because an arbitrary decomposition (we can consider certain diadic $2 + 2$ splitting as particular cases) is not adapted to an N-connection structure.

With respect to N-adapted bases, we shall say that a vector, a tensor and other geometric objects are represented as a distinguished vector (d-vector), a distinguished tensor (d-tensor) and distinguished objects (d-object), respectively. Using frame transforms $\mathbf{g}_{\alpha\beta} = e^{\alpha'}_\alpha e^{\beta'}_\beta g_{\alpha'\beta'}$, any metric g (1) on \mathbf{V} can be written in N-adapted form as a distinguished metric (in brief, d-metric)

$$\mathbf{g} = \mathbf{g}_{\alpha\beta} \mathbf{e}^\alpha \otimes \mathbf{e}^\beta = g_{ij}(u) dx^i \otimes dx^j + g_{ab}(u) \mathbf{e}^a \otimes \mathbf{e}^b. \tag{4}$$

In brief, such an h–v decomposition of a metric structure is parameterized in the form $\mathbf{g} = (h\mathbf{g} = \{g_{ij}\}, v\mathbf{g} = \{g_{ab}\})$.

On nonholonomic manifolds, we can work with a subclass of linear connections $\mathbf{D} = (h\mathbf{D}, v\mathbf{D})$, called distinguished connections, d–connections, preservingm under parallelism, the N–connection splitting. A d–connection is determined by its coefficients $\Gamma^\alpha_{\beta\gamma} = \{L^i_{jk}, \acute{L}^a_{bk}, \acute{C}^i_{jc}, C^a_{bc}\}$, computed with respect to an N-adapted base (3). Linear connections structures which are not adapted to N-connections can also be considered, but they are not preserved under parallelism (2), and satisfy other types of transformation law under frame/coordinate transforms.

For any d-vectors \mathbf{X} and \mathbf{Y} , we can define, in standard form, the torsion d–tensor, \mathcal{T} , the nonmetricity d–tensor, \mathcal{Q} , and the curvature d–tensor, \mathcal{R} , of a \mathbf{D} , respectively, which generally do not depend on \mathbf{g} and/or \mathbf{N} . The formulas are

$$\mathcal{T}(\mathbf{X}, \mathbf{Y}) := \mathbf{D}_X \mathbf{Y} - \mathbf{D}_Y \mathbf{X} - [\mathbf{X}, \mathbf{Y}], \mathcal{Q}(\mathbf{X}) := \mathbf{D}_X \mathbf{g}, \mathcal{R}(\mathbf{X}, \mathbf{Y}) := \mathbf{D}_X \mathbf{D}_Y - \mathbf{D}_Y \mathbf{D}_X - \mathbf{D}_{[\mathbf{X}, \mathbf{Y}]}. \tag{5}$$

In N-adapted coefficients labeled by h- and v-indices, such geometric d-objects are parameterized, respectively, as

$$\begin{aligned} \mathcal{T} &= \{ \mathbf{T}^\gamma_{\alpha\beta} = (T^i_{jk}, T^i_{ja}, T^a_{ji}, T^a_{bi}, T^a_{bc}) \}, \mathcal{Q} = \{ \mathbf{Q}^\gamma_{\alpha\beta} \}, \\ \mathcal{R} &= \{ \mathbf{R}^\alpha_{\beta\gamma\delta} = (R^i_{hjk}, R^a_{bjk}, R^i_{hja}, R^c_{bjc}, R^i_{hba}, R^c_{bea}) \}. \end{aligned}$$

Such coefficients can be computed in explicit form by introducing $\mathbf{X} = \mathbf{e}_\alpha$ and $\mathbf{Y} = \mathbf{e}_\beta$, see (3), and coefficients of a D-connection $\mathbf{D} = \{ \Gamma^\gamma_{\alpha\beta} \}$ into formulas (5).

The AFDM is easy to work with two “preferred” linear connections: the Levi–Civita connection ∇ and the canonical d-connection $\hat{\mathbf{D}}$. Both connections are completely defined by a metric structure \mathbf{g} , following the conditions

$$\mathbf{g} \rightarrow \begin{cases} \nabla : & \nabla \mathcal{Q} = 0 \text{ and } \nabla \mathcal{T} = 0; \\ \hat{\mathbf{D}} : & \hat{\mathcal{Q}} = 0 \text{ and } h\hat{\mathcal{T}} = 0, v\hat{\mathcal{T}} = 0, \end{cases} \tag{6}$$

where the left label ∇ is used for the geometric objects determined by the Levi–Civita (LC) connection. It should be noted here that the N-adapted coefficients of the torsion $\hat{\mathcal{T}}$ are not zero for the case of mixed h- and v-coefficients computed with respect to N-adapted frames (conventionally, we can write this as $hv\hat{\mathcal{T}} \neq 0$, with some nontrivial N-adapted coefficients from the subset $\{T^i_{ja}, T^a_{ji}, T^a_{bi}\}$). Such a torsion $\hat{\mathcal{T}}$ is completely determined by the coefficients of \mathbf{N} and \mathbf{g} (in coordinate frames, such values determine certain generic off-diagonal terms $g_{\alpha\beta}$ which cannot be diagonalized in a finite space–time region $U \subset \mathbf{V}$ by coordinate transforms). We can consider a distortion relation

$$\hat{\mathbf{D}} = \nabla + \hat{\mathbf{Z}},$$

when both linear connections and the distortion tensors $\hat{\mathbf{Z}}$ are completely defined by the geometric data (\mathbf{g}, ∇) , or (in nonholonomic variables) by $(\mathbf{g}, \mathbf{N}, \hat{\mathbf{D}})$.

Contracting the indices of a canonical Riemann d-tensor of $\hat{\mathbf{D}}$, $\hat{\mathcal{R}} = \{ \hat{\mathbf{R}}^\alpha_{\beta\gamma\delta} \}$, we construct a respective canonical Ricci d-tensor, $\hat{\mathcal{R}}ic = \{ \hat{\mathbf{R}}_{\alpha\beta} := \hat{\mathbf{R}}^\gamma_{\alpha\beta\gamma} \}$. The corresponding nontrivial N-adapted coefficients are

$$\hat{\mathbf{R}}_{\alpha\beta} = \{ \hat{R}_{ij} := \hat{R}^k_{ijk}, \hat{R}_{ia} := -\hat{R}^k_{ika}, \hat{R}_{ai} := \hat{R}^b_{aib}, \hat{R}_{ab} := \hat{R}^c_{abc} \}, \tag{7}$$

when the scalar curvature is computed

$$\hat{R} := \mathbf{g}^{\alpha\beta} \hat{\mathbf{R}}_{\alpha\beta} = g^{ij} \hat{R}_{ij} + g^{ab} \hat{R}_{ab}.$$

It should be noted that, generally, $\widehat{\mathbf{R}}_{\alpha\beta} \neq \widehat{\mathbf{R}}_{\beta\alpha}$, even this type of tensor is symmetric to the LC-connection, $R_{\alpha\beta} = R_{\beta\alpha}$. This a nonholonomic deformation and nonholonomic frame effect.

We can introduce the Einstein d-tensor

$$\widehat{\mathbf{E}}_{\alpha\beta} := \widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2} \mathbf{g}_{\alpha\beta} \widehat{R}$$

and consider an effective Lagrangian $\widehat{\mathcal{L}}$, for which the stress–energy momentum tensor, $\widehat{\mathbf{T}}_{\alpha\beta}$, is defined by an N-adapted (with respect to \mathbf{e}_β and \mathbf{e}^α) variational calculus on a nonholonomic manifold $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$,

$$\widehat{\mathbf{T}}_{\alpha\beta} = - \frac{2}{\sqrt{|\mathbf{g}_{\mu\nu}|}} \frac{\delta(\sqrt{|\mathbf{g}_{\mu\nu}|} \widehat{\mathcal{L}})}{\delta \mathbf{g}^{\alpha\beta}}. \tag{8}$$

Following geometric principles, we can postulate the Einstein equations in GR for the data $(\mathbf{g}, \widehat{\mathbf{D}})$, and/or re-write them equivalently for the data (\mathbf{g}, ∇) if additional nonholonomic constraints for zero torsion are imposed

$$\begin{aligned} \widehat{\mathbf{R}}_{\beta\gamma} &= \widehat{\mathbf{Y}}_{\beta\gamma}, \tag{9} \\ \text{and } \widehat{\mathcal{T}} &= 0, \text{ additional condition for } \nabla. \tag{10} \end{aligned}$$

In general, the condition $\widehat{\mathbf{D}}|_{\widehat{\mathcal{T}}=0} = \nabla$ may not have a smooth limit, and such an equation can be considered as a nonholonomic or parametric constraint. Here, we note that the source

$$\widehat{\mathbf{Y}}_{\beta\gamma} := \varkappa(\widehat{\mathbf{T}}_{\alpha\beta} - \frac{1}{2} \mathbf{g}_{\alpha\beta} \widehat{T})$$

is computed with the trace $\widehat{T} := \mathbf{g}^{\alpha\beta} \widehat{\mathbf{T}}_{\alpha\beta}$, and \varkappa should be determined by the Newton constant ${}^{New}G$, as in GR, if we want to study the limits to the Einstein gravity theory. In this work, we shall use the units when ${}^{New}G = 1/16\pi$ and the Planck mass ${}^{Pl}M = (8\pi {}^{New}G)^{-1/2} = \sqrt{2}$. If we do not impose the LC-conditions (10), the system of nonholonomic nonlinear PDEs (9), and similar higher dimension ones, for instance, those with noncommutative and/or supersymmetric variables, can be considered in various classes of MGTs, Finsler–Lagrange gravity, etc.

The values $\widehat{\mathcal{R}}, \widehat{\mathcal{R}}ic$ and \widehat{R} for the canonical d-connection $\widehat{\mathbf{D}}$ are different from the similar ones, $\mathcal{R}, \mathcal{R}ic$ and R , computed for the LC-connection ∇ . Nevertheless, both classes of such fundamental geometric objects are related via distorting relations derived in a unique form for a given metric structure and N-connection splitting. There are at least two priorities that work with $\widehat{\mathbf{D}}$, instead of ∇ . The first one is ensuring that we can find solutions for generalized gravity theories with nontrivial torsion. The second priority is that the equations (9) decouple in very general forms with respect to certain classes of N-adapted frames. The basic idea of the AFDM is to write the Lagrange densities and the resulting fundamental gravitational and matter field equations in terms of such nonholonomic variables, which allows us to decouple and solve nonlinear systems of PDEs. This cannot be done if we use the LC-connection ∇ from the beginning. It is not a d-connection, does not preserve the h - and v -splitting and the condition of zero torsion, $\nabla \mathcal{T} = 0$, under general transformations, and does not allow the equations in general forms to be decoupled. Working with $\widehat{\mathbf{D}}$, we introduce certain “flexibility” in order to apply corresponding geometric techniques for integration PDEs. In such cases, we do not make additional assumptions regarding particular cases for ansatz and connections transforming the fundamental field equations into nonlinear systems of ODEs. Having defined a quite general class of solutions, expressed via generating functions and integration functions and constants, we can impose additional nonholonomic constraints (10), which allows the

extraction of LC-configurations. In this way, we can construct an explicit form of new classes of exact solutions in GR and MGTs, both in (\mathbf{g}, ∇) and $(\mathbf{g}, \mathbf{N}, \tilde{\mathbf{D}})$ variables.

2.2. Finsler–Lagrange Variables in GR and MGTs

On a 4D/Lorentz manifold V , we can introduce Finsler-like variables considering a conventional $2 + 2$ splitting of coordinates $u^\alpha = (x^i, y^a)$ for a nonholonomic fiber structure, where $y = \{y^a\}$, for $a = 3, 4$, are treated as effective fiber coordinates (which are analogous to velocity ones in theories on tangent bundles). In this way, we elaborate a toy model of relativistic Finsler–Lagrange geometry. Let us explain how such constructions provide examples of the above-formulated nonholonomic models of (pseudo) Riemannian geometry. A fundamental function (equivalently, generating function) $V \ni (x, y) \rightarrow L(x, y) \in \mathbb{R}$, i.e., a real valued function (an effective Lagrangian, or a Lagrange density) which is differentiable on $\tilde{V} := V / \{0\}$, with $\{0\}$ being the null section of V , and continuous on the null section of $\pi : V \rightarrow hV$. A relativistic 4D model of a fibered effective Lagrange space $L^{3,1} = (V, L(x, y))$ is determined by a prescribed regular Hessian metric (equivalently, the v-metric)

$$\tilde{g}_{ab}(x, y) := \frac{1}{2} \frac{\partial^2 L}{\partial y^a \partial y^b} \tag{11}$$

is non-degenerate, i.e., $\det |\tilde{g}_{ab}| \neq 0$, and of constant signature. Non-regular configurations can be studied as special cases.

The non-Riemannian total phase space geometries are characterized by nonlinear quadratic line elements

$$ds_L^2 = L(x, y). \tag{12}$$

We can elaborate on geometric and physical theories with a spacetime enabled with a nonholonomic frame and metric, and (non)linear connection structures determined by a nonlinear quadratic line element (12) and related v-metric (11). The geometric objects on $L^{3,1}$ will be labeled by a tilde “~” (for instance, \tilde{g}_{ab}) if they are defined canonically by an effective Lagrange generating function. We write $\tilde{L}^{3,1}$ with tilde in order to emphasize that V is enabled with an effective relativistic Lagrange structure and respective nondegenerate Hessian.

The dynamics of a probing point particle in $\tilde{L}^{3,1}$ are described by Euler–Lagrange equations,

$$\frac{d}{d\tau} \frac{\partial \tilde{L}}{\partial y^i} - \frac{\partial \tilde{L}}{\partial x^i} = 0.$$

These equations are equivalent to the *nonlinear geodesic (semi-spray) equations*

$$\begin{aligned} \frac{d^2 x^i}{d\tau^2} + 2\tilde{G}^i(x, y) &= 0, \\ \text{for } \tilde{G}^i &= \frac{1}{2} \tilde{g}^{ij} \left(\frac{\partial^2 \tilde{L}}{\partial y^i \partial y^j} y^k - \frac{\partial \tilde{L}}{\partial x^i} \right), \end{aligned} \tag{13}$$

where \tilde{g}^{ij} is inverse to \tilde{g}_{ab} (11). In this way, we define a canonical Lagrange N-connection structure

$$\tilde{\mathbf{N}} = \left\{ \tilde{N}_i^a := \frac{\partial \tilde{G}}{\partial y^i} \right\}, \tag{14}$$

determining an effective Lagrange N-splitting $\tilde{\mathbf{N}} : TV = hV \oplus vV$, similar to (2). Using \tilde{N}_i^a from (14), we define effective Lagrange N-adapted (co)frames

$$\tilde{\mathbf{e}}_\alpha = (\tilde{\mathbf{e}}_i = \frac{\partial}{\partial x^i} - \tilde{N}_i^a(x, y) \frac{\partial}{\partial y^a}, e_b = \frac{\partial}{\partial y^b}) \text{ and } \tilde{\mathbf{e}}^\alpha = (\tilde{e}^i = dx^i, \tilde{e}^a = dy^a + \tilde{N}_i^a(x, y) dx^i). \tag{15}$$

Such $\tilde{\mathbf{N}}$ -adapted frames can be considered as results of certain vierbein (frame, for 4D, tetradic) transforms of type $e_\alpha = e_\alpha^\alpha(u) \partial / \partial u^\alpha$ and $e^\beta = e^\beta_\beta(u) du^\beta$. (We can underline the

local coordinate indices in order to distinguish them from arbitrary abstract ones; the matrix $e^{\beta}_{\underline{\beta}}$ is inverse to e^{α}_{α} for orthonormalized bases.).

We can also consider frame transforms $e_{\alpha} = e^{\alpha'}_{\alpha}(u)e_{\alpha'}$, when $\tilde{g}_{ij} = e^i_i e^j_k \tilde{g}'_{i'j'}$ and $\tilde{g}_{ab} = e^{a'}_a e^{b'}_b \tilde{g}'_{a'b'}$ for $\tilde{g}'_{i'j'}$ and $\tilde{g}'_{a'b'}$ being of type (11), define the respective h- and v-components of a d-metric of signature $(+ + + -)$. As a result, we can construct a relativistic Sasaki type d-metric structure

$$\tilde{\mathbf{g}} = \tilde{\mathbf{g}}_{\alpha\beta}(x, y)\tilde{\mathbf{e}}^{\alpha} \otimes \tilde{\mathbf{e}}^{\beta} = \tilde{g}_{ij}(x, y)e^i \otimes e^j + \tilde{g}_{ab}(x, y)\tilde{\mathbf{e}}^a \otimes \tilde{\mathbf{e}}^a. \tag{16}$$

Using respective frame transforms $\mathbf{g}_{\alpha'\beta'} = e^{\alpha}_{\alpha'} e^{\beta}_{\beta'} \tilde{\mathbf{g}}_{\alpha\beta}$ and $\mathbf{g}_{\alpha'\beta'} = e^{\alpha}_{\alpha'} e^{\beta}_{\beta'} \tilde{g}_{\alpha\beta}$, such an effective Lagrange–Sasaki can be represented as a general d-metric (4), or equivalently, as a off-diagonal metric (1),

$$\mathbf{g} = \mathbf{g}_{\alpha'\beta'}(x, y)\mathbf{e}^{\alpha'} \otimes \mathbf{e}^{\beta'} = g_{\underline{\alpha}\underline{\beta}}(x, y)du^{\underline{\alpha}} \otimes du^{\underline{\beta}},$$

where

$$g_{\underline{\alpha}\underline{\beta}} = \begin{bmatrix} g_{ij}(x) + g_{ab}(x, y)N_i^a(x, y)N_j^b(x, y) & g_{ae}(x, y)N_j^e(x, y) \\ g_{be}(x, y)N_i^e(x, y) & g_{ab}(x, y) \end{bmatrix}. \tag{17}$$

Parameterizations of type (17) for metrics are considered in Kaluza–Klein theories, but, in our approach, the N-coefficients are determined by a general or Lagrange N-connection structure.

The Lagrange N-connections $\tilde{\mathbf{N}}$ define an almost complex structure $\tilde{\mathbf{J}}$. Such a linear operator $\tilde{\mathbf{J}}$ acts on $\mathbf{e}_{\alpha} = (e_i, e_b)$ using formulas $\tilde{\mathbf{J}}(\mathbf{e}_i) = -e_{n+i}$ and $\tilde{\mathbf{J}}(e_{n+i}) = \mathbf{e}_i$, and globally defines an almost complex structure $\tilde{\mathbf{J}} \circ \tilde{\mathbf{J}} = -\mathbf{I}$, where \mathbf{I} is the unity matrix. We note that $\tilde{\mathbf{J}}$ is only a (pseudo) almost complex structure for a (pseudo) Euclidean signature. There are omitted tildes written, for instance, \mathbf{J} for arbitrary frame/coordinate transforms.

A Lagrange Neijenhuis tensor field is determined by a Lagrange-generating function introduced as the curvatures of a respective N-connection,

$$\tilde{\Omega}(\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}) := -[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}] + [\tilde{\mathbf{J}}\tilde{\mathbf{X}}, \tilde{\mathbf{J}}\tilde{\mathbf{Y}}] - \tilde{\mathbf{J}}[\tilde{\mathbf{J}}\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}] - \tilde{\mathbf{J}}[\tilde{\mathbf{X}}, \tilde{\mathbf{J}}\tilde{\mathbf{Y}}], \tag{18}$$

for any d-vectors \mathbf{X}, \mathbf{Y} . Such formulas can be written without tilde values if arbitrary frame transforms are considered. In local form, an N-connection is characterized by such coefficients of (18) (i.e., the coefficients of an N-connection curvature):

$$\Omega^a_{ij} = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}. \tag{19}$$

An almost complex structure \mathbf{J} transforms into a standard complex structure for the Euclidean signature if $\Omega = 0$.

Using the Lagrange d-metric $\tilde{\mathbf{g}}$ and d-operator $\tilde{\mathbf{J}}$, we can define the almost symplectic structure $\theta := \tilde{\mathbf{g}}(\tilde{\mathbf{J}}, \cdot)$. Then, we can construct canonical d-tensor fields defined by $L(x, y)$ and N-adapted, respectively, to \tilde{N}_i^a (14) and $\tilde{\mathbf{e}}_{\alpha} = (\tilde{\mathbf{e}}_i, e_b)$ (15):

$$\begin{aligned} \tilde{\mathbf{J}} &= -\delta_i^a e_a \otimes e^i + \delta_a^i \tilde{\mathbf{e}}_i \otimes \tilde{\mathbf{e}}^a \text{ the almost complex structure;} \\ \tilde{\mathbf{P}} &= \tilde{\mathbf{e}}_i \otimes e^i - e_a \otimes \tilde{\mathbf{e}}^a \text{ almost product structure;} \\ \tilde{\theta} &= \tilde{g}_{aj}(x, y)\tilde{\mathbf{e}}^a \wedge e^j \text{ almost symplectic structure.} \end{aligned} \tag{20}$$

We can define the Cartan–Lagrange d-connection $\tilde{\mathbf{D}} = (h\tilde{\mathbf{D}}, v\tilde{\mathbf{D}})$ which, by definition, satisfies the conditions (compare with (6)),

$$\tilde{\mathbf{D}}\tilde{\theta} = 0, \tilde{\mathcal{Q}} = 0 \text{ and } h\tilde{\mathcal{T}} = 0, v\tilde{\mathcal{T}} = 0. \tag{21}$$

The geometric d-objects (16), (20) and (4) can be subjected to arbitrary frame transforms on a Lorentz nonholonomic manifold V , when we can omit tilde on symbols, for instance, by writing geometric data in the form $(\mathbf{g}, \mathbf{J}, \mathbf{P},)$, but we have to preserve the notation $\tilde{\mathbf{D}}$ in all systems of frames/coordinates because such a d-connection is different, for instance, from the LC-connection ∇ .

We can elaborate a relativistic 4D model of Finsler space on a Lorentz manifold V , as an example of Lagrange space when a regular $L = F^2$ is defined by a fundamental (generating) Finsler function $F(x, y)$, also called a Finsler metric, when the nonlinear quadratic element (12) is changed into

$$ds_F^2 = F^2(x, y)$$

and when the following conditions are satisfied: (1) F is a real positive valued function which is differential on $T\tilde{V}$ and continuous on the null section of the projection $\pi : T\tilde{V} \rightarrow V$; (2) a homogeneity condition, $F(x, \lambda y) = |\lambda| F(x, y)$, is imposed for a nonzero real value λ ; (3) the Hessian (11) is defined by F^2 in such a form that in any point $(x_{(0)}, y_{(0)})$, the v-metric is of signature $(+ -)$. The conditions (1)–(3) allow the construction of various types of geometric models with homogenous fiber coordinates, with local anisotropy distinguished on directions. Nevertheless, to extend, for instance, the GR theory in a relativistic covariant form, we need additional assumptions and physical motivations on the type of nonlinear and linear connections we take consideration, as well as how to extract effective quadratic elements, etc.; see details and references in [24,25]. In this work, we consider that we can always prescribe a respective nonholonomic geometric modeling to a Lorentz manifold V a Finsler, or Lagrange, type function using canonical data $(\tilde{L}, \tilde{\mathbf{N}}; \tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}^\alpha; \tilde{g}_{jk}, \tilde{g}_{ab})$, when certain homogeneity conditions can be satisfied for Finsler configurations. For general frame transforms and modified dispersion relations, we do not consider Lagrange- or Finsler-like nonholonomic variables, but can preserve a conventional h- and v-splitting adapted to a N-connection structure with geometric data $(\mathbf{V}, \mathbf{N}; \mathbf{e}_\alpha, \mathbf{e}^\alpha; g_{jk}, g_{ab})$. To elaborate physically realistic gravity models, we need further conventions on the type of linear connection structure (covariant derivative) we shall use for our geometric constructions.

We can always consider distortion relations

$$\hat{\mathbf{D}} = \nabla + \hat{\mathbf{Z}}, \tilde{\mathbf{D}} = \nabla + \tilde{\mathbf{Z}}, \text{ and } \hat{\mathbf{D}} = \tilde{\mathbf{D}} + \mathbf{Z}, \text{ all determined by } (\mathbf{g}, \mathbf{N}) \sim (\tilde{\mathbf{g}}, \tilde{\mathbf{N}}), \tag{22}$$

with distortion d-tensors $\hat{\mathbf{Z}}, \tilde{\mathbf{Z}}$, and \mathbf{Z} , and postulate the (modified) Einstein Equations (9) in various forms

$$\tilde{\mathbf{R}}_{\beta\gamma} = \tilde{\mathbf{Y}}_{\beta\gamma}[\tilde{\mathbf{Z}}, \hat{\mathbf{T}}_{\alpha\beta}], \text{ or} \tag{23}$$

$$R_{\beta\gamma}[\nabla] = \mathbf{Y}_{\beta\gamma}[\hat{\mathbf{Z}}, \hat{\mathbf{T}}_{\alpha\beta}], \tag{24}$$

where the (effective) matter sources are respective functionals on distortions and energy-momentum tensors for matter fields. Such systems of nonlinear PDEs are different and characterized by different types of Bianchi identities, local conservation laws and associated symmetries. Nevertheless, we can establish such classes of nonholonomic frame and distortion structures, with respective equivalence relations

$$(\mathbf{g}, \mathbf{N}, \hat{\mathbf{D}}) \Leftrightarrow (L : \tilde{\mathbf{g}}, \tilde{\mathbf{N}}, \tilde{\mathbf{D}}) \leftrightarrow (\tilde{\theta}, \tilde{\mathbf{P}}, \tilde{\mathbf{J}}, \tilde{\mathbf{D}}) \leftrightarrow [(\mathbf{g}, \nabla)]$$

when the Equations (9), (23) and (24) describe equivalent gravitational and matter field models. Different geometric data have their priorities in constructing explicit different classes of exact/parametric/approximate solutions or for performing certain procedures of quantization and further generalizations of physical theories. If we work with a respective canonical d-connection structure $\hat{\mathbf{D}}$, we can prove a general decoupling property of (9) and construct exact solutions with generic off-diagonal metrics $g_{\underline{\alpha}\underline{\beta}}(u^\gamma)$ (17) represented as d-metrics $\mathbf{g}_{\alpha'\beta'}(x, y)$ (4), when the coefficients of such metrics and associated nonlinear and

linear connection structures depend, in principle, on all space–time coordinates u^γ . We can not decouple the systems of nonlinear PDEs (23), in Lagrange–Finsler variables in general form, and (24), in local coordinates and for the LC-connection. In MGTs with modifications of (23) or (24), even in GR, we are able to find exact solutions for some “special” ansatz of metrics, which, for instance, are diagonalizable and depend only on a radial or time-like coordinate (for instance, for black hole and/or cosmological solutions). In this work, we shall apply the AFDM in order to construct cosmological locally anisotropic solutions in MGTs with (in general, generic off-diagonal) metrics of type $\mathbf{g}_{\alpha'\beta'}$ ($x^i, y^4 = t$). In geometric and analytic form, this is possible if we work with nontrivial N-connection structures and certain variables which are similar to those in Lagrange–Finsler geometry but on Lorentz manifolds. The almost symplectic Lagrange–Finsler variables $(\theta, \hat{\mathbf{P}}, \hat{\mathbf{J}}, \hat{\mathbf{D}})$ allow for the elaboration of deformation quantization and, together with $(\mathbf{g}, \mathbf{N}, \hat{\mathbf{D}})$, allow the introduction of nonholonomic and Finsler-like spinors and, for instance, nonholonomic Einstein–Finsler–Dirac systems. This is not possible if the so-called Berwald– or Chern–Finsler connections are used, because they are not metric-compatible, and self-consistent definitions of locally anisotropic versions of the Dirac equation are a problem.

3. TMTs and Other MGTs in Canonical Nonholonomic Variables

The goal of this section is to show how various classes of MGTs can be extracted from certain effective Einstein gravity theories using nonholonomic or Finsler-like variables. This allows to decouple the gravitational field equations and to generate exact solutions in very general forms, with generic off-diagonal metrics and generalized connections, and with constraints to zero-torsion configurations; see details in [26–33].

In [14–16,26–33,63], different possibilities for modelling different MGTs by imposing corresponding nonholonomic constraints on the metric and canonical d-connection structures and source in (9) were considered. One of the main goals of this work is to prove that, by using corresponding type parameterizations of the effective Lagrangian $\hat{\mathcal{L}}$ in (8), the so-called modified massive gravity theories (in general, with bi-connection and bi-metric structures) can be modeled at TMTs with effective Einstein equations for $\hat{\mathbf{D}}$ when additional constraints $\hat{\mathbf{D}}|_{\hat{\mathcal{T}}=0} = \nabla$ have to be imposed in order to extract LC-configurations.

The actions for equivalent TMT, MGT and nonholonomically deformed Einstein models are postulated

$$\mathcal{S} = ({}^P M)^2 \int d^4u \sqrt{|\hat{\mathbf{g}}|} [\hat{\mathbf{R}} + \hat{\mathcal{L}}] = \tag{25}$$

$$\begin{aligned} \Phi \mathcal{S} + {}^m \mathcal{S} &= \int d^4u \, {}^1\Phi(\mathbf{A}) [\hat{\mathbf{R}} + {}^1L] + \tag{26} \\ &\int d^4u \, {}^2\Phi(\mathbf{B}) \left[{}^2L + \epsilon \mathbf{f}(\tilde{\mathbf{R}}) + (\sqrt{|\mathbf{g}|})^{-1} \Phi(\mathbf{H}) \right] + \int d^4u \sqrt{|\hat{\mathbf{g}}|} \, {}^m \mathcal{L} = \end{aligned}$$

$${}^{F,\mu} \mathcal{S} + {}^m \mathcal{S} = ({}^P M)^2 \int d^4u \left[\sqrt{|\hat{\mathbf{g}}|} \, {}^{F,\mu} \mathcal{L} + \sqrt{|\hat{\mathbf{g}}|} \, {}^m \mathcal{L} \right], \tag{27}$$

where $|\hat{\mathbf{g}}| = \det |\hat{\mathbf{g}}_{\alpha\beta}|$ for a d-metric, $\hat{\mathbf{g}}_{\alpha\beta}$, constructed effectively by a conformal transform of a TMT reference one, $\mathbf{g}_{\alpha\beta}$, (see below, Formula (34)); ΦL defines a class of theories with two independent non-Riemannian volume-forms ${}^1\Phi(A)$ and ${}^2\Phi(B)$ as in [61,62] but with a more general functional for modification, of type $\epsilon \mathbf{f}(\tilde{\mathbf{R}})$, than ϵR^2 if $\hat{\mathbf{D}} \rightarrow \nabla$; the Lagrange density functional ${}^{f,\mu} \mathcal{L} = \mathbf{F}(\tilde{\mathbf{R}})$ is determined similarly to a modified massive gravity, by a mass-deformed scalar curvature [14,15,64–66], (there are various ambiguities and controversies in different approaches to massive gravity when modifications by mass terms are postulated for different Lagrange densities; in this paper, we consider a “toy model” when terms of type $\mathbf{f}(\tilde{\mathbf{R}}, \mu)$ and/or $\mathbf{f}(\mathbf{R}) + \mu \dots$ can modeled by the same MGT, but for different classes of nonholonomic constraint and different classes of solution)

$$\check{\mathbf{R}} := \hat{\mathbf{R}} + 2 \mu^2 (3 - \text{tr} \sqrt{\mathbf{g}^{-1} \mathbf{q}} - \det \sqrt{\mathbf{g}^{-1} \mathbf{q}}), \tag{28}$$

where μ is the graviton’s mass and $\mathbf{q} = \{\mathbf{q}_{\alpha\beta}\}$ is the so-called non-dynamical reference metric; ${}^m\mathcal{L}$ is the Lagrangian for matter fields.

3.1. Nonholonomic Ghost-Free Massive Configurations

The term $\epsilon\mathbf{f}(\tilde{\mathbf{R}})$ in (26) contains possible contributions from a nontrivial graviton mass. Such a theory can be constructed to be ghost-free for very special conditions [14,15]; see explicit results and discussions on possible applications in modern cosmology in Refs [64–66]. In this section, we show how prescribing necessary type nonholonomic configurations such a theory can equivalently be realized as a TMT one (taking equal actions (26) and (27)). For any $(\hat{\mathbf{g}}, \mathbf{N}, \hat{\mathbf{D}})$, we consider the d-tensor $(\sqrt{\hat{\mathbf{g}}^{-1}\mathbf{q}})^\mu{}_\nu$ computed as the square root of $\hat{\mathbf{g}}^{\mu\rho}\mathbf{q}_{\rho\nu}$, where

$$(\sqrt{\hat{\mathbf{g}}^{-1}\mathbf{q}})^\mu{}_\rho(\sqrt{\hat{\mathbf{g}}^{-1}\mathbf{q}})^\rho{}_\nu = \hat{\mathbf{g}}^{\mu\rho}\mathbf{q}_{\rho\nu}, \text{ and } \sum_{k=0}^4 {}^k\beta e_k(\sqrt{\hat{\mathbf{g}}^{-1}\mathbf{q}}) = 3 - \text{tr}\sqrt{\hat{\mathbf{g}}^{-1}\mathbf{q}} - \det\sqrt{\hat{\mathbf{g}}^{-1}\mathbf{q}},$$

for some coefficients ${}^k\beta$. The values $e_k(\mathbf{Y})$ are defined for a d-tensor $\mathbf{Y}^\mu{}_\rho$ and $Y = [Y] := \text{tr}(\mathbf{Y}) = \mathbf{Y}^\mu{}_\mu$, where

$$\begin{aligned} e_0(Y) &= 1, e_1(Y) = Y, 2e_2(Y) = Y^2 - [Y^2], 6e_3(Y) = Y^3 - 3Y[Y^2] + 2[Y^3], \\ 24e_4(Y) &= Y^4 - 6Y^2[Y^2] + 3[Y^2]^2 + 8Y[Y^3] - 6[Y^4]; e_k(Y) = 0 \text{ for } k > 4. \end{aligned}$$

We chose the functional for Lagrange density in (27) in the form ${}^{F,\mu}\mathcal{L} = \mathbf{F}(\tilde{\mathbf{R}})$, where the functional dependence \mathbf{F} is different (in general) from $\mathbf{f}(\tilde{\mathbf{R}})$. For simplicity, we consider Lagrange densities for matter, ${}^m\mathcal{L}$, which only depend on the coefficients of a metric field and not on their derivatives. The energy–momentum d-tensor can be computed via N-adapted variational calculus,

$${}^m\hat{\mathbf{T}}_{\alpha\beta} := -\frac{2}{\sqrt{|\hat{\mathbf{g}}_{\mu\nu}|}} \frac{\delta(\sqrt{|\hat{\mathbf{g}}_{\mu\nu}|} {}^m\mathcal{L})}{\delta\hat{\mathbf{g}}^{\alpha\beta}} = {}^m\mathcal{L}\hat{\mathbf{g}}^{\alpha\beta} + 2\frac{\delta({}^m\mathcal{L})}{\delta\hat{\mathbf{g}}^{\alpha\beta}}. \tag{29}$$

Applying such a calculus to ${}^{F,\mu}\mathcal{S} + {}^m\mathcal{S}$, with ${}^1\mathbf{F}(\tilde{\mathbf{R}}) := d\mathbf{F}(\tilde{\mathbf{R}})/d\tilde{\mathbf{R}}$, see details in [14,15,64–66], we obtain the modified gravitational field equations

$$\hat{\mathbf{R}}_{\mu\nu} = {}^{F,\mu}\hat{\mathbf{Y}}_{\mu\nu}, \tag{30}$$

where ${}^{F,\mu}\hat{\mathbf{Y}}_{\mu\nu} = {}^m\hat{\mathbf{Y}}_{\mu\nu} + {}^f\hat{\mathbf{Y}}_{\mu\nu} + {}^\mu\hat{\mathbf{Y}}_{\mu\nu}$, for

$$\begin{aligned} {}^m\hat{\mathbf{Y}}_{\mu\nu} &= \frac{1}{2M_{\text{p}}^2} {}^m\hat{\mathbf{T}}_{\alpha\beta}, \quad {}^f\hat{\mathbf{Y}}_{\mu\nu} = \left(\frac{\mathbf{F}}{2} \frac{1}{1\mathbf{F}} - \frac{\hat{\mathbf{D}}^2 1\mathbf{F}}{1\mathbf{F}}\right)\hat{\mathbf{g}}_{\mu\nu} + \frac{\hat{\mathbf{D}}_\mu \hat{\mathbf{D}}_\nu 1\mathbf{F}}{1\mathbf{F}}, \\ {}^\mu\hat{\mathbf{Y}}_{\mu\nu} &= -2\mu^2[(3 - \text{tr}\sqrt{\hat{\mathbf{g}}^{-1}\mathbf{q}} - \det\sqrt{\hat{\mathbf{g}}^{-1}\mathbf{q}}) - \frac{1}{2}\det\sqrt{\hat{\mathbf{g}}^{-1}\mathbf{q}}]\hat{\mathbf{g}}_{\mu\nu} + \\ &\quad \frac{\mu^2}{2}\{\mathbf{q}_{\mu\rho}[(\sqrt{\hat{\mathbf{g}}^{-1}\mathbf{q}})^{-1}]^\rho{}_\nu + \mathbf{q}_{\nu\rho}[(\sqrt{\hat{\mathbf{g}}^{-1}\mathbf{q}})^{-1}]^\rho{}_\mu\}. \end{aligned} \tag{31}$$

The field equations for massive gravity (30) are constructed as nonholonomic deformations of the Einstein Equations (9) when the source $\hat{\mathbf{Y}}_{\beta\gamma} \rightarrow {}^{F,\mu}\hat{\mathbf{Y}}_{\mu\nu}$.

3.2. Tmt Massive Configurations with (Broken) Global Scaling Invariance

Let us explain the notations and terms used in the above actions, chosen in such forms that a TMT (26) is equivalent to a massive MGT model (27) when both classes of such theories are encoded via corresponding nonholonomic structures into a nonholonomically deformed Einstein gravity model (25). The non-Riemannian volume-forms (integration

measures on nonholonomic manifold $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$ in (26) are determined by two auxiliary 3-index antisymmetric d-tensor fields $\mathbf{A}_{\alpha\beta\gamma}$ and $\mathbf{B}_{\alpha\beta\gamma}$, when

$${}^1\Phi(\mathbf{A}) := \frac{1}{3!}\varepsilon^{\mu\alpha\beta\gamma}\mathbf{e}_\mu\mathbf{A}_{\alpha\beta\gamma} \text{ and } {}^2\Phi(\mathbf{B}) := \frac{1}{3!}\varepsilon^{\mu\alpha\beta\gamma}\mathbf{e}_\mu\mathbf{B}_{\alpha\beta\gamma}.$$

Nevertheless, for non-triviality of the TMT model, the presence of the 3D auxiliary antisymmetric d-tensor gauge field $\mathbf{H}_{\alpha\beta\gamma}$, when $\Phi(\mathbf{H}) := \frac{1}{3!}\varepsilon^{\mu\alpha\beta\gamma}\mathbf{e}_\mu\mathbf{H}_{\alpha\beta\gamma}$ is crucial. In order to model two flat regions for the inflationary and accelerating universe in some limits, we consider two Lagrange densities for a scalar field

$$\begin{aligned} {}^1L &= -\frac{1}{2}\mathbf{g}^{\mu\rho}(\mathbf{e}_\mu\varphi)(\mathbf{e}_\rho\varphi) - {}^1U(\varphi), \quad {}^1U(\varphi) = {}^1ae^{-q\varphi}; \\ {}^2L &= -\frac{{}^2b}{2}e^{-q\varphi}\mathbf{g}^{\mu\rho}(\mathbf{e}_\mu\varphi)(\mathbf{e}_\rho\varphi) + {}^2U(\varphi), \quad {}^2U(\varphi) = {}^2ae^{-2q\varphi}, \end{aligned} \tag{32}$$

with dimensional positive parameters q , 1a , 2a and a dimensionless one 2b . The action (26) is invariant under global N-adapted Weyl-scale transforms with a positive scale parameter λ , $\mathbf{g}_{\alpha\beta} \rightarrow \lambda\mathbf{g}_{\alpha\beta}$, $\varphi \rightarrow \varphi + q^{-1}\ln\lambda$, $\mathbf{A}_{\alpha\beta\gamma} \rightarrow \lambda\mathbf{A}_{\alpha\beta\gamma}$, $\mathbf{B}_{\alpha\beta\gamma} \rightarrow \lambda^2\mathbf{B}_{\alpha\beta\gamma}$ and $\mathbf{H}_{\alpha\beta\gamma} \rightarrow \mathbf{H}_{\alpha\beta\gamma}$. For holonomic configurations and quadratic functionals on LC-scalar $\mathbf{f}(\check{\mathbf{R}}) \rightarrow R^2$, such a theory is equivalent to that elaborated in [55–57,61,62]. In a more general context, the developments in this work involve non-quadratic nonlinear and nonholonomic functionals and mass gravity deformations via $\check{\mathbf{R}}$ (28), and generic off-diagonal interactions encoded in $\widehat{\mathbf{R}}$.

A variational N-adapted calculus on form fields $\mathbf{A}, \mathbf{B}, \mathbf{H}$ and on d-metric \mathbf{g} (with respect to coordinate bases and for ∇ , being similar to that presented in Section 2 of [61,62]), results in effective gravitational field equations

$$\widehat{\mathbf{R}}_{\mu\nu}[\widehat{\mathbf{g}}_{\alpha\beta}] = {}^{ef}\widehat{\mathbf{Y}}_{\mu\nu} + {}^{F,\mu}\widehat{\mathbf{Y}}_{\mu\nu}, \tag{33}$$

where ${}^{F,\mu}\widehat{\mathbf{Y}}_{\mu\nu}$ is determined by (31) and ${}^{ef}\widehat{\mathbf{Y}}_{\beta\gamma} := \varkappa({}^{ef}\widehat{\mathbf{T}}_{\alpha\beta} - \frac{1}{2}\widehat{\mathbf{g}}_{\alpha\beta}{}^{ef}\widehat{\mathbf{T}})$ is computed using Formulas (8) and (29) for $\mathbf{g}_{\alpha\beta} \rightarrow \widehat{\mathbf{g}}_{\alpha\beta}$ and $\widehat{\mathcal{L}} \rightarrow {}^{ef}\mathcal{L}$, where

$$\begin{aligned} \widehat{\mathbf{g}}_{\alpha\beta} &= \Theta\mathbf{g}_{\alpha\beta}, \text{ for } \Theta = {}^1\chi - {}^2\chi\epsilon^{}{}^1\mathbf{f}({}^1L + {}^1M, \mu); \\ {}^{ef}L &= \Theta^{-1}\left\{{}^1L + {}^1M + {}^2\chi\Theta^{-1}\left[{}^2L + {}^1M + \epsilon^{}{}^1\mathbf{f}({}^1L + {}^1M, \mu)\right]\right\}, \end{aligned} \tag{34}$$

when the conformal factor Θ for the Weyl re-scaling of d-metric is induced by the nonlinear functional in the action

$${}^1\mathbf{f}({}^1L + {}^1M, \mu) = \frac{d\mathbf{f}(\widehat{\mathbf{R}}, \mu)}{d\widehat{\mathbf{R}}} \Big|_{\widehat{\mathbf{R}} = {}^1L + {}^1M} \tag{35}$$

and the two measure functionals ${}^1\chi = {}^1\Phi(\mathbf{A})/\sqrt{|\widehat{\mathbf{g}}_{\mu\nu}|}$ and ${}^2\chi = {}^2\Phi(\mathbf{B})/\sqrt{|\widehat{\mathbf{g}}_{\mu\nu}|}$.

The variations in auxiliary anti-symmetric form fields impose certain constants

$$\mathbf{e}_\mu(\widehat{\mathbf{R}} + {}^1L) = 0, \mathbf{e}_\mu[{}^2L + \epsilon\mathbf{f}(\check{\mathbf{R}}) + \Phi(\mathbf{H})/\sqrt{|\mathbf{g}|}] = 0, \mathbf{e}_\mu[{}^2\Phi(\mathbf{B})/\sqrt{|\widehat{\mathbf{g}}_{\mu\nu}|}] = 0.$$

The nonconstant solutions of such nonholonomic constraints allow to preserve the global Weyl-scale invariance for certain configurations. If we take constant values

$$\widehat{\mathbf{R}} + {}^1L = -{}^1M = \text{const} \text{ and } {}^2L + \epsilon\mathbf{f}(\check{\mathbf{R}}) + \Phi(\mathbf{H})/\sqrt{|\mathbf{g}|} = -{}^2M = \text{const} \tag{36}$$

we select configuration with a nonholonomic dynamical spontaneous breakdown of global Weyl-scale invariance when the condition

$${}^2\Phi(\mathbf{B})/\sqrt{|\widehat{\mathbf{g}}|} = {}^2\chi = \text{const} \tag{37}$$

preserves the scale invariance. There are certain constraints on the scale factor ${}^1\chi = {}^1\Phi(\mathbf{A})/\sqrt{|\widehat{\mathbf{g}}|}$, which can be derived from variations in (26) on $\mathbf{g}_{\mu\nu}$ in N-adapted form. The conditions (36) relate ${}^1\chi$ and ${}^2\chi$, i.e., the integration measures, to traces ${}^{1,2}\mathbf{T} := \mathbf{g}^{\alpha\beta} {}^{1,2}\mathbf{T}_{\alpha\beta}$ of the energy momentum tensors

$${}^{1,2}\mathbf{T}_{\alpha\beta} = \mathbf{g}_{\alpha\beta} {}^{1,2}L - 2\partial({}^{1,2}L)/\partial\mathbf{g}^{\alpha\beta}$$

of Lagrangians for scalar fields (32). (For simplicity, we consider matter actions which only depend on the coefficients of certain effective metric fields and not on their derivatives.) This follows from the N-adapted variation on $\mathbf{g}_{\alpha\beta}$ of the action (26), taken, for simplicity, with zero ${}^m\mathcal{L}$, which results in

$$2 {}^1\chi \left[\widehat{\mathbf{R}}_{\mu\nu}(\mathbf{g}_{\alpha\beta}) + \mathbf{g}_{\mu\nu} {}^1L - {}^1\mathbf{T}_{\mu\nu} \right] - 2 {}^2\chi \left[{}^2\mathbf{T}_{\mu\nu} + \mathbf{g}_{\mu\nu} (\epsilon\mathbf{f}(\widehat{\mathbf{R}}) + {}^2M) - {}^1\mathbf{f} \widehat{\mathbf{R}}_{\mu\nu}(\mathbf{g}_{\alpha\beta}) \right] = 0. \tag{38}$$

Taking the trace of these equations and using (36), we obtain the formula ${}^1\chi = {}^2\chi \frac{{}^2\mathbf{T} + 2 {}^2M}{{}^2\mathbf{T} - 2 {}^1M}$, which does not depend on the type of f -modifications containing possible μ -terms. We conclude that the non-Riemannian integration measures considered above, and the interactions of scalar fields (32), can be modelled as additional distributions on nonholonomic manifold $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$. They modify the conformal factor Θ (34) and can express the field Equations (38) in Einstein like form (33), where ${}^{F,\mu}\widehat{\mathbf{Y}}_{\mu\nu}$ is added as an additional effective matter contribution to the source of scalar fields ${}^{1,2}\mathbf{T}_{\alpha\beta}$.

It should be noted that, using the canonical d-connection, we obtain $\widehat{\mathbf{D}}_{\alpha} \mathbf{T}^{\alpha\beta} = \mathbf{Q}^{\beta} \neq 0$, when $\mathbf{Q}_{\beta}[\mathbf{g}, \mathbf{N}]$ is completely defined by the d-metric and chosen N-connection structure. Considering nonholonomic distortions with $\nabla = \widehat{\mathbf{D}} - \widehat{\mathbf{Z}}$, we obtain standard relations

$$\nabla^{\alpha} (R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R) = 0 \text{ and } \nabla^{\alpha} Y_{\alpha\beta} = 0.$$

A similar property exists in Lagrange mechanics with non-integrable constraints when the standard conservation laws do not hold true. A new class of effective variables and new types of conservation laws can be introduced and, respectively, constructed using Lagrange multiples.

The main conclusion of this section is that various MGTs with two integration measures, possible deformations by mass graviton terms, bi-connection and bi-metric structures can be expressed as nonholonomic deformations of the Einstein equations in the form (9). Different theories are characterized by respective sources (in explicit form, ${}^{F,\mu}\widehat{\mathbf{Y}}_{\mu\nu}$ in (30), or ${}^{\epsilon f}\widehat{\mathbf{Y}}_{\mu\nu} + {}^{F,\mu}\widehat{\mathbf{Y}}_{\mu\nu}$ in (33)). Our next goal is to prove that such effective Einstein equations can be integrated in certain general forms for $\widehat{\mathbf{D}}$ and possible constraints (10) for LC-configurations.

4. Cosmological Solutions in Effective Einstein Gravity and FmgtS

We can generate explicit integral varieties of systems of PDEs of type (9) for d-metrics $\widehat{\mathbf{g}}$ (34) and sources $\widehat{\mathbf{Y}}_{\beta\gamma} = {}^{\epsilon f}\widehat{\mathbf{Y}}_{\mu\nu} + {}^{F,\mu}\widehat{\mathbf{Y}}_{\mu\nu}$ as in (33) which, via frame and coordinate transforms,

$$\widehat{\mathbf{g}}_{\alpha\beta} = e^{\alpha'}_{\alpha}(x^i, y^a) e^{\beta'}_{\beta}(x^i, y^a) \widehat{\mathbf{g}}_{\alpha'\beta'}(x^i, t) \text{ and } \widehat{\mathbf{Y}}_{\alpha\beta} = e^{\alpha'}_{\alpha}(x^i, y^a) e^{\beta'}_{\beta}(x^i, y^a) \widehat{\mathbf{Y}}_{\alpha'\beta'}(x^i, t),$$

for a time like coordinate $y^4 = t$ ($i', i, k, k', \dots = 1, 2$ and $a, a', b, b', \dots = 3, 4$), can be parameterized in the form

$$\begin{aligned} \widehat{\mathbf{g}} &= \widehat{\mathbf{g}}_{\alpha'\beta'} \mathbf{e}^{\alpha'} \otimes \mathbf{e}^{\beta'} = g_i(x^k) dx^i \otimes dx^j + \omega^2(x^k, y^3, t) h_a(x^k, t) \mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^3 &= dy^3 + n_i(x^k, t) dx^i, \mathbf{e}^4 = dt + w_i(x^k, t) dx^i, \end{aligned} \tag{39}$$

for nontrivial

$$\{g_{i'j'}\} = \text{diag}[g_i], g_1 = g_2 = e^{\psi(x^k)}; \{g_{a'b'}\} = \text{diag}[h_a], h_a = h_a(x^k, t);$$

$$N_i^3 = n_i(x^k, t); N_i^4 = w_i(x^k, t);$$

and

$$\hat{Y}_{\alpha'\beta'} = \text{diag}[Y_i; Y_a], \text{ for } Y_1 = Y_2 = \tilde{Y}(x^k) = {}^{ef}\tilde{Y}(x^k) + {}^m\tilde{Y}(x^k) + {}^f\tilde{Y}(x^k) + {}^\mu\tilde{Y}(x^k),$$

$$Y_3 = Y_4 = Y(x^k, t) = {}^{ef}Y(x^k, t) + {}^mY(x^k, t) + {}^fY(x^k, t) + {}^\mu Y(x^k, t). \tag{40}$$

These ansatz for the d-metric and sources are very general, but for an assumption that there are N-adapted frames with respect to which the MGTs interactions have Killing symmetry on $\partial/\partial y^3$ when geometric and physical values do not depend on coordinate y^3 . (It should be noted that it is possible to construct very general classes of generic off-diagonal solutions depending on all spacetime variables in arbitrary finite dimensions; see details and examples in [26–29] for more “non-Killing” configurations. For simplicity, we shall study nonhomogeneous and locally anisotropic cosmological solutions in this work, depending on variables (x^k, t) , with smooth limits to cosmological diagonal configurations depending only on t and very small off-diagonal contributions characterized by a small parameter $\varepsilon, 0 \leq \varepsilon \ll 1$.) We use parameterizations $g_1 = g_2 = e^{\psi(x^i)}$ and $h_a(x^k, t)$ for $i, j, \dots = 1, 2$ and $a, b, \dots = 3, 4$; and N-connection coefficients $N_i^3 = n_i(x^k, t)$ and $N_i^4 = w_i(x^k, t)$. Introducing brief denotations for partial derivatives, like $a^\bullet = \partial_1 a, b' = \partial_2 b, h^* = \partial_4 h = \partial_t h$ and defining the values $\alpha_i = h_3^* \partial_i \omega, \beta = h_3^* \omega^*, \gamma = (\ln |h_3|^{3/2} / |h_4|)^*$ for a generating function

$$\omega := \ln |h_3^* / \sqrt{|h_3 h_4|}|, \text{ we shall also use the value } \Psi := e^\omega, \tag{41}$$

we transform (33) into a nonlinear system of PDEs with decoupling properties for the unknown functions $\psi(x^i), h_a(x^k, t), w_i(x^k, t)$ and $n_i(x^k, t)$,

$$\psi^{\bullet\bullet} + \psi'' = 2 \tilde{Y}, \omega^* h_3^* = 2 h_3 h_4 Y, n_i^{**} + \gamma n_i^* = 0, \beta w_i - \alpha_i = 0. \tag{42}$$

This system possesses another very important property, which allows us to redefine the generating function, $\Psi \longleftrightarrow \tilde{\Psi}$, when $\Lambda(\Psi^2)^* = |Y|(\tilde{\Psi}^2)^*$ and

$$\Lambda \Psi^2 = \tilde{\Psi}^2 |Y| + \int dt \tilde{\Psi}^2 |Y|^* \tag{43}$$

for $\tilde{\Psi} := \exp \tilde{\omega}$ and any prescribed values of effective (for different types of contribution ef, m, f, μ) cosmological constants in $\Lambda = {}^{ef}\Lambda + {}^m\Lambda + {}^f\Lambda + {}^\mu\Lambda$ associated, respectively, with

$$Y(x^k, t) = {}^{ef}Y(x^k, t) + {}^mY(x^k, t) + {}^fY(x^k, t) + {}^\mu Y(x^k, t).$$

For generating off-diagonal cosmological solutions depending on t , we have to consider generating functions, for which $\Psi^* \neq 0$. The Equations (42) for ansatz (39) transform, respectively, into a system of nonlinear PDEs

$$\psi^{\bullet\bullet} + \psi' = 2 \tilde{Y}, \tilde{\omega}^* h_3^* = 2 h_3 h_4 \Lambda, n_i^{**} + \gamma n_i^* = 0, \omega^* w_i - \partial_i \omega = 0 \tag{44}$$

and $\omega^* \partial_i \omega - \omega^* \partial_i \omega = 0$, for the vertical conformal factor.

We have to subject the d-metric and N-connection coefficients to additional constraints (10) in order to satisfy the torsionless conditions, which for the ansatz (39) are written

$$w_i^* = (\partial_i - w_i \partial_4) \ln \sqrt{|h_4|}, (\partial_i - w_i \partial_4) \ln \sqrt{|h_3|} = 0, \partial_i w_j = \partial_j w_i, n_i^* = 0, \partial_i n_j = \partial_j n_i. \tag{45}$$

We can generate exact solutions in TMT, MGT and nonholonomically deformed Einstein theories with respective actions (25), (26) and (27) using integral varieties (The term “integral variety” is used in the theory of differential equations for certain “classes of solutions”, determined by corresponding classes of parameters, generating and integration functions, etc. In GR, we search for an integral variety of solutions of associated systems of PDEs determining, for instance, Einstein spacetimes, black holes, and cosmological solutions. In modified gravity theories, we can establish an analogy for GR if we consider effective models.) of the system of PDEs (44), which can be found in very general forms. Let us briefly explain this geometric formalism elaborated in the framework of the AFDM (see details in [30–33]):

1. The first equation for ψ is the 2D Laplace/d’ Alambert equation which can be solved for any given \tilde{Y} , which allows us to find $g_1 = g_2 = e^{\psi(x^k)}$.
2. Using the second equation in (44) and (41), the coefficients h_a can be expressed as functionals on (Ψ, Y) . We redefine the generating function as in (43) and consider an effective source

$$\Xi := \int dt Y(\tilde{\Psi}^2)^* = {}^{ef}\Xi + {}^m\Xi + {}^f\Xi + {}^\mu\Xi,$$

when ${}^{ef}\Xi := \int dt {}^{ef}Y(\tilde{\Psi}^2)^*$, ${}^m\Xi := \int dt {}^mY(\tilde{\Psi}^2)^*$, ${}^f\Xi := \int dt {}^fY(\tilde{\Psi}^2)^*$, and write

$$h_3 = \frac{\tilde{\Psi}^2}{4({}^{ef}\Lambda + {}^m\Lambda + {}^f\Lambda + {}^\mu\Lambda)} \text{ and } h_4 = \frac{(\tilde{\Psi}^*)^2}{({}^{ef}\Xi + {}^m\Xi + {}^f\Xi + {}^\mu\Xi)}.$$

3. We have to integrate t twice in order to find it in the 3D subset of equations in (44)

$$n_i = {}_1n_i + {}_2n_i \int dt \frac{(\tilde{\Psi}^*)^2}{\tilde{\Psi}^3({}^{ef}\Xi + {}^m\Xi + {}^f\Xi + {}^\mu\Xi)}$$

for some integration functions ${}_1n_i(x^k)$ and ${}_2n_i(x^k)$.

4. The 4th set of equations in (44) are algebraic ones, which allows us to compute

$$w_i = [\omega^*]^{-1} \partial_i \omega = [\Psi^*]^{-1} \partial_i \Psi = [(\Psi^2)^*]^{-1} \partial_i (\Psi)^2 = [\Xi^*]^{-1} \partial_i \Xi.$$

5. We can satisfy the conditions for ω in the second line in (44) if we keep the Killing symmetry on ∂_i and take, for instance, $\omega^2 = |h_4|^{-1}$.

Different types of inhomogeneous cosmological solutions of the system (33) are determined by corresponding classes of effective sources

generating functions:	$\psi(x^k), \tilde{\Psi}(x^k, t), \omega(x^k, y^3, t)$
effective sources:	$\tilde{Y}(x^k); {}^{ef}\Xi(x^k, t), {}^m\Xi(x^k, t), {}^f\Xi(x^k, t), {}^\mu\Xi(x^k, t),$ or ${}^{ef}Y(x^k, t), {}^mY(x^k, t), {}^fY(x^k, t), {}^\mu Y(x^k, t)$
integration cosm. constants:	${}^{ef}\Lambda, {}^m\Lambda, {}^f\Lambda, {}^\mu\Lambda$
integration functions:	${}_1n_i(x^k)$ and ${}_2n_i(x^k)$

We can generate solutions with any nontrivial ${}^{ef}\Lambda, {}^m\Lambda, {}^f\Lambda, {}^\mu\Lambda$ even any, or all, effective source ${}^{ef}Y, {}^mY, {}^fY, {}^\mu Y$ can be zero.

4.1. Inhomogeneous FTMT and MGT Configurations with Induced Nonholonomic Torsion

The solutions with coefficients computed above in 1–5 can be parametrized to describe nonholonomic deformations, $\hat{g}_{\alpha\beta} = e^{\alpha'} e^{\beta'} \hat{g}_{\alpha'\beta'}$, of the Friedman–Lemaître–Robertson–Walker (FLRW) diagonal quadratic element (we can consider spherical symmetry coor-

dinates $u^{\alpha'} = (x^{1'} = r, x^{2'} = \theta, y^{3'} = \varphi, y^{4'} = t)$, or Cartesian ones, $u^{\alpha'} = (x^{1'} = x, x^{2'} = y, y^{3'} = z, y^{4'} = t)$, for a scale factor $\hat{a}(t)$ determining the Hubble constant $H := \hat{a}^*/\hat{a}$.

$$d\hat{s}^2 = \hat{g}_{\alpha'\beta'} du^{\alpha'} du^{\beta'} = \hat{a}^2(t)[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2] - dt^2 \tag{46}$$

into a generic off-diagonal inhomogeneous cosmological metric of type (39) with $g_i = \eta_i e^\psi$ and $h_a = \eta_a \hat{g}_a$ with effective polarization functions $\eta_1 = \eta_2 = a^{-2} e^\psi, \eta_3 = \hat{a}^{-2} h_3, \eta_4 = 1$ and $\hat{h}_3 = h_3/a^2 |h_4|$, when

$$\begin{aligned} ds^2 = & a^2(x^k, t)[\eta_1(x^k, t)(dx^1)^2 + \eta_2(x^k, t)(dx^2)^2] \\ & + a^2(x^k, t)\hat{h}_3(x^k, t)[dy^3 + ({}_1n_i + {}_2n_i \int dt \frac{(\tilde{\Psi}^*)^2}{\tilde{\Psi}^3({}^{ef}\Xi + {}^m\Xi + f\Xi + \mu\Xi)})dx^i]^2 \\ & - [dt + \frac{\partial_i({}^{ef}\Xi + {}^m\Xi + f\Xi + \mu\Xi)}{({}^{ef}\Xi + {}^m\Xi + f\Xi + \mu\Xi)^*} dx^i]^2. \end{aligned} \tag{47}$$

The inhomogeneous scaling factor $a(x^k, t)$ in (47) is related to the generating function $\tilde{\Psi}$ via formula

$$a^2 \hat{h}_3 = \omega^2 h_3 = \frac{h_3}{|h_4|} = \frac{\tilde{\Psi}^2 |{}^{ef}\Xi + {}^m\Xi + f\Xi + \mu\Xi|}{4({}^{ef}\Lambda + {}^m\Lambda + f\Lambda + \mu\Lambda)(\tilde{\Psi}^*)^2}.$$

In general, such target metrics $\hat{g}_{\alpha\beta}(x^k, t)$ determine new classes of cosmological metrics with nontrivial nonholonomically induced torsion computed for $\hat{\mathbf{D}}$. Such modified spacetimes cannot be diagonalized by coordinate transforms if the anholonomy coefficients $W_{\alpha\beta}^\gamma$ are not zero. For trivial gravitational polarizations, $\eta_\alpha = 1$, trivial N-connection coefficients, $N_i^3 = n_i = 0$ and $N_i^4 = w_i = 0$, and for $a(x^k, t) \rightarrow \hat{a}(t)$, we obtain torsionless FLRW metrics. We emphasize that one could not have smooth limits $\hat{g}_{\alpha\beta} \rightarrow \hat{g}_{\alpha\beta}$ for the arbitrary generating function $\tilde{\Psi}$ and any nontrivial effective cosmological constant ${}^{ef}\Lambda, {}^m\Lambda, f\Lambda, \text{ or } \mu\Lambda$, associated with respective mater fields.

We can generate off-diagonal cosmological configurations as “small” deformations with $\eta_\alpha = 1 + \epsilon_\alpha, n_i = \epsilon n_i$ and $w_i = \epsilon w_i$, with $|\epsilon_\alpha|, |\epsilon n_i|, |\epsilon w_i| \ll 1$. In particular, we can only study TMT models if ${}^m\Xi = f\Xi = \mu\Xi = 0$ and ${}^m\Lambda = f\Lambda = \mu\Lambda = 0$ but ${}^{ef}\Upsilon(x^k, t) \neq 0$ and ${}^{ef}\Lambda \neq 0$. off-diagonal cosmological scenarios in massive and bi-metric gravity with nontrivial ${}^m\Xi$ and ${}^m\Lambda$ were studied in our recent works [14,15]. Other classes of MGTs and cosmological models with off-diagonal configurations when f -modified gravity effects, modelled in GR, were studied in [26–33]. The goal of Section 6 is to show how TMT gravity and cosmological models can be associated with certain nonholonomic off-diagonal de Sitter configurations with nontrivial ${}^{ef}\Lambda$ for an effective Einstein–Lagrange spacetime, and such constructions can be generalized to reproduce MGTs and massive gravity.

4.2. Extracting Levi–Civita Cosmological Configurations

Let us show how we can generate, in explicit form, solutions to the system (45) for nonholonomic, generic, off-diagonal configurations with zero torsion. We have to consider certain special classes of generating and integration functions. By straightforward computations, we can check that such conditions are satisfied if we state such conditions for a metric (47) that

$$\begin{aligned} {}_2n_i &= 0 \text{ and } {}_1n_i = \partial_i n(x^k), \text{ for any } n(x^k) \\ \Psi &= \check{\Psi}, \text{ for } (\partial_i \check{\Psi})^* = \partial_i(\check{\Psi}^*) \text{ and find a function } \check{A}(x^k, t) \text{ when} \\ \partial_i w_j &= \partial_j w_i \text{ for } w_i = \check{w}_i = \partial_i \check{\Psi} / \check{\Psi}^* = \partial_i \hat{\Xi} / \hat{\Xi}^* = \partial_i \check{A} \end{aligned} \tag{48}$$

when

$$\Lambda \hat{\Psi}^2 = \hat{\Psi}^2 |Y| + \int dt \hat{\Psi}^2 |Y|^* \text{ and } \hat{\Xi} := \int dt Y (\hat{\Psi}^2)^*$$

are computed with the following formulas (43), except for $\Psi(\tilde{\Psi}) \rightarrow \check{\Psi}(\hat{\Psi})$ and $\tilde{\Psi} \rightarrow \hat{\Psi}$. For certain configurations, we can consider functional dependencies $\hat{\Psi} = \hat{\Psi}(\ln \sqrt{|h_3|})$ and invertible functional dependencies $h_3[\hat{\Psi}[\check{\Psi}]]$. In such cases, we take $h_3(x^k, t)$ as a generating function and consider necessary type functionals $\check{\Psi}[h_3]$ with the property $(\partial_i \check{\Psi})^* = \partial_i (\check{\Psi}^*)$ which are used for defining $\check{w}_i[h_3] = \partial_i \check{\Psi} / \check{\Psi}^* = \partial_i \check{A}[h_3]$.

Putting together the conditions (48), we generate nonhomogeneous cosmological LC-configurations with quadratic linear elements

$$\begin{aligned} ds^2 &= \check{a}^2(x^k, t) [\eta_1(x^k, t)(dx^1)^2 + \eta_2(x^k, t)(dx^2)^2] + \check{a}^2(x^k, t) \hat{h}_3(x^k, t) [dy^3 + (\partial_i n) dx^i]^2 \\ &\quad - [dt + (\partial_i \check{A}) dx^i]^2 \\ &= e^{\psi(x^k)} [(dx^1)^2 + (dx^2)^2] + \frac{\hat{\Psi}^2}{4(e^f \Lambda + {}^m \Lambda + {}^f \Lambda + {}^\mu \Lambda)} [dy^3 + (\partial_i n) dx^i]^2 \\ &\quad - \frac{(\hat{\Psi}^*)^2}{\hat{\Xi}} [dt + (\partial_i \check{A}) dx^i]^2. \end{aligned} \tag{49}$$

The inhomogeneous scaling factor $\check{a}(x^k, t)$ is computed similarly to (47), but using the generating function $\hat{\Psi}$,

$$\check{a}^2 \hat{h}_3 = \frac{\hat{\Psi}^2 |\hat{\Xi}|}{4(e^f \Lambda + {}^m \Lambda + {}^f \Lambda + {}^\mu \Lambda) (\hat{\Psi}^*)^2} \text{ for } \check{\Psi} := e^{\check{\omega}}.$$

Having constructed a class of generic off-diagonal solutions (49), we can impose additional constraints on the generating/integration functions and constants and source in order to explain certain observational cosmological data. For instance, we can fix subclasses of functions $\hat{\Psi} \rightarrow \hat{\Psi}(t), (\partial_i \check{A}) \rightarrow w_i(t)$ etc., describing small deformations of an FLRW metric (46) in a nonlinear parametric way, as well as redefined generating functions (43) and different types of effective source in TMT, MGT and/or massive gravity models.

5. Locally Anisotropic Effective Scalar Potentials and Flat Regions

We study three examples of off-diagonal cosmological solutions reproducing the TMT model with two flat regions of the effective scalar potential studied in [60], than analyse how massive gravity can be modelled as a TMT theory and effective GR, and (in the last subsection) we speculate on non-singular emergent anisotropic universes. The solutions in this section will be constructed to contain nontrivial nonholonomically induced torsion, as for quadratic elements (47). For certain important limits, LC-configurations of type (49) will also be examined.

5.1. Off-Diagonal Interactions and Associated Tmt Models with Two Flat Regions

We chose the nontrivial off-diagonal data in (47) for ${}^m \Lambda = {}^f \Lambda = {}^\mu \Lambda = 0$ and ${}^m Y = {}^f Y = {}^\mu Y = 0$ resulting in ${}^m \Xi = {}^f \Xi = {}^\mu \Xi = 0$, but considering nonzero ${}^e f \Lambda$ and ${}^e f Y$ is taken as a one-Killing configuration, not depending on y^3 in

$${}^e f \hat{Y}_{\beta\gamma} := \varkappa ({}^e f \hat{T}_{\alpha\beta} - \frac{1}{2} \hat{\mathbf{g}}_{\alpha\beta} {}^e f \hat{T})$$

is computed using formula (8) and (29) for $\mathbf{g}_{\alpha\beta} \rightarrow \widehat{\mathbf{g}}_{\alpha\beta}$ and $\widehat{\mathcal{L}} \rightarrow {}^{ef}\mathcal{L}$ for two scalar densities (32) as in (34). We generate solutions of $\widehat{\mathbf{R}}_{\mu\nu}[\widehat{\mathbf{g}}_{\alpha\beta}] = {}^{ef}\widehat{\mathbf{Y}}_{\mu\nu}$ (in a particular case of (33)) for $\widehat{\mathbf{g}}_{\alpha\beta}(x^k, t) = \widehat{\Theta}(x^k, t)\mathbf{g}_{\alpha\beta}(x^k, t)$, parameterized in the form

$$ds^2 = \widehat{\mathbf{g}}_{\alpha\beta}\mathbf{e}^\alpha\mathbf{e}^\beta = \widetilde{a}^2(x^k, t)[\eta_1(x^k, t)(dx^1)^2 + \eta_2(x^k, t)(dx^2)^2 + \widetilde{a}^2(x^k, t)\widehat{h}_3(x^k, t)[dy^3 + (\mathbf{}^1n_i + \mathbf{}^2n_i \int dt \frac{(\widetilde{\Psi}^*)^2}{\widetilde{\Psi}_3} {}^{ef}\Xi)dx^i]^2 - [dt + \frac{\partial_i({}^{ef}\Xi)}{({}^{ef}\Xi)^*}dx^i]^2. \tag{50}$$

The inhomogeneous scaling factor $\widetilde{a}(x^k, t)$ is related to the generating function $\widetilde{\Psi}$ via formula

$$\widetilde{a}^2\widehat{h}_3 = \omega^2h_3 = \frac{h_3}{|h_4|} = \frac{\widetilde{\Psi}^2|{}^{ef}\Xi|}{4({}^{ef}\Lambda)(\widetilde{\Psi}^*)^2}. \tag{51}$$

Choosing a function $\widetilde{\Psi}$, we prescribe a corresponding dependence for $\widehat{\Theta}(x^k, t)$ and, respectively, $\widetilde{a}(x^k, t)$ as follows from the above formulas. Let us speculate on the structure of $\widehat{\Theta}$, which describes off-diagonal generalizations of the model given by Formulas (18)–(23) in [60] on the assumption that the relation (35) for zero-graviton mass and quadratic Ricci scalar curvature has the limit

$$\mathbf{}^1\mathbf{f}(\mathbf{}^1L + \mathbf{}^1M, \mu = 0) = \frac{d\mathbf{f}(\widehat{\mathbf{R}}, \mu = 0)}{d\widehat{\mathbf{R}}} \Big|_{\widehat{\mathbf{R}} = \mathbf{}^1L + \mathbf{}^1M} \rightarrow \mathbf{}^1U - \mathbf{}^1M.$$

In this subsection, we consider $\mathbf{}^1\mathbf{f} \approx \mathbf{}^1U - \mathbf{}^1M$ for a nonhomogeneous $\varphi(x^k, t) \approx \varphi(t)$ in order to construct cosmological TMT models with limits to diagonal two flat regions.

We consider $\widehat{\Theta}$ as a conformal factor Θ in (34) not depending on y^3 , written in explicit form for an Einstein N-adapted frame with effective scalar Lagrangian

$$\begin{aligned} {}^{ef}\widehat{\mathcal{L}} &= \widehat{\Theta}^{-1}\{\mathbf{}^1\widehat{L} + \mathbf{}^1M + \mathbf{}^2\chi\widehat{\Theta}^{-1}[\mathbf{}^2\widehat{L} + \mathbf{}^1M + \epsilon(\mathbf{}^1\mathbf{f})^2]\} \\ &= A(\varphi)X + B(\varphi)X^2 - {}^{ef}U(\varphi), \end{aligned} \tag{52}$$

where we omit cumbersome formulas for $A(\varphi)$ and $B(\varphi)$ in the second line (see similar ones given by Formulas (24) and (25) in [60]), but present

$$\begin{aligned} \mathbf{}^1\widehat{L} &= \widehat{\Theta} X - \mathbf{}^1U \text{ for } \mathbf{}^2\widehat{L} = \frac{\mathbf{}^2b}{\mathbf{}^1a}\widehat{\Theta} \mathbf{}^1U\widehat{X} + \mathbf{}^2U \text{ for } \widehat{X} = -\frac{1}{2}\widehat{\mathbf{g}}^{\alpha\beta}\mathbf{e}_\alpha\varphi\mathbf{e}_\beta\varphi, \\ {}^{ef}U &= \frac{(\mathbf{}^1\mathbf{f})^2}{4\mathbf{}^2\chi[\mathbf{}^2U + \mathbf{}^2M + \epsilon(\mathbf{}^1\mathbf{f})^2]}. \end{aligned} \tag{53}$$

For simplicity, we can construct off-diagonal configurations with $\widehat{h}_3 \simeq 1$ in (51), prescribing a value ${}^{ef}\Lambda$ corresponding to observational data in the accelerating Universe, and computing ${}^{ef}\Xi$ for ${}^{ef}\widehat{\mathcal{L}}$ using formulas

$${}^{ef}\widehat{\mathbf{T}}_{\alpha\beta} := {}^{ef}\widehat{L}\widehat{\mathbf{g}}^{\alpha\beta} + 2\frac{\delta({}^{ef}\widehat{\mathcal{L}})}{\delta\widehat{\mathbf{g}}_{\alpha\beta}} \text{ and } {}^{ef}\widehat{\mathbf{Y}}_{\beta\gamma} := \varkappa({}^{ef}\widehat{\mathbf{T}}_{\alpha\beta} - \frac{1}{2}\widehat{\mathbf{g}}_{\alpha\beta}{}^{ef}\widehat{\mathcal{T}})$$

and constraints of type (40),

$${}^{ef}\widehat{\mathbf{Y}}_{\alpha'\beta'} = \text{diag}[\widehat{\mathbf{Y}}_i; \widehat{\mathbf{Y}}_a], \text{ for } \widehat{\mathbf{Y}}_1 = \widehat{\mathbf{Y}}_2 = {}^{ef}\widehat{\mathbf{Y}}(x^k), \widehat{\mathbf{Y}}_3 = \widehat{\mathbf{Y}}_4 = {}^{ef}\widehat{\mathbf{Y}}(x^k, t).$$

Then, we can compute ${}^{ef}\Xi := \int dt {}^{ef}\widehat{\mathbf{Y}}(\widetilde{\Psi}^2)^*$. Such a problem can be also solved in inverse form for a given $\widetilde{a}(x^k, t)$, when $\widetilde{\Psi}$ has to be defined from an integro-differential Equation (51), $\widetilde{a}^2 = \frac{\widetilde{\Psi}^2 \int dt {}^{ef}\widehat{\mathbf{Y}}(t)(\widetilde{\Psi}^2)^*}{4({}^{ef}\Lambda)(\widetilde{\Psi}^*)^2}$. For cosmological solutions, we can consider $\widetilde{a}(x^k, t) \simeq \widetilde{a}(t)$ and $\widetilde{\Psi}(x^k, t) \simeq \widetilde{\Psi}(t)$, when the generation function $\widetilde{\Psi}(t)$ is prescribed to depend only on time-like coordinate t . The observable effective scaling factor $\widetilde{a}(t)$ is

expressed as a functional on constant ${}^{ef}\Lambda$, on TMT source ${}^{ef}\hat{Y}(t)$ and generating function $\tilde{\Psi}(t)$. For instance, variations in ${}^{ef}\hat{Y}(t)$ are determined by the variation in the second auxiliary 3-index antisymmetric d-tensor field $\mathbf{B}_{\alpha\beta\gamma}$ in ${}^2\Phi(\mathbf{B})$ in the Formula (37). We adapt and write a similar formula with “tilde” values, in order to emphasize that the values are computed for a prescribed value $\tilde{a}(t)$,

$${}^2\Phi(\tilde{\mathbf{B}})/\sqrt{|\tilde{\mathbf{g}}|} = {}^2\tilde{\chi} = const. \tag{54}$$

There are two options to fix a constant ${}^2\tilde{\chi}$: the first one is to choose a function $\tilde{\Psi}$ and/or to modify $\tilde{\mathbf{B}}$ in the second measure. In general, this is a nonlinear effect of the re-definition of generation functions (43), which holds for generic off-diagonal configurations. We can finally prescribe some small off-diagonal corrections but the diagonal values will be re-scaled (we shall maintain “tilde” in order to distinguish such values from similar ones computed from the very beginning, using diagonalized equations).

The main conclusion of this subsection is that by working with generic off-diagonal solutions for effective Einstein Equations (33)—see equivalent Formulas (38)—we can choose generating functions and effective sources that allow us to reproduce, in generalized forms, the properties of TMT gravity theories determined by action (26) and scalar Lagrangians (32). In the next subsection, we prove that such models may be generated to have limits to diagonal two flat regions reproducing accelerating cosmology scenarios.

5.2. Limits to Diagonal Two Flat Regions

Let us consider in ${}^{ef}\hat{L}$ (52) the approximation

$${}^1\mathbf{f}({}^1L + {}^1M, \mu = 0) = \frac{d\mathbf{f}(\hat{\mathbf{R}}, \mu = 0)}{d\hat{\mathbf{R}}} \Big|_{\hat{\mathbf{R}} = {}^1L + {}^1M} \rightarrow {}^1U - {}^1M \tag{55}$$

with $\tilde{\Psi}(t)$ and $\tilde{a}(t)$ resulting in diagonal cosmological solutions with effective FLRW metrics. We approximate the effective potential ${}^{ef}U$ (53) for a prescribed constant ${}^2\tilde{\chi}$ by a relation (54),

$$\begin{aligned} {}^{ef}U &= \frac{({}^1U - {}^1M)^2}{4 {}^2\tilde{\chi} [{}^2U + {}^2M + ({}^1U - {}^1M)^2]} \simeq \\ {}^{ef}\tilde{U} &= \left\{ \begin{array}{ll} [-]\tilde{U} = \frac{({}^1a)^2}{4 {}^2\tilde{\chi} [{}^2a + \epsilon ({}^1a)^2]} & \text{for } \varphi \rightarrow -\infty \\ [+]\tilde{U} = \frac{({}^1M)^2}{4 {}^2\tilde{\chi} [{}^2M + \epsilon ({}^1M)^2]} & \text{for } \varphi \rightarrow +\infty \end{array} \right. \end{aligned}$$

For such diagonal approximations, the A- and B-functions can be computed in explicit form

$$\tilde{A} \simeq \left\{ \begin{array}{l} [-]\tilde{A} = \frac{{}^{2a+\frac{1}{2}}{}^{2b}{}^1a}{{}^{2a+\epsilon}({}^1a)^2} \\ [+]\tilde{A} = \frac{{}^2M}{{}^{2M+\epsilon}({}^1M)^2} \end{array} \right. \quad \text{and} \quad \tilde{B} \simeq \left\{ \begin{array}{l} [-]\tilde{B} = - {}^2\tilde{\chi} \frac{{}^{2b/4-\epsilon}({}^{2a+2b}{}^1a)}{{}^{2a+\epsilon}({}^1a)^2} \\ [+]\tilde{B} = \epsilon {}^2\tilde{\chi} \frac{{}^2M}{{}^{2M+\epsilon}({}^1M)^2} \end{array} \right. .$$

Such values reproduce the results of Section 3 in [60] with two flat regions of the effective potential ${}^{ef}\tilde{U}$, but in our approach, the effective diagonalized metric is of type (49) with $\tilde{a} \simeq \tilde{a}(t)$ for $\eta_\alpha \simeq 1$. This class of diagonalized solutions determined by generating functions contain explicit solutions with an effective scalar field, evolving on the first flat region for large negative φ and describing non-singular “emergent universes” [49–54].

6. Reproducing Modified Massive Gravity as TMTS and Effective GR

The goal of this section is to study solutions to effective Einstein Equations (33) when the source (40) is taken for ${}^m\tilde{Y} = {}^f\tilde{Y} = 0$ and ${}^mY = {}^fY = 0$, i.e.,

$$\begin{aligned} \hat{Y}_{\alpha'\beta'} &= \text{diag}[Y_i; Y_a], \text{ for} \\ Y_1 &= Y_2 = {}^{e\mu}\tilde{Y}(x^k) = {}^{ef}\tilde{Y}(x^k) + {}^{\mu}\tilde{Y}(x^k), \\ Y_3 &= Y_4 = {}^{e\mu}Y(x^k, t) = {}^{ef}Y(x^k, t) + {}^{\mu}Y(x^k, t), \end{aligned}$$

with a left label “ $e\mu$ ” emphasizing that such sources are considered for TMT configurations with a nontrivial mass term μ but zero-matter field configurations and for a possible quadratic ϵR^2 cosmological term. We shall chose such N-adapted frames of reference and generating functions when the TMT gravity model describes modifications to μ^2 terms for nonholonomic ghost-free configurations and corrections to scalar curvature (28) of type $\check{R} \simeq \hat{R} + \tilde{\mu}^2$, where

$$\tilde{\mu}^2 \simeq 2 \mu^2 (3 - \text{tr} \sqrt{\mathbf{g}^{-1}\mathbf{q}} - \det \sqrt{\mathbf{g}^{-1}\mathbf{q}})$$

is determined by the graviton’s mass μ and $\mathbf{q} = \{q_{\alpha\beta}\}$ is the so-called non-dynamical reference metric. For simplicity, we make the assumption that such values can be re-defined as constant for certain choices of the generating functions, effective sources ${}^{ef}Y(x^k, t)$, ${}^{\mu}Y(x^k, t)$ and respective nontrivial constants ${}^{e\mu}\Lambda = {}^{ef}\Lambda + {}^{\mu}\Lambda$.

6.1. Massive Gravity Modifications of Flat Regions

We can integraten generic, off-diagonal forms of TMT systems that are subclasses of solutions (47) when

$$\begin{aligned} ds^2 &= \bar{a}^2(x^k, t)[\eta_1(x^k, t)(dx^1)^2 + \eta_2(x^k, t)(dx^2)^2] + \\ &\bar{a}^2(x^k, t)\hat{h}_3(x^k, t)[dy^3 + ({}_1n_i + {}_2n_i \int dt \frac{(\tilde{\Psi}^*)^2}{\tilde{\Psi}_3({}^{ef}\Xi + {}^{\mu}\Xi)})dx^i]^2 - \\ &[dt + \frac{\partial_i({}^{ef}\Xi + {}^{\mu}\Xi)}{({}^{ef}\Xi + {}^{\mu}\Xi)^*} dx^i]^2, \end{aligned} \tag{56}$$

for

$${}^{ef}\Xi := \int dt {}^{ef}Y(\tilde{\Psi}^2)^*, \quad {}^{\mu}\Xi := \int dt {}^{\mu}Y(\tilde{\Psi}^2)^*.$$

We write $\tilde{\Psi} \rightarrow \bar{\Psi}$ when the generating function is chosen to satisfy the conditions

$$\bar{a}^2\hat{h}_3 = \omega^2 h_3 = \frac{h_3}{|h_4|} = \frac{\bar{\Psi}^2 |{}^{ef}\Xi + {}^{\mu}\Xi|}{4({}^{e\mu}\Lambda)(\bar{\Psi}^*)^2}.$$

In general, such nonhomogeneous locally anisotropic configurations contain nontrivial, nonholonomically induced, canonical d-torsion, which can be constrained to zero for corresponding subclasses of generating functions and sources.

We study off-cosmological solutions depending only on time-like coordinates when $\tilde{a}(x^k, t) \simeq \tilde{a}(t)$ and $\tilde{\Psi}(x^k, t) \simeq \tilde{\Psi}(t)$ and the generation function $\tilde{\Psi}(t)$. The formula relating variations in ${}^{e\mu}Y(t)$ to the variation in the second auxiliary 3-index antisymmetric d-tensor field $\mathbf{B}_{\alpha\beta\gamma}$ in ${}^2\Phi(\mathbf{B})$, a particular case of (37), is given by

$${}^2\Phi(\bar{\mathbf{B}})/\sqrt{|\hat{\mathbf{g}}|} = {}^2\bar{\chi} = {}^2\tilde{\chi} + {}^{\mu}\chi = \text{const},$$

where the constant ${}^\mu\chi$ is zero for $\mu = 0$ and $|{}^\mu\chi| \ll |{}^2\tilde{\chi}|$. Another assumption is that we can formulate a TMT theory corresponding to “pure” μ -deformations of GR, even $\epsilon = 0$. The formula (55) has to be generalized for nontrivial μ when

$${}^1\mathbf{f}({}^1L + {}^1M + {}^\mu M, \mu) = \frac{d\mathbf{f}(\widehat{\mathbf{R}}, \mu)}{d\widehat{\mathbf{R}}} \Big|_{\widehat{\mathbf{R}} = {}^1L + {}^1M} \rightarrow {}^1U - {}^1M - {}^\mu M$$

is a version of generalized Starobinsky relation (35), Formulas (36) and (28) and approximation of type $\widehat{\mathbf{R}} \simeq \widehat{\mathbf{R}} + \widehat{\mu}^2$.

The resulting formulas for effective potential (53) contain additional μ -terms

$$e^\mu U = \frac{({}^1U - {}^1M - {}^\mu M)^2}{4 {}^2\tilde{\chi} [{}^2U + {}^2M + ({}^1U - {}^1M - {}^\mu M)^2]}$$

$$\simeq e^\mu \bar{U} = \begin{cases} [-]\bar{U} = \frac{({}^1a)^2}{4({}^2\tilde{\chi} + {}^\mu\chi) [{}^2a + \epsilon({}^1a)^2]} & \text{for } \varphi \rightarrow -\infty \\ [+]\bar{U} = \frac{({}^1M + {}^\mu M)^2}{4({}^2\tilde{\chi} + {}^\mu\chi) [{}^2M + \epsilon({}^1M)^2]} & \text{for } \varphi \rightarrow +\infty \end{cases}.$$

The A - and B -functions can also contain contributions of μ -terms,

$$\bar{A} \simeq \begin{cases} [-]\bar{A} = \frac{{}^{2a+\frac{1}{2}} {}^{2b} {}^1a}{{}^{2a+\epsilon({}^1a)^2}} \\ [+]\bar{A} = \frac{{}^2M}{{}^{2M+\epsilon({}^1M+{}^\mu M)^2}} \end{cases} \Bigg| \text{ and}$$

$$\bar{B} \simeq \begin{cases} [-]\bar{B} = -({}^2\tilde{\chi} + {}^\mu\chi) \frac{{}^{2b/4-\epsilon({}^{2a+2b} {}^1a)}}{{}^{2a+\epsilon({}^1a)^2}} \\ [+]\bar{B} = \epsilon ({}^2\tilde{\chi} + {}^\mu\chi) \frac{{}^2M}{{}^{2M+\epsilon({}^1M+{}^\mu M)^2}} \end{cases} \Bigg|,$$

when $[-]\bar{A}$ is not modified. We conclude that solutions with nontrivial generating functions for nontrivial massive gravity terms modelled as effective TMT theories may also describe non-singular “emergent universes” [49–54] with corresponding modifications.

6.2. Reconstructing Off-Diagonal Tmt and Massive Gravity Cosmological Models

For the class of solutions (56), we show how we can perform a reconstruction procedure. We introduce a new time coordinate \hat{t} for $t = t(x^i, \hat{t})$ and $\sqrt{|h_4|} \partial t / \partial \hat{t}$, and redefined the scale factor, $\bar{a} \rightarrow \hat{a}(x^i, \hat{t})$, representing the quadratic elements in the form

$$ds^2 = \hat{a}^2(x^i, \hat{t}) [\eta_i(x^k, \hat{t}) (dx^i)^2 + \hat{h}_3(x^k, \hat{t}) (\mathbf{e}^3)^2 - (\hat{\mathbf{e}}^4)^2], \tag{57}$$

$$\text{for } \eta_i = \hat{a}^{-2} e^\psi, \hat{a}^2 \hat{h}_3 = h_3, \mathbf{e}^3 = dy^3 + \partial_k n dx^k, \hat{\mathbf{e}}^4 = d\hat{t} + \sqrt{|h_4|} (\partial_i t + w_i).$$

To model small off-diagonal deformations, we use a small parameter ϵ , $0 \leq \epsilon < 1$, when

$$\eta_i \simeq 1 + \epsilon \chi_i(x^k, \hat{t}), \partial_k n \simeq \epsilon \hat{n}_i(x^k), \sqrt{|h_4|} (\partial_i t + w_i) \simeq \epsilon \hat{w}_i(x^k, \hat{t}) \tag{58}$$

and there are subclasses of generating functions and sources for which $\hat{a}(x^i, \hat{t}) \rightarrow \hat{a}(t), \hat{h}_3(x^i, \hat{t}) \rightarrow \hat{h}_3(\hat{t})$ etc., see details for such a procedure from Section 5 of [67] (see references therein). The analogous TMT massive gravity theory is taken with a source ${}^\mu\hat{\mathbf{Y}}_{\mu\nu}$ (31) and parametrization $\mathbf{f}(\widehat{\mathbf{R}}) = \widehat{\mathbf{R}} + \mathbf{S}({}^\mu\mathbf{T})$, for any N-adapted value

$${}^\mu\mathbf{T} := \mathbf{T} + 2 \mu^2 (3 - \text{tr} \sqrt{\mathbf{g}^{-1} \mathbf{q}} - \det \sqrt{\mathbf{g}^{-1} \mathbf{q}}).$$

Introducing values ${}^1\mathbf{S} := d\mathbf{S}/d{}^\mu\mathbf{T}$ and $\widehat{H} := \hat{a}^*/\hat{a}$ for a limit $\hat{a}(x^i, \hat{t}) \rightarrow \hat{a}(t)$ with $N_i^a = \{n_i, w_i(t)\}$ and effective polarizations $\eta_\alpha(t)$.

In order to test cosmological scenarios, we consider a redshift $1 + z = \hat{a}^{-1}(t)$ for ${}^\mu T = {}^\mu T(z)$ by introducing a new “shift” derivative. For instance, for a function $s(t)$)

$s^* = -(1+z)H\partial_z$. We can derive TMT massive, modified, off-diagonal, deformed FLRW equations using Formulas (63) and (64) in [67], when

$$\begin{aligned} 3\hat{H}^2 + \frac{1}{2}[\mathbf{f}(z) + \mathbf{S}(z)] - \kappa^2\rho(z) &= 0, \\ -3\hat{H}^2 + (1+z)\hat{H}(\partial_z\hat{H}) - \frac{1}{2}\{\mathbf{f}(z) + \mathbf{S}(z) + 3(1+z)\hat{H}^2\} &= 0, \end{aligned} \tag{59}$$

for $\rho(z) \partial_z \mathbf{f} = 0$. We can fix the condition $\partial_z \mathbf{S}(z) = 0$, and rescale the generating function in order to satisfy the condition $\partial_z \mathbf{f} = 0$. We have nonzero densities in certain adapted frames of reference. Here, we note that the functional $\mathbf{S}({}^\mu\mathbf{T})$ encodes effects of mass gravity for the evolution of the energy-density when $\rho = \rho_0 a^{-3(1+\omega)} = \rho_0(1+z)a^{3(1+\omega)}$, when, for the dust matter approximation, ω and $\rho \sim (1+z)^3$. Any FLRW cosmology can be realized in a corresponding class of f -gravity models, which can be re-encoded as a TMT theories using actions of type (25)–(27). Let us introduce $\zeta := \ln a/a_0 = -\ln(1+z)$ as the “e-folding” variable to be used instead of the cosmological time t , and consider

$$\hat{\mathbf{Y}}(x^i, \zeta) = {}^f\mathbf{Y}(x^i, \zeta) + {}^\mu\mathbf{Y}(x^i, \zeta)$$

with dependencies on (x^i, ζ) of generating functions $\partial_\zeta = \partial/\partial\zeta$ with $q^* = \hat{H}\partial_\zeta q$ for any function q .

Repeating all computations leading to Equations (2)–(7) in [68], in our approach for $\mathbf{f}(\check{\mathbf{R}})$, we construct an FLRW-like cosmological model with nonholonomic field equation corresponding to the first FLRW equation

$$\mathbf{f}(\check{\mathbf{R}}) = (\hat{H}^2 + \hat{H} \partial_\zeta \hat{H})\partial_\zeta[\mathbf{f}(\check{\mathbf{R}})] - 36\hat{H}^2 [4\hat{H} + (\partial_\zeta \hat{H})^2 + \hat{H}\partial_{\zeta\zeta}^2 \hat{H}] \partial_{\zeta\zeta}^2 \mathbf{f}(\check{\mathbf{R}}) + \kappa^2\rho.$$

We consider an effective quadratic Hubble rate, $\tilde{\kappa}(\zeta) := \hat{H}^2(\zeta)$, where $\zeta = \zeta(\check{\mathbf{R}})$, we write this equation in the form

$$\mathbf{f} = -18\tilde{\kappa}(\zeta)[\partial_{\zeta\zeta}^2 \tilde{\kappa}(\zeta) + 4\partial_\zeta \tilde{\kappa}(\zeta)] \frac{d^2\mathbf{f}}{d\check{\mathbf{R}}^2} + 6 \left[\tilde{\kappa}(\zeta) + \frac{1}{2}\partial_\zeta \tilde{\kappa}(\zeta) \right] \frac{d\mathbf{f}}{d\check{\mathbf{R}}} + 2\rho_0 a_0^{-3(1+\omega)} a^{-3(1+\omega)\zeta(\check{\mathbf{R}})}. \tag{60}$$

For any off-diagonal cosmological models with quadratic metric elements of type (57) for redefined $t \rightarrow \zeta$ when a functional $\mathbf{f}(\check{\mathbf{R}})$ is used for computing $\hat{\mathbf{Y}}$, the generating function and respective d-metric and N-connection coefficients as solutions of certain effective Einstein spaces for auxiliary connections and effective cosmological constant ${}^{e\mu}\Lambda$. The value $d\mathbf{f}/d\check{\mathbf{R}}$ and higher derivatives vanish for any functional dependence $\mathbf{f}({}^{e\mu}\Lambda)$ because $\partial_\zeta {}^{e\mu}\Lambda = 0$. We conclude that the recovering procedure simplifies substantially, even in TMT theories, by re-scaling the generating function and sources following formulas of type (43).

Now, we speculate on how we can reproduce the Λ CDM era. Using values $\hat{a}(\zeta)$ and $\hat{H}(\zeta)$ determined by an off-diagonal quadratic element (57) and writing analogs of the FLRW equations for Λ CDM cosmology in the form

$$3\kappa^{-2}\hat{H}^2 = 3\kappa^{-2}H_0^2 + \rho_0\hat{a}^{-3} = 3\kappa^{-2}H_0^2 + \rho_0a_0^{-3}e^{-3\zeta},$$

for fixed constant values H_0 and ρ_0 . The second term in this formula describes an inhomogeneous distribution of cold dark matter (CDM). This allows for computation of the effective quadratic Hubble rate and the modified scalar curvature, $\check{\mathbf{R}}$, in the forms $\tilde{\kappa}(\zeta) := H_0^2 + \kappa^2\rho_0a_0^{-3}e^{-3\zeta}$ and

$$\check{\mathbf{R}} = 3\partial_\zeta \tilde{\kappa}(\zeta) + 12\tilde{\kappa}(\zeta) = 12H_0^2 + \kappa^2\rho_0a_0^{-3}e^{-3\zeta}.$$

The solutions of (60) can be found by following [67,68] as Gauss hypergeometric functions. We might denote $\mathbf{f} = F(X) := F(\chi_1, \chi_2, \chi_3; X)$, where, for some constants, A and B ,

$$F(X) = AF(\chi_1, \chi_2, \chi_3; X) + BX^{1-\chi_3}F(\chi_1 - \chi_3 + 1, \chi_2 - \chi_3 + 1, 2 - \chi_3; X).$$

This is the proof that MGTs and various TMT models can indeed describe Λ CDM scenarios without the need for an effective cosmological constant, because we have effective sources, and this follows from the re-scaling property (43) of generic off-diagonal configurations. The Equation (60) transforms into

$$X(1-X)\frac{d^2\mathbf{f}}{dX^2} + [\chi_3 - (\chi_1 + \chi_2 + 1)X]\frac{d\mathbf{f}}{dX} - \chi_1\chi_2\mathbf{f} = 0, \quad (61)$$

for certain constants, for which $\chi_1 + \chi_2 = \chi_1\chi_2 = -1/6$ and $\chi_3 = -1/2$ where $3\zeta = -\ln[\kappa^{-2}\rho_0^{-1}a_0^3(\check{R} - 12H_0^2)]$ and $X := -3 + \check{R}/3H_0^2$.

Finally, we note that the reconstruction procedure can be performed in a similar form for any MGTs and TMT ones which can be modeled, for well-defined conditions, by effective nonholonomic Einstein spaces.

7. Results and Conclusions

7.1. Modified Gravity and Cosmology Theories with Metric Finsler Connections on (Co) Tangent Lorentz Bundles or for Nonholonomic Einstein Manifolds

In the present paper and partner works [26–33], we follow an orthodox point of view that inflation and accelerating cosmological models can be elaborated in the framework of effective Einstein theories via off-diagonal and diagonal solutions for nonholonomic vacuum and non-vacuum configurations determined by generating functions and integration functions and constants. Fixing respective classes of such functions and constants, we can extract various types of modified gravity–matter theories defined in terms of non-Riemannian volume–forms (for instance, in a manifestly globally Weyl-scale invariant form) and with certain modified Lagrange densities of type $f(\check{R})$ including contributions from the Einstein–Hilbert term R , its square R^2 , possible massive gravity μ parametric terms, nonholonomic deformations etc. The principal results are as follows:

1. We defined nonholonomic geometric variables for which various classes of modified gravity theories (MGTs), (generally with nontrivial gravitational mass) can be modelled equivalently as respective two-measure (TMT) [55–57,60–62], bi-connection and/or bi-metric theories. For well-defined nonholonomic constraint conditions, the corresponding gravitational and matter field equations are equivalent to certain classes of generalized Einstein equations with nonminimal connection to effective matter sources and nontrivial nonholonomic vacuum configurations;
2. We stated the conditions when nonholonomic TMT models encode ghost-free massive configurations with (broken) scale invariance and such interactions can be modelled by generic off-diagonal metrics in effective general relativity (GR) and generalizations with induced torsion. Such a nonholonomic geometric technique was elaborated in Finsler geometry in gravity theories and, for a corresponding $2 + 2$ splitting, we can consider Finsler-like variables and work with so-called FTMT models;
3. We developed the anholonomic frame deformation method [30–33], AFDM, in order to generate off-diagonal, generally inhomogeneous and locally anisotropic cosmological solutions in TMT and MGTs. It was proven that the effective Einstein equations for such gravity and cosmological models can be decoupled in general form, which allows for the construction of various classes of exact solutions depending on generating functions and integration functions and constants;
4. We analysed a very important re-scaling property of generating functions with association of effective cosmological constants for different types of modified gravity and matter field interactions, which allow for the definition of nonholonomic variables into

- which the associated systems of nonlinear partial differential equations (PDEs) can be integrated in explicit form when the coefficients of generic off-diagonal metrics and (generalized) nonlinear and linear connections depend on all space–time coordinates;
5. There were stated conditions for generating functions and effective sources when zero-torsion (Levi–Civita, LC) configurations can be extracted in general form, with possible nontrivial limits to diagonal configurations in Λ CDM cosmological scenarios, encoding dark energy and dark matter effects, possible nontrivial zero mass contributions, effective cosmological constants induced by off-diagonal interactions and constrained nonholonomically, to result in nonlinear diagonal effects;
 6. Special attention was devoted to subclasses of generic off-diagonal cosmological solution with effective scalar potentials and two flat regions and limits to the diagonal cosmological TMT scenarios investigated in [61,62] were studied;
 7. We studied possible massive gravity modifications of flat regions and the possible reconstruction of off-diagonal TMT and massive gravity cosmological models. Through corresponding frame transforms and the re-definition of generating functions and nonholonomic variables, we proved that the same geometric techniques are applicable in all such MGTs.

Let us explain why it is important to study exact solutions for off-diagonal and nonlinear gravitational interactions in different MGTs, depending on 2–4 spacetime coordinates, and consider possible the implications for modern cosmology. The gravitational and matter field equations in such theories consist of very sophisticated systems of nonlinear PDEs. It was possible to construct, for instance physically important black hole and cosmological solutions to certain diagonal ansatz, depending on one space/time-like variable modelling (generalized) Einstein spacetimes with two and three Killing symmetries or other types of high-symmetry and asymptotic condition. There were two kinds of motivation for such assumptions: the technical one was that, for diagonalizable ansatz, the systems of nonlinear PDEs transform in “more simple” systems of nonlinear ordinary differential equations (ODEs), which can be integrated in general form. The physical interpretation of such solutions determined by integration constants is more intuitive and natural. Nevertheless, a series of problems arise in modern acceleration cosmology with dark energy and dark matter effects. It became clear that standard diagonal cosmological solutions in GR, together with standard scenarios from particle physics and former elaborated cosmological models, cannot be applied to explain observational cosmological data. A number of MGTs and new cosmological theories have been proposed and developed.

After mathematically selecting some special diagonalizable ansatz with prescribed symmetries, we eliminate other, more general classes of solution, which seem to be important for explaining nonlinear parametric and nonholonomic off-diagonal interactions. This could be related to a new nonlinear physics in gravity and particle field theory which has not been yet investigated. In the past, there were a number of technical restrictions to the construction of such solutions and study of their applications but, at present, there are accessible, advanced numerical, analytic and geometric methods. In this work, we follow a geometric approach developed in [14–16,26–33,63], which allows us to construct exact solutions in different classes of gravity and cosmology theories. Even observational data in modern cosmology can be explained by almost diagonal and homogeneous models; when possible off-diagonal effects and anisotropies are very small, we are not constrained to studying only the solutions to associated systems’ ODEs. For nonlinear gravitational and matter field systems, a well-defined mathematical approach is to generate (if possible, exact) solutions in the most general form, and then to impose additional constraints for diagonal configurations. In result, a number of MGT effects and accelerating cosmology can be explained as standard, except for off-diagonal nonlinear ones in effective GR. Alternative interpretations in the framework of TMT and other type theories are also possible.

7.2. Alternative Finsler Gravity Theories with Metric Non-Compatible Connections

The referee of this work requested “minimal modification” in order to cite and discuss papers [69–74] where some alternative Finsler gravity and geometry models are considered. This is a good opportunity for authors, which allows them to explain their approach, geometric methods, and new results for the construction of new classes of generic, off-diagonal cosmological solutions in more detail, as well as elaborating on applications in non-standard particle physics and modified gravity. To comment on key ideas and constructions in the authors’ works, and compare them to similar ones from the mentioned alternative geometric and cosmological theories, we have to additionally cite [75–79], and references therein. We note that readers should pay attention to reference [24], with respective Introduction and Conclusion sections, and Appendix B (in that work), containing historical remarks and a review of 20 directions on modern generalized Finsler geometry and applications in modern particle physics, modified gravity, and cosmology, mechanics and thermodynamics, information theory, etc. to [24] a number of historical remarks and a review of the last 80 years of research activity are provided, as well as the main achievements in Finsler–Lagrange–Hamilton geometry and its applications in modern physics, gravity, cosmology, mechanics and information theory. The axiomatic part was published in [25]. In the mentioned works, a study of evolution of main research groups on “Finsler geometry and physics” in different countries, and the international collaborations formed is included. The results and bibliography of the conventional 20 directions, and more than 100 sub-directions of research and publications, were reviewed, by the present and other authors, related to Finsler geometry and applications. We also cite the paper [76] and the monograph [79], (for a collection of works on (non-)commutative metric-affine generalized Finsler geometries and nonholonomic supergravity and string theories, locally anisotropic kinetic and diffusion processes, Finsler spinors, etc.), and articles [77,78]. Here, we summarize and discuss such issues:

1. In the abstract and introduction, as well as Section 2.2 of this article, it is emphasized that we do not elaborate a typical work on Finsler gravity and cosmology, but rather provide a cosmological work on Einstein gravity and MGTs, TMTs ones, with two measures/two connections and/or bi-metrics, mass terms, etc., when the constructions are modelled on a Lorentz manifold V of signature $(+++ -)$ with conventional nonholonomic $2 + 2$ splitting. For such theories, the spacetime metrics $g_{\alpha\beta}(x^i, y^a)$ (with $i, j, \dots = 1, 2$ and $a, b, \dots = 3, 4$) are generic off-diagonal and, together with the coefficients of other fundamental geometric objects, depend on all space–time conventional fibred coordinates. Lagrange–Finsler-like variables are introduced to V for “toy” models, when y^a are treated similarly to (co) fiber coordinates on a (co) tangent manifold $(T^*V) TV$, for a prescribed a fundamental Lagrange, $L(x, y)$ (or Finsler, for certain homogeneity conditions $F(x, \beta y) = \beta F(x, y)$, $x = \{x^i\}$ etc., for a real constant $\beta > 0$, when $L = F^2$). This states, for V , a canonical Finsler-like N-connection and nonholonomic (co-)frames structure, which can also be described in coordinate bases, using additional constraints to extract the LC-connection or distorting it to other linear connections determined by the same metric structures. In dual form, we can consider momentum, like p_a -dependencies in $g_{\alpha\beta}(x^i, p_a)$, for a conventional Hamiltonian $H(x, p)$, which can be related to an L via corresponding Legendre transforms. The reason for introducing Finsler-like and other types of nonholonomic variable to a manifold V , or on a tangent bundle TV is that, in so-called nonholonomic canonical variables (with hats on geometric objects), the modified Einstein Equation (9) can be decoupled and integrated in vary general forms. We have to consider some additional nonholonomic constraints (10) in order to extract LC-configurations. This is the main idea of the AFDM [30–33], which was applied in a series of works for constructing a locally anisotropic black hole and cosmological solutions defied by generic off-diagonal metrics and (generalized) connections in Lagrange–Finsler–Hamilton gravity in various limits of (non-)commutative/supersymmetric string/brain theories, massive gravity, TMT models, etc., as we consider in partner works [26–33].

2. One of the formal difficulties in modern Finsler geometry and gravity is that some authors (usually mathematicians) use a different terminology compared to that elaborated by physicists in GR, MGTs, TMTs etc. For instance, a theory of “standard static Finsler spaces”, with a time like Killing field and/or for static solutions of a type of field equation in Finsler gravity is elaborated in [69–71]. Of course, it is possible to prescribe a class of static and a corresponding smooth class of Finsler-generating functions, $F(x, y)$, when semi-spray, N-connections and d-connections, and certain Finsler–Ricci generalized tensors, etc., can be computed for static configurations embedded in locally anisotropic backgrounds. Such constructions can be chosen to have spherical symmetry. However, by introducing and computing corresponding “standard static” Sasaki type metrics of type (16), and their off-diagonal coordinate base equivalents, involving N-coefficients (see the total (phase) space–time metric (17)), we can check that such geometric d-objects (and their corresponding canonical d-connection, or LC-connection) do not solve the (modified) Einstein Equation (9) if the data are the general ones considered in [69–71]. If the d-metric coefficients $g_{\alpha\beta}(x^i, y^a)$ are generic off-diagonal with nontrivial N-connection coefficients, such metrics can be only quasi-stationary following the standard terminology in mathematical relativity and MGTs (when coefficients do not depend on time-like variable, i.e., ∂_t is a Killing symmetry d-vector), but there are nontrivial off-diagonal metric terms because of rotation, N-connections, etc. Stationary metrics of type (16) and/or (17) can be prescribed to describe, for instance, black ellipsoids, which are different from the solutions for Kerr black holes, BHs, because of their more general Finsler local anisotropy. Static configurations with diagonal metrics of Schwarzschild type BHs can be introduced for some trivial N-connection structures (but, in Finsler geometry, this is a cornerstone geometric object). For Finsler-like gravity theories, there are no proofs of BH uniqueness theorems, and it is not clear if such static configurations (for instance, with spherical symmetry) can be stable. Such proofs are sketched for black ellipsoids; see details in [26–33]. Therefore, the existing concepts, definitions, and proofs of “standard” static/stationary/cosmological/stable/nonlinear evolution models, etc., depend on the type of postulated principles for respective concepts and theories of Finsler spacetime.
3. In [72,73,75], certain attempts to elaborate models of Finsler spacetime, geometry and gravity are considered for some types of N-connection and chosen classes of Finsler metric compatible and non-compatible d-connections. In many cases, the Berwald–Finsler d-connection is considered, which is generally noncompatible but can be subjected to certain metrization procedures. Different geometric constructions, with a non-fixed signature for Hessians and sophisticated causality conditions via semi-sprays and generalized nonlinear geodesic configurations, have been proposed and analyzed. In such approaches, there are a series of fundamental unsolved physical and geometric problems in the development of such Finsler theories in self-consistent and viable physical forms. Here, we focus only on the most important issues (for details, critiques, discussions, and motivation regarding Finsler gravity principles, we cite [17,24,25,76,79]):
 - For theories with metric noncompatible connections, for instance, of Chern or Berwald type, there are no unique and simple possibilities to define spinors, conservation laws of type $D_i T^{jk}$, elaborate on supersymmetric and/or non-commutative/nonassociative generalizations, or to consider generalized type classical and quantum symmetries, considering only Finsler type d-connections proposed by some prominent geometers like E. Cartan, S. Chern, B. Berwald etc., and physically un-motivated (effective) energy-momentum tensors with local anisotropy;
 - Physical principles and nonlinear causality schemes elaborated on a base manifold with undetermined lifts, without geometric and physical motivations, on total bundles, depend on the type of Finsler-generating function. Hessians

and nonlinear and linear connections are chosen for elaborating geometric and physical models. A Finsler geometry is not a (pseudo/semi-) Riemannian geometry, where all constructions are determined by the metric and LC-connection structures. For instance, certain constructions with cosmological kinetic/statistical Finsler spacetime in [73,75] are subjected to very complex type conservation laws and nonlinear kinetic/diffusion equations. Those authors have not cited and or applied earlier, locally anisotropic, generalized Finsler kinetic/diffusion/statistical constructions performed for the metric compatible connections studied in [77–79] (N. Voicu was at S. Vacaru’s seminars in Brashov in 2012, on Finsler kinetics, diffusion and applications in modern physics and information theory; see also [33], but, together with her co-authors, do not cite, discuss, or apply such locally anisotropic, metric, compatible and solvable geometric flow, kinetic and geometric thermodynamic theories);

- Various variational principles and certain versions of Finsler modified Einstein equations were proposed and developed in [72,73,75], but such theories have been not elaborated on total bundle spaces, for certain metric compatible Finsler connections. Usually, metric non-compatible Finsler connections were used, when it is not possible to elaborate on certain general methods for the construction of exact and parametric solutions to nonlinear systems of PDEs; for instance, describing locally anisotropic interactions of modified Finsler–Einstein–Dirac–Yang–Mills–Higgs systems. In S. Vacaru and co-authors’ axiomatic approach to relativistic Finsler–Lagrange–Hamilton theories [17,24,25,76], such generalized systems can be studied—for instance, on (co) tangent Lorentz bundles (and on Lorentz manifolds with conventional nonholonomic fibred splitting)—when the AFDM was applied to generate exact and parametric solutions, and certain deformation quantization, gauge-like, etc., schemes were developed;
4. As a result, the authors of [74] concluded their work in a pessimistic fashion: “Finsler geometry is a very natural generalisation of pseudo-Riemannian geometry and there are good physical motivations for considering Finsler spacetime theories. We have mentioned the Ehlers–Pirani–Schild axiomatic and also the fact that a Finsler modification of GR might serve as an effective theory of gravity that captures some aspects of a (yet unknown) theory of Quantum Gravity. We have addressed the somewhat embarrassing fact that there is not yet a general consensus on fundamental Finsler equations, in particular on Finslerian generalisations of the Dirac equation and of the Einstein equation, and not even on the question of which precise mathematical definition of a Finsler spacetime is most appropriate in view of physics. We have seen that the observational bounds on Finsler deviations at the laboratory scale are quite tight. By contrast, at the moment we do not have so strong limits on Finsler deviations at astronomical or cosmological scales.” In that work, there was no discussion or analysis of the approach developed for Lorentz–Finsler–Lagrange–Hamilton, and the nonholonomic manifolds developed by authors of this paper, beginning in 1994 and published in more than 150 papers in prestigious mathematical and physical journals, as well as summarized in three monographs (for reviews, see [24,25,79]).

S. Vacaru’s research group was more optimistic regarding their obtained results and perspectives of Finsler geometry in physics. Having obtained 10 NATO, CERN and DAAD research grants, the group elaborated an axiomatic approach to Finsler–Lagrange–Hamilton gravity theories, using constructions on nonholonomic (co-)tangent Lorentz bundles and Lorentz manifolds, with an N-connection structure and Finsler-like metric compatible connections. They began their activity almost 40 years ago—see the historical remarks, summaries of results and discussions in [17,24,25,76,79], with recent developments in [26–33]. P. Stavrinou (with more than 40 years research experience on Finsler geometry and applications), and his co-authors also published a series of works on modified Finsler gravity and cosmology theories involving tangent Lorentz bundles [18,22,79]. For such classes of modified Finsler geometric flow and gravity theories, a general geometric method

can be used for constructing exact and parametric solutions: the AFDM, with self-consistent extensions to noncommutative and nonassociative, superstring and supergravity, Clifford–Finsler, etc., theories. Together with papers on deformation and other type quantum Finsler–Einstein-gauge gravity theories, which were elaborated and developed in more than 20 research directions for Finsler geometry and applications, this article belongs to an axiomatized and self-consistent direction of mathematical and acceleration cosmology, dark matter and dark energy physics, involving Finsler geometry methods.

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