Quantum State Evolution in an Environment of Cosmological Perturbations

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Abstract: We study the pure and thermal states of quantized scalar and tensor perturbations in various epochs of Universe evolution. We calculate the density matrix of non-relativistic particles in an environment of these perturbations. We show that particle's motion can be described by a stochastic equation with a noise coming from the cosmological environment. We investigate the squeezing of Gaussian wave packets in different epochs and its impact on the noise of quantized cosmological perturbations.

Keywords: geodesic equation; quantum gravity; gravitational environment; squeezed quantum states; stochastic equations

1. Introduction

The study of a system of particles with gravitational interaction is a standard task in an investigation of inhomogeneities and structure formation [1,2]. In such studies, usually only classical gravity is considered. However, the structure formation begins already in the inflationary era [3–6]. The recent discovery of gravitational waves raises hopes for a detection of various phenomena resulting from quantization of gravity [7,8]. In the standard model of the Universe evolution, it is assumed that it begins from a quantum state. The particles created at the end of the inflationary era will evolve in an environment of quantized cosmological perturbations. Hence, formation of inhomogeneities in the form of matter will take place in the environment of quantized perturbations. We can observe the cosmological gravitational perturbations in CMB temperature fluctuations and (possibly) in primordial gravitational waves. The quantum fluctuations are described in a gauge invariant way by (gauge invariant) Bardeen scalar and tensor variables [9–12]. The scalar variable in the inflation era is dominated by the inflaton field. At the end of inflation, the inflaton decays into relativistic particles. The radiation era begins. We assume that in the radiation era the quantum state of the Universe still depends on the scalar and tensor modes of the gravitational field. Moreover, owing to the squeezing during inflation [13–16], it can be described by a Gaussian wave function. Gaussian states are classical in the sense that their Wigner function is positive definite. We assume that the wave function of tensor perturbations in spite of the complex processes taking place in various epochs evolves in a continuous way depending only on the evolution of the scale factor. The wave function of the scalar perturbations is not expected to be continuous in different epochs, but we still work with a Gaussian approximation as it is a consequence of the quadratic approximation to Einstein gravity. The decay of the inflaton creates particles which are moving in the environment of the cosmological perturbations. Such an environment is changing evolution of these particles. In a non-relativistic approximation, we derive the time evolution of the density matrix. We show that this time evolution is determined by a stochastic equation which is a generalization of the equation derived in [17–21] for tensor perturbations (gravitational waves) in the Minkowski metric. There was earlier work on the particle motion in an environment of a quantized metric [22–27] based on the geodesic equation. However, the experience with the motion of a particle

in a gravitational wave [17,21] indicates that the proper approach consists in a study of microscopic quantum effects of relative particle motions near their geodesics through the geodesic deviation equation.

The tensor perturbations arrive to us as primordial gravitational waves. The scalar perturbations are measurable as temperature fluctuations in CMB [2] and as density fluctuations of galaxies [1,28]. We assume that the detector can receive primordial perturbations from the inflationary stage of the Universe evolution (possibly as gravitational waves produced as the second order effect from scalar perturbations [29,30]). After the radiation era and baryonic era, the cosmological perturbations arrive to us at the time interval when the metric can be approximated by a static (Minkowski) metric. The effect of gravitational waves can be studied by means of a stochastic geodesic deviation equation in a weak gravitational field on a flat background. In [20], we studied the interaction of non-relativistic particles with quantum tensor perturbations. We argued after [17] (see also [31]) that the noise from the gravitons can be observed in the wave detector owing to the strong squeezing during inflation. In this paper we extend our results of [20] to quantum scalar and tensor perturbations in an expanding Universe. We suggest that the quantized scalar and tensor perturbations have an effect upon detectors of cosmological perturbations as well as upon formation of inhomogeneities during the radiation domination epoch. These quantum perturbations derived as quantum modifications of the geodesic deviation equation appear in the form of stochastic geodesic deviation equations.

The paper is organized as follows. In Section 2 we introduce our method of representing the environment of oscillators in quantum mechanics. In Section 3, we extend it to quantum field theory. In Section 4, we discuss the scalar perturbations in the inflation era. In Section 5, we study the scalar perturbations after inflation. In Section 6, we obtain Gaussian wave function for scalar perturbation as a solution of the Schrödinger equation. In Section 7, the tensor perturbations and their wave function are discussed. In Section 8, we consider the evolution of the wave function in various cosmological epochs. In Section 9, a non-relativistic particle interacting with cosmological perturbations is discussed. In Section 10, we review our version of the influence functional method in order to derive the density matrix for a particle in an environment of quantum cosmological perturbations. We solve stochastic equations for cosmological perturbations (needed for the calculation of the density matrix) in Section 11. We calculate the density matrix in a simplified model of one-mode approximation in Section 12. The general Gaussian state of the cosmological environment is discussed in Section 13. The particle motion in thermal environment of cosmological perturbations is obtained in Section 14. In Section 15, we summarize our main results and point out some extensions of our work.

Our approach is based on a quantization of the quadratic approximation to Einstein gravity. Such an approach is justified in a classical theory by a linearized coupling of the gravitational modes to the detector as confirmed by the recent discovery of gravitational waves. Until now, there are no indications of the quantum nature of gravitational waves (gravitons) and the relevance of extended theories of gravity (if dark matter and dark energy are accepted). However, recent observations (LIGO/Virgo and Planck2015) evoke the hope to check various methods of quantization as well as some extensions of Einstein gravity. The first category includes: an exponential parameterization [32–34], loop quantization [35], effective field theory [36] and asymptotically save gravity [37]. As possible extensions of Einstein gravity (which eventually could avoid the introduction of dark matter and dark energy), we mention f(R) gravity [38], Brans–Dicke gravity [33,39], non-canonical $P(X, \phi)$ and Horndeski gravity [40,41]. These extensions are particularly interesting in view of the possible measurement of the difference of the light velocity and gravitational waves velocity [42]. We discuss these questions in the last section.

2. Feynman Integral on an Oscillatory Background

Let us consider first a simple model of the Schrödinger equation of quantum mechanics in one dimension with the potential $\frac{m\omega(t)^2 x^2}{2}$ perturbed by a time-dependent potential V_t

$$i\hbar\partial_t\psi_t = (-\frac{\hbar^2}{2m}\nabla_x^2 + \frac{m\omega(t)^2 x^2}{2} + V_t(x))\psi_t. \tag{1}$$

Let ψ_t^g be a solution of the Schrödinger equation with an oscillator potential

$$i\hbar\partial_t\psi_t^g = (-\frac{\hbar^2}{2m}\nabla_x^2 + \frac{m\omega(t)^2 x^2}{2})\psi_t^g. \tag{2}$$

Let us write the solution of Equation (1) in the form

$$\psi_t = \psi_g^t\chi_t. \tag{3}$$

Inserting χ_t from Equation (3) into Equations (1) and (2), we find that χ_t satisfies the equation

$$\partial_t\chi_t = \frac{i\hbar}{2m}\nabla_x^2\chi_t + \frac{i\hbar}{m}(\nabla_x\ln\psi_g^t)\nabla_x\chi_t - \frac{i}{\hbar}V_t\chi_t \tag{4}$$

with the initial condition

$$\chi_0 = \psi_0(\psi_g^0)^{-1} \tag{5}$$

expressed by the initial conditions for ψ_t and ψ_g^t.

Equation (4) can be considered as the diffusion equation with the imaginary diffusion constant $\frac{i\hbar}{m}$, a time-dependent drift $\frac{i\hbar}{m}\nabla_x\ln\psi_g^t$ and the potential (killing rate) $\frac{i}{\hbar}V_t$.

The solution of Equation (4) is determined by the solution of the Langevin equation

$$dq_s = \frac{i\hbar}{m}\nabla\ln\psi_g^{t-s}(q_s)ds + \sqrt{\frac{i\hbar}{m}}db_s. \tag{6}$$

Here, the Brownian motion b_s is defined as the Gaussian process with the covariance

$$E[b_t b_s] = min(t, s). \tag{7}$$

The solution of Equation (1) is expressed [43,44] by the Feynman–Kac formula

$$\chi_t(x) = E\left[\exp\left(-\frac{i}{\hbar}\int_0^t ds V_{t-s}(q_s(x))\right)\chi_0(q_t(x))\right], \tag{8}$$

where $q_t(x)$ is the solution of Equation (6) with the initial condition $q_0(x) = x$ and the expectation value is over the paths of the Brownian motion.

A derivation of the Feynman integral (8) has been discussed previously [45,46]. An extension of the real diffusion processes [43] to a complex domain with an application to the Feynman integral is studied in [47–49].

As the simplest case, we consider the ground state solution of Equation (2) (with a constant ω)

$$\psi_g(x) = \left(\frac{\pi\hbar}{m\omega}\right)^{-\frac{1}{4}}\exp(-\frac{m\omega}{2\hbar}x^2). \tag{9}$$

The stochastic Equation (6) reads

$$dq = -i\omega q dt + \sqrt{\frac{i\hbar}{m}}db. \tag{10}$$

A simple calculation gives

$$\int dx|\psi^g(x)|^2 E[q_t(x)q_{t'}(x)] = \frac{\hbar}{2m\omega}\exp(-i\omega|t - t'|). \tag{11}$$

The rhs of Equation (11) is the expectation value of the time-ordered product of Heisenberg picture position operators in the ground state of the harmonic oscillator.

3. Quantum Field Theory

We consider the canonical field theory of a scalar massless field with the Hamiltonian (we set the velocity of light $c = 1$)

$$H = \frac{1}{2} \int d\mathbf{x} \Big(\Pi^2 + (\nabla \phi)^2 + v(t)\phi^2 \Big) + \int d\mathbf{x} V_t(\phi), \tag{12}$$

where $\Pi(\mathbf{x})$ is the canonical momentum ($v(t)$ is a certain function which is specified below),

$$[\phi(\mathbf{x}), \Pi(\mathbf{y})] = i\hbar \delta(\mathbf{x} - \mathbf{y}). \tag{13}$$

We solve the Schrödinger equation

$$i\hbar \partial_t \Psi = H\Psi. \tag{14}$$

Let

$$\Psi_t = \psi_g^t \chi_t, \tag{15}$$

where ψ_g^t is the solution of the Schrödinger equation for free field theory

$$i\hbar \partial_t \psi_g^t = \frac{1}{2} \int d\mathbf{x} \Big(\Pi^2 + (\nabla \phi)^2 + v(t)\phi^2 \Big) \psi_g^t. \tag{16}$$

Then, χ satisfies the equation (an infinite dimensional version of Equation (6))

$$\hbar \partial_t \chi = \int d\mathbf{x} \Big(-\frac{i}{2}\Pi^2 - i(\Pi \ln \psi_g^t))\Pi - iV_t(\phi) \Big) \chi, \tag{17}$$

where

$$\Pi(\mathbf{x}) = -i\hbar \frac{\delta}{\delta \phi(\mathbf{x})}. \tag{18}$$

It follows from Equation (8) that the solution of Equation (17) can be expressed as

$$\chi_t(\phi) = E\Big[\exp\Big(-\frac{i}{\hbar} \int_0^t ds V_{t-s}(\phi_s) \Big) \chi_0\Big(\phi_t(\phi)\Big) \Big], \tag{19}$$

where $\phi_s(\phi)$ is the solution of the stochastic equation

$$d\phi_s(\mathbf{x}) = i\hbar \frac{\delta}{\delta \phi(\mathbf{x})} \ln \psi_g^{t-s} ds + \sqrt{i\hbar} dW_s(\mathbf{x}) \tag{20}$$

with the initial condition ϕ. $E[...]$ denotes an expectation value with respect to the Wiener process (Brownian motion) defined by the covariance

$$E\Big[W_t(\mathbf{x})W_s(\mathbf{y}) \Big] = min(t, s)\delta(\mathbf{x} - \mathbf{y}). \tag{21}$$

Let us consider the simplest example: the free field. Then, the ground state is

$$\psi_g = \Big(\det(\frac{\pi}{\omega}) \Big)^{-\frac{1}{4}} \exp(-\frac{1}{2\hbar}\phi\omega\phi). \tag{22}$$

where

$$\omega = \sqrt{-\triangle}$$

Equation (20) reads

$$d\phi_t = -i\omega\phi_t dt + \sqrt{i\hbar} dW. \tag{23}$$

The solution is

$$\phi_t(\phi) = \exp(-i\omega t)\phi + \sqrt{i\hbar}\int_0^t \exp(-i\omega(t-s))dW_s. \tag{24}$$

For Fourier transforms (in other words, in the one mode approximation), Equation (23) reads

$$d\phi_t(\mathbf{k}) = -i|\mathbf{k}|\phi_t(\mathbf{k})dt + \sqrt{i\hbar}dW(\mathbf{k}). \tag{25}$$

In subsequent sections, we use the same notation for $\mathbf{x}$ and $\mathbf{k}$ functions. The $\mathbf{k}$-representation is useful for a smooth transition from one-mode approximations to infinite modes.

4. Scalar Perturbations in the Era of Inflation

A metric perturbation of the flat conformal metric (with the conformal time $\tau = \int a^{-1}(t)dt$, where t is the cosmic time)

$$ds^2 = a^2(d\tau^2 - d\mathbf{x}^2)$$

in a special gauge (conformal Newtonian gauge with no anisotropic stress) takes the form

$$ds^2 = a^2((1+2\psi)d\tau^2 - ((1-2\psi)\delta_{jk} + h_{jk})dx^j dx^k). \tag{26}$$

We consider a single field inflaton model of inflation (for the formalism with multiple scalar fields, see [50]). Then, according to the authors of [10,11,14,50,51], the action for scalar cosmological perturbations in the inflationary era (in conformal time) is

$$S = \frac{1}{2}\int dx\left((\varphi')^2 - (\nabla\varphi)^2 + z^{-1}z''\varphi^2\right), \tag{27}$$

where

$$\varphi = a\Phi \tag{28}$$

and Φ is a gauge invariant variable linear in the scalar metric perturbation ψ and in the inflaton perturbation.

The Lagrangian equations of motion following from the action (27) are

$$(\partial_\tau^2 - \nabla^2 - z^{-1}z'')\varphi = 0 \tag{29}$$

where z can be expressed by the scale factor a [51] $z = a\sqrt{\gamma}$ with

$$\gamma = 1 - a^2(\partial_\tau a)^{-2}\partial_\tau(a^{-1}\partial_\tau a).$$

During an inflation in a scalar potential U in the slow-roll approximation [50], we have

$$z^{-1}z'' = (H_c a)^2(2 + 5\epsilon - 3\eta), \tag{30}$$

where H_c is the Hubble variable in the cosmic time and

$$\epsilon = \frac{1}{16\pi G}\left(\frac{U'}{U}\right)^2,$$

$$\eta = \frac{1}{8\pi G}\frac{U''}{U}.$$

$z^{-1}z''$ in the approximation of an almost exponential expansion (i.e., for small ϵ and η) is $2\tau^{-2}$ (as $(H_c a)^2 \simeq \tau^{-2}$). Hence, in this approximation, Equation (29) reads

$$(\partial_\tau^2 + k^2 - 2\tau^{-2})\varphi = 0. \tag{31}$$

The Hamiltonian (12) for the action (27) is

$$H = \frac{1}{2} \int d\mathbf{x} \left(\Pi^2 + (\nabla \varphi)^2 - z^{-1} z'' \varphi^2 \right). \tag{32}$$

The model is quantized in a standard way by a realization of the canonical commutation relations with $\Pi(\mathbf{x})$ defined in Equation (18) (now, $v = -z^{-1} z''$ in Equation (13)).

5. Scalar Acoustic Environment

During inflation, the inflaton field is dominant in Φ (28), but, when the inflation stops, the inflaton decays and the reheating begins (radiation era). In such a case, in the gauge invariant variable Φ, the scalar perturbations of the metric become dominant. We assume that the scalar perturbations evolve adiabatically (constant entropy) according to the equation for the gauge invariant scalar metric perturbations (with no anisotropic stress and with the flat spatial background metric) [10,11]

$$\frac{d^2}{d\tau^2} \Phi + 3(1 + c_s^2) \mathcal{H} \frac{d}{d\tau} \Phi + c_s^2 \triangle \Phi + (2 \frac{d}{d\tau} \mathcal{H} + (1 + 3c_s^2) \mathcal{H}^2) \Phi = 0, \tag{33}$$

where $\mathcal{H} = a^{-1} \frac{d}{d\tau} a$, c_s is the acoustic velocity approximately equal $\sqrt{w}$ and $p = w\rho$ where p is the pressure and ρ is the density in the energy–momentum tensor on the rhs of Einstein equations. The effect of the decay of the inflanton at the end of the inflation era could be described by a modification of Equation (33) by a friction term $\gamma \partial_\tau \Phi$ [52]. We assume that either γ is negligible or Equation (33) describes the evolution of the scalar perturbation after the decay of the inflaton.

We consider power-law expansion in a conformal time

$$a = C\tau^\alpha. \tag{34}$$

We introduce

$$\Phi = \tau^r \phi, \tag{35}$$

where

$$r = -\frac{3}{2}(1 + w)\alpha, \tag{36}$$

then the Fourier transform of Equation (33) can be expressed as

$$\frac{d^2}{d\tau^2} \phi + c_s^2 k^2 \phi - \kappa \tau^{-2} \phi = 0, \tag{37}$$

where $\kappa \tau^{-2} = z^{-1} z''$ of Equation (29) with

$$\kappa = \frac{9}{4}(1 + w)^2 \alpha^2 + \frac{1}{2}\alpha - \frac{3}{2} w\alpha - (1 + 3w)\alpha^2. \tag{38}$$

Equation (37) is an analog of Equation (29) with $\triangle \to c_s^2 \triangle$. The solution of Equation (37) can be expressed by the cylinder functions Z_ν

$$\phi = \sqrt{k\tau} Z_{p - \frac{1}{2}}(c_s k\tau) \tag{39}$$

where

$$\kappa = p(p - 1) \tag{40}$$

The Hamiltonian corresponding to Equation (37) is

$$H = \frac{1}{2} \int d\mathbf{x} \left(\Pi^2 + c_s^2 (\nabla \phi)^2 - \kappa \tau^{-2} \phi^2 \right), \tag{41}$$

i.e., $z^{-1} z'' \to \kappa \tau^{-2}$ in Equation (32).

6. Gaussian Solution of the Schr ödinger Equation for Scalar Perturbations

We look for a solution of the Schrödinger Equation (14) (with the Hamiltonian (32) or (41)) in the form

$$\psi_t^g = N \exp\left(\frac{i}{2\hbar}\phi\Gamma_\phi(\tau)\phi + \frac{i}{\hbar}J_\tau\phi\right), \tag{42}$$

where Γ_ϕ (we denote Γ in the scalar case with an index ϕ in order to distinguish it from the one for the tensor perturbations in the next sections; we skip ϕ if there is no danger of confusion) is an operator defined by a bilinear form $\Gamma_\phi(\mathbf{x} - \mathbf{y})$. Inserting ψ_t^g into the Schrödinger Equation (14) with the Hamiltonian (32), we obtain equations for N, Γ, J (in Fourier space)

$$i\hbar\partial_\tau \ln N = \tfrac{1}{2}\int d\mathbf{k}J(\mathbf{k})J(-\mathbf{k}) - \tfrac{i\hbar}{2}\delta(0)\int d\mathbf{k}\Gamma(\mathbf{k}), \tag{43}$$

where $\Gamma(\mathbf{k})$ is the Fourier transform of $\Gamma(\mathbf{x})$,

$$\partial_\tau J = -\Gamma J, \tag{44}$$

$$\partial_\tau \Gamma = -\Gamma^2 - c_s^2 k^2 + z^{-1}z''. \tag{45}$$

The term $\delta(\mathbf{0})$ in the normalization factor (43) results from an infinite sum of oscillator energies. It could be made finite by a regularization of the Hamiltonian (32), but this is irrelevant for calculations of the expectation values (because the normalization factor cancels). If we define

$$u(\tau) = \exp\left(\int^\tau ds\Gamma_s\right), \tag{46}$$

then $\Gamma(\mathbf{k}) = u^{-1}\partial_\tau u$ where u satisfies the equation

$$(\partial_\tau^2 + c_s^2 k^2 - z^{-1}z'')u(k) = 0. \tag{47}$$

With the result (30) in the inflation era ($c_s = 1$), this equation reads

$$(\partial_\tau^2 + k^2 - (2 + 5\epsilon - 3\eta)\tau^{-2})u(k) = 0. \tag{48}$$

The solution is

$$u = C_1\sqrt{k\tau}Z_{p_1-\frac{1}{2}}(k\tau) + C_2\sqrt{k\tau}Z_{p_2-\frac{1}{2}}(k\tau), \tag{49}$$

where p_1 and p_2 are the solutions of the quadratic equation

$$p(p - 1) = 2 + 5\epsilon - 3\eta$$

and Z_ν are the cylinder functions. The solutions (49) enter the formula for the free field quantization in the Heisenberg picture with the Bunch–Davis vacuum and in the formula for the spectrum of scalar perturbations [53,54].

For the acoustic perturbations (33), we have the Hamiltonian (41). The Gaussian wave function (42) is the solution of the Schrödinger Equation (14) if $\Gamma = u^{-1}\partial_s u$ where

$$\frac{d^2}{d\tau^2}u + c_s^2 k^2 u - \kappa\tau^{-2}u = 0. \tag{50}$$

In the era of radiation domination ($\alpha = 1$), we insert $w = \frac{1}{3}$ for relativistic particles, then Equation (50) reads

$$\frac{d^2}{d\tau^2}\phi + \frac{1}{3}k^2\phi - 2\tau^{-2}\phi = 0. \tag{51}$$

The Hamiltonian is

$$H = \frac{1}{2}\int d\mathbf{x}\left(\Pi^2 + \frac{1}{3}(\nabla\phi)^2 - 2\tau^{-2}\phi^2\right). \tag{52}$$

The equation for u determining Γ reads

$$(\partial_\tau^2 + \frac{1}{3}k^2 - 2\tau^{-2})u(k) = 0. \tag{53}$$

7. Schrödinger Wave Function for Tensor Perturbations

The quadratic action for tensor (transverse, traceless) perturbations is [10] (where τ is the conformal time)

$$S = \frac{1}{2}\int d\tau d\mathbf{x} a^2 (\partial_\tau h_{ij}\partial_\tau h_{ij} - \nabla h_{ij}\nabla h_{ij}). \tag{54}$$

Let us decompose h_{ij} in polarization tensors e_{ij}^ν

$$h_{ij} = a^{-1}e_{ij}^\nu h_\nu, \tag{55}$$

then

$$S = \int d\tau d\mathbf{x}\left(\partial_\tau h^\nu \partial_\tau h^\nu + h^\nu(\triangle + a''a^{-1})h^\nu\right). \tag{56}$$

The Hamiltonian is

$$H = \frac{1}{2}\int d\mathbf{x}\left((\Pi^\nu)^2 - h^\nu(\triangle + a''a^{-1})h^\nu\right), \tag{57}$$

where Π^ν is the canonical momentum. After quantization,

$$H = \frac{1}{2}\int d\mathbf{x}\left(-\hbar^2\frac{\delta^2}{\delta h^\nu(\mathbf{x})^2} + h^\nu(-\triangle - a''a^{-1})h^\nu\right). \tag{58}$$

The Schrödinger equation

$$i\hbar\partial_\tau \Psi = H\Psi$$

has a Gaussian solution (where Γ_h is an integral operator with the kernel $\Gamma_h(\mathbf{x}-\mathbf{y})$)

$$\psi_\tau^g = N(\tau)\exp(\frac{i}{2\hbar}h^\nu\Gamma_h(\tau)h^\nu) \tag{59}$$

if (we skip the index h)

$$\partial_\tau\Gamma + \Gamma^2 + (k^2 - a''a^{-1})\Gamma = 0 \tag{60}$$

and

$$\partial_\tau\ln N = -\frac{1}{2}\delta(\mathbf{0})\int d\mathbf{k}\Gamma(\mathbf{k}). \tag{61}$$

Note that, if Γ is a continuous function of τ, then $N(\tau)$ (hence also ψ_τ^g) is a continuous function of τ. As shown in the next section, Γ can be continuous between different epochs of the expansion, but the derivative of Γ has a discontinuity between the inflationary, radiation and baryonic epochs. Let as in the scalar case

$$\Gamma = u^{-1}\partial_\tau u \tag{62}$$

Then,

$$\partial_\tau^2 u + (k^2 - a''a^{-1}.)u = 0 \tag{63}$$

On the boundaries of various epochs, the Schrödinger equation needs some correction terms (barriers) because of the discontinuity of a''. In the de Sitter space (inflation era), $a = \exp(H_c t)$ (in the cosmic time t), $\tau = -H_c^{-1}\exp(-H_c t)$ and $a = -(H_c\tau)^{-1}$ (in conformal time, where H_c is the Hubble constant in the cosmic time) then

$$\partial_\tau^2 u + (k^2 - 2\tau^{-2})u = 0. \tag{64}$$

From Friedmann equations, if $p = w\rho$ (where ρ is the density and p the pressure), then $a \simeq t^{\frac{2}{3(1+w)}}$. In the radiation era, $w = \frac{1}{3}$, then $a \simeq \tau$, hence in Equation (63)

$$\partial_\tau^2 u + k^2 u = 0 \tag{65}$$

with the general solution

$$u_r(\tau) = k\sigma_r \cos(k\tau) + k\delta_r \sin(k\tau). \tag{66}$$

We consider further on mostly $\sigma \neq 0$ and $\delta \neq 0$. We may let $\delta = 0$, but then, to obtain a normalizable Gaussian wave packet, we must shift the argument of cosine by a complex number $\alpha - i\gamma$. Then, as in [4] ($u = k\cos(k\tau + \alpha - i\gamma)$),

$$\Gamma = -k\tan(k\tau + \alpha - i\gamma) = k\Big(i\sinh(2\gamma) - \sin(k\tau + \alpha)\cos(k\tau + \alpha)\Big)$$
$$\times \Big(\cosh(2\gamma) + \tfrac{1}{2}\cos(2k\tau + 2\alpha)\Big)^{-1}. \tag{67}$$

Note that $\Im(\Gamma) \simeq 2k\gamma$ for a small γ (squeezing). This form of Γ is useful if we wish to represent the squeezing as explicitly proportional to γ (for a small γ).

In the baryonic era when $w = 0$ ("dust") then $a \simeq \tau^2$, hence again $a''a^{-1} = 2\tau^{-2}$. Thus, we have the same Equation (64) as for the exponential (inflationary) expansion (when we use the approximation $\epsilon = \eta = 0$).

The solution of Equation (64) (inflationary era) is ($p_1 = 2, p_2 = -1$ in Equation (49))

$$u = \tau^{-1}\Big(\sigma(k\tau\cos(k\tau) - \sin(k\tau)) + \delta(k\tau\sin(k\tau) + \cos(k\tau))\Big) \tag{68}$$

It can be seen that for the solution (66) as well as (68) Γ depends only on $R = \delta\sigma^{-1}$. We can obtain normalizable solutions of the Schrödinger equation in the inflationary era with $\delta = 0$ shifting the arguments of sin and cos by a complex factor. Thus, instead of (68), we can write a solution of Equation (64) in the form

$$u = k\cos(k\tau + \alpha - i\gamma) - \tau^{-1}\sin(k\tau + \alpha - i\gamma). \tag{69}$$

The solution of the acoustic Equations (33), (37) and (50) is obtained from Equations (68)–(69) with $k\tau \to c_s k\tau$. Thus, Equation (53) (describing scalar perturbations in the radiation era) for an acoustic wave of a relativistic fluid has a solution analogous to Equation (68):

$$u = \tau^{-1}\Big(\sigma(k\tau\cos(\tfrac{1}{\sqrt{3}}k\tau) - \sin(\tfrac{1}{\sqrt{3}}k\tau)) + \delta(k\tau\sin(\tfrac{1}{\sqrt{3}}k\tau) + \cos(\tfrac{1}{\sqrt{3}}k\tau))\Big). \tag{70}$$

An analog of Equation (69) is

$$u = k\cos(\frac{1}{\sqrt{3}}k\tau + \alpha - i\gamma) - \tau^{-1}\sin(\frac{1}{\sqrt{3}}k\tau + \alpha - i\gamma). \tag{71}$$

These solutions can be used in Section 13 for a calculation of the density matrix in the environment of scalar perturbations in the radiation era.

We view the time evolution of the Gaussian wave function of tensor perturbations as a continuous process through various epochs of Universe evolution. We wish to follow the Gaussian wave function starting from the inflationary era. For this purpose, we need to choose the expansion scale $a(\tau)$ in a continuous way. We use [13]

$$a(\tau) = \tau^{-1}K\tau_1^{-1}$$

in the inflationary era when $-\infty < \tau < \tau_1 < 0$

$$a(\tau) = -\tau_1^{-2}(\tau - 2\tau_1)K\tau_1^{-1}$$

in the radiation era when $\tau_1 < \tau < \tau_2$ and

$$a(\tau) = -\frac{1}{4}(\tau + \tau_2 - 4\tau_1)^2\tau_1^{-2}(\tau_2 - 2\tau_1)^{-1}K\tau_1^{-1} \tag{72}$$

for $\tau > \tau_2$ (baryonic era).

$a(\tau)$ is continuous together with its first derivative. Hence, $\Gamma(\tau)$ can be glued together in a continuous way (so that the wave function is continuous). $\frac{d^2a}{d\tau^2}$ is discontinuous (does not exist at the transition points between different eras). In such a case, Equation (63) (equivalent to the Schrödinger Equation (14)) requires an interpretation. Equation (63) is similar to the Schrödinger equation in one dimension with a discontinuous potential $a^{-1}a''$. When $a^{-1}a''$ is discontinuous between different epochs, we must impose continuity conditions upon u_s as in the Schrödinger equation on the line with discontinuous barriers.

8. Evolution of the Quantum Gaussian State in Various Cosmological Epochs

In this section, we investigate whether squeezing of the wave function for scalar and tensor perturbations [13–15] $\Im i\Gamma \simeq -\tau^2$ (at small τ) achieved in the inflation era continues to the subsequent epochs of the Universe evolution assuming that the Gaussian wave function is continuous between different epochs (it satisfies the Schrödinger equation with the quadratic Hamiltonians in the corresponding epochs). The squeezing is relevant for the noise intensity in the equations of motion of quantum particles [17], as shown in Section 13. We suggest that the expansion of the Universe itself (not the physical processes in various epochs) has the major impact on the wave function evolution at least for the tensor perturbations. This can be justified by the fact that in the first-order approximation the gravitational waves created during inflation interact neither with the scalar perturbations nor with the matter created after inflation. For the scalar perturbations, this assumption may be questionable because the inflanton decays between inflation and radiation era so that in the radiation era its contribution to Φ of Equation (33) is diminishing. Finally, only the scalar metric perturbation remains in Φ. However, in the second-order perturbative calculations, the scalar perturbations can produce the gravitational waves [29,30]. Such non-linear effects cannot be described by a Gaussian approximation. The quantum state varies with $\Gamma(\tau)$ where $\Gamma(\tau)$ is determined by u. We begin the evolution of the state $\psi_\tau(h)$ in the inflationary era. Then, Equations (59) and (62)–(64) apply. We assume that when the inflation stops at τ_1 then the radiation era begins. Then, the evolution of $\psi_\tau(h)$ is determined by Equation (66). After the recombination at τ_2, the photons decouple. We assume that in this era $w = 0$. Then, again, Equation (64) is satisfied (but now $a \simeq \tau^2$).

For a continuous evolution of the wave function $\psi_\tau(h)$ we need the continuity conditions for $\Gamma(\tau)$ at the start τ_1 of the radiation era and at the beginning τ_2 of the baryonic era. Let us denote $R = \sigma^{-1}\delta$ (the functions Γ are defined by R. During the inflation we have (from Equation (68))

$$
\begin{aligned}
\Gamma_i(\tau) = \Big(&- k^2\sin(k\tau) - k\tau^{-1}\cos(k\tau) + \tau^{-2}\sin(k\tau) \\
&+ R_i k^2\cos(k\tau) - R_i k\tau^{-1}\sin(k\tau) - R_i\tau^{-2}\cos(k\tau)\Big) \times \\
&\Big(k\cos(k\tau) - \tau^{-1}\sin(k\tau) + R_i k\sin(k\tau) + R_i\tau^{-1}\cos(k\tau)\Big)^{-1}
\end{aligned} \tag{73}
$$

For small $k\tau$ (large cosmic time t), we obtain

$$\Gamma_i(\tau) = \tau^{-1}\Big(-1 + (k\tau)^2 - \frac{1}{R_i}(k\tau)^3\Big) \tag{74}$$

From Equation (74), the real part of $i\Gamma_i$ is small (squeezing [14]), whereas the imaginary part of $i\Gamma_i$ is large (classical WKB behaviour [14]). The assumption that at $\tau \to -\infty$ the state ψ_τ^g tends to the vacuum requires $R_i = i$. For $R_i = i$, we have from Equation (73) exactly

$$\Gamma_i = ik\left(1 + i(k\tau)^{-3}\right)\left(1 + (k\tau)^{-2}\right)^{-1} \tag{75}$$

which agrees with the approximate Formula (74).

In the beginning of the radiation era at time τ_1 (from Equation (66)),

$$\Gamma_r(\tau_1) = \left(-k^2 \sin(k\tau_1) + k^2 R_r \cos(k\tau_1)\right)\left(k\cos(k\tau_1) + kR_r \sin(k\tau_1)\right)^{-1} \tag{76}$$

If the wave function $\psi_\tau^g(h)$ is to be continuous between the inflation era and the radiation era, then, from $\Gamma_i(\tau_1) = \Gamma_r(\tau_1)$, we get

$$R_r = -\left(k\sin(k\tau_1) + \Gamma_i \cos(k\tau_1)\right)\left(\Gamma_i \sin(k\tau_1) - k\cos(k\tau_1)\right)^{-1} \tag{77}$$

and, subsequently, we can express R_r by R_i

$$
\begin{aligned}
R_r = &\left(k\tau_1 - \sin(k\tau_1)\cos(k\tau_1) - R_i(k^2\tau_1^2 - \cos^2(k\tau_1))\right) \\
&\times\left(-k^2\tau_1^2 + \sin^2(k\tau_1) - k\tau_1 R_i - \sin(k\tau_1)\cos(k\tau_1)R_i\right)^{-1}
\end{aligned}
\tag{78}
$$

At small $k\tau_1$, this gives

$$R_r \simeq -\frac{1}{2k\tau_1} + \frac{i}{4}(k\tau_1)^2 \tag{79}$$

If $k\tau_1$ is large, then from Equation (78)

$$R_r \simeq R_i \tag{80}$$

$\kappa\tau_1$ is large in the baryonic era. As shown in Section 13 (see also [20]), large R ensures big noise. To have a large noise in the radiation era (and subsequently in the baryonic era), we need big R_i in the inflation era.

As in the inflation era and in the baryonic era the same Equation (64) applies, we have the same continuity conditions in the passage between different eras (Equation (72)). Equation (77) remains true in the baryonic era (for $\tau > \tau_2$) when we replace R_i by R_b (the squeeze parameter in the baryonic era) but now τ is large. It follows from Equation (80) that $R_b \simeq R_r$.

9. Particle Interacting with Scalar and Tensor Perturbations in the Post-Inflationary Era

We are interested in the motion of a particle in the gravitational field of cosmological perturbations. The particles ("baryons" or dark matter particles) appear at the end of the inflation era. The environment of the cosmological perturbations may have an impact on the clustering (formation of inhomogeneities) of such particles. The environment of tensor perturbations (quantized gravitational waves) may be detected in LIGO/Virgo detector, as suggested in [17]. This is a system of mirrors such that the interference takes place depending on the distance of the mirrors. Let us consider a system of two particles (mirrors) with masses $m' >> m$ in a free falling frame (e.g., in satellites as in prospective LISA gravitational wave detector). The distance between the particles depends on the metric $ds^2 = g_{\mu\nu}dX^\mu dX^\nu$. The action for the light (m) particle is

$$S = -m\int\sqrt{g_{\mu\nu}dX^\mu dX^\nu}$$

where the metric is defined in Equation (26). If we are to compute averages over the metric, we need to expand the action in a perturbation series in weak gravitational perturbations.

To do it in a covariant way, we choose the Fermi coordinates between the space-like separated geodesics of the particles m' and m. Now, $X^\mu = (\tau, \mathbf{X})$, where $\mathbf{X}$ are the Fermi coordinates between the neighboring geodesics of the two particles. Calculating the square root in S in the lowest order with the metric (26), we obtain [17,21]

$$S = \frac{m}{2} \int d\tau a (\frac{d\mathbf{X}}{d\tau})^2 - \frac{m}{2} \int d\tau a R_{0l0r} X^r X^l,$$

where $R_{\mu\nu\alpha\beta}$ is the Riemannian tensor. Inserting the expression for the Riemannian tensor with a linear approximation (27) for the metric, we obtain the action

$$S_I = \frac{m}{2} \int d\tau a \frac{dX^k}{d\tau} \frac{dX^k}{d\tau} - \lambda m \int d\tau \psi(\tau, \mathbf{X}(\tau)) \frac{d^2}{d\tau^2} a \mathbf{X}^2$$
$$+ \frac{1}{2} m\lambda \int d\tau h_{jk}(\tau, \mathbf{X}(t)) \frac{d^2}{d\tau^2}(a X^k X^j), \tag{81}$$

where $\lambda^2 = 8\pi G$ (we rescale $\psi \to \lambda\psi, h \to \lambda h$ so that the quadratic Einstein gravitational action for the perturbations is the same as the one for the free massless scalar field of Section 3).

In a general metric perturbation, the action should depend on gauge invariant variables. Hence, $\psi \to \Phi$ in Equation (81). According to Equation (35), $\phi = \tau^{\frac{3}{2}(1+w)\alpha}\Phi$ so the action (81) takes the form

$$S_I = \frac{m}{2} \int d\tau a \frac{dX^k}{d\tau} \frac{dX^k}{d\tau} - \lambda m \int d\tau \tau^{-\frac{3}{2}(1+w)\alpha} \phi(\tau, \mathbf{X}(\tau)) \frac{d^2}{d\tau^2} a \mathbf{X}^2$$
$$+ \frac{1}{2} m\lambda \int dt h_{jk}(\tau, \mathbf{X}(\tau)) \frac{d^2}{d\tau^2}(a X^k X^j), \tag{82}$$

where ϕ satisfies Equation (37) and its quantum evolution is determined by the Hamiltonian (41) (in the radiation era, $w = \frac{1}{3}$, $\alpha = 1$ and $\phi = \tau^2\Phi$). The tensor field h_{ij} is expressed by h^ν (55) with the Hamiltonian (57)

The interaction of a particle with the gravitational waves in Equation (82) is the same (for $a = 1$) as the one in [17,21] (the scalar terms have been considered in [55]). Equation (82) defines a linear coupling model with the coupling

$$V = \int d\mathbf{x}(\phi f + h_{rl} f^{rl}), \tag{83}$$

where

$$f(\mathbf{x}) = -m\lambda \int d\tau \tau^{-\frac{3}{2}(1+w)\alpha} \delta(\mathbf{x} - \mathbf{X}(\tau)) \frac{d^2}{d\tau^2} a \mathbf{X}^2 \tag{84}$$

and

$$f^{rl}(\mathbf{x}) = \frac{1}{2} m\lambda \int d\tau \phi \delta(\mathbf{x} - \mathbf{X}(\tau)) \frac{d^2}{d\tau^2} \left(a X^r(\tau) X^l(\tau) \right). \tag{85}$$

We apply the formula for the linear coupling in order to calculate the density matrix of quantum particles.

From the action (82), neglecting the dependence of fields on $\mathbf{X}$ (as we do in subsequent sections for the quantized fields), we obtain an equation of motion for a particle in a classical solution ψ^{cl} and h^{cl}_{jk} of the Einstein equations for the gravitational perturbations

$$\frac{d}{d\tau} a \frac{dX_k}{d\tau} + \lambda \frac{d^2 h^{cl}_{jk}}{d\tau^2} a X^j + 2\lambda \frac{d^2 \psi^{cl}}{d\tau^2} a X_k = 0. \tag{86}$$

10. Linear Coupling to an Oscillatory Environment

We are interested in quantum mechanics of particles interacting with quantized cosmological perturbations. The quantized tensor perturbations are expected to be detectable [17] as gravitons. The quantum scalar perturbations are analogs of phonons in solid state physics or plasmons in the physics of plasma. We expect that they have an impact on detectors of cosmological perturbations (e.g., that they can be measurable as the noise as suggested in [17] for tensor perturbations). We approximate the non-linear Einstein action

by its quadratic term in scalar and tensor perturbations. For both the scalar perturbations (84) and the tensor perturbations (85), the interaction with a particle takes place through a linear coupling of an oscillatory system with the coordinates of the particle. In one-mode approximation and with a linearized interaction, there are no substantial differences between the scalar and the tensor terms. We consider a model of a system with a Lagrangian $\mathcal{L}_X$ described by a coordinate X interacting linearly with an oscillator q (q can be either the scalar or the tensor mode). We denote the particle part of the interaction by $f_s(X)$. We have the Lagrangian

$$\mathcal{L} = \mathcal{L}_X + \frac{1}{2}((\frac{dq}{ds})^2 - \omega^2 q^2) + q_s f_s(X). \tag{87}$$

We could quantize the interaction (87) in the Heisenberg picture solving Lagrange equations with quantized oscillators q. Such an approach for an analogous electromagnetic interaction is developed in [56,57]. One obtains a quantum system $\mathcal{L}_X$ with a noise expressed by the oscillator creation and annihilation operators. Similar treatment of the particle–gravity interaction is discussed in [21]. Then, the noise does not depend on the quantum state of the oscillators but the correlation functions of the noise must be calculated in a particular quantum state of these oscillators.

In the approach of Section 2, the noise depends on the state of the oscillatory background. When we calculate the expectation values of the observables of the X-system (which are independent of the q-variables), according to quantum mechanics, the calculation is reduced to an evaluation of the trace in the mixed state ρ_t

$$Tr_q(_t|\Phi><\Phi|_t) \equiv \rho_t \tag{88}$$

where Tr_q is the trace over the states of the q- subsystem and $|\Phi>$ is the pure state of the system (87). We consider an initial state of the product form $\Phi(q, X) = \psi_0^g(q)\chi_0(q)\phi(X)$. According to Equation (3), it evolves into $\psi_t^g(q)\chi_t(q)\phi_t(q, X)$ where

$$\chi_t(q)\phi_t(q, X) = E[\chi_0(q_t(q))\phi(q_t(q), X)]$$

where in $E[\chi_0(q_t(q))\phi(q_t, X)]$ the evolution of the $\chi_0\phi$ state is expressed by the ordinary Feynman integral over the X paths and the expectation value over oscillator paths of Section 2 (so $q_t(q)$ is the stochastic process (10)). When the initial state of the oscillator is fixed as $\psi_0^g(q)$ (as in Sections 12 and 13), the average (88) is reduced to a q-average over $|\psi_t^g|^2$. For the thermal state of Section 14, the average (88) is over all states of the oscillator with a proper Gibbs weight.

In this paper, we consider the system of particles and cosmological fluctuations. We do no measurements on gravitational fluctuations. Nevertheless, these fluctuations have some impact on the motion of quantum particles. As shown in the following sections, the effect of the fluctuations upon the particle's motion can be described classically as a friction and noise. The averaging applies also to classical fields (including classical gravitational fluctuations). It would not make much difference whether we derived the density matrix considering, e.g., classical background of gravitational waves or coherent states of quantized gravitational waves. In the Appendix of [20], we showed that the assumption that gravitational waves have classical thermal distribution leads to the same friction and noise as the high temperature limit in the average (88) over the quantum Gibbs distribution. Nevertheless, for lower temperature, the noise and friction are $\hbar$-dependent. The effect of a classical cosmological background is weak (proportional to the Newton constant), whereas we can expect a strong detectable noise from quantum squeezed states as discussed in Section 13.

Our approach can be considered as a tool for a calculation of Feynman–Vernon influence functional [58]. Using the stochastic representation of Section 2, the density matrix ρ of the system $\mathcal{L}_X$ is obtained as an average over the environment of the oscillator in the state ψ_t^g

$$\rho_t(X, X') = \int dx \mathcal{D}X \mathcal{D}X' |\psi_t^g(x)|^2 \exp(-\frac{i}{\hbar} \int ds \mathcal{L}_X + \frac{i}{\hbar} \int ds \mathcal{L}_{X'}) \overline{\phi_i}(X_t(X)) \phi_i(X_t(X'))$$
$$E\left[\exp\left(\frac{i}{\hbar} \int_0^t q_s f_{t-s}(X'_s) \right) \chi_i(q_t(x)) \exp\left(-\frac{i}{\hbar} \int_0^t q_s^* f_{t-s}(X_s) \right) \overline{\chi_i}(q_t^*(x)) \right]. \tag{89}$$

where $\mathcal{D}X$ means the Feynman integral over the particle's trajectories, $\mathcal{D}X'$ is an integral over independent Feynman paths X' and $*$ when acting on functions means the complex conjugation and when applied to the stochastic process (10) means a complex conjugation of an independent version of the process (10).

For a Gaussian variable q_s, we have (for any number α_s)

$$E[\exp(\alpha_s q_s)] = \exp\left(\alpha_s < q_s > + \frac{1}{2} < (\alpha_s q_s - \alpha_s < q_s >)^2 > \right).$$

This equation can easily be generalized to $\int ds \alpha_s q_s$. If $\chi_i = 1$, the expectation value in Equation (89) is

$$E\left[\exp\left(\frac{i}{\hbar} \int_0^t ds q_s f_{t-s}(X_s) \right) \right] = \exp\left(\frac{i}{\hbar} \int_0^t ds < q_s > f_{t-s}(X_s) \right)$$
$$\exp\left(-\frac{1}{2\hbar^2} \int_0^t ds ds' E[(q_s - < q_s >)(q_{s'} - < q_{s'} >)] f_{t-s}(X_s) f_{t-s'}(X_{s'}) \right). \tag{90}$$

11. Solution of the Stochastic Equation for Scalar and Tensor Perturbations

According to the results of Sections 2 and 10, the calculation of the Feynman integral is reduced to the calculation of expectation values over solutions of stochastic equations. For the scalar perturbation with ψ_g^t, Equations (42) and (20) have the solution (with the initial condition ϕ at $\tau = \tau_0$)

$$\phi_s = \frac{u_{\tau-s}}{u_{\tau-\tau_0}} \phi - u_{\tau_0} u_{\tau-s} J_0 \int_{\tau_0}^s u_{\tau-t}^{-2} dt + \sqrt{i\hbar} u_{\tau-s} \int_{\tau_0}^s u_{\tau-t}^{-1} dW_t, \tag{91}$$

where u_s is a solution of Equation (47) (in subsequent sections we denote by the same symbol solutions of Equation (47) which are different depending on the choice of c_s and $z^{-1} z''$ in Equation (47)). We have

$$E[(\phi_s - < \phi_s >)(\phi_{s'} - < \phi_{s'} >)] = i\hbar u_{\tau-s} u_{\tau-s'} \int_{\tau_0}^{min(s,s')} u_{t-\tau}^{-2} d\tau. \tag{92}$$

With V linear in ϕ in Equation (19), we can calculate the expectation value in Equation (90) explicitly using Equations (91) and (92).

Linearized gravity decomposed in polarization components h^ν (55) has the Hamiltonian (57) which is the same as the one for two independent scalar fields. Hence, the solution of the Schrödinger equation of the linearized Einstein gravity is the product of the solutions for the scalar fields h_ν (a generalization of Equation (59) with a source term J which can describe coherent states of the gravitational waves; we assume that, owing to the rotation invariance, Γ_h does not depend on ν)

$$\psi_g^\tau(h) = A(\tau) \exp\left(\frac{i}{2\hbar} (h^\nu \Gamma_h(\tau) h^\nu + 2 J_\tau^\nu h_\nu) \right), \tag{93}$$

where Γ is an integral operator with the kernel $\Gamma(\tau, \mathbf{x} - \mathbf{y})$. As in Section 6, we find that the Fourier transform of $\Gamma(\tau, \mathbf{x} - \mathbf{y})$ can be expressed as $\Gamma(\tau, \mathbf{k}) = u(\mathbf{k})^{-1} \partial_\tau u(\mathbf{k})$ and $J_\tau = J_0 u_0 u_\tau^{-1}$. Then, Equation (20) takes the form

$$dh_s^\nu = -\Gamma(\tau - s) h_s^\nu ds - J_{\tau-s}^\nu ds + \sqrt{i\hbar} dW_s^\nu,$$

where

$$E[W_t^\alpha(\mathbf{x}) W_s^\beta(\mathbf{y})] = min(t, s) \delta^{\alpha\beta} \delta(\mathbf{x} - \mathbf{y}).$$

Expressing Γ and J in terms of u (Equation (62)), we have (we suppress the index ν)

$$dh_s = -\partial_\tau \ln u_{\tau-s} h_s ds - J_0 u_0 u_{\tau-s}^{-1} ds + \sqrt{i\hbar} dW_s. \tag{94}$$

Equation (94) has the solution (with the initial condition h at $\tau = \tau_0$)

$$h_s(h) = \frac{u_{\tau-s}}{u_{\tau-\tau_0}} h - u_{\tau_0} u_{\tau-s} J_0 \int_{\tau_0}^s u_{\tau-t}^{-2} dt + \sqrt{i\hbar} u_{\tau-s} \int_{\tau_0}^s u_{\tau-t}^{-1} dW_t. \tag{95}$$

We have

$$E[(h_s^\alpha - <h_s^\alpha>)(h_{s'\beta} - <h_{s'\beta}>)] = i\hbar \delta^{\alpha\beta} u_{\tau-s} u_{\tau-s'} \int_{\tau_0}^{min(s,s')} u_{t-\tau}^{-2} d\tau, \tag{96}$$

where for the solution (66) ($R = \sigma^{-1}\delta$)

$$\int_0^s u_{t-\tau}^{-2} d\tau = k^{-4}(1+R^2)^{-1}(\Gamma_h(t-s) - \Gamma_h(t)) \tag{97}$$

with

$$\Gamma_h(s) = k\Big(-\sin(ks) + R\cos(ks)\Big)\Big(\cos(ks) + R\sin(ks)\Big)^{-1} \tag{98}$$

whereas for the solution (68)

$$\begin{aligned}
\sigma k^3 \int_t^{t'} u_\tau^{-2} d\tau &= (kt'\sin(kt') + \cos(kt'))\Big((\delta kt' - \sigma)\sin(kt') + (\sigma kt' + \delta)\cos(kt')\Big)^{-1} \\
&- (kt\sin(kt) + \cos(kt))\Big((\delta kt - \sigma)\sin(kt) + (\sigma kt + \delta)\cos(kt)\Big)^{-1} \equiv \sigma k^{-1}(\Gamma(t) - \Gamma(t'))
\end{aligned} \tag{99}$$

We use the notation Γ at the rhs of Equation (99) in order to comply with the formulas for the correlation functions of ϕ_s in Equations (92) and (97). For the scalar perturbations in the radiation era when we have the solution u_s (of Equation (70)), the result (99) still applies with $k \to c_s k$ with $c_s = \frac{1}{\sqrt{3}}$.

If $\delta = 0$ and $u = k\cos(k\tau + \alpha - i\gamma)$, then

$$\int_t^{t'} u_s^{-2} ds = k^{-3}\Big(\tan(kt' + \alpha - i\gamma) - \tan(kt + \alpha - i\gamma)\Big) = k^{-4}(\Gamma_h(t) - \Gamma_h(t')), \tag{100}$$

where Γ is defined in Equation (67). In the analogous formula in the inflation era,

$$\begin{aligned}
k^3 \int_t^{t'} u_\tau^{-2} d\tau &= \Big(kt'\sin(kt' + \alpha - i\gamma) + \cos(kt' + \alpha - i\gamma)\Big)\Big(-\sin(kt' + \alpha - i\gamma) + kt'\cos(kt' + \alpha - i\gamma)\Big)^{-1} \\
&- \Big(kt\sin(kt + \alpha - i\gamma) + \cos(kt + \alpha - i\gamma)\Big)\Big(-\sin(kt + \alpha - i\gamma) + kt\cos(kt + \alpha - i\gamma)\Big)^{-1} \\
&\equiv k(\Gamma_h^i(t) - \Gamma_h^i(t'))
\end{aligned} \tag{101}$$

Formulas (97) and (99)–(101) allow calculating the evolution of the density matrix (88) for the interaction of particles with the cosmological perturbations.

12. One Mode Approximation

In our linearized model (87), q is either ϕ or h_{jk} and f is defined in Equations (84) and (85). In coordinate space (with an infinite number of modes), if ψ_g is the ground state (22), then we have

$$\int d\phi |\psi_g(\phi)|^2 \exp(iF\phi) = \exp(-\frac{\hbar}{4} F\omega^{-1}F)$$

for any function F and

$$\omega^{-1}(\mathbf{x}, \mathbf{y}) = D(\mathbf{x}, \mathbf{y}) = (2\pi)^{-3} \int d\mathbf{k}k^{-1} \exp(i\mathbf{k}(\mathbf{x} - \mathbf{y})) \tag{102}$$

For the scalar perturbation, we set q_s as

$$\phi_s(\mathbf{k}) = \exp(-iks)\phi(\mathbf{k}) + \sqrt{i\hbar}\int_0^s \exp(-ik(s-\tau))dW_\tau(\mathbf{k}) \tag{103}$$

To exhibit the method without an involvement with cumbersome formulas we perform the functional integration for one mode first (repeating for the convenience of the reader some of the calculations in [20]). We begin with the simplest case of the background of one single oscillator in the ground state (9). The average of one mode $\phi(\mathbf{k})$ in Equation (89) is calculated as

$$\int d\phi(\mathbf{k})\exp\left(-\tfrac{1}{\hbar}\phi(\mathbf{k})^*k\phi(\mathbf{k})\right)\exp(i\phi(\mathbf{k})^*F(\mathbf{k}) + iF(\mathbf{k})\phi^*(\mathbf{k}))) \\ = \exp(-\hbar F(\mathbf{k})^*k^{-1}F(\mathbf{k})). \tag{104}$$

We calculate the expectation value (89) for the density matrix of the X-system assuming that the oscillator is in the ground state and we do not calculate expectation values of any oscillator observables. According to Equations (90)–(92), we obtain (we use the solution (103), assume that the initial condition $\chi_i = 1$ and take only one component of $\mathbf{X}$)

$$\begin{aligned} \rho_\tau &\simeq \int dx \mathcal{D}X \mathcal{D}X' \\ &\exp(-\tfrac{kx^2}{\hbar})E\left[\exp\left(\tfrac{i}{\hbar}\int_0^\tau(\tfrac{m}{2}\tfrac{dX}{ds}\tfrac{dX}{ds} - \tfrac{m}{2}\tfrac{dX'}{ds}\tfrac{dX'}{ds} - \tfrac{i}{\hbar}\int_0^\tau(q_{\tau-s}f_s - q_{\tau-s}^*f_s')ds)\right)\right] \\ &= \int dx \mathcal{D}X \mathcal{D}X'\exp\left(\tfrac{i}{\hbar}\int_0^\tau(\tfrac{m}{2}\tfrac{dX}{ds}\tfrac{dX}{ds} - \tfrac{m}{2}\tfrac{dX'}{ds}\tfrac{dX'}{ds}\right. \\ &\exp(-\tfrac{kx^2}{\hbar})\exp\left(-\tfrac{i}{\hbar}\int_0^\tau(x\exp(-ik(\tau-s))f_s - x\exp(i\omega(\tau-s))f_s')ds\right) \\ &\exp\left(-\tfrac{1}{2\hbar^2}\int_0^\tau dsds'\left(E[(q_{\tau-s}- <q_{\tau-s}>)(q_{\tau-s'}- <q_{\tau-s'}>)]f_sf_{s'}\right.\right. \\ &\left.\left.+E[(q_{\tau-s}^*- <q_{\tau-s}^*>)(q_{\tau-s'}^*- <q_{\tau-s'}^*>)]f_s'f_{s'}'\right)\right), \end{aligned} \tag{105}$$

here $f_s' = f_s(X')$. In Equation (105), we have (this is the special case of Equation (92) with $u_s = \exp(iks)$)

$$\begin{aligned} E[(q_{\tau-s}- <q_{\tau-s}>)(q_{\tau-s'}- <q_{\tau-s'}>)] \\ = \tfrac{\hbar}{2k}\left(\exp(-ik|s-s'|) - \exp(-ik(2\tau-s-s'))\right). \end{aligned} \tag{106}$$

If the oscillator is in a time-dependent state, then we should insert the solution (66) (or (69)) in the Feynman formula (88). Hence, instead of Equation (105), we have

$$\begin{aligned} \int dx| \exp(i\tfrac{\Gamma(\tau)x^2}{2\hbar})|^2E\left[\exp\left(\tfrac{i}{\hbar}\int_0^\tau(q_sf_{\tau-s} - q_s^*f_{\tau-s}')ds\right)\right] \\ = \int dx\exp(i\tfrac{\Gamma(\tau)x^2}{2\hbar})\exp(-i\tfrac{\Gamma^*(\tau)x^2}{2\hbar})\exp\left(\tfrac{-i}{\hbar}\int_0^\tau(<q_{\tau-s}>f_s- <q_{\tau-s}^*>f_s')ds\right) \\ \exp\left(-\tfrac{1}{2\hbar^2}\int_0^\tau dsds'\left(E[(q_{\tau-s}- <q_{\tau-s}>)(q_{\tau-s'}- <q_{\tau-s'}>)]f_sf_{s'}\right.\right. \\ \left.\left.+E[(q_{\tau-s}^*- <q_{\tau-s}^*>)(q_{\tau-s'}^*- <q_{\tau-s'}^*>)]f_s'f_{s'}'\right)\right), \end{aligned} \tag{107}$$

where

$$\begin{aligned} E[(q_{\tau-s}- <q_{\tau-s}>)(q_{\tau-s'}- <q_{\tau-s'}>)] = i\hbar u_su_{s'}\int_0^{min(\tau-s,\tau-s')}dtu(\tau-t)^{-2} \\ = -i\hbar k^{-2}u_su_{s'}(\sigma^2 + \delta^2)^{-1}(\Gamma(\tau) - \Gamma(max(s,s'))). \end{aligned} \tag{108}$$

In our model, q_s is ϕ_s for the scalar perturbation and

$$f_s(\mathbf{k}) = -m\lambda s^{-\frac{3}{4}(1+w)\alpha}\exp(i\mathbf{k}\mathbf{X}_s)\frac{d^2}{ds^2}a\mathbf{X}^2 \tag{109}$$

In the tensorial case, q_s is h_s^{rl} and

$$f_s^{rl}(\mathbf{k}) = \frac{m}{2}\lambda\exp(i\mathbf{k}\mathbf{X}_s)\frac{d^2}{ds^2}aX^rX^l \tag{110}$$

The calculation of the x integral in Equations (105) and (107) leads to a quadratic functional of f_s and $f'_{s'}$ in the exponential. In the case (105) of the ground state of the one-dimensional oscillator, the scalar fluctuations give (now, $a = 1$ and $\alpha = 0$)

$$\rho_\tau \simeq \exp\left(\frac{i}{\hbar} \int_0^\tau \left(\frac{m}{2} \frac{dX}{ds} \frac{dX}{ds} - \frac{m}{2} \frac{dX'}{ds} \frac{dX'}{ds} \right) \right)$$
$$\exp\left(-\frac{1}{4\hbar k} \int_0^t ds\, ds' \left(f_s f_{s'} \exp(-ik|s-s'|) + f'_s f'_{s'} \exp(ik|s-s'|) - 2 f_{s'} f'_s \exp(ik(s-s')) \right) \right). \tag{111}$$

We write

$$Q = \frac{1}{2}(X + X') \tag{112}$$

$$y = X - X'.$$

We expand the exponential in Equation (111) in y. We obtain

$$\rho_t = \int \mathcal{D}Q \mathcal{D}y \exp\left(\frac{i}{\hbar} \int_0^\tau ym \frac{d^2 Q}{ds^2} \right)$$
$$\exp\left(-\frac{1}{\hbar k} \int_0^\tau ds' \int_0^{s'} ds \left(-i \sin(k(s-s'))\left(\frac{d^2}{ds^2} yQ \frac{d^2}{ds'^2} Q^2 \right. \right. \right. \tag{113}$$
$$\left. \left. \left. - \frac{d^2}{ds^2} yQ \frac{d^2}{ds'^2} yQ \cos(k(s-s')) \right) \rho_0(Q_\tau, y_\tau) \right.$$

The term linear in y modifies the equation of motion of the Q coordinate. The term quadratic in y is a noise acting upon the particle [25].

In the expression (107) of the time-dependent reference state, we obtain

$$\rho_\tau \simeq \exp\left(-\frac{i}{2\hbar}(\Gamma(\tau) - \Gamma^*(\tau))^{-1} \left(\int_0^\tau (u_\tau^{-1} u_s f_s - u_\tau^{*-1} u_s^* f'_s) ds \right)^2 \right.$$
$$-\frac{i}{2\hbar k^2} \int_0^\tau ds \int_0^{s'} ds' \left(u_s u_{s'} (\sigma^2 + \delta^2)^{-1}(\Gamma(\tau) - \Gamma(max(s,s'))) f_s f_{s'} \right. \tag{114}$$
$$\left. \left. - u_s^* u_{s'}^* (\sigma^{*2} + \delta^{*2})^{-1}(\Gamma^*(\tau) - \Gamma^*(max(s,s'))) f'_s f'_{s'} \right) ds\, ds' \right).$$

We expand (114) in y. After the expansion in y until the second-order terms in Equation (105), we obtain

$$\int \mathcal{D}Q \mathcal{D}y \exp\left(\frac{i}{\hbar} \int_0^\tau y(m\frac{d^2 Q}{ds^2} + L(Q) + \frac{i}{2\hbar} My) \right) \rho_0(Q,y)$$
$$= \int \mathcal{D}Q \mathcal{D}y \exp\left(\frac{i}{\hbar} \int_0^\tau (y\tilde{L} + \frac{i}{2\hbar} yMy) \right) \rho_0(Q,y)$$
$$= \int \mathcal{D}Q \mathcal{D}y \exp\left(-\frac{1}{2\hbar^2}(y - i\hbar M^{-1}\tilde{L}) M (y - i\hbar M^{-1}\tilde{L}) - \frac{1}{2}\tilde{L} M^{-1}\tilde{L} \right) \rho_0(Q,y),$$

where by L we denote a functional of Q, M is an operator and by $y\tilde{L}$ we denote the term proportional to y. We introduce $\tilde{Q} = M^{-\frac{1}{2}}\tilde{L}$; then, $\tilde{Q}$ is a Gaussian variable which has the white noise distribution that can be represented as $\partial_s b_s$. It can be seen that the equation $\tilde{Q} = M^{-\frac{1}{2}}\tilde{L}$ can be expressed as the stochastic equation

$$m\frac{d^2 Q}{ds^2} + L(Q) = M^{\frac{1}{2}} \partial_s b_s. \tag{115}$$

The calculation of ρ_τ involves an average over solutions of the stochastic Equation (115). In general, there is still the Gaussian integral over y so that the expression for the density matrix can be obtained in the form of an integral over the solutions of the stochastic Equation (115) and over the y terms resulting from an expansion in y of $\rho_0(Q,y)$ (this is an expansion in $\hbar$).

13. General Gaussian Environment of Cosmological Perturbations

In this section, we consider a general Gaussian time-dependent state of scalar and tensor perturbations. These perturbations are generated by independent scalar fields ϕ, h^ν. The difference between scalar and tensor perturbations is in the way they couple to particle velocities (Equations (109)–(110)). The action (82) (together with the gravitational action) in

Fourier space takes the form of a sum over $\mathbf{k}$ modes (in the interaction of a particle with cosmological perturbations, we neglect the dependence of $h(s, \mathbf{x})$ on spatial coordinates)

$$
\begin{aligned}
S = \int ds \Big(&\tfrac{1}{2} \int d\mathbf{k} h_\nu^*(s, \mathbf{k})(-\partial_s^2 - k^2 - a^{-1}a'')h_\nu(s, \mathbf{k}) \\
&+ \int d\mathbf{k} \phi(s, \mathbf{k})(-\partial_s^2 - c_s^2 k^2 - \kappa s^{-2})\phi(s, \mathbf{k}) \\
&+ \tfrac{1}{2} m a \frac{dX_r}{ds}\frac{dX_r}{ds} + (2\pi)^{-\frac{3}{2}} \int d\mathbf{k} h_\nu(s, \mathbf{k}) f_s^\nu(\mathbf{k}) + (2\pi)^{-\frac{3}{2}} \int d\mathbf{k} \phi_s(\mathbf{k}) f_s(\mathbf{k}) \Big),
\end{aligned}
\tag{116}
$$

where $h_\nu = a^{-1} e_{rl}^\nu h^{rl}$, $f^\nu(\mathbf{k}) = e_{rl}^\nu f^{rl}(\mathbf{k})$ and $f^{rl}(\mathbf{k})$, $f(\mathbf{k})$ are defined in Equations (109)–(110).

We consider a solution of the Schrödinger equation in an expanding universe which has the Gaussian form

$$
\psi_\tau^g(h, \phi) = \exp\Big(\frac{i}{2\hbar}\int d\mathbf{k} h^\alpha \Gamma_h(\tau) h^\alpha + \frac{i}{2\hbar}\int d\mathbf{k} \phi \Gamma_\phi(\tau)\phi\Big).
\tag{117}
$$

As discussed in Section 8, in an expanding universe, $\Gamma(\tau)$ in ψ_τ^g can dramatically change in time so that $\Im i\Gamma \simeq -\tau^2$ (squeezing) [13–15]. We show in this section that calculating expectation values in terms of the density matrix (obtained by averaging over $|\psi_\tau^g|^2$) according to Equations (88) and (89) leads to a large noise on the basis of Equation (115).

The calculation of the functional integral with the action (116) in an environment of cosmological perturbations (117) is reduced (according to Section 10) to a calculation of expectation values with respect to the stochastic processes ϕ_s and h_s^ν. These stochastic processes and their correlation functions are defined by the solutions u_s of Equation (47) with various c_s (for the scalar perturbations) and $z^{-1}z''$. The general result for a calculation of the expectation values is contained in Equation (90) but the complexity of the detailed formulas depends on the complexity of the solution u_s. In the remaining part of this section, we write down explicitly the expressions for the environment of the ϕ field which is in the ground state in Equation (117) or in the time dependent (squeezed) state with $z^{-1}z'' = 0$ (Minkowski space-time). We outline the calculations for the scalar perturbations in the radiation era when $z^{-1}z'' = 2\tau^{-2}$. For the tensor perturbations, we have explicit elementary solutions for u_s and h_s^ν and their correlations (Equation (99)) so that we can calculate the density matrix in the environment of the tensor perturbations exactly in all epochs of the universe evolution.

The scalar part of the contribution to the density matrix in the non-expanding metric $a = 1$ (described by the solution (66)) or in the radiation era (described by the solution (70)) is

$$
\begin{aligned}
\rho_\tau \simeq \exp\Big(&-\tfrac{1}{2\hbar}\int d\mathbf{k}(\Gamma(\tau) - \Gamma(\tau)^*)^{-1}\Big(\int_0^\tau (u_\tau^{-1}u_s f_s - u_\tau^{*-1}u_s^* f_s')ds\Big)^2 \\
&-\tfrac{i}{2\hbar}\int d\mathbf{k} k^{-4}\int_0^\tau \Big(u_s u_{s'}(\sigma^2 + \delta^2)^{-1}(\Gamma(\tau) - \Gamma(\mu)))f_s f_{s'} \\
&-u_s^* u_{s'}^*(\sigma^{*2} + \delta^{*2})^{-1}(\Gamma^*(\tau) - \Gamma^*(\mu))f_s' f_{s'}'\Big)ds ds'\Big).
\end{aligned}
\tag{118}
$$

where $\mu = max(s, s')$ and in the radiation era $(\sigma^2 + \delta^2)^{-1}\Gamma$ should be replaced by Γ from Equation (99), f_s is defined in Equation (109) where we neglect $\exp(i\mathbf{k}\mathbf{X})$.

The tensor part in the expectation values (105) and (107) with an infinite number of modes has been calculated in [20] (for $a = 1$). We have sums of the form

$$
f_s^\nu f_{s'}^\nu = \Lambda_{mn;rl} f_s^{mn} f_{s'}^{rl},
\tag{119}
$$

where $\Lambda_{rl;mn} = e_{rl}^\nu e_{mn}^\nu$. The result of an averaging over angles leads to an insertion $f \to q^{rl}$ in Equation (118) where

$$
q^{rl} = \frac{m}{2} a^{-1} \frac{d^2}{ds^2}\Big(X^r X^l - \frac{2}{3}\delta^{rl}\mathbf{X}^2\Big)a.
\tag{120}
$$

Hence, the tensor contribution to the expectation value (118) is

$$
\begin{aligned}
\rho_\tau \simeq \exp\Big(&-\tfrac{i}{2\hbar}\int d\mathbf{k}(\Gamma(\tau)-\Gamma(\tau)^*)^{-1}\Big(\int_{\tau_0}^\tau (u_\tau^{-1}u_s f_s^\alpha - u_\tau^{*-1}u_s^* f_s'^\alpha)ds\Big)^2 \\
&-\tfrac{i\lambda^2}{2\hbar}\tfrac{4\pi}{5}\int d\mathbf{k}k^{-2}\int_{\tau_0}^\tau \Big(u_s u_{s'}(\sigma^2+\delta^2)^{-1} \\
&(\Gamma(\tau)-\Gamma(\mu))q_s^{rl}q_{s'}^{rl} - u_s^* u_{s'}^*(\sigma^{*2}+\delta^{*2})^{-1}(\Gamma^*(\tau)-\Gamma^*(\mu))q_s'^{rl}q_{s'}'^{rl}\Big)dsds'\Big).
\end{aligned}
\tag{121}
$$

where u_s in general would be the solution of Equation (63) but in the static metric and in the radiation era $a^{-1}a'' = 0$, hence we have the solution (66). We could use Formula (121) for tensor perturbations in the inflationary and baryonic era when $a^{-1}a'' = 2\tau^{-2}$. Then, the function Γ is defined in Equation (99) or (101). The factor $\frac{4\pi}{5}$ in Equation (121) comes from the average over angles in the $d\mathbf{k}$ integration. The behavior of ρ_τ depends on the complex function $R(k) = \sigma(k)\delta(k)^{-1}$ in Equation (121). We can see that the final noise term can be large because of the squeezing factor $(\Gamma-\Gamma^*)^{-1}$ in Equation (121). Explicitly, the term with $(\Gamma-\Gamma^*)^{-1}$ coming from the scalar perturbation is

$$
\begin{aligned}
\exp\Big(&-\tfrac{i\lambda^2}{2\hbar}4\pi\int d\mathbf{k}k^2(\Gamma(\tau)-\Gamma(\tau)^*)^{-1}\int_{\tau_0}^\tau dsds'\Big(u_\tau^{-2}u_s u_{s'}f_s f_{s'} \\
&+u_\tau^{*-2}u_s^* u_{s'}^* f_s' f_{s'}' - u_\tau^{-1}u_\tau^{*-1}u_s^* u_{s'}f_s' f_{s'} - u_\tau^{-1}u_\tau^{*-1}u_s u_{s'}^* f_s f_{s'}'\Big)\Big).
\end{aligned}
\tag{122}
$$

For a comparison, let us first calculate the expression (122) for the ground state (22) of the scalar field. Then, $\frac{\delta}{\sigma} = i$, $\Gamma = ik$, $u_s = \exp(iks)$ and in the integral (122) we obtain (this is an infinite mode version of Equation (111))

$$
\begin{aligned}
\rho_\tau \simeq \exp\Big(&-\tfrac{1}{2\hbar}\int d\mathbf{k}k^{-1}\int_{\tau_0}^\tau dsds' \\
&\times\Big(\exp(-ik|s-s'|)f_{(s)}f_{s'}) + \exp(ik|s-s'|)f'(s)f'(s') \\
&-\exp(-ik(s-s'))(f_s f_{s'}' + f_s' f_{s'})\Big).
\end{aligned}
\tag{123}
$$

After the expansion (112), Equation (123) gives a term linear in y:

$$
-4i\tfrac{\lambda^2}{2\hbar}\int d\mathbf{k}k^{-1}\int_{\tau_0}^\tau ds'\tfrac{d^2}{ds'^2}Q^r y^r\int_0^{s'}ds\tfrac{d^2}{ds^2}\mathbf{Q}^2\sin(k(s'-s)).
\tag{124}
$$

Representing $\sin(ks)$ in Equation (124) as $-k^{-1}\partial_s\cos(ks)$, we integrate over k obtaining $\partial_s\delta(s-s')$. This is a local radiation damping term which coincides with the one which is obtained in Section 14 (Equation (144)) for thermal gravitational perturbations. The noise resulting from Equation (123) can be read from the quadratic part in Equation (123)

$$
\rho_\tau \simeq \exp\Big(-\tfrac{\lambda^2 m^2}{8\hbar}\int d\mathbf{k}k^{-1}\int_{\tau_0}^\tau ds'\int_{\tau-0}^{s'}ds\Big(-4\tfrac{d^2}{ds'^2}Q^r y^r\tfrac{d^2}{ds^2}y^l Q^l\cos(k(s-s'))\Big)\rho_0(Q_t,y_t)
\tag{125}
$$

For a time dependent ψ_t^g, we work out Equation (121) in more explicit form (in the Minkowski space-time $a = 1$) calculating

$$
\Gamma(\tau)-\Gamma(\tau)^* = k^3(R-R^*)(u_\tau u_\tau^*)^{-1},
\tag{126}
$$

$$
\Gamma(\tau)-\Gamma(s) = k^3 u_\tau^{-1}u_s^{-1}\sin(k(s-\tau))(1+R^2).
\tag{127}
$$

Using Equations (126) and (127), we obtain from Equation (122)

$$
\begin{aligned}
\rho_\tau \simeq \exp\Big(&-\tfrac{i}{2\hbar}\int dsds'\int d\mathbf{k}k^{-3} \\
&\times\Big((R-R^*)^{-1}(u_s u_{s'}u_\tau^{-1}u_\tau^* f_s f_{s'} + u_s^* u_{s'}^* u_\tau^{*-1}u_\tau f_s' f_{s'}' - 2u_s u_{s'}^* f_s f_{s'}') \\
&-u_s u_{s'}u_\tau^{-1}u_\mu^{-1}\sin(k(\mu-\tau))f_s f_{s'} + u_s^* u_{s'}^* u_\tau^{*-1}u_\mu^{*-1}\sin(k(\mu-\tau))f_s' f_{s'}'\Big)\Big),
\end{aligned}
\tag{128}
$$

where $\mu = max(s,s')$.

In an expansion in **y** (112) of the scalar contribution, the term that modifies equations of motion (a phase factor) is

$$
\begin{aligned}
\exp\Big(& -\tfrac{i}{2\hbar} \int ds ds' \int d\mathbf{k} k^{-3} \\
& \times \Big((R - R^*)^{-1}(u_\tau^* u_s + u_\tau u_s^*)(u_\tau^{-1} u_{s'} - u_\tau^{*-1} u_{s'}^*) \\
& - \sin(\mu - \tau))(u_s u_{s'} u_\tau^{-1} u_\mu^{-1} + u_s^* u_{s'}^* u_\tau^{*-1} u_\mu^{*-1}) \Big) \tfrac{d^2}{ds'^2} Q^n y^n \tfrac{d^2}{ds^2} \mathbf{Q}^2 \Big) \Big).
\end{aligned}
\tag{129}
$$

The scalar contribution to the noise in the time dependent environment ψ_τ^g ($a = 1$) can be written as (the term in Equation (128) quadratic in y)

$$
\begin{aligned}
\exp\Big(& -\tfrac{i}{2\hbar} \int d\mathbf{k} k^{-3} \int ds ds' \\
& \times \Big((R - R^*)^{-1}(u_\tau^* u_s + u_\tau u_s^*)(u_\tau^{-1} u_{s'} + u_\tau^{*-1} u_{s'}^*) \\
& - \sin(k(\mu - \tau))(u_s u_{s'} u_\tau^{-1} u_\mu^{-1} - u_s^* u_{s'}^* u_\tau^{*-1} u_\mu^{*-1}) \Big) \Big(\tfrac{d^2}{ds'^2} Q^l y^l \tfrac{d^2}{ds^2} Q^n y^n \Big) \Big) \\
\equiv & \exp(-\tfrac{1}{2\hbar^2}\mathbf{y} M \mathbf{y}).
\end{aligned}
\tag{130}
$$

Equation (121) simplifies if $\Gamma(\tau) \simeq const$. Set in Equation (66) $S\delta = i\sigma$ (S may depend on k). Then, $\Gamma(0) = ikS^{-1}$. We have a real Gaussian function in Equation (93) as an initial state. If S is large and $(k\tau)^{-1} >> S >> k\tau$ with $k\tau << 1$, then to Equation (118) only the term (122) contributes, where $u_s \simeq \cos(ks)$. The density matrix is a product of scalar and tensor terms (we assume that $\Gamma_\phi = \Gamma_h$)

$$
\begin{aligned}
\rho_\tau \simeq & \exp\Big(-\tfrac{1}{4\hbar} \int d\mathbf{k} \tfrac{S}{k} \Big(\int_{\tau_0}^\tau (\cos(k\tau))^{-1} \cos(ks)(f_s - f_s') ds \Big)^2 \Big) \\
& \times \exp\Big(-\tfrac{1}{4\hbar} \int d\mathbf{k} \tfrac{S}{k} \Big(\int_{\tau_0}^\tau (\cos(k\tau))^{-1} \cos(ks)(f_s^\alpha - f_s'^\alpha) ds \Big)^2 \Big).
\end{aligned}
\tag{131}
$$

The integration over the angles $k^{-1}\mathbf{k}$ of $\epsilon_{rl}^\alpha \epsilon_{mn}^\alpha$ is expressed by $< \Lambda_{rl;mn} >$ where $\Lambda_{rl;mn}$ is defined in Equation (119) and

$$
< \Lambda_{ij;mn} > = 4\pi\Big(\tfrac{1}{5}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) - \tfrac{2}{15}\delta_{ij}\delta_{nm} \Big).
$$

Hence, finally, when we put together the scalar and tensor terms, we obtain in Equation (131)

$$
\begin{aligned}
\rho_\tau \simeq \exp\Big(& -\tfrac{\lambda^2 m^2}{16\hbar} 4\pi \int dk k S(k) \Big(\int_{\tau_0}^\tau ds ds' (\cos(k\tau))^{-2} \cos(ks) \cos(ks') \\
& \times \Big(\tfrac{1}{4\pi} < \Lambda_{rl;mn} > \tfrac{d^2}{ds^2}(Q^r y^l + Q^l y^r) \tfrac{d^2}{ds'^2}(Q^m y^n + Q^n y^m) + 8\delta_{rl}\delta_{mn}) \tfrac{d^2}{ds'^2} Q^l y^l \tfrac{d^2}{ds^2} Q^n y^n \Big) \Big).
\end{aligned}
\tag{132}
$$

Equation (132) gives the spectrum of the noise as $8\pi G S k$.

It is useful to express the results (128)–(132) on the evolution of the density matrix (in Minkowski background space-time) with the representation (67) of Γ (with $u_s = k\cos(ks + \alpha - i\gamma)$). Now

$$
\Gamma(\tau) - \Gamma^*(\tau) = ik \sinh(2\gamma)\Big(\cosh^2\gamma - \sin^2(k\tau) \Big)^{-1}
\tag{133}
$$

and

$$
\Gamma(\tau) - \Gamma(\tau') = k \sin(k(\tau' - \tau))\Big(\cos(k\tau + \alpha - i\gamma)\cos(k\tau' + \alpha - i\gamma) \Big)^{-1}
\tag{134}
$$

Equation (118) for the scalar perturbation reads

$$
\begin{aligned}
\rho_\tau \simeq \exp\Big(& \tfrac{1}{2\hbar} \int d\mathbf{k}\, k^{-1} \Big(\sinh(2\gamma)(\cosh^2\gamma - \sin^2(k\tau + \alpha)) \Big)^{-1} \\
& \times \Big(\int_{\tau_0}^{\tau} (u_\tau^{-1} u_s f_s - u_\tau^{*-1} u_s^* f_s') ds \Big)^2 \\
& - \tfrac{i}{2\hbar} 4\pi \int dk\, k^{-3} \int_{\tau_0}^{\tau} ds\, ds' \Big(u_s u_{s'} \sin(k\mu - k\tau) \times \\
& \Big(\cos(k\tau + \alpha - i\gamma) \cos(k\mu + \alpha - i\gamma) \Big)^{-1} f_s f_{s'} \\
& - u_s^* u_{s'}^* \sin(k\mu - k\tau) \Big(\cos(k\tau + \alpha + i\gamma) \cos(k\mu + \alpha + i\gamma) \Big)^{-1} f_s' f_{s'}' \Big) \Big),
\end{aligned}
\tag{135}
$$

where $\mu = max(s, s')$. We have

$$
\begin{aligned}
u_\tau^{-1} u_s f_s - u_\tau^{*-1} u_s^* f_s' = \lambda m \Big(& \cosh^2\gamma - \sin^2(k\tau + \alpha) \Big)^{-1} \times \\
& \Big(i \sin(k(s - \tau)) \sinh(2\gamma) \tfrac{d^2}{ds^2} (\mathbf{Q}^2 + \tfrac{1}{4}\mathbf{y}^2) \\
& + (\cos(k(s - \tau)) \cosh(2\gamma) + \cos(k(s + \tau))) \tfrac{d^2}{ds^2} \mathbf{Q}\mathbf{y} \Big)
\end{aligned}
\tag{136}
$$

Hence, for small γ, the term (122) is dominating. Its contribution to the noise is

$$
\begin{aligned}
\rho_\tau \simeq \exp\Big(& -\tfrac{\lambda^2 m^2}{2\hbar} \int \tfrac{d\mathbf{k}}{k} \int ds\, ds' \Big(\sinh(2\gamma) \Big)^{-1} (\cosh^2\gamma - \sin^2(k\tau + \alpha))^{-1} \\
& \times \Big((\cos(k(s - \tau)) \cosh(2\gamma) + \cos(k(s + \tau))) \tfrac{d^2}{ds^2} \mathbf{Q}\mathbf{y} \\
& \times (\cos(k(s' - \tau)) \cosh(2\gamma) + \cos(k(s' + \tau))) \tfrac{d^2}{ds'^2} \mathbf{Q}\mathbf{y} \Big) \Big).
\end{aligned}
\tag{137}
$$

It follows from Equation (137) that the noise can be large if γ is small.

We can calculate the density matrix in the radiation era when $a(s) = s$ with the contribution of the scalar and tensor perturbations. We use the results (70) and (99). We do not write down these complicated expressions. Let us mention only the contribution of tensor perturbations to the noise. Thus, the contribution of the tensor perturbations to the quadratic part of the density matrix in the radiation era $(a(\tau) = \tau)$ is expressed as the following modification of Equation (132):

$$
\begin{aligned}
\rho_\tau \simeq \exp\Big(& -\tfrac{\lambda^2 m^2}{16\hbar} \int dk\, k\, S(k) \Big(\int_{\tau_0}^{\tau} ds\, ds' (\cos(k\tau))^{-2} \cos(ks) \cos(ks') ((a(s)a(s'))^{-1} \\
& \times <\Lambda_{rl;mn}> \tfrac{d^2}{ds^2} a(Q^r y^l + Q^l y^r) \tfrac{d^2}{ds'^2} a(Q^m y^n + Q^n y^m) \Big).
\end{aligned}
\tag{138}
$$

14. Thermal Perturbations

In this section, we first consider the constant metric $a = 1$ (Minkowski space-time) for scalar and tensor perturbations. In such a case the Hamiltonian (32) for the scalar field as well as the Hamiltonian (58) for radiation coincide with the Hamiltonians of free massless relativistic scalar fields. During the radiation era the Hamiltonian in Equation (58) for the tensor field in the conformal time also consists of a sum of two scalar free massless fields (12) with $v = V = 0$ (as $a'' = 0$). For the scalar perturbations, this does not happen unless $a = 1$. We consider first the thermal scalar perturbation on a Minkowski space-time. The geodesic deviation in thermal gravitational waves h^{rl} has been studied in [20]. At the end of this section we derive a modification of the geodesic deviation equation in a thermal state in the radiation era. The Lagrangian (27) and (83) with an infinite number of modes in the coordinate space is

$$
\mathcal{L} = \tfrac{1}{2} \int d\mathbf{x}\, \phi(s, \mathbf{x})(-\partial_s^2 + \triangle)\phi(s, \mathbf{x}) + \int d\mathbf{x}\, \phi f.
\tag{139}
$$

The evolution of the density is obtained from Equation (88) where the partial trace over the oscillator states involves all eigenstates of energy ϵ_n with the weight factor $\exp(-\beta\epsilon_n)$

with $\beta = \frac{1}{k_B T}$ where k_B is the Boltzmann constant and T is the temperature. This partial trace has been calculated in [20,59,60] with the result

$$
\begin{aligned}
\rho_\tau(\mathbf{X}, \mathbf{X}') &= \int \mathcal{D}X \mathcal{D}X' \exp\left(\frac{im}{2\hbar} \int_0^\tau ds\left(\frac{dX_r}{ds}\frac{dX_r}{ds} - \frac{dX_r'}{ds}\frac{dX_r'}{ds}\right)\right) \\
&\exp\left(\frac{1}{\hbar^2}\int_0^\tau ds \int_0^s ds'\left((f-f')C(f+f') - (f-f')A(f-f')\right)\right)\rho_0(\mathbf{X}_\tau, \mathbf{X}_\tau').
\end{aligned}
\tag{140}
$$

In (140), we have a decomposition of the finite temperature propagator D into the real and imaginary parts $D = A + iC$

$$
A(\mathbf{x} - \mathbf{x}', s - s') = 2\hbar(2\pi)^{-3}\int \frac{d\mathbf{k}}{2k}\cos(\mathbf{k}(\mathbf{x} - \mathbf{x}'))\cos(k(s-s'))\coth(\frac{\hbar\beta k}{2}),
\tag{141}
$$

$$
C(\mathbf{x} - \mathbf{x}', s - s') = 2\hbar(2\pi)^{-3}\int \frac{d\mathbf{k}}{2k}\cos(\mathbf{k}(\mathbf{x} - \mathbf{x}'))\sin(k(s-s')).
\tag{142}
$$

In Equations (141) and (142), we neglect the $\mathbf{x}$ dependence of the propagators and average over the angles. Then, the k-integral $d\mathbf{k} \simeq 4\pi dk k^2$ in the high temperature limit $\beta\hbar \to 0$ of A in Equation (141) gives $\delta(s - s')$. In C (142) we write (as in [25]) $\sin(k(s-s')) = -k^{-1}\partial_s\cos(k(s-s'))$. Then, integrating over k, we obtain $\partial_s\delta(s - s')$. In such a case, the formula for the density matrix in the limit $\beta\hbar \to 0$ is

$$
\begin{aligned}
\rho_\tau(\mathbf{X}, \mathbf{X}') &\simeq \int \mathcal{D}X\mathcal{D}X' \exp\left(\frac{i}{2\hbar}\int_{\tau_0}^\tau ds\left(\frac{dX_r}{ds}\frac{dX_r}{ds} - \frac{dX_r'}{ds}\frac{dX_r'}{ds}\right)\right) \\
&\exp\left(-\frac{i}{2\pi\hbar}\int_{\tau_0}^\tau ds\left((f-f')\partial_s(f+f') - \frac{1}{2\pi\hbar^2\beta}(f-f')(f-f')\right)\right).
\end{aligned}
\tag{143}
$$

We expand Equation (143) around Q. In the exponential (143), the term linear in y becomes

$$
y_n\left(-m\frac{d}{ds}\frac{dQ_n}{ds} + \frac{2\lambda^2 m^2}{\pi}Q_n\frac{d^5}{ds^5}\mathbf{Q}^2\right).
\tag{144}
$$

The contribution of the tensor perturbations to the density matrix has been calculated in [20]. It follows from Equation (140) with $f_s \to f_s^{rl}$. After an expansion in y, the term linear in y reads (we omit the contribution of classical solutions appearing in Equation (86))

$$
y_n\left(-m\frac{d}{ds}\frac{dQ_n}{ds} + 16Gm^2 Q_n\frac{d^5}{ds^5}\mathbf{Q}^2 + \frac{4Gm^2}{5}Q_l\frac{d^5}{ds^5}(\frac{1}{3}Q_r Q_r \delta_{nl} - Q_n Q_l)\right)
\tag{145}
$$

The term quadratic in y is the noise term. For low temperature, we obtain in general the non-local and non-Markovian stochastic Equation (115). In the high temperature limit $\beta\hbar \to 0$, the calculation of the density matrix is reduced to an expectation value over the solutions of the stochastic differential equation

$$
\begin{aligned}
&-\frac{d^2 Q_n}{ds^2} + 16GmQ_n\frac{d^5}{ds^5}\mathbf{Q}^2 + \frac{8\pi Gm}{10\pi}Q^l\frac{d^5}{ds^5}(\frac{1}{3}Q_r Q_r\delta_{nl} - Q_n Q_l) \\
&= m^{-1}\sqrt{2G}\beta^{-\frac{1}{2}}(M^{\frac{1}{2}})_{nr}\partial_s b_s^r,
\end{aligned}
\tag{146}
$$

(the tensor term on the lhs of Equation (146) coincides with the one derived in [61]). As explained in the derivation of Equation (115), the term quadratic in y defines the operator M. From Equation (143) (after an insertion of tensor perturbations), we obtain that M is an operator defined by the bilinear form

$$
\begin{aligned}
2G\beta^{-1}y^r M^{rl}y^l &= \frac{m^2\lambda^2}{4\pi}\beta^{-1}\int ds\left(\frac{d^2}{ds^2}(Q^j y^l)\frac{d^2}{ds^2}(Q^j y^l)\right. \\
&+ \frac{d^2}{ds^2}(Q^j y^l)\frac{d^2}{ds^2}(Q^l y^j) - \frac{2}{3}\frac{d^2}{ds^2}(Q^j y^j)\frac{d^2}{ds^2}(Q^l y^l)\Big) \\
&+ 8Gm^2\beta^{-1}\int ds\frac{d^2}{ds^2}y^n Q^n\frac{d^2}{ds^2}y^l Q^l = \frac{m^2\lambda^2}{4\pi}\beta^{-1}y^r Q^k \mathcal{M}_{rk;ln}y^l Q^n
\end{aligned}
\tag{147}
$$

where

$$
\mathcal{M}_{rk;ln}(s, s') = (< \frac{1}{4\pi}\Lambda_{rk;ln} > +8\delta_{rk}\delta_{ln})\partial_s^2\partial_{s'}^2\delta(s - s')
\tag{148}
$$

As in [20], we could derive Equation (146) together with the noise (148) from the classical Gibbs distribution of gravitational waves. The crucial check of the quantum nature

of gravity would involve a comparison of the experimental measurement of noise with the truly quantum spectrum $k \coth\left(\frac{\hbar\beta k}{2}\right)$ following from Equation (141).

In the radiation era $(a(\tau) = \tau)$, there are minor changes in our formula when applied to tensor perturbations. As follows from Equation (116) the kinetic term changes into $\frac{1}{2}m\lambda a\left(\frac{d\mathbf{X}}{ds}\right)^2$ and f^{rl} into $a^{-1}\frac{d^2}{ds^2}aQ^rQ^l$ so that the tensor noise term in Equation (147) is replaced by

$$
\begin{aligned}
\frac{m^2\lambda^2}{4\pi}\beta^{-1}\int ds\, a^{-2}\Big(& \frac{d^2}{ds^2}(aQ^jy^l)\frac{d^2}{ds^2}(aQ^jy^l) \\
&+\frac{d^2}{ds^2}(aQ^jy^l)\frac{d^2}{ds^2}(aQ^ly^j)-\frac{2}{3}\frac{d^2}{ds^2}(aQ^jy^j)\frac{d^2}{ds^2}(aQ^ly^l)\Big)
\end{aligned}
\tag{149}
$$

The lhs of Equation (146) resulting from tensor perturbations is changed as

$$
-\frac{d}{ds}a\frac{dQ_n}{ds}+\frac{8\pi Gm}{10\pi}Q^l\frac{d^3}{ds^3}a^{-1}\frac{d^2}{ds^2}a\left(\tfrac{1}{3}Q_rQ_r\delta_{nl}-Q_nQ_l\right)
\tag{150}
$$

It follows that we can associate a definite temperature to the tensor perturbations in the radiation era. We obtain the radiation damping (150) and the noise (149) which is proportional to the temperature β^{-1}. We cannot do this for scalar perturbations. The contribution of the scalar perturbations to the density matrix in the radiation era can be treated by means of the methods of Section 13 with (from Equation (109)) $f_s(\mathbf{k}) = -m\lambda s^{-2}\frac{d^2}{ds^2}s\mathbf{X}^2$. There is no thermal state for scalar perturbations in the radiation era. It is remarkable that the Hamiltonian (52) for the scalar perturbations in the radiation era is equal to the one for the tensor perturbations (gravitons) (57) in the baryonic era $(a \simeq \tau^2$ when the velocity of light is replaced by the acoustic velocity $c_s = \frac{1}{\sqrt{3}}$. For large conformal time, the term $a^{-1}a'' \simeq 2\tau^{-2}$ is negligible. This suggests that, at the beginning of the baryonic era (recombination time), we could have the thermal state for tensor perturbations with the temperature T and the thermal state for scalar perturbations with the temperature $T\sqrt{3}$.

The spectrum of the noise following from Equation (141) is $8\pi Gk\hbar\coth\left(\frac{\hbar\beta k}{2}\right)$, which at low temperature is $8\pi Gk$ and at high temperature $8\pi G\beta^{-1}$.

15. Discussion

Our study is based on the quadratic approximations to the Hamiltonian of tensor and scalar perturbations of Einstein gravity. We explore the Schrödinger wave function as a solution of the Schrödinger equation in various epochs of universe evolution. We are interested in the squeezing of the wave function in various epochs. The squeezing of the wave function is relevant for the motion of particles in the environment of cosmological perturbations because it determines the intensity of the quantum noise. We calculate the density matrix resulting from an average over quantized cosmological perturbations in a thermal state and in a general Gaussian state. We show that the evolution of the density matrix can be described by a stochastic equation of radiation damping. The quantum modification of the equation of motion can influence the formation of inhomogeneities in the early stages of universe evolution. It can be seen that the squeezing has little effect on the strength of the friction term in radiation damping but can substantially enforce the noise term. As earlier pointed out by Parikh, Wilczek and Zahariade, the quantized tensor perturbations could be detected in gravitational wave detectors as a specific noise. We show in this paper that this noise is modified by scalar perturbations. The determination of the exact form of the quantum noise in detectors may be important for distinguishing it from other sources of noise. Besides the gravitational wave detectors the scalar cosmological perturbations should have an impact on primordial black holes formation, CMB temperature fluctuations and on galaxies distribution.

In view of the prospective development of the detection of cosmological perturbations, it is interesting to extend our method to modified theories of gravity and non-perturbative quantizations. The question arises whether the quadratic approximation to the Hamiltonian and Gaussian approximation to the wave function of the gravitational perturbations applies

in these theories. If so, then the treatment in Section 10 of the interaction of gravitational perturbations with a non-relativistic particle (detector) can be used to obtain a stochastic equation for the radiation damping. Some of the modified theories of gravity (Horndeski and $P(X, \phi)$) predict the speed of propagation of gravitational waves different from the speed of light. This (hypothetical) difference is carefully studied in present day observations. The results on the particle interaction with cosmological perturbations in such theories could further be used for a selection of a proper model on the basis of detection experiments. We did not insert numerical values in our mathematical results. Numerical estimates for solutions of the evolution equations are needed for a comparison with observations. Such detailed numerical studies are postponed to a prospective research.

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