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Renormalizability of Alternative Theories of Gravity: Differences between Power Counting and Entropy Argument

Francesco Bajardi [1,2,](#), Francesco Bascone [1,2] and Salvatore Capozziello [1,2,3,4]

1   Dipartimento di Fisica "E. Pancini", Università di Napoli "Federico II", Via Cinthia, I-80126 Napoli, Italy; francesco.bascone@unina.it (F.B.); capozziello@na.infn.it (S.C.)
2   Istituto Nazionale di Fisica Nucleare (INFN), sez. di Napoli, Via Cinthia 9, I-80126 Napoli, Italy
3   Scuola Superiore Meridionale, Largo San Marcellino 10, I-80138 Napoli, Italy
4   Laboratory for Theoretical Cosmology, Tomsk State University of Control Systems and Radioelectronics (TUSUR), 634050 Tomsk, Russia
*   Correspondence: francesco.bajardi@unina.it

Abstract: It is well known that General Relativity cannot be considered under the standard of a perturbatively renormalizable quantum field theory, but asymptotic safety is taken into account as a possibility for the formulation of gravity as a non-perturbative renormalizable theory. Recently, the entropy argument has however stepped into the discussion claiming for a "no-go" to the asymptotic safety argument. In this paper, we present simple counter-examples, considering alternative theories of gravity, to the entropy argument as further indications, among others, on the possible flows in the assumptions on which the latter is based. We considered different theories, namely curvature-based extensions of General Relativity as $f(R)$, $f(\mathcal{G})$, extensions of teleparallel gravity as $f(T)$, and Horava–Lifshitz gravity, working out the explicit spherically symmetric solutions in order to make a comparison between power counting and the entropy argument. Even in these cases, inconsistencies were found.

Keywords: renormalization; Bekenstein–Hawking entropy; modified gravity

1. Introduction

One of the main issues arising when considering a quantum theory of gravity is the non-renormalizability of General Relativity (GR). More precisely, GR cannot be formulated as a perturbatively renormalizable quantum field theory. This can be easily understood by simple power-counting arguments. However, starting from the Weinberg results [1], researchers started looking for non-perturbative renormalizable quantum field formulations of GR. This latter case is usually referred to as asymptotic safety (see [2–5] for reviews on the subject). This belief relies on the possibility that there may be another non-trivial UV fixed point in the renormalization group flow of the gravitational field with respect to which GR can be considered as a perturbation by some relevant operator. In this case, the non-renormalizability of gravity would just mean that we are expanding in the wrong variables, and power counting fails for this reason. However, the fixed point of asymptotic safety scenario is gauge dependent; this issue can be addressed by considering a generalization of the standard effective action, the so-called gauge-fixing independent effective action. Although it formally yields the same S-matrix as the standard one, it is independent of the choice of the gauge condition [6].

Starting from holographic aspects of gravity, seen as incompatible with the aim of turning GR into a local quantum field theory, one may be led to ask whether there could be some no-go result obstructing the asymptotic safety scenario. In [7] (but the idea behind was presented earlier in [8]) for example, the so-called entropy argument was adopted as an explicit argument against GR renormalizability in any spacetime dimension. The crux of the argument relies on the assumption that the high-energy spectrum of GR is

dominated by black holes, in the sense that once enough energy is concentrated in a spacetime region, a black hole will form, claiming that the high-energy behavior of any n-dimensional renormalizable quantum field theory should be that of a Conformal Field Theory (CFT) in the same number of dimensions. Technically, this means that the entropy of a generic conformal field theory and the entropy given by the Bekenstein–Hawking formula should have the same energy-scaling behavior.

Even though this argument appears reasonable from different points of view, its validity is still under debate. In fact, there are several problematic physical assumptions on which the argument is based. The clearest exposition of the issues and the debate under discussion to our current knowledge is the one given in [9]. The main assumption of the entropy argument is the black hole dominance. This is surely reasonable from our low-energy expectation, but it is indeed borrowed from the low-energy approximation, so that it is legitimate to wonder to what extent this expectation should be trusted. In fact, at the Planck scale, physics could be different from what we expect, and furthermore, our understanding of micro black holes is still dominated by conjectures. Shomer [7] pointed out that this counter argument should not hold since the asymptotic safety scenario is based on the assumption that gravity is a valid low-energy approximation of some quantum field theory. Therefore, arguing against the non-renormalizability of gravity is like arguing against the validity of the Bekenstein–Hawking formula. However, this is another source of concern. Indeed, there is no a priori reason to expect the Bekenstein–Hawking formula to hold at Planck scales, and even more if one considers the recent literature concerning quantum corrections to the formula [10–13].

Specifically, the statement that the n-dimensional Bekenstein–Hawking entropy is asymptotically the same as that in n-dimensional CFT, in the high-energy regime, it suffers some inconsistencies. Indeed, this is basically due to the fact that, if we have a field of mass M and take the energy to be $\mathcal{E} \gg M$, almost all of the energy will be momentum, and the mass will be negligible, resulting in a UV CFT fixed point. However, because of this, it is more appropriate to consider the entropy argument only as a necessary condition. This latter consideration makes it only useful as a way to understand if a QFT is non-renormalizable.

What is usually done in GR is to consider the energy-scaling behavior of an n-dimensional CFT as $\mathscr{S} \sim \mathcal{E}^{\frac{n-1}{n}}$ and compare it with the energy scaling of the entropy of a Schwarzschild black hole (considering, for instance, asymptotically flat spacetimes) via the Bekenstein–Hawking formula for the entropy:

$$\mathscr{S} = \frac{A}{4G} \sim \frac{r_H^{n-2}}{G} \sim \mathcal{E}^{\frac{n-2}{n-3}}, \tag{1}$$

with G being the Newton constant, A the black hole surface area, and r_H the horizon radius. In this case, it is evident that there is no dimension where the two formulas agree. According to this result, it follows that n-dimensional GR is non-renormalizable. The same can be done for a negative cosmological constant, where the usual AdS/CFT correspondence is recovered as a manifestation of holography. On the other hand, the positive cosmological constant (de Sitter) is instead peculiar and cannot be addressed in this way.

In this paper, we considered counterexamples to this approach, analyzing four different modified theories of gravity and checking their coupling constant mass dimension via power counting (around a Gaussian fixed point, the superficial degree of divergence must give the correct information about the renormalizability of quantum field theory; the other way around is prevented by the possibility that cancellations occur). Then, we compared the results with the entropy argument, showing that the approach manifests a clear non-agreement in the comparison. Interestingly, we also showed that spherically symmetric solutions of modified theories (e.g., $f(R) = R^k$ and $f(\mathcal{G}) = \mathcal{G}^k$) occur when the power-law parameter k is linked to the dimension such that the theory results in being power-counting renormalizable.

Throughout the paper, we use natural units: $\hbar = c = k_B = e^+ = 1$.

2. A Summary of Extended and Modified Theories of Gravity

As a general remark, we dealt with *modified gravity* [14] when some basic assumption of GR, as the Equivalence Principle, the metric connection, or Lorentz invariance is not requested at the foundation of the theory. On the other hand, *extended gravity* [15] means that GR is recovered in some limit or as a particular choice in the action. To give an example, Teleparallel Equivalent General Relativity (TEGR), based on the torsion scalar T, is a modified theory of gravity, while $f(R)$ gravity, based on the curvature scalar R, is a straightforward extension of GR.

This section aims to outline the main properties of some modified/extended theories of gravity, in order to deal with their spherically symmetric solutions and to compute power counting and Bekenstein–Hawking entropy in view of a possible renormalizability of gravitational interaction. We mainly analyze four different theories, namely $f(R)$, $f(\mathcal{G})$, $f(T)$, and Horava–Lifshitz gravity. The idea of extending GR started due to the need to explain many inconsistencies at the astrophysical and cosmological level, but also for better adapting gravity with other fundamental interactions. With regards to the large scale, most modified/extended theories explain the accelerating expansion of the universe by introducing new curvature or torsion contributions, instead of dark matter or dark energy [16–21]. These geometric terms are also considered for the early universe inflationary paradigm [22–25]. The simplest case is $f(R)$ gravity, since it represents a natural extension of Einstein's GR to a generic function of the Ricci scalar curvature R, not required to be linear in the Einstein–Hilbert action. By relaxing the hypothesis of second-order field equations, it is possible to construct an action of the form $S = \int \sqrt{-g} f(R) \, d^4x$. By varying with respect to the metric, one gets the following field equations:

$$f_R(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R(R) = 0. \tag{2}$$

The presence of higher order terms yields several interesting aspects, as well as massive gravitational waves [26–28] or modifications of the Newtonian potential [29–31]. Other theories leading to higher than fourth-order field equations can be constructed by the general action:

$$S = \int \sqrt{-g} f(R, \Box R, \Box^2 R, ... \Box^k R) \, d^4x, \tag{3}$$

where $\Box$ is the d'Alembert operator. A detailed treatment can be found in [32–34]. Another possibility is to deal with other higher order curvature invariants, such as $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$. Among possible extensions involving these second-order scalars, a particular combination gives rise to the topological surface term $\mathcal{G}$, defined as:

$$\mathcal{G} \equiv R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}, \tag{4}$$

the so-called *Gauss–Bonnet* topological invariant. In four dimensions, it represents the Pfaffian of the curvature form of the Levi–Civita connection, and due to the generalized Gauss–Bonnet theorem, its integral over the manifold provides the Euler characteristic of the manifold. For this reason, in four dimensions, it does not provide any contribution to the dynamics. Nevertheless, a function of $\mathcal{G}$ is topologically trivial only in less than 3 + 1 dimensions and can be used in the gravitational action in order to study the dynamics of the system. Varying $f(\mathcal{G})$ with respect to the metric leads to the field equations:

$$\frac{1}{2}g_{\mu\nu}f(\mathcal{G}) - \left(2RR_{\mu\nu} - 4R_{\mu\alpha}R^\alpha{}_\nu + 2R_\mu{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma} - 4R^{\alpha\beta}R_{\mu\alpha\nu\beta}\right)f_\mathcal{G}(\mathcal{G})$$
$$+ \left[2R\nabla_\mu\nabla_\nu + 4G_{\mu\nu}\Box - 4(R^\rho_\nu\nabla_\mu + R^\rho_\mu\nabla_\nu)\nabla_\rho + 4g_{\mu\nu}R^{\rho\sigma}\nabla_\rho\nabla_\sigma - 4R_{\mu\alpha\nu\beta}\nabla^\alpha\nabla^\beta\right]f_\mathcal{G}(\mathcal{G}) = 0. \tag{5}$$

An important advantage of this formulation of gravity is given by the absence of ghost modes in gravitational waves, as discussed in [28]; even in cosmology, the theory yields many interesting predictions, which alleviate the GR issues at infrared scales [35–39].

Another theory we considered is TEGR and its extensions. It is mainly aimed at solving problems occurring in GR by treating gravity as a gauge theory of the translation group in the flat tangent spacetime. The flat and curved spacetime have a point-by-point correspondence, which is mathematically represented by a tetrad field through the relation:

$$g_{\mu\nu}e_a^\mu e_b^\nu = \eta_{ab}, \tag{6}$$

where the Latin indices label the flat spacetime, while the Greek indices label the curved spacetime. The fundamental field is represented by torsion, while curvature vanishes identically after neglecting the spin connections. The action reads as:

$$S = \frac{1}{16\pi G} \int d^4x \, |e| \, T, \tag{7}$$

where e is the determinant of tetrad fields e_μ^a and T is the torsion scalar, defined as:

$$T = S^{\rho\mu\nu} T_{\rho\mu\nu}, \tag{8}$$

with $S^{\rho\mu\nu}$ being the so-called "superpotential":

$$S^{\rho\mu\nu} = K^{\mu\nu\rho} - g^{\rho\nu}T^{\sigma\mu}{}_\sigma + g^{\rho\mu}T^{\sigma\nu}{}_\sigma, \tag{9}$$

$K^{\rho\mu\nu}$ the "contorsion tensor":

$$K^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}\left(T_{\mu\lambda\nu} + T_{\nu\lambda\mu} + T_{\lambda\mu\nu}\right), \tag{10}$$

and $T^{\rho\mu\nu}$ the "torsion tensor":

$$T^\rho{}_{\mu\nu} = 2\Gamma^\rho{}_{[\mu\nu]}. \tag{11}$$

When discarding the curvature and dealing only with torsion, the Christoffel connection can be written in terms of tetrad fields as:

$$\Gamma^\alpha{}_{\mu\nu} = e_a^\alpha \partial_\mu e_\nu^a \tag{12}$$

which is called the Weitzenböck connection.

The main properties of TEGR and its applications to cosmology can be found in [40,41]. Treating gravity as a gauge theory of the translation group might solve some ultraviolet inconsistencies, though it does not provide any further prediction with respect to GR at the large scales. However, an extension of the teleparallel action (7) can be considered in the action through a function of the torsion scalar [14], namely:

$$S = \int d^4x \, |e| \, f(T). \tag{13}$$

By varying the action with respect to the tetrad fields, second-order field equations arise, that is:

$$\frac{1}{e}\partial_\mu(e \, e_a^\rho S_\rho{}^{\mu\nu})f_T(T) - e_a^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}f_T(T) + e_a^\rho S_\rho{}^{\mu\nu}(\partial_\mu T)f_{TT}(T) + \frac{1}{4}e_a^\nu f(T) = 0 \tag{14}$$

Though the teleparallel field equations are exactly equivalent to those of Einstein's GR, $f(R)$ theory and $f(T)$ theory are different. Nevertheless, extended TEGR provides several cosmological predictions not contemplated by GR, as well as modifications to the ΛCDM model [42], the explanation of the accelerated universe expansion without dark energy [14,17,43,44], and models for the inflationary paradigm [25].

To conclude, the last theory of gravity we considered is the *Horava–Lifshitz* gravity, firstly proposed by Horava in [45,46]; it represents a quantum gravity approach aimed to extend GR to the UV scales. According to this theory, the Lorentz invariance is a low-energy

symmetry, which breaks at the fundamental level, whose spacetime structure turns out to be anisotropic. Moreover, as pointed out in [47–51], it can provide a solution for the divergences in the two-loop effective action of GR, since it is power counting renormalizable. With a generic line element:

$$ds^2 = N^2 dt^2 - g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right) \tag{15}$$

the Horava–Lifshitz four-dimensional action reads:

$$S = \int d^3x\, dt\, \sqrt{-g}\left\{\frac{2}{\kappa^2}\left(K_{ij}K^{ij} - \lambda K^2\right) - \frac{\kappa^2}{2w^4}\left(\nabla_i R_{jk}\nabla^i R^{jk} - \nabla_i R_{jk}\nabla^j R^{ik} - \frac{1}{8}\nabla_i R\nabla^i R\right)\right\} \tag{16}$$

with

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right) \qquad K^2 = g_{ij}K^{ij} \tag{17}$$

where λ, ω, and κ are coupling constants. To develop the entropy analysis, in Section 3.4, we use the spherically symmetric solution of the above action, provided in [52]. For further reading, see [53–56] and the reference therein.

The above ones are the theories that we considered in view of the issue of the renormalizability of gravitational interaction.

3. Spherical Symmetry

Let us consider now the spherical symmetry for the above theories of gravity to calculate the power counting analysis and the Bekenstein–Hawking entropy.

3.1. $f(R)$ Gravity

Let us solve the $f(R)$ field equations in n-dimensional spherically symmetric spacetime starting from the action:

$$S = \int \sqrt{|g|} f(R)\, d^n x. \tag{18}$$

The above action can be recast in terms of Lagrange multiplier λ as:

$$S = \int \left[\sqrt{|g|} f(R) - \lambda(R - \tilde{R})\right] d^n x, \tag{19}$$

$\tilde{R}$ being the spherically symmetric scalar curvature expression. By assuming an interval of the form:

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\Omega_{n-2}^2, \tag{20}$$

the n-dimensional Ricci scalar can be written as [57,58]:

$$\tilde{R} = e^{-\lambda}\left(\nu'' - \frac{\nu'^2}{2} - \frac{\nu'\lambda'}{2}\right) + \frac{(n-2)e^{-\lambda}}{r}(\nu' - \lambda') - \frac{(n-2)(n-3)}{r^2}(1 - e^{-\lambda}), \tag{21}$$

where the prime denotes the derivative with respect to r and $d\Omega_{n-2}^2$ is the $n-2$ sphere volume element. As a check, we see that by setting $n = 4$, Equation (21) reduces to the well-known four-dimensional case. By varying the action (19) with respect to R, we find Lagrange multiplier λ:

$$\frac{\delta S}{\delta R} = |g| f_R(R) - \lambda = 0 \qquad \rightarrow \qquad \lambda = |g| f_R(R), \tag{22}$$

$f_R(R)$ being the first derivative of $f(R)$ with respect to the scalar curvature. Replacing Equations (21) and (22) in Equation (19), the action becomes:

$$S = \int \left\{ e^{\frac{\nu+\lambda}{2}}[f(R) - Rf_R(R)] + e^{\frac{\nu-\lambda}{2}}r^{n-2}\left(\nu'' + \frac{\nu'^2}{2} - \frac{\nu'\lambda'}{2}\right)f_R(R) \right.$$
$$\left. + (n-2)e^{\frac{\nu-\lambda}{2}}r^{n-3}(\nu'-\lambda')f_R(R) + (n-2)(n-3)e^{\frac{\nu+\lambda}{2}}r^{n-4}(1-e^{-\lambda})f_R(R) \right\}d^n x. \tag{23}$$

By integrating second derivatives and neglecting the surface terms, the point-like Lagrangian finally takes the form:

$$\mathcal{L} = e^{\frac{\nu+\lambda}{2}}[f(R) - Rf_R(R)] - (n-2)e^{\frac{\nu-\lambda}{2}}r^{n-3}\lambda'f_R(R) + (n-2)(n-3)e^{\frac{\nu+\lambda}{2}}r^{n-4}(1-e^{-\lambda})f_R(R) - e^{\frac{\nu-\lambda}{2}}r^{n-2}R'\nu'f_{RR}(R). \tag{24}$$

To find out exact solutions of the equations of motion, we focus on the function $f(R) = f_0 R^k$, with k being a real number parameter. As soon as $k = 1$, Einstein's GR is recovered. This choice is motivated by symmetry considerations [59,60], which may be helpful in reducing the dynamics. Specifically, as demonstrated in [60], a power-law form of $f(R)$ gravity is compatible with the existence of Noether symmetries. The Euler–Lagrange equations provide the following solution in n dimensions:

$$e^\nu = e^{-\lambda} = 1 - \frac{2(c_2 - c_1 n)}{n(n-1)(n-2)}\frac{1}{r^{n-3}} + \frac{r^2 R_0}{n(n-1)} \qquad k = \frac{n}{2} \qquad R(r) = R_0, \tag{25}$$

with c_1, c_2 integration constants. Interestingly, it is worth noticing that the n-dimensional solution of the spherically symmetric field equations occurs analytically for $k = n/2$. The only exception is for $k = 1$, where there exists a solution also in four dimensions. As pointed out in the next section, the value $k = n/2$ is the limit for the renormalizability of R^k gravity. Note also that the $n = 4$ limit provides:

$$e^\nu = e^{-\lambda} = \frac{4c_1 - c_2 + 12r + r^3 R_0}{12r}, \tag{26}$$

so that R_0 plays the role of the cosmological constant. Assuming $c_1 = (c_2 - 24GM)/4$, the solution can be recast as:

$$e^\nu = e^{-\lambda} = 1 - \frac{2GM}{r} + \frac{R_0}{12}r^2, \tag{27}$$

which is a Schwarzschild–de Sitter-like solution [60].

3.2. $f(\mathcal{G})$ Gravity

Let us consider now a spherically symmetric background in n-dimensional $f(\mathcal{G})$ gravity. In order to get the point-like Lagrangian, we start from the action:

$$S = \int \left[\sqrt{-g}f(\mathcal{G}) - \lambda(\mathcal{G} - \tilde{\mathcal{G}})\right]d^n x; \tag{28}$$

with $\tilde{\mathcal{G}}$ being the spherically symmetric expression of the Gauss–Bonnet scalar (4), written in terms of the interval (20), namely [61]:

$$\tilde{\mathcal{G}} = \frac{e^{-2\lambda}}{r^4}(n-2)(n-3)\{4r\lambda'e^{\lambda/2}[(e^\lambda - 1)(n-4) + (e^\lambda - 3)e^{\nu/2}r\nu']$$
$$+ (e^\lambda - 1)[(e^\lambda - 1)(n-4)(n-5) - 4(n-4)e^{\nu/2}r\nu' - 4r^2 e^{\nu/2}\nu'^2 - 4r^2 e^{\nu/2}\nu'']\}. \tag{29}$$

Note that the Gauss–Bonnet term vanishes for $n \leq 3$, while it turns into a surface term for $n = 4$. After varying the action with respect to $\mathcal{G}$ and integrating the terms containing higher spatial derivatives, the point-like Lagrangian takes the form:

$$\mathcal{L} = e^{-2\lambda}r^{n-6}\{e^{\frac{1}{2}(5\lambda+\nu)}r^4(f - \mathcal{G}f_{\mathcal{G}}) + (e^\lambda - 1)(n-2)(n-3)[(n-4)(n-5)e^{(\nu+\lambda)/2}f_{\mathcal{G}}(e^\lambda - 1)$$
$$+ 4e^{3\lambda/2}r\lambda'] + 4e^{\lambda/2+\nu}r^2\mathcal{G}'\nu'f_{\mathcal{G}\mathcal{G}}\}. \tag{30}$$

By choosing a power-law function of the form $f(\mathcal{G}) = f_0 \mathcal{G}^k$, suggested by symmetry considerations, the solution of the Euler–Lagrange equations turns out to be [61]:

$$e^\nu = e^{-\lambda} = 1 \pm r^{\frac{5-n}{2}} \sqrt{\frac{4c_1(n-1)}{120\binom{n}{n-5}}} \pm r^2 \sqrt{\frac{\mathcal{G}_0(n-4)}{120\binom{n}{n-5}}} \qquad \mathcal{G}(r) = \mathcal{G}_0 \qquad k = n/4, \quad (31)$$

with c_1, $\mathcal{G}_0$ constants. Equation (31) is a de Sitter-like solution for $f(\mathcal{G})$ gravity, so that $\sqrt{\mathcal{G}_0}$ can be intended as a cosmological constant and c_1 as a dimensional coupling with mass dimension $c_1 \sim M^2$. Similar to $f(R)$ gravity, the constraint $k = n/4$ can be considered to obtain analytic solutions. Note that setting $n = 4$, the only possible solution occurs for $k = 1$, which does not provide any contribution to the dynamics. Here as well, the value $k = n/4$ is the limit value for the renormalizability of $f(\mathcal{G}) = \mathcal{G}^k$ gravity. As a matter of fact, being constructed by second-order curvature invariants, the renormalizability limit of $\mathcal{G}^k$ gravity can be easily inferred by the definition of the Gauss–Bonnet scalar. This link between analytic solutions and the power-counting renormalizability of modified theories of gravity represents an interesting feature, which deserves to be discussed in detail.

3.3. $f(T)$ Teleparallel Gravity

$f(T)$ gravity is another interesting case. Here, the approach is not straightforward as in the previous extended theories. In fact, the interval yields different sets of tetrad fields, which must be selected according to the postulate of "good tetrads" [62]. For this reason, let us focus on the four-dimensional case starting from the action:

$$S = f_0 \int |e| \, T^k \, d^4x. \quad (32)$$

As a starting point, among all the possible sets of tetrad fields providing the interval (20), it seems reasonable to choose:

$$e^a_\mu = \text{diag}(e^{\frac{\nu}{2}}, e^{\frac{\lambda}{2}}, r, r\sin\theta). \quad (33)$$

It yields the following torsion scalar [14]:

$$\tilde{T} = \frac{e^{-2\lambda}(1 + r\nu')}{r^2}, \quad (34)$$

and after some computations, two classes of solutions are:

$$ds^2 = \frac{c_1}{r} dt^2 - \frac{r}{c_2 - 4r} dr^2 - r^2 d\Omega^2 \quad (35)$$

and:

$$ds^2 = \left(\frac{c_4}{c_5 r^8 - 2c_3} \right)^{\frac{1}{8}} dt^2 - \left(\frac{2c_3 - c_5 r^8}{2r^6} \right) dr^2 - r^2 d\Omega^2. \quad (36)$$

Notice that both solutions are independent of the parameter k labeling the power-law function $f(T) = f_0 T^k$. This is due to the fact that the diagonal set of tetrads (33) constrains the free parameter k to be $k = \frac{1}{2}$. Therefore, the function $f(T)$ is also further constrained to $f(T) = f_0 \sqrt{T}$. Moreover, solution (36) can be obtained only after the imposition $T(r) \sim$ const. Another possibility is to consider a non-diagonal set of tetrads of the form [14]:

$$e^a_\mu = \begin{pmatrix} e^{\frac{\nu}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{\lambda}{2}} \sin\theta \cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & e^{\frac{\lambda}{2}} \sin\theta \sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & e^{\frac{\lambda}{2}} \cos\theta & -r\sin\theta & 0 \end{pmatrix}, \quad (37)$$

whose determinant is $|e| = e^{\frac{\nu+\lambda}{2}} r^2 \sin\theta$. Under this assumption, the torsion scalar becomes:

$$T = \frac{2e^{-\lambda}(e^{\frac{\lambda}{2}} - 1)(e^{\frac{\lambda}{2}} - 1 - rv')}{r^2}. \tag{38}$$

The Lagrange multiplier method allows finding a suitable point-like Lagrangian of the form:

$$\mathcal{L} = e^{\frac{\nu-\lambda}{2}} \left\{ e^\lambda r^2 [f(T) - Tf_T(T)] + 2f_T(T)(e^{\frac{\lambda}{2}} - 1)(e^{\frac{\lambda}{2}} - 1 - rv') \right\}. \tag{39}$$

admitting as a possible solution:

$$ds^2 = \left(\frac{r}{\ell}\right)^m dt^2 - \frac{1}{n^2} dr^2 - r^2 d\Omega^2, \tag{40}$$

with n and m constants of integration. Furthermore, a relation among k, n, and m occurs, i.e.,

$$f(T) = f_0 T^k \qquad k = \frac{m^2 + 4(n-1)^2}{4(2+m-2n)} \neq 0. \tag{41}$$

This solution is directly linked to the *ansatz* in (37), while other teleparallel black hole solutions can be found in [63–70].

3.4. Horava–Lifshitz Gravity

The last alternative theory of gravity we discuss is the four-dimensional Horava–Lifshitz theory. Let us take into account the action (16) in the spherically symmetric background (20) with $n = 2$. This choice, as pointed out in [52,71], yields the solution:

$$e^\nu = e^{-\lambda} = 1 + (\omega - \Lambda)r^2 - \sqrt{r[\omega(\omega - 2\Lambda)r^3 + \beta]}, \tag{42}$$

with β and ω constants. Equivalently, by means of the assumption $\beta = 4\omega GM$ and $\Lambda = 0$, the solution can be written as:

$$e^\nu = 1 + \omega r^2 - \omega r^2 \sqrt{1 + \frac{4GM}{\omega r^3}}, \tag{43}$$

which, in the second-order expansion $\frac{4GM}{\omega r^3} \ll 1$, provides a Schwarzschild-like interval.

4. Power-Counting Analysis

The renormalizability of the above theories of gravity can be developed according to a power-counting analysis. As a first remark of this section, we point out that the analysis of the coupling constant mass dimension can provide only an indication about the renormalizability. Indeed, most of the above treated modified theories of gravity might be not-renormalizable when more detailed aspects are considered, as well the propagator homogeneity in the momenta space [72]. Detailed discussions on the renormalizability of $f(R)$ theories of gravity can be found e.g., in [73–76]. One of the first attempts towards a renormalization scheme was pursued by Stelle in [77]. He showed that the gravitational action can be renormalized only when all possible combinations of second-order curvature invariants are considered along with the Hilbert-Einstein action. Adding terms with more than four derivatives, indeed, makes the theory finite after a certain order in the loop expansion, but does not remove the divergences at the one-loop order.

However, in this case, we follow the same prescription as [7] with the aim to find inconsistencies between the two approaches. Moreover, power counting can suggest possible candidates for a perturbatively renormalizable theory.

We first consider the Einstein–Hilbert action and its $f(R)$ extension, which is a good starting point for dealing with theories constructed by the scalar curvature and its deriva-

tives. Then, we find the renormalizability limit of the modified Gauss–Bonnet gravity and, finally, we apply the same approach to teleparallel actions. For this purpose, the actions must be rewritten in terms of a sum between a free term, acting as a Gaussian fixed point, and other higher perturbative contributions. In this way, it is possible to check the mass dimension of the coupling constant of each term. The first step is to recast the metric tensor as a perturbation of the Minkowski background, i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{44}$$

so the Ricci scalar can be written in terms of $\eta_{\mu\nu}$ and $h_{\mu\nu}$. Considering the Levi–Civita connection:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}(\eta^{\rho\lambda} - h^{\rho\lambda})(\partial_{\nu}h_{\lambda\mu} + \partial_{\mu}h_{\nu\lambda} - \partial_{\lambda}h_{\mu\nu}) \tag{45}$$

and the form of the scalar curvature:

$$R = (\eta^{\mu\nu} - h^{\mu\nu})(\nabla_{\rho}\Gamma^{\rho}_{\mu\nu} - \nabla_{\nu}\Gamma^{\rho}_{\mu\rho}), \tag{46}$$

it turns out that the perturbed Ricci scalar depends on the second derivatives of $h_{\mu\nu}$ multiplied by $h_{\mu\nu}$ itself. Since we were not interested in writing the exact form of R, rather to check the mass dimension of each term, a naive procedure permits recasting the Einstein–Hilbert action as:

$$S = \frac{1}{\chi^2} \int \Big[(\partial\partial h) + (\partial h \partial h) + h(\partial h \partial h) + h^2(\partial h \partial h) + ... \Big] d^4x, \tag{47}$$

with χ being a generic coupling constant. Notice that the notation does not take into account constant factors, while it only considers the amount of h and ∂h occurring in each term. For instance, the quantity $\partial h \partial h$ includes all terms made of the product of the first derivatives of the tensor $h_{\mu\nu}$. By an appropriate gauge choice, it is always possible to delete the first term. Thus, by means of the reparameterization $h = \chi\tilde{h}$, the action can be written as:

$$S = \int \Big[(\partial\tilde{h}\partial\tilde{h}) + \chi\tilde{h}(\partial\tilde{h}\partial\tilde{h}) + \chi^2\tilde{h}^2(\partial\tilde{h}\partial\tilde{h}) + ... \Big] d^4x. \tag{48}$$

In this form, the action accounts for a perturbation of the Gaussian fixed point $\int(\partial\tilde{h})^2 d^n x$, which is the expression needed in order to check the coupling constant mass dimension of any terms. To this aim, there is no need to study the mass dimension of higher order terms, since it can be inferred from the mass dimension of the first one. Because of the negative mass dimension of χ, the above action shows the reason why GR is not a renormalizable field theory.

Considering the linearized form of R in Equation (48), the prescription can be easily extended to the more general action:

$$S = \frac{1}{\chi^2} \int \sqrt{|g|} R^k \, d^n x. \tag{49}$$

Now, by the expansion (47), the action (49) (up to first order) reads as:

$$
\begin{aligned}
S &= \frac{1}{\chi^2} \int [(\partial h \partial h) + h(\partial h \partial h)]^k d^n x = \frac{1}{\chi^2} \int \sum_{i=0}^{k} \binom{k}{i} (\partial h \partial h)^{k-i} h^i (\partial h \partial h)^i \, d^n x = \frac{1}{\chi^2} \int \sum_{i=0}^{k} \binom{k}{i} (\partial h \partial h)^k h^i \, d^n x \\
&= \frac{1}{\chi^2} \int \left[(\partial h \partial h)^k + k h (\partial h \partial h)^k + \frac{k(k-1)}{2} h^2 (\partial h \partial h)^k + ... \right] d^n x \\
&= \frac{1}{\chi^2} \int \left[(\partial h \partial h)^k \left(1 + kh + \frac{k(k-1)}{2} h^2 + ... + h^k \right) \right] d^n x \sim \frac{1}{\chi^2} \int \left[(\partial h \partial h)^k + k h (\partial h \partial h)^k \right] d^n x.
\end{aligned} \tag{50}
$$

Similar to the previous case, to check the coupling constant dimension, we can introduce the definition $h = (\chi)^{\frac{1}{k}}\tilde{h}$, so that the action takes the form:

$$S = \int \left[(\partial\tilde{h}\partial\tilde{h})^k + k\,\chi^{\frac{1}{k}}\tilde{h}(\partial\tilde{h}\partial\tilde{h})^k \right] d^n x. \tag{51}$$

From the initial action (49), it is possible to infer the mass dimension of χ in n-dimensional R^k gravity, that is $\chi \to [M^{k-\frac{n}{2}}]$. Therefore, power-law $f(R)$ gravity, with $f(R)$, must satisfy at least one of the following conditions:

$$1 - \frac{n}{2k} \geq 0 \quad \to \quad k < 0 \,\vee\, k \geq \frac{n}{2}. \tag{52}$$

to be power-counting renormalizable. Thus, according to the power-counting argument, only $f(R) = R^2$ gravity is renormalizable in four dimensions. In addition, all power-law models of the form $f(R) = R^k$, with $k > 2$, are super-renormalizable.

By similar reasons, it is straightforward to show that higher order theories with action $S = \int \sqrt{|g|}\square^\ell R^k \, d^n x$ must satisfy the conditions:

$$\begin{cases} k \geq \dfrac{n}{2} - \ell \,\vee\, k \leq 0 & \text{if } n > 2\ell, \\[3mm] k \leq \dfrac{n}{2} - \ell \,\vee\, k \geq 0 & \text{if } n < 2\ell, \end{cases} \tag{53}$$

to be at least renormalizable. Notice that this result holds for any $\ell \in \mathbb{Z}$, which means that even negative powers of the operator $\square$ can occur in the action. In this case, the corresponding theory turns out to be non-local [78,79]. The most general higher order action containing one single term that depends on the d'Alembert operator and on the scalar curvature is:

$$S = \frac{1}{\chi^2} \int \sqrt{|g|} R^k \square^\ell R^p \, d^n x. \tag{54}$$

In this case, the mass dimension of the coupling constant is $[\chi] = M^{k+\ell+p-\frac{n}{2}}$, and the first-order linearization of the action (54) reads as:

$$S = \frac{1}{\chi^2} \int \left[(\partial h\partial h)^k + h(\partial h\partial h)^k \right] \left\{ \square^\ell (\partial h\partial h)^p + \square^\ell [h(\partial h\partial h)^p] \right\} d^n x. \tag{55}$$

The reparameterization of h, which allows writing the action in the form required by power-counting analysis, is $h = \tilde{h}\chi^{\frac{1}{p+k}}$. Therefore, Equation (55) becomes:

$$S = \int (\partial\tilde{h}\partial\tilde{h})^k \square^\ell (\partial\tilde{h}\partial\tilde{h})^p \, d^n x + \int \left\{ (\partial\tilde{h}\partial\tilde{h})^k \square^\ell [\tilde{h}(\partial\tilde{h}\partial\tilde{h})^p] + \tilde{h}(\partial\tilde{h}\partial\tilde{h})^k \square^\ell (\partial\tilde{h}\partial\tilde{h})^p \right\} \chi^{\frac{1}{p+k}} \, d^n x. \tag{56}$$

The conditions that makes the theory power-counting renormalizable are therefore:

$$\begin{cases} k \geq \dfrac{n}{2} - p - \ell \,\vee\, k \leq -p & \text{if } n > 2\ell, \\[3mm] k \leq \dfrac{n}{2} - p - \ell \,\vee\, k \geq -p & \text{if } n < 2\ell. \end{cases} \tag{57}$$

This general result can be extended to actions containing higher order curvature invariants. For this purpose, we considered as equivalent all those scalar quantities with the same mass dimensions. This is the case of the Gauss–Bonnet term $\mathcal{G}$ and the quadratic scalar curvature R^2. In the modified Gauss–Bonnet gravity with action:

$$S = \frac{1}{\chi^2} \int \sqrt{|g|}\mathcal{G}^k \square^\ell \mathcal{G}^p \, d^n x, \tag{58}$$

the relation (57) can be recast as:

$$\begin{cases} k \geq \dfrac{n}{4} - p - \ell \ \vee k \leq -p & \text{if } n > 4\ell, \\[3mm] k \leq \dfrac{n}{4} - p - \ell \ \vee k \geq -p & \text{if } n < 4\ell. \end{cases} \tag{59}$$

Specifically, in the limit $\ell = p = 0$, the condition $k > \frac{n}{4}$ is recovered. As a consequence, the four-dimensional action:

$$S = \frac{1}{\chi^2} \int \sqrt{-g} \, \mathcal{G}^k \, d^4 x, \tag{60}$$

is not-renormalizable only in the range $0 < k < 1$.

To conclude, the last example is given by the action:

$$S = \frac{1}{\chi^2} \int \sqrt{-g} R^k \mathcal{G}^m \, d^n x, \tag{61}$$

considered, e.g., in [37,80]. The resulting power-counting analysis can be inferred from the previous cases, so that the theory turns out to be (super-)renormalizable if:

$$k \geq \frac{n}{4} - m \ \vee k \leq -m \tag{62}$$

It is worth pointing out the analogy with the teleparallel case, where the linearization of the metric is replaced by the expansion of the tetrad fields around the unity matrix, namely:

$$e^a_\mu = \delta^a_\mu + h^a_\mu. \tag{63}$$

By means of Equations (63) and (12), the first-order teleparallel action can be written as:

$$S = \frac{1}{\chi^2} \int (\partial h \partial h + h \partial h \partial h) d^n x, \tag{64}$$

which is formally equivalent to the first-order Einstein–Hilbert action. This implies that the same results, provided in Equation (57), hold also in this case, and a separate treatment is not needed. This is mainly due to the mass dimension of the torsion scalar T, which turns out to be the same as that of the scalar curvature.

5. Renormalizability via the Bekenstein–Hawking Entropy Argument

The renormalizability of theories of gravity can be studied also by comparing their asymptotic behavior to that of a CFT in the appropriate dimension. To introduce the approach, let us first treat the case of n-dimensional GR with the cosmological constant, whose spherically symmetric solution reads:

$$ds^2 = \left(1 - \frac{\eta_n GM}{r^{n-3}} - \frac{\Lambda}{3} r^2\right) dt^2 - \frac{1}{1 - \frac{\eta_n GM}{r^{n-3}} - \frac{\Lambda}{3} r^2} dr^2 - r^2 d\Omega^2_{n-2}, \tag{65}$$

η_n being a constant depending on the spacetime dimension. The Bekenstein–Hawking formula $\mathscr{S} = \frac{A}{4G}$ can be used to infer the relation between $\mathscr{S}$ and the energy $\mathcal{E}$. According to the entropy argument, if such dependence is the same as that of an n-dimensional CFT, the theory turns out to be renormalizable. For a CFT, the energy turns out to scale as:

$$\frac{\mathscr{S}}{\mathcal{E}} \sim \frac{(RT)^{n-1}}{R^{n-1} T^n} \quad \rightarrow \quad \mathscr{S} \sim \mathcal{E}^{\frac{n-1}{n}}. \tag{66}$$

so that, in four dimensions, the condition $\mathscr{S} \sim \mathcal{E}^{\frac{3}{4}}$ is satisfied. To compare Equation (66) with the entropy provided by n-dimensional Schwarzschild black holes, we must dis-

tinguish the cases $\Lambda r^2 \ll 1$ and $\Lambda r^2 \gg 1$. In the former case, the time-like component simply becomes:

$$e^\nu = 1 - \frac{\eta_n GM}{r^{n-3}}, \tag{67}$$

so that the horizon sits at $\tilde{r} \sim GM^{\frac{1}{n-3}}$. Therefore, the Bekenstein–Hawking entropy has the following trend:

$$\mathscr{S} = \frac{A}{4G} \sim \frac{r^{n-2}}{G} \sim \mathcal{E}^{\frac{n-2}{n-3}}. \tag{68}$$

By comparing Equation (68) with Equation (66), it turns out that there are no dimensions in which GR can be renormalized. On the other hand, in the latter case ($\Lambda r^2 \gg 1$), no terms can be neglected in the time-like component of the interval (65), and the solution can be written as:

$$1 - \frac{\eta_n GM}{r^{n-3}} - \frac{\Lambda}{3} r^2 = \frac{r^{n-3} - \eta_n GM - r^{n-1}\frac{\Lambda}{3}}{r^{n-3}} = \frac{\Lambda}{3} \left(\frac{3\frac{r^{n-1}}{r^2\Lambda} - 3\frac{\eta_n GM}{\Lambda} - r^{n-1}}{r^{n-3}} \right). \tag{69}$$

Considering that, for the first term, the condition $\dfrac{r^{n-1}}{r^2\Lambda} \ll 1$ must hold, similar computations as before show that the horizon is:

$$\tilde{r} \sim \left(\frac{GM}{\Lambda} \right)^{\frac{1}{n-1}} \tag{70}$$

and the entropy scales as:

$$\mathscr{S} \sim \frac{r^{n-2}}{G} \sim \mathcal{E}^{\frac{n-2}{n-1}}. \tag{71}$$

Here as well, according to the entropy argument, the Schwarzschild–de Sitter theory cannot be renormalized in any dimensions, in agreement with the power-counting analysis in Section 4. However, as will be shown in the next section, this represents the only case in which the two approaches agree.

5.1. The $f(R)$ Case

Let us now follow a similar procedure for spherically symmetric solutions coming from GR extensions, starting from $f(R)$ gravity. As shown in the previous section, $f(R)$ theory admits the following field equations' solution:

$$e^\nu = e^{-\lambda} = \frac{c_2(n-4)r^{3-n} - 24GMnr^{3-n} + 2(n-2)(n^2 - n + r^2 R_0)]}{2n(n-1)(n-2)} \qquad k = \frac{n}{2}, \tag{72}$$

where the constant c_1 can be identified with GM. By setting $c_2 = 0$, the solution becomes:

$$e^\nu = e^{-\lambda} = \frac{r^{3-n}\left(-12\frac{GMn}{R_0} + \frac{n(n-1)(n-2)}{R_0}r^{n-3} + r^{n-1} \right)}{2n(n-1)(n-2)} \qquad k = \frac{n}{2} \tag{73}$$

and under the assumption $\dfrac{n(n-1)(n-2)}{R_0}r^{n-3} \ll 1$, the horizon sits at:

$$\tilde{r} \sim M^{\frac{1}{n-1}}. \tag{74}$$

By applying the Bekenstein–Hawking formula and comparing the energy scale with Equation (68), we notice that the entropy argument imposes the theory to be non-renormalizable regardless of the value of n. This result disagrees with the power-counting analysis in Section 4. Furthermore, noticing that the solution of the n-dimensional modified $f(R)$

gravity can be recast as an Einstein–de Sitter solution, the horizon must be the same as Equation (70).

5.2. The $f(\mathcal{G})$ Case

With the purpose of studying the Bekenstein–Hawking entropy provided by $f(\mathcal{G}) = f_0 \mathcal{G}^k$ gravity, let us rewrite the spherically symmetric solution (31) as:

$$e^\nu = \frac{r^{\frac{n-5}{2}} \pm \eta_{1,n} M \pm r^{\frac{n-1}{2}} \eta_{2,n}}{r^{\frac{n-5}{2}}}, \tag{75}$$

with $\eta_{1,n}$ and $\eta_{2,n}$ being constants depending on the dimension. In the limit $r^{\frac{n-5}{2}} \ll r^{\frac{n-1}{2}} \eta_{2,n}$, the horizon and the entropy scale as:

$$\tilde{r} \sim M^{\frac{2}{n-1}} \qquad \mathscr{S} = \frac{A}{4G} \sim \frac{r^{n-2}}{G} \sim \mathcal{E}^{\frac{2(n-2)}{n-1}}. \tag{76}$$

This means that the entropy argument provides no values of k and n for which $f(\mathcal{G}) = f_0 \mathcal{G}^k$ gravity is renormalizable. On the other hand, the analysis in Section 4 suggests that the theory is power-counting renormalizable for any $k \geq n/4$, at odds with the entropy analysis.

5.3. The Teleparallel Case

In the teleparallel case, the power-law solution (40) has no event horizon besides $r = 0$, so that the comparison between the two approaches cannot be applied. Let us therefore consider the solution (36), coming from a diagonal set of tetrad fields. In this case, the torsion scalar is constant: $T = T_0$. Due to the general structure of the theory, some difficulties arise in recasting the ratio $\dfrac{c_3}{c_5}$ as a mass term. This is due to the presence of torsion to label the spacetime, which implies that the spin of the compact object must be somewhere considered. However, following [14] and identifying the constant c_5 with T_0, the horizon sits at:

$$\tilde{r} \sim T_0^{\frac{1}{8}}. \tag{77}$$

Since the energy is linearly proportional to the torsion, the entropy scales as:

$$\mathscr{S} \sim \frac{A}{4G} \sim \mathcal{E}^{\frac{1}{4}}, \tag{78}$$

and suggests that the modified teleparallel gravity with function $f(T) = f_0 T^k$ is not renormalizable for any k. On the contrary, according to the power-counting renormalization scheme, the function $f(T) = f_0 T^k$ is renormalizable for any $k > 2$. Here as well, the two approaches are not in agreement.

5.4. The Horava–Lifshitz Case

To conclude, we showed that also the analysis of Horava–Lifshitz gravity provides disagreements between the entropy and the power counting arguments. Though we did not consider Horava–Lifshitz gravity in Equation (4), we took for granted the result provided in [47], where the author showed that the theory is power-counting renormalizable.

Considering the spherically symmetric solution provided in [52], the horizon is:

$$\tilde{r} = GM[1 - \sqrt{1 - 1/(2\omega GM^2)}]. \tag{79}$$

In the limit $2\omega GM^2 \gg 1$, the horizon turns out to have the same form as in GR, namely $\tilde{r} \sim 2GM$, so that the theory can be considered non-renormalizable according to the entropy argument. In general, without assuming any further constraints, the entropy

$\mathcal{S} \sim \tilde{r}^{n-2}$ behaves differently than that of CFT, denying the validity of this method to test the renormalizability of alternative theories of gravity.

6. Conclusions

In this paper, we addressed some concerns claiming for "no-go" arguments to the asymptotic safety of GR, also taking into account power-counting analysis of modified theories of gravity as counterexamples. In particular, we addressed the analysis to the so-called entropy argument [7], which is naturally based on physical intuition. However, the intuition behind it seems unjustified when applied to energy scales far beyond the scales of the theories from which they come, as pointed out in [9]. One of the main premises of the entropy argument is the presupposition of the validity of the Bekenstein–Hawking entropy formula at high-energy scales, but there is no a priori reason to expect the Bekenstein–Hawking entropy to be a well-defined quantity at Planck scales. Furthermore, assuming the Bekenstein–Hawking formula, working at high energies, means that its validity is exact, but quantum corrections to the formula, as shown, e.g., in [10,11], could be a strong indication that this assumption is not justified. Another assumption on which the entropy argument is based is that of black hole dominance at high energies, extrapolated from the classical theory. Unfortunately, so far, very little knowledge concerns mini and quantum black holes.

In summary, we briefly introduced some modified theories of gravity and wrote down the related explicit spherically symmetric solutions. Then, we performed a naive power counting analysis, which at least in these cases, can give sufficient conditions. This was done in order to make a comparison with the entropy argument. What we derived is that, even for these simple cases, the two methods disagree. Moreover, as pointed out throughout the paper, $f(R) = R^k$ field equations can be analytically solved in an n-dimensional spherically symmetric background only when the relation $k = n/2$ occurs. This result is even more interesting if compared to $f(\mathcal{G})$ gravity. Indeed, it turns out that, in the same background, $f(\mathcal{G}) = \mathcal{G}^k$ gravity admits analytic solutions only when $k = n/4$.

Of course, this is not a proof of the overall validity of the entropy argument, which has to be better investigated in modified gravity, as we will do in future papers. However, the approach is in agreement with the results provided by Ref. [81], according to which relation of entropy density versus energy density must be adopted to check renormalizability.

We also aim to study the power counting renormalizability and unitarity of other modified theories of gravity, such as $R^2 + R_{\mu\nu}R^{\mu\nu}$, which is scale invariant and multiplicatively renormalizable, but may have some problems with unitarity.

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