Minisuperspace Quantization of $f(T, B)$ Cosmology

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Abstract: We discuss the quantization in the minisuperspace for the generalized fourth-order teleparallel cosmological theory known as $f(T, B)$. Specifically we focus on the case where the theory is linear on the torsion scalar, in that consideration we are able to write the cosmological field equations with the use of a scalar field different from the scalar tensor theories, but with the same dynamical constraints as that of scalar tensor theories. We use the minisuperspace description to write for the first time the Wheeler-DeWitt equation. With the use of the theory of similarity transformations we are able to find exact solutions for the Wheeler-DeWitt equations as also to investigate the classical and semiclassical limit in the de Broglie -Bohm representation of quantum mechanics.

1. Introduction

Modified theories of gravity have drawn the attention of cosmologists the last years because they provide geometric mechanisms for the explanation of the cosmological observations [1–4]. The common feature of the modified theories of gravity is the introduction of geometric invariants in the Einstein-Hilbert action such that the new field equations admit additional dynamical terms which drive the dynamics in order to explain the observations. The most simple modification of the Einstein-Hilbert Action which has been proposed in the literature is the R^2-gravity in which the quadratic Ricci scalar term has been introduced [5]. That specific modification is the geometric mechanism for one of the well-known inflationary models [6,7]. Generalizations of the latter modification lead to the so-called $f(R)$-theory in which the gravitational Action Integral is a function f of the Ricci scalar [8]. $f(R)$-theory is a higher-order theory while with the use of a Lagrangian multiplier it can be written as a scalar-tensor theory [9–11]. However, the use of the Ricciscalar to modify General Relativity is not the unique approach which has been studied in the literature.

The formulation of the teleparallel equivalent of General Relativity (TEGR) is based on the use of the curvature-less Weitzenböck connection instead of using the torsion-less Levi-Civita connection. The Lagrangian of TEGR is torsion scalar T. Contrary to the Ricciscalar for General Relativity [12,13]. A generalization of TEGR gravity is the $f(T)$-theory [14] which is inspired by the $f(R)$-theory. In contrary to $f(R)$-theory, $f(T)$-theory is a second-order theory, since torsion tensor includes only products of first derivatives. However, while $f(T)$-theory is a second-order as General Relativity there are various differences. However, the Ricciscalar and the Torsion scalar are not the only invariants which have been proposed in the literature, we refer the reader to [15–29] and references therein.

In this work we are interested in the modified theory of gravity known as $f(T, B)$ theory, where T is the torsion scalar and B is the boundary term defined as $B = T + R$ [30]. Because B includes second derivatives, $f(T, B)$ is a fourth-order theory of gravity. In general is different from that of $f(R)$ theory. Specifically the latter is recovered for $f(T, B) = f(B − T)$. Moreover, in the simple case when $f_{,BB} = 0$, the theory reduces to that of $f(T)$ teleparallel gravity [31], thus in this study we shall consider the case where $f_{,BB} \neq 0$. Some recent analysis on $f(T, B)$ gravity can be found in [32–34] where in [35] the modified theory is tested for the solution of the H_0 tension.

A special case of the $f(T, B)$ theory which has been studied before is that in which $f_{,TT} = 0$, $f_{,TB} = 0$, which means that f is a linear function of T, that is, $f(T, B) = T + F(B)$ [36]. In the latter consideration in the case of Friedmann–Lemaître–Robertson–Walker (FLRW) universe, the theory can be written as a scalar field theory, but not a scalar tensor theory, with the same number of dynamical constraints as the $f(R)$-theory. That observation leads to field equations which can be described by a point-like Lagrangian with the same number of constraint equations as $f(R)$-gravity which means that there is a minisuperspace description for the theory. The general asymptotic behaviour as also the stability of some important cosmological solutions such are the scaling or the de Sitter solutions have been studied before in [36,37]. Moreover, the integrability properties of the field equations for this modified theory of gravity were investigated in [38].

We make use of the existence of the minisuperspace for the latter $f(T, B)$ theory in cosmological studies such that to perform a quantization following the minisuperspace quantization which leads to the Wheeler-DeWitt (WdW) equation [39]. The WdW equation is actually in general a hyperbolic functional differential equation on a spatial superspace with infinite degrees of freedom. However, when there exists a minisuperspace description the infinite degrees of freedom reduce to a finite number and the WdW equation is represented as a single equation for all the points of the spatial hypersurface. The WdW has been investigated for various modified f-theories of gravity [40–44] but not for a teleparallel f-theory before. Recently, in the de Broglie-Bohm representation of quantum mechanics it was found that in the semiclassical limit of the WdW equation for the Szekeres universe the field equations are modified by a quantum potential such that the Szekeres universe does not remain silent in the early universe [45]. Moreover, in scalar field theory, the same approach gives a mechanism in which terms of a pressureless fluid are introduced in the field equations [46]. An interesting discussion and critique on the WdW equation can be found in [47]. The plan of the paper is as follows.

In Section 2, we present the cosmological model of our consideration, we reproduce previous results and we show how the $f(T, B)$ theory can be written with the use of a Lagrange multiplier into a scalar field theory with a minisuperspace description. In Section 3, we write the WdW equation for the theory of our analysis and we apply the theory of similarity transformations in order to constrain the unknown functional form of the theory such that the similarity transformations which lead to the existence of exact solutions. The complete classification for the similarity transformations with generators point symmetries is presented. Furthermore, the one-dimensional optimal system is derived. The latter is used to write all the exact wavefunctions. Furthermore, in Section 4 we investigate the classical limit in the WKB approximate where we find the analytic solutions for the classical gravitational field equations In addition we investigate the quantum potentiality of the theory according to the Bohmian representation of quantum mechanics. Finally in Section 5, we summarize our results and we draw our conclusions.

2. $f(T, B)$ Cosmology

Consider $\mathbf{e}_i(x^\mu)$ to be the vierbein fields, which are the dynamical variables of teleparallel gravity. Vierbein fields form an orthonormal basis for the tangent space at each point P with coordinates, $P(x^\mu)$, of the manifold. Hence, $g(e_i, e_j) = \mathbf{e}_i \cdot \mathbf{e}_i = \eta_{ij}$, where η_{ij} is the line element of four-dimensional Minkowski spacetime. In a coordinate basis the vierbeins are expressed as $e_i = h_i^\mu(x)\partial_i$, from where it follows that the metric of the spacetime is expressed as $g_{\mu\nu}(x) = \eta_{ij} h_\mu^i(x) h_\nu^j(x)$.

The main characteristic of the teleparallel gravity is the curvatureless Weitzenböck connection $\hat{\Gamma}^\lambda{}_{\mu\nu} = h_a^\lambda \partial_\mu h_\nu^a$ from where we can define the nonnull torsion tensor, Refs. [48,49] $T^\beta_{\mu\nu} = \hat{\Gamma}^\beta{}_{\nu\mu} - \hat{\Gamma}^\beta{}_{\mu\nu} = h_i^\beta(\partial_\mu h_\nu^i - \partial_\nu h_\mu^i)$. On the other hand, the Lagrangian density of the teleparallel gravity, from which is the scalar $T = S_\beta{}^{\mu\nu} T^\beta{}_{\mu\nu}$, where $S_\beta{}^{\mu\nu}$ is defined as $S_\beta{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}{}_\beta + \delta_\beta^\mu T^{\theta\nu}{}_\theta - \delta_\beta^\nu T^{\theta\mu}{}_\theta)$. $K^{\mu\nu}{}_\beta$ is the contorsion tensor and equals the difference

between the Levi-Civita connections in the holonomic and the nonholonomic frame and it is defined by the nonnull torsion tensor, $T^{\mu\nu}{}_\beta$, as $K^{\mu\nu}{}_\beta = -\frac{1}{2}(T^{\mu\nu}{}_\beta - T^{\nu\mu}{}_\beta - T_\beta{}^{\mu\nu})$.

$f(T, B)$ gravity is an extension of teleparallel theory. $f(T, B)$ is a fourth-order theory where the Action integral is a function of scalar T and of the boundary term $B = 2e_\nu^{-1}\partial_\nu\left(eT_\rho{}^{\rho\nu}\right)$, which is defined as $B = T + R$, where R is the Ricciscalar. Specifically, the Action integral is defined as

$$S \equiv \frac{1}{16\pi G}\int d^4xe[f(T, R + T)] + S_m \equiv \frac{1}{16\pi G}\int d^4xe[f(T, B)] + S_m, \tag{1}$$

with $e = \det(e_\mu^i) = \sqrt{-g}$ and S_m describes the additional matter sources.

Variation with respect to the vierbein fields of (1) provides the field equations [31]

$$4\pi G e\mathcal{T}_a^{(m)\lambda} = \frac{1}{2}eh_a^\lambda(f_{,B})^{;\mu\nu}g_{\mu\nu} - \frac{1}{2}eh_a^\sigma(f_{,B})_{;\sigma}{}^{;\lambda} + \frac{1}{4}e(Bf_{,B} - f)h_a^\lambda + (eS_a{}^{\mu\lambda})_{,\mu}f_{,T}$$
$$+ e\left((f_{,B})_{,\mu} + (f_{,T})_{,\mu}\right)S_a{}^{\mu\lambda} - ef_{,T}T^\sigma{}_{\mu a}S_\sigma{}^{\lambda\mu}, \tag{2}$$

where $\mathcal{T}_a^{(m)\lambda}$ is the energy-momentum tensor of the matter source. When $f(T, B)$ is linear on B, i.e., $f(T, B) = f(T) + f_1 B$ the latter equations take the form of $f(T)$ teleparallel gravity

With the use of the Einstein tensor G_a^λ the field equations are expressed as

$$4\pi G e\mathcal{T}_a^{(m)\lambda} = ef_{,T}G_a^\lambda + \left[\frac{1}{4}(Tf_{,T} - f)eh_a^\lambda + e(f_{,T})_{,\mu}S_a{}^{\mu\lambda}\right] + \tag{3}$$
$$+ \left[e(f_{,B})_{,\mu}S_a{}^{\mu\lambda} - \frac{1}{2}e\left(h_a^\sigma(f_{,B})_{;\sigma}{}^{;\lambda} - h_a^\lambda(f_{,B})^{;\mu\nu}g_{\mu\nu}\right) + \frac{1}{4}eBh_a^\lambda f_{,B}\right]$$

or

$$ef_{,T}G_a^\lambda = 4\pi G e\mathcal{T}_a^{(m)\lambda} + 4\pi G e\mathcal{T}_a^{(DE)\lambda}, \tag{4}$$

that is,

$$eG_a^\lambda = G_{eff}\left(e\mathcal{T}_a^{(m)\lambda} + e\mathcal{T}_a^{(DE)\lambda}\right), \tag{5}$$

in which now $G_{eff} = \frac{4\pi G}{f_{,T}}$, is an effective varying gravitational constant.

We have defined as $\mathcal{T}_a^{(DE)\lambda}$ the effective energy momentum tensor which attributes the additional dynamical terms which follows from the modified Action Integral,

$$4\pi G e\mathcal{T}_a^{(DE)\lambda} = -\left[\frac{1}{4}(Tf_{,T} - f)eh_a^\lambda + e(f_{,T})_{,\mu}S_a{}^{\mu\lambda}\right] + \tag{6}$$
$$- \left[e(f_{,B})_{,\mu}S_a{}^{\mu\lambda} - \frac{1}{2}e\left(h_a^\sigma(f_{,B})_{;\sigma}{}^{;\lambda} - h_a^\lambda(f_{,B})^{;\mu\nu}g_{\mu\nu}\right) + \frac{1}{4}eBh_a^\lambda f_{,B}\right].$$

The geometric energy momentum tensor reads $\mathcal{T}_a^{(DE)\lambda} = \mathcal{T}_a^{(T)\lambda} + \mathcal{T}_a^{(B)\lambda}$ in which [36]

$$4\pi G e\mathcal{T}_a^{(T)\lambda} = -\left[\frac{1}{4}(Tf_{,T})eh_a^\lambda + e(f_{,T})_{,\mu}S_a{}^{\mu\lambda}\right] \tag{7}$$

and $\mathcal{T}_a^{(B)\lambda}$ is given by the expression

$$4\pi G e\mathcal{T}_a^{(B)\lambda} = -\left[e(f_{,B})_{,\mu}S_a{}^{\mu\lambda} - \frac{1}{2}e\left(h_a^\sigma(f_{,B})_{;\sigma}{}^{;\lambda} - h_a^\lambda(f_{,B})^{;\mu\nu}g_{\mu\nu}\right) + \frac{1}{4}(eBf_{,B} - f)h_a^\lambda\right]. \tag{8}$$

2.1. The $f(T, B) = T + F(B)$ Theory

In this work we are interested in the case where f is a linear function of T, that is, $f(T, B) = T + F(B)$. In that case, the only geometric fluid components which survive are

the one of $\mathcal{T}_a^{(B)\lambda}$ while $G_{eff} = 4\pi G$. In addition, by using a Lagrange multiplier the extra degrees of freedom have been attributed to a scalar field. It is important to mention that this scalar field does not belong to the family of scalar-tensor theories [11].

According to the cosmological principle in large scales the universe is isotropic and homogeneous and described by the Friedmann–Lemaître–Robertson–Walker (FLRW) line element [50]

$$ds^2 = -N^2(t)dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{9}$$

where $a(t)$ is the scale factor and describes the radius of the three-dimensional Euclidean space and $N(t)$ is the lapse function. Furthermore, from the cosmological principle we select the observer to be $u^\mu = \frac{1}{N}\delta_t^\mu$ such that $u^\mu u_\mu = -1$.

For the vierbein we consider the following diagonal frame $h_\mu^i(t) = diag(N(t), a(t), a(t), a(t))$ from which we calculate

$$T = \frac{6}{N^2}\left(\frac{\dot{a}}{a}\right)^2, \quad B = \frac{6}{N^2}\left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} - \frac{\dot{a}\dot{N}}{aN}\right). \tag{10}$$

We define the new variables $\phi = \Phi_{,B}(B)$ and $V(\phi) = BF(B)_{,B} - F(B)$, then the energy momentum $\mathcal{T}_a^{(B)\lambda}$ (8) can be written in the equivalent form

$$4\pi G e \mathcal{T}_a^{(B)\lambda} = -\left[e(F(B)_{,BB})B_{;\mu}S_a{}^{\mu\lambda} - \frac{1}{2}e\left(h_a^\sigma\left(F(B)_{,B}\right)_{;\sigma}{}^{;\lambda} - h_a^\lambda\left(F(B)_{,B}\right)^{;\mu\nu}g_{\mu\nu}\right) + \frac{1}{4}e\left(F\beta(B)_{,B} - F(B)\right)h_a^\lambda\right] \tag{11}$$

or

$$4\pi G e \mathcal{T}_a^{(B)\lambda} = -\left[e\phi_{,\mu}S_a{}^{\mu\lambda} - \frac{1}{2}e\left(h_a^\sigma(\phi)_{;\sigma}{}^{;\lambda} - h_a^\lambda(\phi)^{;\mu\nu}g_{\mu\nu}\right) + \frac{1}{4}V(\phi)h_a^\lambda\right]. \tag{12}$$

Hence for $N(t) = 1$ the gravitational field equations are [36,37]

$$3H^2 = 3H\dot{\phi} + \frac{1}{2}V(\phi) + \rho_m, \tag{13}$$

$$2\dot{H} + 3H^2 = \ddot{\phi} + \frac{1}{2}V(\phi) - p_m \tag{14}$$

while it follows the constraint equation

$$\frac{1}{6}V_{,\phi} + \dot{H} + 3H^2 = 0 \tag{15}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble function.

Minisuperspace Description

The gravitational field Equations (13)–(15) are derived by the point-like singular Lagrangian function [36]

$$\mathcal{L}(N, a, \dot{a}, \phi, \dot{\phi}) = -\frac{6}{N}a\dot{a}^2 + \frac{6}{N}a^2\dot{a}\dot{\phi} - Na^3V(\phi) + L_m, \tag{16}$$

which is a minisuperspace description for the theory. In particular Equation (13) follows from the variation with respect to the variable N, $\frac{\partial L}{\partial N} = 0$, while the rest second-order equations follow by the variation with respect to the scale factor and the scalar field. Moreover, we have assumed that L_m denotes the Lagrangian component of the additional matter source ρ_m, p_m. We proceed by assuming that the additional matter source is an ideal gas, that is $p_m = w_m\rho_m$ with equation of state parameter $\dot{\rho}_m + 3(1 + w_m)H\rho_m = 0$, that is $\rho_m = \rho_{m0}a^{-3(1+w_m)}$.

Hence, from Lagrangian (16) we can define the momentum

$$p_a = -\frac{12}{N}a\dot{a} + \frac{6}{N}a^2\dot{\phi} \,, \; p_\phi = \frac{6}{N}a^2\dot{a},$$ (17)

which can be used to write the Hamiltonian of the field equations

$$\mathcal{H} = N\left(\frac{p_a p_\phi}{3a^2} + \frac{p_\phi^2}{3a^3} + a^3 V(\phi) + 2\rho_{m0}a^{-3w_m}\right),$$ (18)

while from (13) it follows $\mathcal{H} = 0$. Moreover, the field equations can be written in the equivalent form

$$\dot{a} = N\frac{p_\phi}{3a^2} \,, \; \dot{\phi} = N\frac{p_a}{3a^2} + \frac{2}{3}Np_\phi \,, \; \dot{p}_\phi = -Na^3 V_{,\phi} \,,$$ (19)

$$\dot{p}_a = N\frac{p_a p_\phi}{6a^3} - 3a^2 NV(\phi) - 6\rho_{m0}Na^{-3w_m - 1} \,.$$ (20)

The existence of the minisuperspace description for this generalized teleparallel model is essential in order to proceed with the quantization of the theory. Moreover because the quantum effects refer to the very early universe we assume that there is not any contribution of the matter source in the field equations, that is, we assume that $\rho_m = 0$.

3. Wheeler-DeWitt Equation

From the point-like Lagrangian (16) we define the minisuperspace

$$ds^2 = -12ada^2 + 12a^2 dad\phi$$ (21)

which is a two-dimensional space with Ricciscalar $R_{(2)} = 0$, which means that it is the two-dimensional flat space.

In general, the WdW equation is defined with the use of the conformal invariant Laplace operator $\hat{L}_\gamma = \Delta_\gamma + \frac{n-2}{4(n-1)}R_\gamma$ where γ_{ij} remarks the minisuperspace metric, Δ_γ is the Laplace operator, R_γ is the Ricciscalar of γ and $n = \dim\gamma$. For the two-dimensional minisuperspace of the theory of our consideration it follows that $\hat{L}_\gamma = \Delta_\gamma$. In this case the WdW equation is equivalent to the classical quantization $[x, p] = \delta_{ij}$ and replace p with the operator $p = i\frac{\partial}{\partial q}$ in the Hamiltonian Equation (18). Therefore, we write the WdW equation

$$W \equiv \left(\frac{1}{3a^3}\left(a\frac{\partial^2}{\partial a\partial\phi} + \frac{\partial^2}{\partial\phi^2}\right) - a^3 V(\phi)\right)\Psi(a, \phi) = 0,$$ (22)

where $\Psi(a, \phi)$ is the wavefunction of the universe.

In order to solve the latter partial differential equation we investigate for specific functions of the scalar field potential $V(\phi)$ in which we can define differential operators which leave the wavefunction invariant. Specifically, we shall investigate the existence of one-parameter point transformations which keep the WdW equation invariant. The infinitesimal generator of the one-parameter point transformation will be called a Lie symmetry.

3.1. Quantum Operators

Consider the vector field $\mathbf{X} = \xi^a(a, \phi, \Psi)\partial_a + \xi^\phi(a, \phi, \Psi)\partial_\phi + \eta(a, \phi, \Psi)\partial_\Psi$ defined in the jet space $\{a, \phi, \Psi\}$, which is the generator of the infinitesimal one-parameter point transformation $P \to P'$ defined as [51,52]

$$(a', \phi', \Psi') = (a, \phi, \Psi) + \varepsilon\left(\xi^a(a, \phi, \Psi), \xi^\phi(a, \phi, \Psi), \eta(a, \phi, \Psi)\right),$$ (23)

in which ε is an infinitesimal parameter. Then equation (22) will remain invariant under the action of the point transformation if and only if

$$\lim_{\varepsilon \to 0} \frac{W(a', \phi', \Psi') - W(a, \phi, \Psi)}{\varepsilon} = 0 \qquad (24)$$

or equivalently $L_X W = \mu W$, mod$W = 0$, where $\mathcal{L}_X$ is the Lie derivative with respect to the vector field **X** and μ is a function which should be determined. When the latter condition is true, field **X** is called Lie symmetry for the differential equation W.

For the conformal Laplace equation it was found that the generic Lie symmetry vector **X** is of the form [53]

$$\mathbf{X} = \xi^i \left(y^k \right) \partial_i + \left[\frac{(2-n)}{2} \psi \Psi + \alpha_0 \Psi + \beta(a, \psi) \right] \partial_\Psi, \qquad (25)$$

in which $\xi^i \left(y^k \right) = \left(\xi^a(a, \phi), \xi^\phi(a, \phi) \right)$ is a conformal Killing vector field of the minisuperspace γ_{ij}, with conformal factor $\psi(a, \phi)$, that is, $L_\xi \gamma_{ij} = 2\psi \gamma_{ij}$ and $\psi = \frac{1}{n} \nabla_{(\gamma)j} \xi^j$. Moreover, the conformal Killing vector $\xi^i \left(y^k \right)$ and the effective potential function $V_{eff}(a, \phi) = a^3 V(\phi)$ are constraint as $L_\xi V_{eff} + 2\psi V_{eff} = 0$. Moreover, a_0 is a constant, while $\beta(a, \psi)$ denotes the infinity number of solutions of the original conformal Laplace equation. These two vector fields indicate that the differential equation is linear. The vector field $\beta(a, \phi) \partial_\Psi$ is a trivial symmetry vector and has not any application on the construction of similarity solutions. Thus we shall omit it in the following analysis.

The main application of Lie point symmetries in partial differential equations is the determination of similarity transformations which can be used to reduce the number of indepedent variables for the equation. Indeed, we find the point transformations in which the conformal vector field ξ^i is written in normal coordinates that is, we search the transformation $y^k \to y^J$ in which the Lie symmetry vector is written in the normal form [51,52]

$$\mathbf{X} = \xi^i \left(x^k \right) \partial_i + \left[\frac{(2-n)}{2} \psi \Psi + \alpha_0 \Psi \right] \partial_\Psi. \qquad (26)$$

Now there two ways to proceed with the application of the symmetry vector. The two approaches provide the same result, that is, they are equivalent.

The first approach is the derivation of the zero-order invariants for the symmetry vector which follow by the solution of the system

$$\frac{dy^J}{1} = \frac{d\Psi}{\left(\frac{2-n}{2} \psi + \alpha_0 \right) \Psi}, \qquad (27)$$

that is y^b, $\Psi \left(y^b, y^J \right) = \Phi \left(y^b \right) \exp \left[\int \left(\frac{2-n}{2} \psi + \alpha_0 \right) dy^J \right]$. Therefore by defining y^b to be the new independent variables and $\Phi \left(y^b \right)$ the dependent variable we end with a new differential equation known as reduced equation.

On the other hand, for partial differential equations every Lie symmetry is equivalent to the Lie-Bäcklund vector field $\hat{X} = \left(\Psi_J - \left(\frac{2-n}{2} \psi + a_0 \right) \Psi \right) \partial_\Psi$. A symmetry vector transforms solutions under solutions; that is, if Ψ is a solution then $\hat{X} \Psi = a_1 \Psi$. from where there is defined the quantum operator

$$\Psi_J - \left(\frac{2-n}{2} \psi + \alpha_0 \right) \Psi = \alpha_1 \Psi. \qquad (28)$$

which provides $\Psi \left(y^b, y^J \right) = \Phi \left(y^b \right) \exp \left[\int \left(\frac{2-n}{2} \psi + \alpha \right) dy^J \right]$, with $\alpha = \alpha_0 + \alpha_1$. Hence, it is clear that the approaches are equivalent. Moreover, for our consideration in which $n = 2$, it

follows that the quantum operator (28) reads $\Psi_J - \alpha\Psi = 0$, which provides the reduction $\Psi\left(y^b, y^J\right) = \Phi\left(y^b\right)\exp\left(\alpha y^J\right)$.

We apply the symmetry condition (24) for Equation (22) and we find the following functional forms of $V(\phi)$ in which there exist Lie symmetries which keep the wavefunction Ψ invariant.

The scalar field potentials are derived to be

$$V_I(\phi) = V_0 e^{-\lambda\phi}, \tag{29}$$

$$V_{II}(\phi) = V_0\left(e^\phi + V_1 e^{(1+\kappa)\phi}\right)^{-2-\frac{6}{\kappa}} e^{\frac{5+(\kappa+1)^2}{\kappa}\phi}. \tag{30}$$

The Lie symmetries for the WdW equation for the scalar field potential $V_I(\phi)$ are

$$X_1 = a^{-\lambda}e^{\lambda\phi}\partial_\phi, \ X_2 = \lambda a\partial_a + \partial_\phi, \ X_3 = a^{\lambda-6}\left(a\partial_a + \partial_\phi\right), \tag{31}$$

while for the potential function $V_{II}(\phi)$ the Lie symmetry vector is

$$X_4 = V_1 a^{1+\kappa}\partial_a + a^\kappa\left(V_1 + e^{-\kappa\phi}\right)\partial_\phi, \tag{32}$$

while in both cases the WdW equation admits the trivial symmetry vector $X_\Psi = \Psi\partial_\Psi$.

From the symmetry vectors we can construct the corresponding operators

$$\hat{Q}_1 = a^{-\lambda}e^{\lambda\phi}\frac{\partial}{\partial\phi}, \tag{33}$$

$$\hat{Q}_2 = \lambda a\frac{\partial}{\partial a} + \frac{\partial}{\partial\phi}, \tag{34}$$

$$\hat{Q}_3 = a^{\lambda-6}\left(a\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\phi}\right), \tag{35}$$

$$\hat{Q}_4 = V_1 a^{1+\kappa}\frac{\partial}{\partial a} + a^\kappa\left(V_1 + e^{-\kappa\phi}\right)\frac{\partial}{\partial\phi}. \tag{36}$$

For the potential $V_I(\phi)$ the WdW equation admits more than one quantum operators, thus the natural question which follows is, how many independent quantum operators can be constructed. Indeed, if we consider general linear operator $\hat{Q} = \rho_1\hat{Q}_1 + \rho_2\hat{Q}_2 + \rho_3\hat{Q}_3$ we should define the values of the coefficients ρ_1, ρ_2, ρ_3 which lead to independent similarity solutions. The problem is equivalent with the derivation of the one-dimensional optimal system for the WdW Equation (22).

By definition, for the three-dimensional Lie algebra G_3 with elements $\{X_1, \ X_2, X_3\}$ and structure constants C_{BC}^A, we define the two symmetry vectors [51,52]

$$\mathbf{Z} = \sum_{i=1}^{3}\rho_i X_i, \ \mathbf{Y} = \sum_{i=1}^{3}\zeta_i X_i, \ \rho_i, \ \zeta_i \text{ are coefficient constants.} \tag{37}$$

Then we shall say that the vector fields $\mathbf{Z}$ and $\mathbf{Y}$ are equivalent and provide the same similarity transformation if $\mathbf{Y} = \sum_{j=i}^{n} Ad(\exp(\epsilon_i X_i))\mathbf{Z}$ or $W = c\mathbf{Z}$, $c = const$ that is $\zeta_i = c\rho_i$. The operator $Ad(\exp(\epsilon X_i))X_j$ is defined as

$$Ad(\exp(\epsilon X_i))X_j = X_j - \epsilon\left[X_i, X_j\right] + \frac{1}{2}\epsilon^2\left[X_i, \left[X_i, X_j\right]\right] + \dots \tag{38}$$

and it is called the adjoint representation, which has the property $Ad(\exp(\epsilon X))X = X$. Therefore, the derivation of all the independent Lie symmetries and their independent linear combination lead to the one-dimensional optimal system.

For the Lie algebra G_3 we calculate the Adjoint-representations

$$Ad(\exp(\epsilon X_1))X_2 = X_2 - \epsilon\lambda(\lambda - 6)X_1 \,, \quad Ad(\exp(\epsilon X_1))X_3 = X_3 \,, \tag{39}$$

$$Ad(\exp(\epsilon X_2))X_1 = e^{\epsilon\lambda(\lambda-6)}X_1 \,, \quad Ad(\exp(\epsilon X_2))X_3 = e^{-\epsilon\lambda(\lambda-6)}X_3 \,, \tag{40}$$

$$Ad(\exp(\epsilon X_3))X_1 = X_1 \,, \quad Ad(\exp(\epsilon X_3))X_2 = X_2 + \epsilon\lambda(\lambda - 6)X_3. \tag{41}$$

Consequently, the one-dimensional system consists by the one-dimensional Lie algebras $\{X_1\}$, $\{X_2\}$, $\{X_3\}$, $\{X_1 \pm X_3\}$, from which it follows that the quantum operators that we should consider in order to find all the possible independent solutions are $\{\hat{Q}_1\}$, $\{\hat{Q}_2\}$, $\{\hat{Q}_3\}$, $\{\hat{Q}_1 + \hat{Q}_3\}$.

3.2. Potential Function $V_I(\phi)$

For the scalar field potential $V_I(\phi)$, with the use of the operator $\hat{Q}_1$ we define the constraint equation $\hat{Q}_1\Psi = q_1\Psi$, thus from the WdW equation we find

$$\Psi_1(a,\phi) = \Psi_1^0 \exp\left(\frac{q_1}{\lambda}a^\lambda e^{-\lambda\phi} + \frac{3V_0}{q_1}\frac{a^{6-\lambda}}{6-\lambda}\right). \tag{42}$$

Similarly with the use of the operator $\hat{Q}_2$, that is, $\hat{Q}_2\Psi = q_2\Psi$ we find the similarity solution

$$\Psi_2(a,\phi) = a^{-\frac{q_2(\lambda-3)}{\lambda(\lambda-6)}} e^{\frac{q_2\phi}{2(\lambda-6)}} \left(\Psi_2^{01} J_{\frac{q_2}{\lambda(\lambda-6)}}\left(-2\sqrt{\frac{3V_0}{\lambda(6-\lambda)}}a^3 e^{-\frac{\lambda}{2}\phi}\right) + \Psi_2^{02} Y_{\frac{q_2}{\lambda(\lambda-6)}}\left(-2\sqrt{\frac{3V_0}{\lambda(6-\lambda)}}a^3 e^{-\frac{\lambda}{2}\phi}\right)\right), \tag{43}$$

where J, Y are the Bessel functions.

Furthermore, from the constraint equation $\hat{Q}_3\Psi = q_3\Psi$ we derive the wavefunction

$$\Psi_3(a,\phi) = \Psi_3^0 \exp\left(\frac{3V_0}{q_3\lambda}a^\lambda e^{-\lambda\phi} + \frac{q_3}{6-\lambda}a^{6-\lambda}\right). \tag{44}$$

Moreover, from the operator $\hat{Q}_1 + \hat{Q}_3$ we construct the constraint equation $(\hat{Q}_1 + \hat{Q}_3)\Psi = q^+\Psi$ which with the use of the WdW equation provides the wavefunction

$$\Psi_+(a,\phi) = \exp\left(\frac{q^+}{\lambda-6}a^{6-\lambda}\right)\left(\Psi_+^{01}\exp\left(\Delta_+\left(a^\lambda e^{-\lambda\phi}(\lambda-6) - a^{6-\lambda}\right)\right) + \Psi_+^{02}\exp\left(\Delta_-\left(a^\lambda e^{-\lambda\phi}(\lambda-6) - a^{6-\lambda}\right)\right)\right). \tag{45}$$

where $\Delta_\pm = -\frac{-q^+ \pm \sqrt{(q^+)^2 - 12V_0}}{2\lambda(\lambda-6)}$.

3.3. Potential Function $V_{II}(\phi)$

We continue our analysis with the derivation of the similarity solution for the WdW equation for the scalar field potential $V_{II}(\phi)$. Thus, with the use of the unique admitted operator $\hat{Q}_4$ we define the constraint equation $\hat{Q}_4\Psi = q_4\Psi$. In order to write the solution we prefer to work in normal coordinates, hence we perform the change of variables

$$a = x^{-\frac{1}{\kappa}} \,, \quad e^{-\kappa\phi} = \frac{V_1 x}{e^{\kappa y} - x}. \tag{46}$$

In the new variables the WdW equation becomes

$$\left(\kappa V_1 e^{y(6+\kappa)}\frac{\partial^2}{\partial x\partial y} + V_1 e^{6y}\frac{\partial^2}{\partial y^2} - \kappa e^{6y}V_1\frac{\partial}{\partial y} + 3V_0\right)\Psi(x,y) = 0 \tag{47}$$

while the constraint equation is simplified in the simplest form $\left(\frac{\partial}{\partial x} - q_4\right)\Psi(x,y) = 0$. Hence the similarity solution is

$$\Psi(x,y) = e^{-q_4 x} U(y) \tag{48}$$

in which $U(y)$ solve the differential equation

$$\left(-q_4 \kappa V_1 e^{y(6+\kappa)}\frac{\partial}{\partial y} + V_1 e^{6y}\frac{\partial^2}{\partial y^2} - \kappa e^{6y}V_1\frac{\partial}{\partial y} + 3V_0\right)U(y) = 0 \tag{49}$$

where a special solution for $q_4 = 0$ is

$$U(y) = e^{\frac{\kappa}{2}y}\left(U_1 J_{\frac{\kappa}{3}}\left(\sqrt{\frac{V_0}{3V_1}}e^{-3y}\right) + U_2 Y_{\frac{\kappa}{3}}\left(\sqrt{\frac{V_0}{3V_1}}e^{-3y}\right)\right). \tag{50}$$

On the other hand for $q_4 \neq 0$ for $\kappa = -6$ the closed form solution of $U(y)$ is expressed in terms of Kummer's $M(\alpha, \beta, x)$ and Tricomi's $U(\alpha, \beta, x)$ functions, such as,

$$U(y) = e^{-6y}\left(U_1 M\left(\alpha, 2, q_4 e^{-6y}\right) + U_2 U\left(\alpha, 2, q_4 e^{-6y}\right)\right), \; \alpha = -1 + \frac{V_0}{12 q_4 V_1}. \tag{51}$$

4. Semi-Classical Limit

In the Madelung representation [54] of the complex-wave function of the universe $\Psi(a, \phi) = \Omega(a, \phi)e^{\frac{i}{\hbar}S(a,\phi)}$, the real part of the WdW Equation (22) reads

$$\frac{1}{3a^3}\left(a\left(\frac{\partial S}{\partial a}\right)\left(\frac{\partial S}{\partial \phi}\right) + \left(\frac{\partial S}{\partial \phi}\right)^2\right) + a^3 V(\phi) - \frac{\hbar^2}{2\Omega}\Delta_\gamma(\Omega) = 0, \tag{52}$$

where in the limit $\hbar^2 \to 0$, the Hamilton-Jacobi equation of the gravitational field equations is recovered. The additional term, $V_Q = -\frac{\hbar^2}{2\Omega}\Delta_\gamma(\Omega)$ which depends on the amplitude of the wavefunction $\Psi(a, \phi)$ is called the quantum potential in the de Broglie-Bohm representation of quantum mechanics [55,56].

We continue our analysis by studying first the classical limit of the of the WKB approximation, while secondly we investigate the case in which the effects of the quantum potential are assumed nonzero.

4.1. Classical Limit

In this section we study the classical limit without any quantum potential term and we derive the solution of the Hamilton-Jacobi equation. The latter is used to simplify the field equations.

4.2. Potential Function $V_I(\phi)$

For the potential function $V_I(\phi)$ from the closed-form solutions of the WdW equation we derived before we can easily see that the solution of the Hamilton-Jacobi Equation (52) is

$$S(a,\phi) = \frac{1}{\lambda}a^\lambda e^{-\lambda\phi} + 3V_0\frac{a^{6-\lambda}}{6-\lambda}. \tag{53}$$

Hence, we calculate $p_\phi = -a^\lambda e^{-\lambda\phi}, \; p_a = a^{\lambda-1}e^{-\lambda\phi} + 3V_0 a^{5-\lambda}$ where the field equations are reduced to the following system of first-order ordinary differential equations

$$\dot{a} = -\frac{N}{3}a^{\lambda-2}e^{-\lambda\phi}, \tag{54}$$

$$\dot{\phi} = \frac{N}{3}\left(a^{\lambda-3}e^{-\lambda\phi} + 3V_0 a^{3-\lambda} - a^\lambda e^{-\lambda\phi}\right). \tag{55}$$

Without loss of generality we assume $N(t) = -3a^{2-\lambda}e^{\lambda\phi}$, which provides $a(t) = t$ and

$$\dot{\phi} = \frac{t^{\lambda-3}e^{-\lambda\phi} + 3V_0 t^{3-\lambda} - t^2 e^{-\lambda\phi}}{t^{\lambda-2}e^{-\lambda\phi}}, \tag{56}$$

which can be integrated explicitly.

$$\frac{d\phi}{da} = -\left(a^{-1} + 3V_0 a^{-1}e^{\lambda\phi} - a^2\right)$$

4.3. Potential Function $V_I(\phi)$

As far as the second scalar field potential in the new coordinates $\{x, y\}$ is concerned the Hamilton-Jacobi equation reads

$$\left(\left(\kappa V_1\left(\frac{\partial S}{\partial x}\right)\left(\frac{\partial S}{\partial y}\right) + V_1\left(\frac{\partial S}{\partial y}\right)^2\right) - V_0 e^{-\kappa y - 6y}\right) = 0 \tag{57}$$

where we have replaced $N = 3\left(x^{1+\frac{3}{\kappa}}(1 - xe^{-\kappa y})\right)^{-1}$. Moreover in the new coordinates we have $\dot{x} = p_y$, $\dot{y} = \kappa V_1 p_x + 2V_1 p_y$.

From (57) we derive the solution for the Hamilton-Jacobi equation

$$S(x, y) = \frac{I_0}{2}(2x - \kappa y) - \frac{\sqrt{\kappa^2 V_1 I_0^2 + 4V_1 V_0 e^{Ky}}}{V_1 K} + \frac{1}{2}\ln\left(\frac{\kappa V_1 I_0 + \sqrt{\kappa^2 V_1 I_0^2 + 4V_1 V_0 e^{Ky}}}{\kappa V_1 I_0 - \sqrt{\kappa^2 V_1 I_0^2 + 4V_1 V_0 e^{Ky}}}\right),$$

where $K = -(6 + \kappa)$ and I_0 is the conservation law corresponding to the symmetry vector X_4. Therefore

$$\dot{x} = \frac{2V_0 e^{Ky}}{\kappa V_1 V_0 - \sqrt{V_1\left(\kappa^2 I_0^2 V_1\right) + 4V_0 e^{Ky}}}, \tag{58}$$

$$\dot{y} = \kappa V_1 I_0 + \frac{4V_0 V_1 e^{Ky}}{\kappa V_1 V_0 - \sqrt{V_1\left(\kappa^2 I_0^2 V_1\right) + 4V_0 e^{Ky}}}. \tag{59}$$

In the simple case where $I_0 = 0$ the Hamilton-Jacobi equation provides $S(x, y) = -\frac{2}{KV_1}\sqrt{V_1 V_0 e^{Ky}}$, from which the reduced system follows,

$$\dot{x} = -\sqrt{\frac{V_0}{V_1}}e^{\frac{K}{2}y}, \quad \dot{y} = 2\sqrt{V_1 V_0}e^{\frac{K}{2}y} \tag{60}$$

with closed-form solution $y(t) = -\frac{1}{K}\ln\left(V_0 V_1 K^2(t - t_0)^2\right)$ and $x(t) = -\frac{1}{KV_1}\ln(t - t_0) + x_0$.

4.4. Quantum Potentiality

For the derivation of the semi-classical solution in the de Broglie-Bohm representation of quantum mechanics, in the wavefunction, $\Psi(a, \phi) = \Omega(a, \phi)e^{\frac{i}{\hbar}S(a, \phi)}$, $S(a, \phi)$ it is assumed to be the solution of the modified Hamilton-Jacobi equation, which is used to reduce the field equations into a system of two first-order ordinary differential equations.

4.5. Potential Function $V_I(\phi)$

For the wavefunctions which correspond to $V_I(\phi)$ a nonconstant amplitude $\Omega(a, \phi)$ follows from the wavefunction $\Psi_2(a, \phi)$ as expressed by Equation (43). For $\Psi_2^{02} = 0$ and in the limit in which $a^3 e^{-\frac{\lambda}{2}\phi} \to \infty$, the wavefunction is approximated by $\Psi_2(a, \phi) =$

$\sqrt{\frac{2}{\pi Z}} e^{\frac{q_2 \phi}{2(\lambda-6)} - \frac{q_2(\lambda-3)}{\lambda(\lambda-6)} \ln a} \cos\left(Z - \frac{q_2}{\lambda(\lambda-6)} \frac{\pi}{2} - \frac{\pi}{4}\right)$ where $Z = -2\sqrt{\frac{3V_0}{\lambda(6-\lambda)}}$, while q_2 is assumed to be an imaginary number, i.e., $q_2 = i|q_2|$.

Hence, $\Omega(a, \phi) = \sqrt{\frac{2}{\pi Z}}$ from where we end with the quantum potential term $V_Q(a, \phi)$ = $\frac{\lambda(\lambda-6)}{96} a^{-3}$, which means that when $a^3 e^{-\frac{\lambda}{2}\phi} \to \infty$, a term which corresponds to an ideal gas with equation of state parameter $w = 1$, that is, a stiff fluid component, is introduced in the field equations as a quantum correction. Now the quantum correction is important for small values of a, that is when $e^{-\frac{\lambda}{2}\phi} \to \infty$.

4.6. Potential Function $V_{II}(\phi)$

In a similar approach, for potential $V_{II}(\phi)$ and for $q_4 = 0$, for the wavefunction (50) in the limit $e^{-3y} \to \infty$, we can define the amplitude $\Omega(x, y) = \Omega_0 e^{\frac{\kappa+3}{2}y}$. Hence the quantum correction term is derived to be $V_Q(x, y) = \frac{V_1}{8}(R^2 - 9)e^{6y}$; however because that is true in the limit $e^{-3y} \to \infty$ easily it follows that $\left(e^{3y}\right)^2 \to 0$, that is, the quantum potential tern can be neglected.

5. Conclusions

In this study, we focused on the quantization of an extended higher-order teleparallel cosmological theory. In particular we considered the so-called $f(T, B)$ gravity and its special form $f(T, B) = T + F(B)$. In the latter scenario the cosmological field equations can be described by a point-like Lagrangian with the same number of dynamical constraints with that of scalar tensor theory. Indeed, with the use of a Lagrange multiplier the higher-order derivatives can be attribute in a scalar field. However, the latter is different from that of scalar-tensor theories.

Because the point-like Lagrangian of the cosmological theory has the 2 + 1 degrees of freedom, they are the scale factor $a(t)$, the scalar field $\phi(t) = F_{,B}(B(t))$ and the lapse function $N(t)$. Consequently, we can define the WdW equation by quantize the Hamilton function for the point-like Lagrangian, that is, we performed a minisuperspace quantization of the theory. According to our knowledge, this is the first minisuperspace quantization in modified teleparallel theories in the literature. There are some previous studies in the literature on the minisuperspace quantization $f(T)$ theory, however in these studies the authors did not considered all the degrees of freedom and the constraint equations on their quantization approach.

In order to solve the WdW equation, we applied the theory of similarity transformations. Specifically we investigated the functional forms for the potential $V(\phi) = F_{,B}B - F$, where point symmetries exists. The latter symmetries were used for the construction of quantum operators. Furthermore, we were able to find the classical limit for this models, that is, we solved the gravitational field equations. Finally, we investigated the existence of quantum corrections for the field equations in the semi-classical limit as it is given by the de Broglie -Bohm representation of quantum mechanics.

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