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Optimal Perturbation Technique within the Asymptotic Iteration Method for Heavy-Light Meson Mass Splittings

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Abstract: For further insight into the perturbation technique within the framework of the asymptotic iteration method (PAIM), we suggest this method to be used as an alternative method to the traditional well-known perturbation techniques. We show by means of very simple algebraic manipulations that PAIM can be directly applied to obtain the symbolic expectation value of any perturbed potential piece without using the eigenfunction of the unperturbed problem. One of the fundamental advantages of PAIM is its ability to extract a reference unperturbed potential piece or pieces from the total Hamiltonian which can be solved exactly within AIM. After all, one can easily compute the symbolic expectation values of the remaining potential pieces. As an example, the present scheme is applied to the semi-relativistic wave equation with the harmonic-oscillator potential implemented with the Fermi–Breit potential terms. In particular, the non-trivial symbolic expectation values of the Dirac delta function, and the momentum-dependent orbit–orbit coupling terms are successfully calculated. Results are then used, as an illustration, to compute the semi-relativistic s-wave heavy-light meson masses. We obtain good agreement with experimental data for the meson mass splittings $c\bar{u}$, $c\bar{d}$, $c\bar{s}$, $b\bar{u}$, $b\bar{d}$, $b\bar{s}$.

Keywords: optimal perturbation technique within the framework of the asymptotic iteration method; semi-relativistic wave equation; Fermi–Breit potential; masses of heavy-light quark-antiquark structures

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1. Introduction

The nonexistence of a precise analytical solution to Schrödinger-like wave-equations with different potential models has led the study of such wave-equations to be one of the most common theoretical laboratories for investigating the validity of various methods based on perturbative and non-perturbative approaches. The literature is full of these methods, such as the Rayleigh–Schrödinger perturbation method [1], the moment method [2], the analytic continued fraction theory [3], the Hill determinant [4], the super-symmetric quantum mechanics (SUSYQM) [5], the shifted 1/N expansion method [6], and other many methods [7,8].

The perturbation technique within the framework of the asymptotic iteration method (PAIM) previously introduced for solving Schrödinger-like wave-equations including Coulomb plus linear plus Oscillator potential models [9] is one such method.

In this work, our aim was to extend PAIM, and we would like to show how PAIM can be directly applied to obtain the symbolic expectation values of any perturbed potential piece within the same mathematical process of the PAIM.

Contrary to the traditional well-known perturbation techniques, there is no need in PAIM to find the base eigenfunction of the unperturbed problem. In addition, PAIM is different from other perturbation approaches [1,6], as it lays no limitations on the coupling constants or the quantum numbers included in the phenomenological potential models. Furthermore, it avoids attaching to numerical computations at the initial stages. Above

all, we will see that PIAM is valid with the same form for each of the ground and excited energy states.

Simply, PAIM looks, as the other methods, lie it is the solution of the fundamental problems in theoretical physics $H \mid \Psi\rangle = E \mid \Psi\rangle$, where H is the total Hamiltonian of the system, E is the energy, and Ψ is the wavefunction. The solution of this equation for a general observable H is in general very difficult, excepted, however, for very limited simple potentials with a high degree of symmetry where an analytic solution is possible. Therefore, for a large class of un-symmetrical potentials which are in fact representing realistic physical systems, the total Hamiltonian is usually re-written as the sum of two components: $H = H_0 + \omega V$. One of the fundamental advantage of PAIM in this respect is that it facilitates choosing a reference Hamiltonian H_0 from several available options in H that can be easily solved exactly within the asymptotic iteration method (AIM) [10–12].

In this regard, a reasonable problem to tackle the above issue could be the semi-relativistic wave-Equation (SR). This equation is used to solve the semi-relativistic bound-state structures, which in fact represents a standard non-covariant approximation to the covariant Bethe–Salpeter Equation [13,14].

As a practical application, we will investigate the masses of the heavy-light meson structures using the SR wave-equation within the PAIM procedure. The Hamiltonian for the quark–antiquark structure consists of the SR wave equation with the harmonic-oscillator potential model, implemented with the Fermi–Breit potential terms plus a constant term.

To start with, we consider the transformed SR wave-equation as a semi-relativistic Schrödinger-like wave-equation with the inclusion of relativistic corrections up to order $(v/c)^2$ in the non-local kinetic energy term. Thus, it can be easily solved with any additional potential pieces by applying the PAIM directly. Interestingly, the transformed SR wave-equation is reduced to a form almost the same as the Schrödinger wave-equation with an extra self-induced non-separable energy-dependent harmonic-oscillator field. The presence of this term makes the solution to this equation generally more difficult to be solved within the available traditional well-known perturbation techniques [1,15].

With this in mind, this paper is arranged as follows. Section 2 is dedicated to the theoretical framework; and total inter-quark Hamiltonian is introduced. We will discuss the several available options that can be easily chosen for H_0 and be solved exactly by the AIM. Therein, we discuss in detail the technique for obtaining the total energy eigenvalues, including the perturbed Fermi–Breit potential pieces with the exclusion of use the base eigenfunctions of the unperturbed problem. Section 3 is dedicated to the numerical calculations, and the discussion of results. Finally, we present our summary with concluding remarks on the suggested technique and on its results.

2. Theoretical Framework: The Total Inter-Quark Hamiltonian

In the semi-relativistic framework, the effective Hamiltonian which describes the internal structure of the s-wave mesons is assumed to be of the following form [16]:

$$H = (p_1^2 + m_1^2\,)^{1/2} + (p_2^2 + m_2^2\,)^{1/2} + \frac{1}{2}Kr^2 + H^{FB}(r) + H_{eve}, \tag{1}$$

$$H^{FB}(r) = (\alpha Q_1 Q_2 - \frac{4}{3}\alpha_s)\left[\frac{1}{\mid r \mid} - \frac{1}{2m_1 m_2}\left(\frac{p_1.p_2}{\mid r \mid} + \frac{r.(r.p_1)p_2}{\mid r^3 \mid}\right)\right. \tag{2}$$

$$\left. - \frac{\pi}{2}\delta(r)\left(\frac{1}{m_1^2} + \frac{1}{m_2^2} + \frac{16 s_1.s_2}{3m_1 m_2}\right)\right], \tag{3}$$

where Q_i, m_i, p_i, and s_i are the charge, mass, momentum, and spin of the i-th quark or antiquark, respectively. r is the inter-quark distance between quark–antiquark, and $H^{FB}(r)$ is the Fermi–Breit potential terms. H_{eve} is everything else, which we assume to be the flavor-, spin-, and r-independent parts of the Hamiltonian, respectively.

2.1. Analytical Eigenenergy for SR Wave Equation with the Harmonic-Oscillator Potential via PAIM

As we noted earlier, one of the advantages of PAIM is that it facilitates choosing a reference Hamiltonian H_0 from the several available options in H that can be easily solved exactly within the AIM [12]. Looking at Equation (1), the first option one may choose H_0 to include the un-perturbed $\sim 1/r$ or $\sim r^2+\sim 1/r$ pieces. The second option is to choose the reference Hamiltonian H_0 to only include the un-perturbed $\sim r^2$ piece. This is the only case which we are willing to discuss it in this work. Accordingly, the masses of the $Q\bar{q}$ mesons can be estimated as the sum of the energy eigenvalues obtained from the solution of SR wave- equation with the harmonic-oscillator potential, plus the energy eigenvalues which are obtained from the expectation values of the remaining potential terms within the mathematical procedure of PAIM. Therefore, herein, we start by simplifying the semi-relativistic wave-equation including the harmonic-oscillator potential:

$$\left[(p^2+m_1^2\,)^{1/2}+(p^2+m_2^2\,)^{1/2}+\frac{1}{2}Kr^2-M_{_{SA}}\right]\psi(r)=0, \tag{4}$$

where m_1 and m_2 are the two quark–antiquark masses, and $M_{_{SA}}$ is the total spin averaged mass.

To facilitate the solution process for this equation, we start the investigation by removing the non-locality in the kinetic energy operators by the direct expansion of the square root-operator in powers of $(v/c)^2$ up to two terms. This technique leads to a Schrödinger-like wave-equation with some relativistic dynamics:

$$\left\{\frac{p^2}{2\mu}-\frac{1}{2\eta}[E_{n\ell}^2+\frac{1}{4}K^2r^4-KE_{n\ell}r^2]+\frac{1}{2}Kr^2\right\}\psi_{n\ell}(r)=E_{n\ell}\psi_{n\ell}(r), \tag{5}$$

where $E_{n\ell}=M_{SA}-m_1-m_2$, $\mu=m_1m_2/(m_1+m_2)$ is the reduced mass, $\nu=m_1^3m_2^3/(m_1^3+m_2^3)$ is a useful parameter, and $\eta=\nu/\mu^2$.

For our later discussions, in order to obtain the Fermi–Breit potential energy terms, it is helpful to re-write Equation (4) as

$$\left\{\frac{p^2}{2\mu}+V_{eff}(r)\right\}\psi_{n\ell}(r)=E_{n\ell}\psi_{n\ell}(r), \tag{6}$$

where $V_{eff}(r)=\frac{1}{2}Kr^2-\frac{1}{2\eta}\left(E_{n\ell}^2+\frac{1}{4}K^2r^4-KE_{n\ell}r^2\right)$.

Expressing the operator p^2 in spherical polar coordinates, and for states of definite orbital angular momentum l, we define $\psi_{n\ell}(r)=r^{-1}R_{n\ell}(r)$. This will transform Equation (4) to a relativistic radial Schrödinger-like equation with non-separable energy term (in units $\hbar=c=1$):

$$\left[\frac{d^2}{dr^2}-\frac{\ell(\ell+1)}{r^2}-\mu K(1+\frac{E_{n\ell}}{\eta})r^2+\frac{\mu K^2}{4\eta}r^4\right]R_{n\ell}(r)=-2\mu(\frac{E_{n\ell}^2}{2\eta}+E_{n\ell})R_{n\ell}(r). \tag{7}$$

To transform Equation (6) to an appropriate form for PAIM, it is worthwhile observe the asymptotic behavior of the radial wave function $R_{n\ell}(r)$ at $r\longrightarrow 0$ and at $r\longrightarrow\infty$. This procedure suggests that $R_{n\ell}(r)$ must look like:

$$R_{n\ell}(r)=r^{(\ell+1)}e^{-\varrho_{n\ell}r^2/2}g_{n\ell}(r), \tag{8}$$

with the unknown energy-dependent parameter $\varrho_{n\ell}$. This parameter represents the size of the suggested wave function, and to be determined from PAIM. In the following, as we shall later see that this parameter will play an important role in explaining the mass splittings of the heavy-light mesons, as well as in predicting the inter-quark distances.

Substituting the trial wave function of Equation (7) into Equation (6), performing the mathematics, and considering:

$$\zeta_{n\ell}^2 = \mu K(1+\frac{E_{n\ell}}{\eta}),\ \ \varepsilon_{n\ell} = 2\mu(\frac{E_{n\ell}^2}{2\eta}+E_{n\ell}),\ \ A_{n\ell} = 2(\ell+1)\varrho_{n\ell}, and \gamma^2 = \frac{\mu}{4\eta}K^2. \tag{9}$$

As a result, the new function $g_{n\ell}(r)$ would satisfy a new second-order homogenous linear differential equation which is appropriate to the application of the PAIM method:

$$g_{n\ell}^{''}(r) = \lambda_0(r,\omega)g_{n\ell}^{'}(r) + s_0(r,\omega)g_{n\ell}(r), \tag{10}$$

where $\lambda_0(r,\omega) = 2\Big(\varrho_{n\ell}r - \frac{(\ell+1)}{r}\Big)$, and we propose $s_0(r,\omega)$ to be expressed as a sum of two parts:

$$s_0(r,\omega) = s_0^0(r,\omega) + s_0^1(r,\omega), \tag{11}$$

with $s_0^0(r,\omega) = \big((\zeta_{n\ell}^2 - \varrho_{n\ell}^2)r^2 + \varrho_{n\ell} + A_{n\ell} - \varepsilon_{n\ell}\big)$.

In the present scheme, $s_0^0(r,\omega)$ is chosen as a reference potential, in such a way that it has an accurate solution in AIM [12]. However, $s_0^1(r,\omega) = \omega\big(-\gamma^2 r^4 + H^{FB}(r)\big)$ is representing the remaining potential pieces, and it is to be treated using PAIM, where ω is an artificially introduced perturbation expansion parameter to be set equal to one at the end of the computations.

Following the systematic technique of PAIM, the energy eigenvalues $\varepsilon_{n\ell}$ are firstly solved by satisfying the asymptotic convergency criteria of the AIM; that is for the sufficiently large iteration number $i = 0, 1, 2, \ldots$.:

$$\delta_i(r,\omega) \equiv s_i(r,\omega)\lambda_{i+1}(r,\omega) - s_{i+1}(r,\omega)\lambda_i(r,\omega) = 0, \tag{12}$$

where:

$$\lambda_i(r,\omega) = \lambda_{i-1}^{'}(r,\omega) + s_{i-1}(r,\omega) + \lambda_0(r,\omega)\lambda_{i-1}(r,\omega), \tag{13}$$

and:

$$s_i(r,\omega) = s_{i-1}^{'}(r,\omega) + s_0(r,\omega)\lambda_{i-1}(r,\omega). \tag{14}$$

In order to obtain the leading energy term $\varepsilon_{n\ell}^{(0)}$, one must only switch off ω in $s_0^1(r,\omega)$, and then choose $\zeta_{n\ell}^2 = \varrho_{n\ell}^2$ in the $s_0^0(r,\omega)$ term. This procedure would lead to an exactly solvable eigenvalue problem in the framework of AIM:

$$g_{n\ell}^{''}(r) = 2\left(\varrho_{n\ell}r - \frac{(\ell+1)}{r}\right)g_{n\ell}^{'}(r) + (\varrho_{n\ell} + A_{n\ell} - \varepsilon_{n\ell})g_{n\ell}(r), \tag{15}$$

and the roots of $\delta_i^{(0)}(r,0) = 0$ are written in terms of $\zeta_{n\ell}$ like:

$$\varepsilon_{n\ell}^{(0)} = (4n+1)\zeta_{n\ell} + 2\zeta_{n\ell}(\ell+1),\ \text{for}\ \begin{cases} n = 0,1,2,\ldots, \\ \ell = 0,1,2,\ldots. \end{cases} \tag{16}$$

To find the total energy

$$\varepsilon_{n\ell} = \varepsilon_{n\ell}^{(0)} + \omega\varepsilon_{n\ell}^{(1)} + \omega^2\varepsilon_{n\ell}^{(2)} + \omega^3\varepsilon_{n\ell}^{(3)} + \omega^4\varepsilon_{n\ell}^{(4)} + \ldots, \tag{17}$$

we switched on ω in the $s_0^1(r,\omega)$ term, and then we expanded $\delta_i(r,\omega)$ of Equation (11) around $\omega = 0$:

$$\delta_i(r,\omega) = \delta_i(r,0) + \frac{\omega}{1!}\frac{\partial\delta_i(r,\omega)}{\partial\omega}|_{\omega=0} + \frac{\omega^2}{2!}\frac{\partial^2\delta_i(r,\omega)}{\partial\omega^2}|_{\omega=0} + \frac{\omega^3}{3!}\frac{\partial^3\delta_i(r,\omega)}{\partial\omega^3}|_{\omega=0} + \ldots. \tag{18}$$

According to the PAIM technique, $\delta_i(r,\omega)$ should be equal to zero; if this to be correct for each ω value, each term of the series in Equation (17) should be equal to zero.

Thus, a quantitative estimate for the $\varepsilon_{n\ell}$ expansion terms can be obtained by comparing the terms with the same order of ω in Equations (16) and (17). Consequently, it is clear from the technique of PAIM that the roots of $\delta_i^{(0)}(r,0)=0$ provide us with the leading zeroth-order contribution energy terms $\varepsilon_{n\ell}^{(0)}$. Similarly, the roots of $\delta_i^{(1)}(r,\omega)=0$ provide us with first-order correction terms to $\varepsilon_{n\ell}^{(1)}$, etc. Therefore, the final solution for the total energy $\varepsilon_{n\ell}$ in conjunction with Equation (16) within the PAIM framework is:

$$\varepsilon_{n\ell}=(4n+1)\zeta_{n\ell}+2\zeta_{n\ell}(\ell+1)+\omega\varepsilon_{n\ell}^{(1)}+\omega^2\varepsilon_{n\ell}^{(2)}+\omega^3\varepsilon_{n\ell}^{(3)}+\ldots. \tag{19}$$

For clarity, in the next section, we will show the application of the PAIM technique to find the $\varepsilon_{n\ell}^{(1)}$ contribution term.

2.2. The Explicit Perturbed Symbolic Energy Expressions via PAIM

To determine $\varepsilon_{n\ell}^{(1)}$, the perturbed symbolic energy expressions for $s_0^1(r,\omega)=\omega\left(-\gamma^2r^4+H^{FB}(r)\right)$. One should simply add the terms of $s_0^1(r,\omega)$, one by one, single-separated in Equation (14), after then replace $\varepsilon_{n\ell}$ with $\varepsilon_{n\ell}^{(0)}+\omega\varepsilon_{n\ell}^{(1)}$, so that for the first piece, the terms $\lambda_0(r,\omega)$ and $s_0(r,\omega)$ of Equation (14) reads

$$\begin{aligned}\lambda_0(r,\omega) &= 2\left(\varrho_{n\ell}r-\frac{(\ell+1)}{r}\right),\\ s_0(r,\omega) &= \left(\varrho_{n\ell}+A_{n\ell}-\varepsilon_{n\ell}^{(0)}+\omega\varepsilon_{n\ell}^{(1)}-\omega\gamma^2r^4\right),\end{aligned} \tag{20}$$

and then one should terminate the iterations by using the termination condition $\delta_i^{(1)}(r,\omega)=\omega\frac{\partial\delta_i(r,\omega)}{\partial\omega}|_{\omega=0}=0$ as a solution to Equation (19). The first root of the resulting solutions give us the first-order energy $\varepsilon_{n\ell}^{(1)}$ for γ^2r^4 term:

$$\langle\gamma^2r^4\rangle=\frac{15}{4}\frac{\gamma^2}{\zeta_{n\ell}^2}. \tag{21}$$

Herein, the calculations are given up to the first-order in terms of γ and $\zeta_{n\ell}$, where it is observed that higher-order corrections are of negligible contributions.

At this stage, we can write the energy $\varepsilon_{n\ell}$ for $n=0$ and $\ell=0$ up to the first-order as

$$\varepsilon_{00}=3\zeta_{00}-\frac{15}{4}\frac{\gamma^2}{\zeta_{00}^2}+\ldots. \tag{22}$$

Similarly, one can obtain the other perturbed symbolic energy expressions for the Fermi–Breit potential terms. To do that, one should remove the γ^2r^4 term from Equation (19), and then replace the Fermi–Breit coulomb $\frac{1}{|r|}$ term, and continue iterating with the same procedure as before.

By the same way, one can also obtain the perturbed symbolic energy eigenvalue for the Dirac delta $\delta(r)$ Fermi–Breit potential function via PAIM. Replacing the Dirac delta function with its equivalence of the square modulus of the wave-function at the origin $\mid\psi(0)\mid^2$, where $\delta(r)=\mid\psi(0)\mid^2$. The only s-wave states $\ell=0$ have a non-zero value of the wave-function at the origin: $\mid\psi(0)\mid^2=\frac{\mu}{2\pi}\langle V_{eff}^{'}(r)\rangle$. Therefore, the expectation value of $\langle\delta(r)\rangle=\frac{\mu}{2\pi}\langle Kr-\frac{1}{2\eta}\left(K^2r^3-2KE_{n\ell}r\right)\rangle$. Furthermore, the calculations of the expectation value of the momentum-dependent orbit–orbit coupling term $\langle\left(\frac{p_1.p_2}{|r|}+\frac{r.(r.p_1)p_2}{|r^3|}\right)\rangle$ is straightforward within the PAIM using $p^2=2\mu^2\Big(E_{n\ell}-V_{eff}(r)\Big)$.

Below, we explicitly give the expectation values of the perturbed symbolic radial Fermi–Breit potential terms:

$$\begin{aligned} \frac{1}{r} \equiv \langle \frac{1}{\mid r \mid} \rangle &= 2(\frac{\zeta_{n\ell}}{\pi})^{1/2}, \\ \langle \delta(r) \rangle &= (\frac{\zeta_{n\ell}}{\pi})^{3/2}\Big(1-4\gamma^2\Big), \\ \langle \Big(\frac{p_1.p_2}{\mid r \mid} + \frac{r.(r.p_1)p_2}{\mid r^3 \mid} \Big) \rangle &= -4\pi(\frac{\zeta_{n\ell}}{\pi})^{3/2}\Big(1+\gamma^2\Big). \end{aligned} \tag{23}$$

It is clear that the expectation values of $\langle \delta(r) \rangle$, and the momentum-dependent orbit–orbit coupling term $\langle \Big(\frac{p_1.p_2}{|r|} + \frac{r.(r.p_1)p_2}{|r^3|} \Big) \rangle$ are relativistically modified by extra factors $\big(1-4\gamma^2\big)$ and $\big(1+\gamma^2\big)$, respectively. However, these extra factors do not exist in the traditional perturbation method [16]. Thus, the expectation value of the Fermi–Breit potential can be expressed as

$$\langle H^{FB}(r) \rangle = (\alpha Q_1 Q_2 - \frac{4}{3}\alpha_s)\Big[\frac{1}{r} + \frac{\pi}{4m_1 m_2}\frac{(1+\gamma^2)}{r^3} \tag{24}$$

$$- \frac{\pi}{16}\frac{(1-4\gamma^2)}{r^3}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2} + \frac{16 s_1.s_2}{3m_1 m_2}\right)\Big], \tag{25}$$

Finally, the explicit expression for the masses of the s-wave heavy-light mesons with $(n = \ell = 0)$ then reads:

$$M = M_{sA} + \langle H^{FB}(r) \rangle + E_0, \tag{26}$$

bearing in mind that $M_{sA} = m_1 + m_2 + E_{00}$; $E_{00} = \eta\left(\frac{\zeta_{00}^2}{\mu K} - 1\right)$, and $E_0 = \langle H_{eve} \rangle$.

3. Numerical Results, and Discussion

It is useful here to clarify that the obtained formulas in Equations (15), (21), (22), (23), and (24) provide remarkably accurate and simple analytical expressions to explain the mass splittings of the heavy-light pseudoscalar and vector mesons. To find these masses, we first have to solve the energy-dependent size parameter ζ_{00}. Recalling Equations (8) and (22), and after some algebra, these two equations explicitly yield:

$$\zeta_{00}^6 - 12\gamma^2\zeta_{00}^3 - (\mu K)^2\zeta_{00}^2 + 15\gamma^4 = 0. \tag{27}$$

The positive real root of Equation (25) is assumed to be the acceptable solution to this equation. Substituting this solution into Equation (24), then the different meson masses with the different structures can be calculated. Typically, in numerical computations, this approach needs various input parameters that one should fix and fit during the computational process. For the constituent quark masses, we choose roughly $m_u = (0.432 \pm 0.001)\ GeV$, $m_d = (0.434 \pm 0.001)\ GeV$, $m_c = (1.944 \pm 0.001)\ GeV$, $m_b = (5.297 \pm 0.001)\ GeV$, and $K = 0.14 \pm 0.01\ GeV^3$. These input values were taken from [16]. However, the errors which we included in the input parameters did not changing the results, thus we ensure the interested readers that the obtained results in Table 1 in these regions also work.

Firstly, we focus our attention here on the calculated results of the s-wave meson mass splittings given in Table 1. We started with the pseudoscalar meson $D^0(c\bar{u})$ mass with $\langle \mathbf{S_1} \cdot \mathbf{S_2} \rangle = -\frac{3}{4}$. To proceed further, one should find the positive root of Equation (25); $\zeta_{00}^6 - 0.03256\ \zeta_{00}^3 - 0.00245\ \zeta_{00}^2 + 0.00011 = 0$, so that $\zeta_{00} = 0.332986\ GeV^2$. After Equation (24), we obtain $M_{sA} = m_c + m_u + E_{00} = 3.1680\ GeV$. Plugging all these results into Equation (24), and considering the standard value $\alpha_s(m_c^2) = 0.31 \pm 0.01$, we then obtain the mass of $D^0(c\bar{u}) = 1.8651\ GeV$ by fitting $E_0 = -0.943 \pm 0.001\ GeV$. Once the value of E_0 is obtained,

one can obtain the mass of the vector meson $D^{*0}(c\bar{u}) = 2.007\ GeV$ with $\langle \mathbf{S_1} \cdot \mathbf{S_2} \rangle = \frac{1}{4}$, etc. However, the strange constituent quark mass in our procedure is the only fitted mass: $m_s = (0.580 \pm 0.001)\ GeV$—which is also consistent with the constituent strange quark mass available in the literature [17].

Table 1 shows an explicit list of numerical computations for the predicted heavy-light pseudoscalar and vector mesons, together with predicted inter-quark distances, so that our findings may be reproduced by the reader. The experimental values in Table 1 are those of the Particle Data Group [18]. From Table 1, one can observe that there is a satisfactory agreement with the experimental results, as well as with the work of other researchers [16].

Table 1. The computed masses of the heavy-light pseudoscalar and vector mesons in MeV, with their interquark-distances in units of GeV^{-1}.

Meson	M(Our Work)	M(Expt.) [18]	M [16]	Interquark Distance
Pseudoscalar mesons				
$D^0(c\bar{u})$	1865.1	1864.83 ± 0.05	1863.8	1.53579
$D^+(c\bar{d})$	1870.5	1869.65 ± 0.05	1868.7	1.53584
$D_s^+(c\bar{s})$	1969.7	1968.34 ± 0.07	1969.7	1.53409
$B^-(b\bar{u})$	5280.3	5279.34 ± 0.12	5282.2	1.45768
$B^0(b\bar{d})$	5279.3	5279.65 ± 0.12	5280.6	1.45751
$B_s(b\bar{s})$	5330.8	5366.88 ± 0.14	5343.9	1.44283
Vector mesons				
$D^{*0}(c\bar{u})$	2007.0	2006.85 ± 0.05	2008.7	1.53579
$D^{*+}(c\bar{d})$	2010.0	2010.26 ± 0.05	2011.4	1.53584
$D_s^*(c\bar{s})$	2075.7	2112.20 ± 0.40	2096.9	1.53409
$B^{*-}(b\bar{u})$	5340.2	5324.70 ± 0.21	5333.9	1.45768
$B^{*0}(b\bar{d})$	5339.2	5331.30 ± 4.7	5332.6	1.45751
$B_s^*(b\bar{s})$	5377.2	5415.40 ± 1.5	5393.1	1.44283

4. Conclusions

In the present work, we presented a deeply theoretical investigation on the perturbation technique within the framework of the asymptotic iteration method (PAIM). We proposed this method as an alternative method to the traditional well-known perturbation techniques. We showed by means of very simple algebraic manipulations that PAIM can be directly applied to obtain the symbolic expectation values of any perturbed potential piece without using the eigenfunction of the unperturbed problem. Thus, one of the most interesting aspects of PAIM was to show how it is an easy way to extract a reference unperturbed potential piece or pieces from the total Hamiltonian which can be solved exactly within AIM. After that, one can easily apply PAIM to compute the symbolic expectation values of the remaining potential terms.

For clarity, we examined with PIAM the semi-relativistic energy eigenvalues of the SR wave-equation with the harmonic-oscillator potential implemented with the Fermi–Breit potential terms to obtain the mass splittings of the heavy-light pseudoscalar and vector mesons. Although we limited ourselves to one illustrative example, however, the range of application of the method is rather large and appears to be straightforward.

It is worth noting here that the PAIM approach which we described in the current paper with its analytical expressions is a lot more helpful than the pure numerical computations. Moreover, we see that the expectation values of $\langle \delta(r) \rangle$, and the momentum-dependent orbit–orbit coupling terms $\langle \left(\frac{p_1.p_2}{|r|} + \frac{r.(r.p_1)p_2}{|r^3|} \right) \rangle$ are relativistically modified by an extra factors $\left(1 - 4\gamma^2\right)$ and $\left(1 + \gamma^2\right)$, respectively. This is a new piece of information. However, these extra factors do not exist in the traditional perturbation methods [16].

Finally, in this work, we showed that it was a very easy task to use the PAIM without needing to be concerned with the ranges of the couplings in the potential. Furthermore, the

degree of precision of our results can be significantly enhanced by raising the perturbation order in the energy expansion series by one more step, without any technical difficulty. Altogether, it is hoped that this approach and its findings will give us a valuable future hint and also enrich our knowledge in investigating the fascinating diquark–antidiquark problem in any tetraquark configuration.

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