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Review

BRST and Superfield Formalism—A Review

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Abstract: This article, which is a review with substantial original material, is meant to offer a comprehensive description of the superfield representations of BRST and anti-BRST algebras and their applications to some field-theoretic topics. After a review of the superfield formalism for gauge theories, we present the same formalism for gerbes and diffeomorphism invariant theories. The application to diffeomorphisms leads, in particular, to a horizontal Riemannian geometry in the superspace. We then illustrate the application to the description of consistent gauge anomalies and Wess–Zumino terms for which the formalism seems to be particularly tailor-made. The next subject covered is the higher spin YM-like theories and their anomalies. Finally, we show that the BRST superfield formalism applies as well to the $N = 1$ super-YM theories formulated in the supersymmetric superspace, for the two formalisms go along with each other very well.

Keywords: gauge invariant theories; BRST and anti-BRST symmetry; superfield formalism; diffeomorphism invariance; consistent gauge anomalies; HS gauge symmetry; $N = 1$ super-Yang-Mills

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1. Introduction

The discovery of the BRST symmetry in gauge field theories, refs. [1–4], is a fundamental achievement in quantum field theory. This symmetry is not only the building block of the renormalization programs, but it has opened the way to an incredible number of applications. Besides gauge field theories, all theories with a local symmetry are characterized by a BRST symmetry: theories of gerbes, sigma models, topological field theories, string and superstring theories, to name the most important ones. Whenever a classical theory is invariant under local gauge transformations, its quantum counterpart has a BRST-type symmetry that governs its quantum behavior. Two main properties characterize the BRST symmetry. The first is its group theoretical nature: performing two (different) gauge transformations one after the other and then reversing the order of them does not lead to the same result (unless the original symmetry is Abelian), but the two different results are related by a group theoretical rule. This is contained in the nilpotency of the BRST transformations.

The second important property of the BRST transformations is nilpotency itself. It is inherited, via the Faddeev–Popov quantization procedure, from the anticommuting nature of the ghost and anti-ghost fields. This implies that, applying twice the same transformation, we obtain 0. The two properties together give rise to the Wess–Zumino consistency conditions, a fundamental tool in the study of anomalies. It must be noted that while the first property is classical, the second is entirely quantum. In other words, the BRST symmetry is a quantum property.

It was evident from the very beginning that the roots of BRST symmetry are geometrical. The relevant geometry was linked at the beginning to the geometry of the principal fiber bundles [5–8] (see also [9]). Although it is deeply rooted in it, the BRST symmetry is rather connected to the geometry of infinite dimensional bundles and groups, in particular, the Lie group of gauge transformations [10,11]. One may wonder where this infinite dimensional geometry is stored in a perturbative quantum gauge theory. The answer is, in the anticommutativity of the ghosts and in the nilpotency of the BRST transformations themselves, together with their group theoretical nature. There is a unique way to synthesize these quantum properties, and this is the superfield formalism. The BRST symmetry calls for the introduction of the superfield formulation of quantum field theories. One might even dare say that the superfield formalism is the genuine language of a quantum gauge theory. This is the subject of the present article, which is both a review of old results and a collection of new ones, with the aim of highlighting the flexibility of the superfield approach to BRST symmetry (it is natural to extend it to include also the anti-BRST symmetry [12–14]). Here, the main focus is on the algebraic aspects and on the ample realm of applications, leaving the more physical aspects (functional integral and renormalization) for another occasion. We will meet general features—we can call them universal—which appear in any application and for any symmetry. One is the so-called horizontality conditions, i.e., the vanishing of the components along the anticommuting directions, which certain quantities must satisfy. Another is the so-called Curci–Ferrari conditions [12], which always appear when both (non-Abelian) BRST and anti-BRST symmetries are present.

Before passing to a description of how the present review is organized, let us comment on the status of anti-BRST. It is an algebraic structure that comes up naturally as a companion to the BRST one, but it is not necessarily a symmetry of any gauge-fixed action. It holds, for instance, for linear gauge fixing, and some implications have been studied to some extent in [14] and also in [15–17]. However, it is fair to say that no fundamental role for this symmetry has been uncovered so far, although it is also fair to say that the research in this field has never overcome a preliminary stage[1]. In this review, we consider BRST and anti-BRST together in the superfield formalism but whenever it is more convenient and expedient to use only the BRST symmetry, we focus only on it.

We start in Section 2 with a review of the well-known superfield formulation of BRST and anti-BRST of non-Abelian gauge theories, which is obtained by enlarging the spacetime with two anticommuting coordinates, ϑ and $\bar{\vartheta}$. Section 3 is devoted to gerbe theories, which are close to ordinary gauge theories. After a short introduction, we show that it is simple and natural to reproduce the BRST and anti-BRST symmetries with the superfield formalism. As always, when both BRST and anti-BRST are involved, we come across specific CF conditions. The next two sections are devoted to diffeomorphisms. Diffeomorphisms are a different kind of local transformation; therefore, it is interesting to see, first of all, if the superfield formalism works. In fact, in Section 4, we find horizontality and CF conditions for which BRST and anti-BRST transformations are reproduced by the superfield formalism. We show, however, that the super-metric, i.e., the metric with components in the anticommuting directions, is not invertible. So a super-Riemannian geometry is not possible in the superspace but, in exchange, we can define a horizontal super-geometry, with Riemann and Ricci tensors defined on the full superspace. In Section 5 we deal with frame superfields and define fermions in superspace. In summary, there are no obstructions to formulate quantum gravitational theories in the superspace.

The second part of the paper concerns applications of the superfield method to some practical problems, notably to anomalies. Consistent anomalies are a perfect playground for the superfield method, as we show in Section 6. We show that not only are all the formulas concerning anomalies in any even dimension easily reproduced, but in fact, the superfield formalism seems to be tailor-made for them. A particularly sleek result is the way one can extract Wess–Zumino terms from it. In Section 7, we apply the superfield formalism to HS-YM-like theories. After a rather detailed introduction to such novel models, we show that the superfield method fits perfectly well and is instrumental in deriving the form of anomalies, which would otherwise be of limited access. Section 8 is devoted to the extension of the superfield method in still another direction—that of supersymmetry. We show, as an example, that the supersymmetric superspace formulation of $N = 1$ SYM theory in 4D can be easily enlarged by extending the superspace with the addition of $\vartheta, \bar{\vartheta}$, while respecting the supersymmetric geometry (constraints). In Section 9, we make some concluding remarks and comments on some salient features of our present work.

The appendices contain auxiliary materials, except the first (Appendix A), which might seem a bit off topic with respect to the rest of the paper. We deem it useful to report in order to clarify the issue of the classical geometric description of the BRST symmetry. As mentioned above, this description is possible. However, one must formulate this problem in the framework of the geometry of the infinite dimensional groups of gauge transformations (which are, in turn, rooted in the geometry of principal fiber bundles). The appropriate mathematical tool is the evaluation map. One can easily see how the superfield method formulation parallels the geometrical description.

Finally, let us add that this review covers only a part of the applications of the superfield approach that have appeared in the literature. We must mention [21–35] for further extensions of the method and additional topics not presented here. A missing subject in this paper, as well as, to the best of our knowledge, in the present literature, is the exploration of the possibility to extend the superfield method to the Batalin–Vilkovisky approach to field theories with local symmetries.

Notations and Conventions. The superspace is represented by super-coordinates $X^M = \{x^\mu, \vartheta, \bar{\vartheta}\}$, where x^μ ($\mu = 0, 1, \ldots, d-1$) are ordinary commuting coordinates, while ϑ and $\bar{\vartheta}$ are anticommuting: $\vartheta^2 = \bar{\vartheta}^2 = \vartheta\bar{\vartheta} + \bar{\vartheta}\vartheta = 0$, but commute with x^μ. We make use of a generalized differential geometric notation: the exterior differential $d = \frac{\partial}{\partial x^\mu} dx^\mu$ is generalized to $\tilde{d} = d + \frac{\partial}{\partial \vartheta} d\vartheta + \frac{\partial}{\partial \bar{\vartheta}} d\bar{\vartheta}$. Correspondingly, mimicking the ordinary differential geometry, we introduce super-forms; for instance, $\widetilde{\omega} = \omega_\mu(x) dx^\mu + \omega_\vartheta(x) d\vartheta + \omega_{\bar{\vartheta}}(x) d\bar{\vartheta}$, where ω_μ are ordinary commuting intrinsic components, while $\omega_\vartheta, \omega_{\bar{\vartheta}}$ anticommute with each other and commute with ω_μ. In the same tune, we introduce also super-tensors, such as the super-metric; see Section 4.4. As far as commutativity properties (gradings) are concerned, the intrinsic components of forms and tensors on one side and the symbols $d^\mu, d\vartheta, d\bar{\vartheta}$, on the other constitute separate, mutually commuting sets. When $\tilde{d}$ acts on a super-function $\widetilde{F}(X)$, it is understood that the derivatives act on it from the left to form the components of a 1-super-form:

$$\tilde{d}\widetilde{F}(X) = \frac{\partial}{\partial x^\mu} \widetilde{F}(X) dx^\mu + \frac{\partial}{\partial \vartheta} \widetilde{F}(X) d\vartheta + \frac{\partial}{\partial \bar{\vartheta}} \widetilde{F}(X) d\bar{\vartheta} \tag{1}$$

When it acts on a 1-super-form, it is understood that the derivatives act on the intrinsic components from the left, and the accompanying symbol $dx^\mu, d\vartheta, d\bar{\vartheta}$ becomes juxtaposed to the analogous symbols of the super-form from the left to form the combinations $dx^\mu \wedge dx^\nu, dx^\mu \wedge d\vartheta, d\vartheta \wedge d\vartheta, d\vartheta \wedge d\bar{\vartheta}, \ldots$, with the usual rule for the spacetime symbols, and $dx^\mu \wedge d\vartheta = -d\vartheta \wedge dx^\mu, dx^\mu \wedge d\bar{\vartheta} = -d\bar{\vartheta} \wedge dx^\mu$, but $d\vartheta \wedge d\bar{\vartheta} = d\bar{\vartheta} \wedge d\vartheta$, and $d\vartheta \wedge d\vartheta$ and $d\bar{\vartheta} \wedge d\bar{\vartheta}$ are non-vanishing symbols. In a similar way, one proceeds with higher degree super-forms. More specific notations will be introduced later when necessary.

2. The Superfield Formalism in Gauge Field Theories

The superfield formulation of the BRST symmetry in gauge field theories was proposed in [15]; for an earlier version, see [36,37]. Here, we limit ourselves to a summary. Let us consider a generic gauge theory in d dimensional Minkowski spacetime M, with connection $A_\mu^a T^a$ ($\mu = 0, 1, \ldots, \mathsf{d}-1$), valued in a Lie algebra $\mathfrak{g}$ with anti-hermitean generators T^a, such that $[T^a, T^b] = f^{abc} T^c$. In the following, it is convenient to use the more compact form notation and represent the connection as a one-form $A = A_\mu^a T^a dx^\mu$. The curvature and gauge transformation are as follows:

$$F = dA + \frac{1}{2}[A, A] \qquad \text{and} \qquad \delta_\lambda A = d\lambda + [A, \lambda], \tag{2}$$

with $\lambda(x) = \lambda^a(x) T^a$ and $d = dx^\mu \frac{\partial}{\partial x^\mu}$. The infinite dimensional Lie algebra of gauge transformations and its cohomology can be formulated in a simpler and more effective way if we promote the gauge parameter λ to an anticommuting ghost field $c = c^a T^a$ and define the BRST transform as follows[2]:

$$\mathfrak{s} A \equiv dc + [A, c], \qquad \mathfrak{s}\, c = -\frac{1}{2}[c, c]. \tag{3}$$

As a consequence of this, we have the following:

$$\boldsymbol{\mathcal{F}} \equiv (d - \mathfrak{s})(A + c) + \frac{1}{2}[A + c, A + c] = F, \tag{4}$$

which is sometime referred to as the *Russian formula* [7,15,38,39] . Equation (A10) is true, provided we assume the following:

$$[A, c] = [c, A]. \tag{5}$$

i.e., if we assume that c behaves like a one-form in the commutator with ordinary forms and with itself. It can be, in fact, related to the Maurer–Cartan form in G. This explains its anticommutativity.

A very simple way to reproduce the above formulas and properties is by enlarging the space to a superspace with coordinates (x^μ, ϑ), where ϑ is anticommuting, and promoting the connection A to a one-form superconnection $\widetilde{A} = \phi(x, \vartheta) + \phi_\vartheta(x, \vartheta) d\vartheta$ with the following expansions:

$$\phi(x, \vartheta) = \mathsf{A}(x) + \vartheta \Gamma(x), \qquad \phi_\vartheta(x, \vartheta) = \mathsf{c}(x) + \vartheta G(x), \tag{6}$$

and two-form supercurvature

$$\widetilde{F} = \widetilde{d}\widetilde{A} + \frac{1}{2}[\widetilde{A}, \widetilde{A}], \qquad \widetilde{F} = \Phi(x, \vartheta) + \Phi_\vartheta(x, \vartheta) d\vartheta + \Phi_{\vartheta\vartheta}(x, \vartheta) d\vartheta \wedge d\vartheta, \tag{7}$$

with $\Phi(x, \vartheta) = \mathsf{F}(x) + \vartheta \Lambda(x)$ and $\widetilde{d} = d + \frac{\partial}{\partial \vartheta} d\vartheta$. Notice that since $\vartheta^2 = 0$, $d\vartheta \wedge d\vartheta \neq 0$, while $dx^\mu \wedge d\vartheta = -d\vartheta \wedge dx^\mu$. Then, we impose the 'horizontality' condition:

$$\widetilde{F} = \Phi(x, \vartheta), \qquad \text{i.e.,} \qquad \Phi_\vartheta(x, \vartheta) = 0 = \Phi_{\vartheta\vartheta}(x, \vartheta). \tag{8}$$

The last two conditions imply the following:

$$\Gamma(x) = dc(x) + [\mathsf{A}(x), c(x)], \qquad G(x) = -\frac{1}{2}[\mathsf{c}(x), \mathsf{c}(x)].$$

Moreover, $\Lambda(x) = [\mathsf{F}(x), \mathsf{c}(x)]$.

This means that we can identify $\mathsf{c}(x) \equiv c(x)$, $\mathsf{A} \equiv A$, $\mathsf{F} \equiv F$, and the ϑ translation with the BRST transformation $\mathfrak{s}$, i.e., $\mathfrak{s} \equiv \frac{\partial}{\partial \vartheta}$. In this way all the previous transformations, including Equation (5)—which, at first sight, is strange looking—are naturally explained. It is also possible to push further the use of the superfield formalism by noting that, after imposing the horizontality condition, we have the following:

$$\widetilde{A} = e^{-\vartheta c} A\, e^{\vartheta c} + e^{-\vartheta c} \widetilde{d} e^{\vartheta c}, \qquad \widetilde{F} = e^{-\vartheta c} F\, e^{\vartheta c}. \tag{9}$$

A comment is in order concerning the horizontality condition (HC). This condition is suggested by the analogy with the principal fiber bundle geometry. In the total space of a principal fiber bundle, one can define horizontal (or basic) forms. These are forms with no components in the vertical direction: for instance, given a connection, its curvature is horizontal. In our superfield approach, the ϑ coordinate mimics the vertical direction, as the curvature $\widetilde{F}$ does not have components in that direction. This horizontality principle can be extended also to other quantities, for instance, to covariant derivatives of matter fields and, in general, to all quantities that are invariant under local gauge transformations.

2.1. Extension to Anti-BRST Transformations

The superfield representation of the BRST symmetry with one single anticommuting variable is, in general, not sufficient for ordinary Yang–Mills theories because gauge fixing requires, in general, other fields besides A_μ and c. For instance, in the Lorenz gauge, the Lagrangian density takes the following form:

$$\mathcal{L}_{YM} = -\mathrm{tr}\left(\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + A_\mu\partial^\mu B - \partial^\mu\bar{c}D_\mu c + \frac{\alpha}{2}B^2\right), \tag{10}$$

where two new fields are introduced, the antighost field $\bar{c}(x)$ and the Nakanishi–Lautrup field $B(x)$. It is necessary to enlarge the algebra (3) as follows:

$$\mathfrak{s}\,\bar{c} = B, \qquad \mathfrak{s}B = 0, \tag{11}$$

in order to obtain a symmetry of (10). At this point, $\mathcal{L}_{YM}$ is invariant under a larger symmetry, whose transformations, besides (3) and (11), are the anti-BRST ones:

$$\bar{\mathfrak{s}}\,A = d\bar{c} + [A,\bar{c}], \qquad \bar{\mathfrak{s}}\,\bar{c} = -\frac{1}{2}[\bar{c},\bar{c}], \qquad \bar{\mathfrak{s}}\,c = \bar{B}, \qquad \bar{\mathfrak{s}}\,\bar{B} = 0, \tag{12}$$

provided the following:

$$B + \bar{B} + [c,\bar{c}] = 0. \tag{13}$$

This is the Curci–Ferrari condition, ref. [12].

The BRST and anti-BRST transformation are nilpotent and anticommute:

$$\mathfrak{s}^2 = 0, \qquad \bar{\mathfrak{s}}^2 = 0, \qquad \mathfrak{s}\,\bar{\mathfrak{s}} + \bar{\mathfrak{s}}\,\mathfrak{s} = 0. \tag{14}$$

The superfield formalism applies well to this enlarged symmetry, provided we introduce another anticommuting coordinate, $\bar{\vartheta}$: $\bar{\vartheta}^2 = 0, \vartheta\bar{\vartheta} + \bar{\vartheta}\vartheta = 0$. Here, we do not repeat the full derivation as in the previous case but simply introduce the supergauge transformation [15,40–43]:

$$U(x,\vartheta,\bar{\vartheta}) = \exp\left[\vartheta\bar{c}(x) + \bar{\vartheta}c(x) + \vartheta\bar{\vartheta}(B(x) + [c(x),\bar{c}(x)])\right], \tag{15}$$

and generate the following superconnection:

$$\widetilde{A}(x,\vartheta,\bar{\vartheta}) = U(x,\vartheta,\bar{\vartheta})^\dagger\left(\widetilde{d} + A(x)\right)U(x,\vartheta,\bar{\vartheta}), \tag{16}$$

where $\widetilde{d} = d + d\vartheta\frac{\partial}{\partial\vartheta} + d\bar{\vartheta}\frac{\partial}{\partial\bar{\vartheta}}$ and the hermitean operation is defined as follows:

$$\vartheta^\dagger = \vartheta, \quad \bar{\vartheta}^\dagger = -\bar{\vartheta}, \qquad (c^a)^\dagger = c^a, \qquad (\bar{c}^a)^\dagger = -\bar{c}^a,$$

while the $B^a(x), \bar{B}^a(x)$ are real. Then, the superconnection is the following:

$$\widetilde{A}(x,\vartheta,\bar{\vartheta}) = \Phi(x,\vartheta,\bar{\vartheta}) + \eta(x,\vartheta,\bar{\vartheta})d\bar{\vartheta} + \bar{\eta}(x,\vartheta,\bar{\vartheta})d\vartheta. \tag{17}$$

The one-form Φ is the following:

$$\Phi(x,\vartheta,\bar{\vartheta}) = A(x) + \vartheta D\bar{c}(x) + \bar{\vartheta}Dc(x) + \vartheta\bar{\vartheta}(DB(x) + [Dc(x),\bar{c}(x)]), \tag{18}$$

where D denotes the covariant differential: $Dc = dc + [A, c]$, etc., and the anticommuting functions $\eta, \bar{\eta}$ are the following:

$$\eta(x, \vartheta, \bar{\vartheta}) = c(x) + \vartheta\, \bar{B}(x) - \frac{1}{2} \bar{\vartheta}\, [c(x), c(x)] + \vartheta\bar{\vartheta}\, [\bar{B}(x), c(x)], \tag{19}$$

$$\bar{\eta}(x, \vartheta, \bar{\vartheta}) = \bar{c}(x) - \frac{1}{2} \vartheta\, [\bar{c}(x), \bar{c}(x)] + \bar{\vartheta}\, B(x) + \vartheta\bar{\vartheta}\, [\bar{c}(x), B(x)], \tag{20}$$

together with the condition (13). One can verify that the supercurvature $\widetilde{F}$ satisfies the following horizontality condition:

$$\widetilde{F}(x, \vartheta, \bar{\vartheta}) = d\Phi(x, \vartheta, \bar{\vartheta}) + \frac{1}{2} [\Phi(x, \vartheta, \bar{\vartheta}), \Phi(x, \vartheta, \bar{\vartheta})]. \tag{21}$$

The BRST transformation correspond to $\bar{\vartheta}$ translations and the anti-BRST to ϑ ones:

$$\mathfrak{s} = \frac{\partial}{\partial \bar{\vartheta}}\bigg|_{\vartheta=0}, \qquad \bar{\mathfrak{s}} = \frac{\partial}{\partial \vartheta}\bigg|_{\bar{\vartheta}=0}. \tag{22}$$

At the end of this short review, it is important to highlight an important fact. As anticipated, above the Lagrangian density, (10) is invariant under *both* the BRST and anti-BRST transformations—(3), (11) and (12)—provided that (13) is satisfied. However, while the Lagrangian density contains a specific gauge fixing, the BRST and anti-BRST *algebras* (when they hold) are independent of any gauge-fixing condition. We can change the gauge fixing, but the BRST and anti-BRST algebras (when they are present), as well as their superfield representation, are always the same. These algebras can be considered the *quantum versions* of the original classical gauge algebra. A classical geometrical approach based on fiber bundle geometry was originally proposed in [5,8]. Subsequently, the nature of the BRST transformations was clarified in [10,11]. In fact, it is possible to uncover the BRST algebra in the geometry of principal fiber bundles, particularly in terms of the evaluation map as shown in Appendix A. However, while classical geometry is certainly the base of classical gauge theories, it becomes very cumbersome and actually intractable for perturbative quantum gauge theories. On the other hand, in dealing with the latter, anticommuting ghost and antighost fields and (graded) BRST algebra seem to be the natural tools. Therefore, as noted previously, one may wonder whether the natural language for a quantum gauge field theory is, in fact, the superfield formalism. We leave this idea for future developments.

Here ends our short introduction of the superfield formalism in gauge field theories, which was historically the first application. Later on, we shall see a few of its applications. Now, we would like to explore the possibility to apply this formalism to other local symmetries. The first example, and probably the closest to the one presented in this section, is a theory of gerbes. A gerbe is a mathematical construct, which, in a sense, generalizes the idea of gauge theory. From the field theory point of view, the main difference with the *latter* is that it is not based on a single connection but, besides one-forms, it contains also other forms. Here, we consider the simplest case, an Abelian 1-gerbe; see [44,45].

3. 1-Gerbes

Let us recall a few basic definitions. A 1-gerbe [46–53] is a mathematical object that can be described with a triple (B, A, f), formed by the 2-form B, 1-form A and 0-form f, respectively. These are related in the following way. Given a covering $\{U_i\}$ of the manifold M, we associate to each U_i a 2-form B_i. On a double intersection $U_i \cap U_j$, we have $B_i - B_j = dA_{ij}$. On the triple intersections $U_i \cap U_j \cap U_k$, we must have $A_{ij} + A_{jk} + A_{ki} = df_{ijk}$ (B_i denotes B in U_i, A_{ij}denotes A in $U_i \cap U_j$, etc.). Finally, on the quadruple intersections $U_i \cap U_j \cap U_k \cap U_l$, the following integral cocycle condition must be satisfied by f:

$$f_{ijl} - f_{ijk} + f_{jkl} - f_{ikl} = 2\,\pi\, n, \qquad n = 0, 1, 2, 3\ldots\ldots\ldots \tag{23}$$

This integrality condition does not concern us in our Lagrangian formulation but it has to be imposed as an external condition.

Two triples, represented by (B, A, f) and (B', A', f'), respectively, are said to be gauge equivalent if they satisfy the following relations:

$$B'_i = B_i + dC_i \qquad \text{on} \quad U_i, \tag{24}$$

$$A'_{ij} = A_{ij} + C_i - C_j + d\lambda_{ij} \qquad \text{on} \quad U_i \cap U_j, \tag{25}$$

$$f'_{ijk} = f_{ijk} + \lambda_{ij} + \lambda_{ki} + \lambda_{jk} \qquad \text{on} \quad U_i \cap U_j \cap U_k, \tag{26}$$

for the 1-form C and the 0-form λ.

Let us now define the BRST and anti-BRST transformations corresponding to these geometrical transformations. It should be recalled that, while the above geometric transformations are defined on (multiple) neighborhood overlaps, the BRST and anti–BRST transformations, in quantum field theory, are defined on a single local coordinate patch. These (local, field-dependent) transformations are the means for QFT to record the underlying geometry.

The appropriate BRST and anti-BRST transformations are as follows:

$$\begin{array}{ccc} \mathfrak{s}\, B = dC, & \mathfrak{s}\, A = C + d\lambda, & \mathfrak{s}\, f = \lambda + \mu, \\ \mathfrak{s}\, C = -dh, & \mathfrak{s}\, \lambda = h, & \mathfrak{s}\, \mu = -h, \\ \mathfrak{s}\, \bar{C} = -K, & \mathfrak{s}\, \bar{K} = d\rho, & \mathfrak{s}\, \bar{\mu} = -g, \\ \mathfrak{s}\, \bar{\beta} = -\bar{\rho}, & \mathfrak{s}\, \bar{\lambda} = g, & \mathfrak{s}\, \bar{g} = \rho, \end{array} \tag{27}$$

together with $\mathfrak{s}\, [\rho, \bar{\rho}, g, K_\mu, \beta] = 0$, and

$$\begin{array}{ccc} \bar{\mathfrak{s}}\, B = d\bar{C}, & \bar{\mathfrak{s}}\, A = \bar{C} + d\bar{\lambda}, & \bar{\mathfrak{s}}\, f = \bar{\lambda} + \bar{\mu}, \\ \bar{\mathfrak{s}}\, \bar{C} = +d\bar{h}, & \bar{\mathfrak{s}}\, \bar{\lambda} = -\bar{h}, & \bar{\mathfrak{s}}\, \bar{\mu} = -\bar{h}, \\ \bar{\mathfrak{s}}\, C = +\bar{K}, & \bar{\mathfrak{s}}\, K = -d\bar{\rho}, & \bar{\mathfrak{s}}\, \bar{\mu} = \bar{g}, \\ \bar{\mathfrak{s}}\, \beta = +\rho, & \bar{\mathfrak{s}}\, \lambda = -\bar{g}, & \bar{\mathfrak{s}}\, g = -\bar{\rho}, \end{array} \tag{28}$$

while $\bar{\mathfrak{s}}\, [\bar{\beta}, \bar{g}, \bar{K}, \mu, \rho, \bar{\rho}] = 0$.

In these formulas, $C, \bar{C}$ are anticommuting 1-forms, and $K, \bar{K}$ are commuting 1-forms. The remaining fields are scalars, which are commuting if denoted by Latin letters and anticommuting if denoted by Greek letters.

It can be easily verified that $(\mathfrak{s} + \bar{\mathfrak{s}})^2 = 0$ if the following constraint is satisfied:

$$\bar{K} - K = d\,\bar{g} - d\,g. \tag{29}$$

This condition is both BRST and anti-BRST invariant. It is the analogue of the Curci–Ferrari condition in non-Abelian 1-form gauge theories, and we refer to it with the same name.

Before we proceed to the superfield method, we would like to note that the above realization of the BRST and anti-BRST algebra is not the only possibility. In general, it may be possible to augment it by the addition of a sub-algebra of elements that are all in the kernel of both s and $\bar{s}$, or, if it contains such a sub-algebra, the latter could be moded out. For instance, in Equations (27) and (28), ρ and $\bar{\rho}$ form an example of this type of subalgebra. It is easy to see that ρ and $\bar{\rho}$ can be consistently set equal to 0.

The Superfield Approach to Gerbes

We introduce superfields, whose lowest components are B, A and f.

$$\begin{aligned} \widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}_{MN}(X)dX^M \wedge dX^N &= \mathcal{B}_{\mu\nu}(X)dx^\mu \wedge dx^\nu + \mathcal{B}_{\mu\vartheta}(X)dx^\mu \wedge d\vartheta + \mathcal{B}_{\mu\bar\vartheta}(X)dx^\mu \wedge d\bar\vartheta, \\ &\quad + \mathcal{B}_{\vartheta\vartheta}(X)d\vartheta \wedge d\vartheta + \mathcal{B}_{\bar\vartheta\bar\vartheta}(X)d\bar\vartheta \wedge d\bar\vartheta + \mathcal{B}_{\vartheta\bar\vartheta}(X)d\vartheta \wedge \bar\vartheta, \quad (30) \\ \widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_M(X)dX^M &= \mathcal{A}_\mu(X)dx^\mu + \mathcal{A}_\vartheta(X)d\vartheta + \mathcal{A}_{\bar\vartheta}(X)d\bar\vartheta, \quad (31) \\ \widetilde{\mathsf{f}}(X) &= f(x) + \vartheta\bar\phi(x) + \bar\vartheta\phi(x) + \vartheta\bar\vartheta F(x). \quad (32) \end{aligned}$$

where X denotes the superspace point and $X^M = (x^\mu, \vartheta, \bar\vartheta)$, the superspace coordinates. All the intrinsic components are to be expanded like (32). Then, we impose the horizontality conditions. There are two, which are as follows:

$$\begin{aligned} \tilde d\,\widetilde{\mathcal{B}} &= d\mathcal{B}, & \mathcal{B} &= \mathcal{B}_{\mu\nu}(X)dx^\mu \wedge dx^\nu, \quad (33) \\ \widetilde{\mathcal{B}} - \tilde d\,\widetilde{\mathcal{A}} &= \mathcal{B} - d\mathcal{A}, & \mathcal{A} &= \mathcal{A}_\mu(X)dx^\mu. \quad (34) \end{aligned}$$

The first is suggested by the invariance of $H = dB$ under $B \to B + d\Lambda$, where Λ is a 1-form, and the second by the invariance of $B - dA$ due to the transformations $B \to B + d\Sigma$, $A \to A + \Sigma$, where Σ is also a 1-form.

Using the second, we can eliminate many components of $\widetilde{\mathcal{B}}$ in favor of the components of $\widetilde{\mathcal{A}}$:

$$\begin{aligned} \mathcal{A}_\mu(X) &= A_\mu(x) + \vartheta\,\bar\alpha_\mu(x) + \bar\vartheta\,\alpha_\mu(x) + \vartheta\bar\vartheta\,\mathsf{A}_\mu(x), \quad (35) \\ \mathcal{A}_\vartheta(X) &= \gamma(x) + \vartheta\,\bar e(x) + \bar\vartheta\,e(x) + \vartheta\bar\vartheta\,\Gamma(x), \quad (36) \\ \mathcal{A}_{\bar\vartheta}(X) &= \bar\gamma(x) + \vartheta\,\bar a(x) + \bar\vartheta\,a(x) + \vartheta\bar\vartheta\,\bar\Gamma(x). \quad (37) \end{aligned}$$

Imposing (34), $\widetilde{\mathcal{B}}$ takes the following form:

$$\begin{aligned} \mathcal{B}_{\mu\nu}(X) &= B_{\mu\nu}(x) + \vartheta\,\bar\beta_{\mu\nu}(x) + \bar\vartheta\,\beta_{\mu\nu}(x) + \vartheta\bar\vartheta\,M_{\mu\nu}(x), \quad (38) \\ \mathcal{B}_{\mu\vartheta}(X) &= \big(-\bar\alpha_\mu(x) + \partial_\mu\gamma(x)\big) + \vartheta\,\partial_\mu\bar e(x) + \bar\vartheta\,\big(\partial_\mu e(x) - \mathsf{A}_\mu(x)\big) + \vartheta\bar\vartheta\,\partial_\mu\Gamma(x), \quad (39) \\ \mathcal{B}_{\mu\bar\vartheta}(X) &= \big(-\alpha_\mu(x) + \partial_\mu\bar\gamma(x)\big) + \vartheta\,\partial_\mu\big(\bar a(x) + \mathsf{A}_\mu(x)\big) + \bar\vartheta\,\partial_\mu a(x) + \vartheta\bar\vartheta\,\partial_\mu\bar\Gamma(x), \quad (40) \\ \mathcal{B}_{\vartheta\vartheta}(X) &= \bar e(x) + \bar\vartheta\,\Gamma(x), \quad (41) \\ \mathcal{B}_{\bar\vartheta\bar\vartheta}(X) &= a(x) - \vartheta\,\bar\Gamma(x), \quad (42) \\ \mathcal{B}_{\vartheta\bar\vartheta}(X) &= (e(x) + \bar a(x)) + \bar\vartheta\,\bar\Gamma(x) - \vartheta\Gamma(x), \quad (43) \end{aligned}$$

where all the component fields on the RHSs are so far unrestricted. If we now impose (33), we obtain the following further restrictions:

$$\begin{aligned} \bar\beta_{\mu\nu}(x) &= -(d\beta)_{\mu\nu}(x) = (d\bar\alpha)_{\mu\nu}(x), \quad (44) \\ \beta_{\mu\nu}(x) &= -(d\bar\beta)_{\mu\nu}(x) = (d\alpha)_{\mu\nu}(x), \quad (45) \\ M_{\mu\nu}(x) &= (d\mathsf{A})_{\mu\nu}(x), \quad (46) \end{aligned}$$

where $\beta, \bar\beta, \mathsf{A}$ denote 1-forms with components $\beta_\mu(x), \bar\beta_\mu(x), \mathsf{A}_\mu(x)$, respectively.

We also consider, instead of $\widetilde{\mathcal{A}}$, the superfield $\widetilde{\mathcal{A}} - \tilde d\,\widetilde{\mathsf{f}}$, and, in particular, we replace A with $A' = A - df$.

From the previous equations, we can read off the BRST transformations of the independent component fields. Dropping the argument (x) and using the form notation for the BRST transformations, we have the following:

$$\begin{array}{ccc} \mathfrak{s}B = d\bar\alpha, & \mathfrak{s}A' = \bar\alpha - d\bar\phi, & \mathfrak{s}f = \bar\phi, \\ \mathfrak{s}\alpha = -d\mathsf{A}, & \mathfrak{s}\gamma = \bar e, & \mathfrak{s}e = -\Gamma, \\ \mathfrak{s}\bar\gamma = \bar a, & \mathfrak{s}a = -\bar\Gamma, & \mathfrak{s}\phi = -F, \end{array} \quad (47)$$

all the other $\mathfrak{s}$ transformations being *trivial*. For the anti-BRST transformations, we have the following:

$$
\begin{array}{ccc}
\bar{\mathfrak{s}}B = d\alpha, & \bar{\mathfrak{s}}A' = \alpha - d\phi, & \bar{\mathfrak{s}}f = \phi, \\
\bar{\mathfrak{s}}\bar{\alpha} = \mathcal{A}, & \bar{\mathfrak{s}}\bar{\gamma} = a, & \bar{\mathfrak{s}}\bar{a} = \bar{\Gamma}, \\
\bar{\mathfrak{s}}\gamma = e, & \bar{\mathfrak{s}}\bar{e} = \bar{\Gamma}, & \bar{\mathfrak{s}}\bar{\phi} = F,
\end{array} \tag{48}
$$

All the other anti-BRST transformations are *trivial*.

The system (47) and (48) differs from (27) and (28) only by field redefinitions. Let us set the following:

$$
C = \bar{\alpha} + d\gamma, \qquad \lambda = -\gamma - \bar{\phi}, \tag{49}
$$

$$
\overline{C} = \alpha - d\bar{\gamma}, \qquad \bar{\lambda} = \bar{\gamma} - \phi. \tag{50}
$$

Then, the first equation of (47) and the first of (48) become the following:

$$
\begin{array}{ccc}
\mathfrak{s}B = dC, & \mathfrak{s}A' = C + d\lambda, & \mathfrak{s}f = \lambda + \gamma, \\
\bar{\mathfrak{s}}\bar{B} = d\overline{C}, & \bar{\mathfrak{s}}A' = \overline{C} + d\bar{\lambda}, & \bar{\mathfrak{s}}f = \bar{\lambda} - \bar{\gamma}.
\end{array} \tag{51}
$$

Next, we define the following:

$$
K_\mu = \mathcal{A}_\mu + \partial_\mu \bar{a}, \qquad \overline{K}_\mu = \mathcal{A}_\mu + \partial_\mu e. \tag{52}
$$

The remaining s and $\bar{s}$ transformations become the following:

$$
\begin{array}{ccc}
\mathfrak{s}C = d\bar{e}, & \mathfrak{s}\lambda = -\bar{e}, & \mathfrak{s}\gamma = \bar{e}, \\
\mathfrak{s}\overline{C} = -K, & \mathfrak{s}\overline{K} = -d\Gamma, & \mathfrak{s}\bar{\gamma} = \bar{a}, \\
\mathfrak{s}a = -\overline{\Gamma}, & \mathfrak{s}\bar{\lambda} = \bar{a} + F, & \mathfrak{s}a = -\bar{\Gamma},
\end{array} \tag{53}
$$

and

$$
\begin{array}{ccc}
\bar{\mathfrak{s}}\overline{C} = -da, & \bar{\mathfrak{s}}\bar{\lambda} = a, & \bar{\mathfrak{s}}\bar{\gamma} = a, \\
\bar{\mathfrak{s}}C = \overline{K}, & \bar{\mathfrak{s}}K = -d\overline{\Gamma}, & \bar{\mathfrak{s}}\gamma = e, \\
\bar{\mathfrak{s}}\bar{e} = \Gamma, & \bar{\mathfrak{s}}\lambda = -e - F, & \bar{\mathfrak{s}}\bar{a} = \bar{\Gamma}.
\end{array} \tag{54}
$$

Moreover, we have the following CF-like condition:

$$
\overline{K} - K = d(e - \bar{a}). \tag{55}
$$

These relations coincide with those of the 1-gerbe, provided that we make the following replacements: $\gamma \to \mu$, $\bar{\gamma} \to -\bar{\mu}$, $a \to -\bar{\beta}$, $\bar{a} \to g$, $e \to \bar{g}$, $\bar{e} \to -\beta$ and $\Gamma \to -\rho$, $\overline{\Gamma} \to \bar{\rho}$.

There is only one difference: the presence of F in two cases in the last lines of both (53) and (54). This is an irrelevant term, as it belongs to the kernel of both s and $\bar{s}$.

Remark 1. *One can also impose the horizontality condition $\widetilde{\mathcal{A}} - \tilde{d}\tilde{\mathsf{f}} = \mathcal{A} - d\,\mathsf{f}$, but this does not change much the final result: in fact, the resulting 1-gerbe algebra is the same.*

4. Diffeomorphisms and the Superfield Formalism

After the successful extension of the superfield formalism to gerbes, we wish to deal with an entirely different type of symmetry: the diffeomorphisms. Our aim is to answer a few questions:

- Is the superfield formalism applicable to diffeomorphisms?
- What are the horizontality conditions for the latter?
- What are the CF conditions?
- Can we generalize the Riemannian geometry to the superspace?

In the sequel, we will answer all these questions. The answer to the last question will be partly negative, because an inverse supermetric does not exist. Nevertheless, it is possible to develop a superfield formalism in the horizontal (commuting) directions.

The first proposal of a superfield formalism for diffeomorphisms was made by [54–56]. Here, we present another approach, presented in [57], closer in spirit to the standard (commutative) geometrical approach.

Diffeomorphisms, or general coordinate transformations, are given in terms of generic (smooth) functions of x^μ:

$$x^\mu \to x'^\mu = f^\mu(x).$$

An infinitesimal diffeomorphism is defined by means of a local parameter $\xi^\mu(x)$: $f^\mu(x) = x^\mu - \xi^\mu(x)$. In a quantized theory, this is promoted to an anticommuting field, and the BRST transformations for a scalar field, a vector field, the metric and ξ, respectively, are the following:

$$\delta_\xi \varphi = \xi^\lambda \partial_\lambda \varphi, \tag{56}$$
$$\delta_\xi A_\mu = \xi^\lambda \partial_\lambda A_\mu + \partial_\mu \xi^\lambda A_\lambda, \tag{57}$$
$$\delta_\xi g_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \xi^\lambda g_{\lambda\nu} + \partial_\nu \xi^\lambda g_{\mu\lambda}, \tag{58}$$
$$\delta_\xi \xi^\mu = \xi^\lambda \partial_\lambda \xi^\mu, \tag{59}$$

It is easy to see that these transformations are nilpotent. We wish now to define the analogs of anti-BRST transformations. To this end, we introduce another anticommuting field, $\bar\xi$, and a $\delta_{\bar\xi}$ transformation, which transforms a scalar, vector, the metric and $\bar\xi$ in just the same way as δ_ξ (these transformations are not rewritten here). In addition, we have the following cross-transformations:

$$\delta_\xi \bar\xi^\mu = b^\mu, \qquad \delta_{\bar\xi} \xi^\mu = \bar b^\mu, \tag{60}$$
$$\delta_\xi b^\mu = 0, \qquad \delta_\xi \bar b^\mu = -\bar b \cdot \partial \xi^\mu + \xi \cdot \partial \bar b^\mu, \tag{61}$$
$$\delta_{\bar\xi} \bar b^\mu = 0, \qquad \delta_{\bar\xi} b^\mu = -b \cdot \partial \bar\xi^\mu + \bar\xi \cdot \partial b^\mu, \tag{62}$$

It follows that the overall transformation $\delta_\xi + \delta_{\bar\xi}$ is nilpotent:

$$(\delta_\xi + \delta_{\bar\xi})^2 = 0.$$

4.1. The Superfield Formalism

Our aim now is to reproduce the above transformations by means of the superfield formalism. The superspace coordinates are $X^M = (x^\mu, \vartheta, \bar\vartheta)$, where $\vartheta, \bar\vartheta$ are the same anticommuting variables as above. A (super)diffeomorphism is represented by a superspace transformation $X^M = (x^\mu, \vartheta, \bar\vartheta) \to \tilde X^M = (F^\mu(X^M), \vartheta, \bar\vartheta)$, where[3],

$$F^\mu(X^M) = f^\mu(x) - \vartheta\, \bar\xi^\mu - \bar\vartheta\, \xi^\mu(x) + \vartheta\bar\vartheta\, h^\mu(x). \tag{63}$$

Here, $f^\mu(x)$ is an ordinary diffeomorphism, $\xi, \bar\xi$ are the generic anticommuting functions introduced before, and h^μ is a generic commuting one.

The horizontality condition is formulated by selecting appropriate *invariant geometric expressions* in ordinary spacetime and identifying them with the same expressions extended to the superspace. To start, we work out explicitly the case of a scalar field.

4.2. The Scalar

The diffeomorphism transformation properties of an ordinary scalar field are as follows:

$$\tilde\varphi(f^\mu(x)) = \varphi(x^\mu). \tag{64}$$

Now, we embed the scalar field φ in a superfield[4]

$$\Phi(X) = \varphi(x) + \vartheta\,\bar{\beta}(x) + \bar{\vartheta}\,\beta(x) + \vartheta\bar{\vartheta}\,C(x), \tag{65}$$

The BRST interpretation is $\delta_\xi = \frac{\partial}{\partial\vartheta}\Big|_{\vartheta=0}, \delta_{\bar{\xi}} = \frac{\partial}{\partial\bar{\vartheta}}\Big|_{\bar{\vartheta}=0}$. The horizontality condition, suggested by (64), is the following:

$$\Phi(F(X)) = \varphi(x). \tag{66}$$

Using (63) with $f(x^\mu) = x^\mu$, this becomes the following:

$$\begin{aligned} \Phi(F(X)) &= \varphi(x) - (\vartheta\,\bar{\xi}(x) + \bar{\vartheta}\,\xi(x) - \vartheta\bar{\vartheta}\,h(x))\cdot\partial\varphi(x) + \vartheta\big(\bar{\beta}(x) - \bar{\vartheta}\xi(x)\cdot\partial\bar{\beta}(x)\big) \\ &\quad + \bar{\vartheta}\big(\beta(x) - \vartheta\bar{\xi}\cdot\partial\beta(x)\big) + \vartheta\bar{\vartheta}\big(C(x) - \bar{\xi}^\mu\xi^\nu\partial_\mu\partial_\nu\varphi(x)\big) \\ &= \varphi(x) + \vartheta\big(\bar{\beta}(x) - \bar{\xi}\cdot\partial\varphi(x)\big) + \bar{\vartheta}\big(\beta(x) - \xi\cdot\partial\varphi(x)\big) \\ &\quad + \vartheta\bar{\vartheta}\big(C(x) - \xi\cdot\partial\bar{\beta}(x) + \bar{\xi}\cdot\partial\beta(x) + h(x)\cdot\partial\varphi(x) - \bar{\xi}^\mu\xi^\nu\partial_\mu\partial_\nu\varphi(x)\big), \end{aligned} \tag{67}$$

where $\cdot$ denotes index contraction. Then, (66) implies the following:

$$\begin{aligned} \beta(x) &= \xi\cdot\partial\varphi(x), \qquad \bar{\beta}(x) = \bar{\xi}\cdot\partial\varphi(x), \\ C(x) &= \xi\cdot\partial\bar{\beta}(x) - \bar{\xi}\cdot\partial\beta(x) - \xi\bar{\xi}\partial^2\varphi(x) - h(x)\cdot\partial\varphi(x), \end{aligned} \tag{68}$$

where $\xi\bar{\xi}\partial^2\varphi(x) = \xi^\mu\bar{\xi}^\nu\partial_\mu\partial_\nu\varphi(x)$.

Now, the BRST interpretation implies the following:

$$\delta_{\bar{\xi}}\varphi(x) = \bar{\beta}(x) = \bar{\xi}\cdot\partial\varphi(x), \qquad \delta_\xi\varphi(x) = \beta(x) = \xi\cdot\partial\varphi(x), \tag{69}$$

and $\delta_\xi\bar{\beta}(x) = C(x), \delta_{\bar{\xi}}\beta(x) = -C(x)$.

Inserting β and $\bar{\beta}$ into C in (68), we obtain the following:

$$\delta_\xi\delta_{\bar{\xi}}\varphi = b\cdot\partial\varphi - \bar{\xi}\cdot\partial\xi\cdot\partial\varphi - \bar{\xi}\xi\partial^2\varphi. \tag{70}$$

This coincides with the expression of C, (68), if

$$h^\mu = -b^\mu + \xi\cdot\partial\bar{\xi}^\mu. \tag{71}$$

Likewise,

$$\delta_{\bar{\xi}}\delta_\xi\varphi = \bar{b}\cdot\partial\varphi - \xi\cdot\partial\bar{\xi}\cdot\partial\varphi - \xi\bar{\xi}\partial^2\varphi, \tag{72}$$

which coincides with the expression of $-C$, (68), if the following holds:

$$h^\mu = \bar{b}^\mu - \bar{\xi}\cdot\partial\xi^\mu. \tag{73}$$

Equating (71) with (73) we obtain the following:

$$h^\mu(x) = -b^\mu(x) + \xi(x)\cdot\partial\bar{\xi}^\mu(x) = \bar{b}^\mu(x) - \bar{\xi}(x)\cdot\partial\xi^\mu(x), \tag{74}$$

which is possible if and only if the following CF condition is satisfied:

$$b^\mu + \bar{b}^\mu = \xi^\lambda\partial_\lambda\bar{\xi}^\mu + \bar{\xi}^\lambda\partial_\lambda\xi^\mu,$$

This condition is consistent, for applying δ_ξ and $\delta_{\bar{\xi}}$ to both sides produces the same result. As we shall see, this condition is, so to speak, universal: it appears whenever BRST and anti-BRST diffeomorphisms are involved, and it is the only required condition.

4.3. The Vector

We now extend the previous approach to a vector field. In order to apply the horizontality condition, we must first identify the appropriate expression. This is a 1-superform:

$$\mathbb{A} \equiv \mathcal{A}_M(X)dX^M = A_\mu(X)dx^\mu + A_\vartheta(X)d\vartheta + A_{\bar{\vartheta}}(X)d\bar{\vartheta}, \tag{75}$$

where

$$\begin{aligned} \mathcal{A}_\mu(X) &= A_\mu(x) + \vartheta\,\bar{\phi}_\mu(x) + \bar{\vartheta}\phi_\mu(x) + \vartheta\bar{\vartheta}B_\mu(x), & (76)\\ \mathcal{A}_\vartheta(X) &= \chi(x) + \vartheta\bar{C}(x) + \bar{\vartheta}C(x) + \vartheta\bar{\vartheta}\psi(x), & (77)\\ \mathcal{A}_{\bar{\vartheta}}(X) &= \omega(x) + \vartheta\bar{D}(x) + \bar{\vartheta}D(x) + \vartheta\bar{\vartheta}\rho(x). & (78) \end{aligned}$$

According to our prescription, horizontality means the following:

$$\mathcal{A}_M(\tilde{X})\tilde{d}\tilde{X}^M = A_\mu(x)dx^\mu, \tag{79}$$

where $\tilde{d} = \frac{\partial}{\partial x^\mu}dx^\mu + \frac{\partial}{\partial\vartheta}d\vartheta + \frac{\partial}{\partial\bar{\vartheta}}d\bar{\vartheta}$. Thus, we obtain the following:

$$\begin{aligned} \tilde{d}\tilde{X}^M &= \Big(dx^\mu - \vartheta\,\partial_\lambda\bar{\xi}^\mu dx^\lambda - \bar{\vartheta}\,\partial_\lambda\xi^\mu dx^\lambda + \vartheta\bar{\vartheta}\partial_\lambda h^\mu dx^\lambda \\ &\qquad -(\bar{\xi}^\mu - \bar{\vartheta}h^\mu)d\vartheta - (\xi^\mu + \vartheta h^\mu)d\bar{\vartheta},\, d\vartheta,\, d\bar{\vartheta}\Big). \end{aligned} \tag{80}$$

It remains for us to expand the LHS of (79). The explicit expression can be found in Appendix B. The commutation prescriptions are the following: $x^\mu, \vartheta, \bar{\vartheta}, \xi^\mu$ commute with dx^μ and $d\vartheta, d\bar{\vartheta}$; and $\xi^\mu, \bar{\xi}^\mu$ anticommute with $\vartheta, \bar{\vartheta}$. From (A21), we obtain the following identifications:

$$\begin{aligned} \phi_\mu &= \xi\cdot\partial A_\mu + \partial_\mu\xi^\lambda A_\lambda, & (81)\\ \bar{\phi}_\mu &= \bar{\xi}\cdot\partial A_\mu + \partial_\mu\bar{\xi}^\lambda A_\lambda, & (82)\\ B_\mu &= \xi\cdot\partial\bar{\phi}_\mu - \bar{\xi}\cdot\partial\phi_\mu - \xi\bar{\xi}\cdot\partial^2 A_\mu + \partial_\mu\bar{\xi}^\lambda\xi\cdot\partial A_\lambda - \partial_\mu\xi^\lambda\bar{\xi}\cdot\partial A_\lambda \\ &\quad -\partial_\mu\bar{\xi}^\lambda\phi_\lambda + \partial_\mu\xi^\lambda\bar{\phi}_\lambda - h\cdot\partial A_\mu - \partial_\mu h\cdot A, & (83) \end{aligned}$$

$$\begin{aligned} \chi &= A_\mu\bar{\xi}^\mu, & (84)\\ C &= -\xi\cdot\partial A_\mu\bar{\xi}^\mu + \phi_\mu\bar{\xi}^\mu + \xi\cdot\partial\chi - h\cdot A, & (85)\\ \bar{C} &= -\bar{\xi}\cdot\partial A_\mu\bar{\xi}^\mu + \bar{\phi}_\mu\bar{\xi}^\mu + \bar{\xi}\cdot\partial\chi, & (86)\\ \psi &= \xi\bar{\xi}\cdot\partial^2 A_\mu\bar{\xi}^\mu - \xi\cdot\partial\bar{\phi}_\mu\bar{\xi}^\mu + \bar{\xi}\cdot\partial\phi_\mu\bar{\xi}^\mu + B_\mu\bar{\xi}^\mu - \xi\bar{\xi}\cdot\partial^2\chi + \xi\cdot\partial\bar{C} - \bar{\xi}\cdot\partial C & (87)\\ &\quad -\bar{\xi}\cdot\partial A\cdot h + \bar{\phi}\cdot h - h\cdot\partial\chi + h\cdot\partial A_\mu\bar{\xi}^\mu, \end{aligned}$$

and

$$\begin{aligned} \omega &= A_\mu\xi^\mu, & (88)\\ D &= -\xi\cdot\partial A_\mu\xi^\mu + \phi_\mu\xi^\mu + \xi\cdot\partial\omega, & (89)\\ \bar{D} &= -\bar{\xi}\cdot\partial A_\mu\xi^\mu + \bar{\phi}_\mu\xi^\mu + \bar{\xi}\cdot\partial\omega + h\cdot A, & (90)\\ \rho &= \xi\bar{\xi}\cdot\partial^2 A_\mu\xi^\mu - \xi\cdot\partial\bar{\phi}_\mu\xi^\mu + \bar{\xi}\cdot\partial\phi_\mu\xi^\mu + B_\mu\xi^\mu - \xi\bar{\xi}\cdot\partial^2\omega + \xi\cdot\partial\bar{D} - \bar{\xi}\cdot\partial D. & (91)\\ &\quad -\xi\cdot\partial A\cdot h + \phi\cdot h - h\cdot\partial\omega + h\cdot\partial A_\mu\xi^\mu. \end{aligned}$$

One can see that

$$\phi_\mu = \delta_\xi A_\mu, \qquad \bar{\phi}_\mu = \delta_{\bar{\xi}} A_\mu, \qquad B_\mu = \delta_{\bar{\xi}}\phi_\mu = -\delta_\xi\bar{\phi}_\mu. \tag{92}$$

$$D = \delta_\xi\omega, \qquad \bar{D} = \delta_{\bar{\xi}}\omega, \qquad \rho = -\delta_{\bar{\xi}} D = \delta_\xi\bar{D}, \tag{93}$$

and

$$C = \delta_{\xi}\chi, \qquad \bar{C} = \delta_{\bar{\xi}}\chi, \qquad \psi = -\delta_{\bar{\xi}}C = \delta_{\xi}\bar{C}, \tag{94}$$

provided

$$h^{\mu}(x) = -b^{\mu}(x) + \xi(x)\cdot\partial\bar{\xi}^{\mu}(x) = \bar{b}^{\mu}(x) - \bar{\xi}(x)\cdot\partial\xi^{\mu}(x), \tag{95}$$

which is possible if and only if the following CF condition is satisfied:

$$b^{\mu} + \bar{b}^{\mu} = \xi^{\lambda}\partial_{\lambda}\bar{\xi}^{\mu} + \bar{\xi}^{\lambda}\partial_{\lambda}\xi^{\mu},$$

In particular, ρ can be rewritten as follows:

$$\begin{aligned} \rho \quad = \quad & \xi\cdot\partial\bar{\xi}\cdot\partial A_{\mu}\xi^{\mu} - \bar{\xi}\cdot\partial\xi\cdot\partial A_{\mu}\xi^{\mu} + \bar{\xi}\cdot\partial\xi^{\mu}\,\xi\cdot\partial A_{\mu}\xi^{\mu} \\ & -\xi\cdot\partial\bar{\xi}^{\mu}\,\xi\cdot\partial A_{\mu}\xi^{\mu} + \xi\bar{\xi}\cdot\partial^{2}A_{\mu}\xi^{\mu} - \xi\cdot\partial A\cdot h + \phi\cdot h - h\cdot\partial\omega. \end{aligned}$$

4.4. The Metric

The most important field for theories invariant under diffeomorphisms is the metric $g_{\mu\nu}(x)$. To represent its BRST transformation properties in the superfield formalism, we embed it in a supermetric $G_{MN}(X)$ and form the symmetric 2-superdifferential as follows:

$$\mathbb{G} = G_{MN}(X)\tilde{d}X^{M} \vee \tilde{d}X^{N}, \tag{96}$$

where $\vee$ denotes the symmetric tensor product and

$$\begin{aligned} G_{\mu\nu}(X) &= g_{\mu\nu}(x) + \vartheta\,\bar{\Gamma}_{\mu\nu}(x) + \bar{\vartheta}\Gamma_{\mu\nu}(x) + \vartheta\bar{\vartheta}V_{\mu\nu}(x), \\ G_{\mu\vartheta}(X) &= \gamma_{\mu}(x) + \vartheta\,\bar{g}_{\mu}(x) + \bar{\vartheta}g_{\mu}(x) + \vartheta\bar{\vartheta}\Gamma_{\mu}(x) = G_{\vartheta\mu}(X), \\ G_{\mu\bar{\vartheta}}(X) &= \bar{\gamma}_{\mu}(x) + \vartheta\,\bar{f}_{\mu}(x) + \bar{\vartheta}f_{\mu}(x) + \vartheta\bar{\vartheta}\bar{\Gamma}_{\mu}(x) = G_{\bar{\vartheta}\mu}(X), \end{aligned} \tag{97}$$

$$\frac{1}{2}G_{\vartheta\bar{\vartheta}}(X) = g(x) + \vartheta\,\bar{\gamma}(x) + \bar{\vartheta}\gamma(x) + \vartheta\bar{\vartheta}G(x) = -\frac{1}{2}G_{\bar{\vartheta}\vartheta}(X), \tag{98}$$

while $G_{\vartheta\vartheta}(X) = 0 = G_{\bar{\vartheta}\bar{\vartheta}}(X)$, because the symmetric tensor product becomes antisymmetric for anticommuting variables: $d\vartheta \vee d\vartheta = 0 = d\bar{\vartheta} \vee d\bar{\vartheta}, d\vartheta \vee d\bar{\vartheta} = -d\bar{\vartheta} \vee d\vartheta$.

The horizontality condition is obtained by requiring the following:

$$\widetilde{G}_{MN}(\tilde{X})\tilde{d}\tilde{X}^{M} \vee \tilde{d}\tilde{X}^{N} = g_{\mu\nu}(x)dx^{\mu} \vee dx^{\nu}. \tag{99}$$

The explicit expression of the LHS of this equation can be found again in Appendix B, from which the following identification follows:

$$\Gamma_{\mu\nu} = \xi\cdot\partial g_{\mu\nu} + \partial_{\mu}\xi^{\lambda}g_{\lambda\nu} + \partial_{\nu}\xi^{\lambda}g_{\mu\lambda} = \delta_{\xi}g_{\mu\nu}, \tag{100}$$

$$\bar{\Gamma}_{\mu\nu} = \bar{\xi}\cdot\partial g_{\mu\nu} + \partial_{\mu}\bar{\xi}^{\lambda}g_{\lambda\nu} + \partial_{\nu}\bar{\xi}^{\lambda}g_{\mu\lambda} = \delta_{\bar{\xi}}g_{\mu\nu}, \tag{101}$$

$$\begin{aligned} V_{\mu\nu} \quad = \quad & -\xi\bar{\xi}\cdot\partial^{2}g_{\mu\nu} + \xi\cdot\partial\bar{\Gamma}_{\mu\nu} + \partial_{\mu}\xi^{\lambda}\bar{\Gamma}_{\lambda\nu} + \partial_{\nu}\xi^{\lambda}\bar{\Gamma}_{\mu\lambda} - \bar{\xi}\cdot\partial\Gamma_{\mu\nu} - \partial_{\mu}\bar{\xi}^{\lambda}\Gamma_{\lambda\nu} - \partial_{\nu}\bar{\xi}^{\lambda}\Gamma_{\mu\lambda} \\ & +\partial_{\mu}\bar{\xi}^{\lambda}\xi\cdot\partial g_{\lambda\nu} + \partial_{\nu}\bar{\xi}^{\lambda}\xi\cdot\partial g_{\mu\lambda} - \partial_{\mu}\xi^{\lambda}\bar{\xi}\cdot\partial g_{\lambda\nu} - \partial_{\nu}\xi^{\lambda}\bar{\xi}\cdot\partial g_{\mu\lambda} + \partial_{\mu}\bar{\xi}^{\lambda}\partial_{\nu}\xi^{\rho}g_{\lambda\rho} \\ & +\partial_{\nu}\bar{\xi}^{\lambda}\partial_{\mu}\xi^{\rho}g_{\lambda\rho} - h\cdot\partial g_{\mu\nu} - \partial_{\mu}h^{\lambda}g_{\lambda\nu} - \partial_{\nu}h^{\lambda}g_{\mu\lambda} \\ = \quad & \delta_{\xi}\bar{\Gamma}_{\mu\nu} = -\delta_{\bar{\xi}}\Gamma_{\mu\nu}. \end{aligned} \tag{102}$$

Moreover,

$$\begin{aligned}
\gamma_\mu &= g_{\mu\nu}\bar\xi^\nu, && (103)\\
g_\mu &= \partial_\mu\xi^\lambda g_{\lambda\nu}\bar\xi^\nu + \partial_\nu\xi^\lambda g_{\mu\lambda}\bar\xi^\nu + \xi\cdot\partial g_{\mu\nu}\bar\xi^\nu + g_{\mu\nu}\xi\cdot\partial\bar\xi^\nu - g_{\mu\nu}h^\nu = \delta_\xi\gamma_\mu, && (104)\\
\bar g_\mu &= \bar\xi\cdot\partial g_{\mu\nu}\bar\xi^\nu + \partial_\mu\bar\xi^\lambda g_{\lambda\nu}\bar\xi^\nu = \delta_{\bar\xi}\gamma_\mu, && (105)\\
\Gamma_\mu &= -\xi\bar\xi\cdot\partial^2\gamma_\mu + \xi\cdot\partial\bar g_\mu - \partial_\mu\bar\xi^\lambda g_\lambda - \bar\xi\cdot\partial g_\mu + \partial_\mu\xi^\lambda\bar g_\lambda + \partial_\mu\bar\xi^\lambda\xi\cdot\partial\gamma_\lambda - \partial_\mu\xi^\lambda\bar\xi\cdot\partial\gamma_\lambda\\
&\quad -\bar\xi\cdot\partial h^\nu g_{\mu\nu} - \partial_\mu h^\lambda g_{\lambda\nu}\bar\xi^\nu + \bar\Gamma_{\mu\nu}h^\nu - h\cdot\partial\gamma_\mu - g_{\nu\lambda}h^\nu\partial_\mu\bar\xi^\lambda\\
&\quad +\partial_\nu\xi^\lambda\bar\Gamma_{\mu\lambda}\bar\xi^\nu - \partial_\nu\bar\xi^\lambda\Gamma_{\mu\lambda}\bar\xi^\nu + \bar\xi\cdot\partial\bar\xi^\lambda\,\xi\cdot\partial g_{\mu\lambda}\\
&\quad -\bar\xi\cdot\partial\xi^\lambda\,\bar\xi\cdot\partial g_{\mu\lambda} - \bar\xi\cdot\partial\xi^\lambda\, g_{\lambda\rho}\partial_\mu\bar\xi^\rho + \bar\xi\cdot\partial\bar\xi^\lambda\, g_{\lambda\rho}\partial_\mu\xi^\rho = -\delta_{\bar\xi}g_\mu \;=\; \delta_\xi\bar g_\mu, && (106)
\end{aligned}$$

and

$$\begin{aligned}
\bar\gamma_\mu &= g_{\mu\nu}\xi^\nu, && (107)\\
\bar f_\mu &= \partial_\mu\bar\xi^\lambda g_{\lambda\nu}\xi^\nu + \partial_\nu\bar\xi^\lambda g_{\mu\lambda}\xi^\nu + \bar\xi\cdot\partial g_{\mu\nu}\xi^\nu + g_{\mu\nu}\bar\xi\cdot\partial\xi^\nu + g_{\mu\nu}h^\nu = \delta_{\bar\xi}\bar\gamma_\mu, && (108)\\
f_\mu &= \xi\cdot\partial g_{\mu\nu}\xi^\nu + \partial_\mu\xi^\lambda g_{\lambda\nu}\xi^\nu = \delta_\xi\bar\gamma_\mu, && (109)\\
\bar\Gamma_\mu &= -\xi\cdot\partial g_{\mu\nu}h^\nu - \partial_\mu h^\lambda g_{\lambda\nu}\xi^\nu + \Gamma_{\mu\nu}h^\nu - h\cdot\partial\bar\gamma_\mu - \partial_\mu h^\lambda\bar\gamma_\lambda - \xi\cdot\partial h^\lambda g_{\lambda\mu}\\
&\quad +\partial_\mu\xi_\lambda\bar f^\lambda + \xi\cdot\partial\bar f_\mu - \bar\xi\cdot\partial f_\mu - \xi^\lambda\bar\xi^\rho\partial_\lambda\partial_\rho\bar\gamma_\mu - \partial_\mu\bar\xi^\lambda\partial_\lambda\xi^\rho\bar\gamma_\rho\\
&\quad -\bar\xi\cdot\partial\bar\gamma_\lambda\partial_\mu\xi^\lambda + \partial_\mu\bar\xi^\lambda\xi\cdot\partial\xi^\rho g_{\lambda\rho} - \xi\cdot\partial\bar\xi^\rho\partial_\rho\xi^\lambda g_{\lambda\mu} + \xi\cdot\partial\xi^\rho\partial_\rho\bar\xi^\lambda g_{\lambda\mu}\\
&= \delta_\xi\bar f_\mu \;=\; -\delta_{\bar\xi}f_\mu. && (110)
\end{aligned}$$

Finally, we obtain the following:

$$\begin{aligned}
g &= g_{\mu\nu}\big(\bar\xi^\mu\xi^\nu - \xi^\mu\bar\xi^\nu\big) = 2g_{\mu\nu}\bar\xi^\mu\xi^\nu, && (111)\\
\gamma &= 2\xi\cdot\partial g_{\mu\nu}\bar\xi^\mu\xi^\nu + 2g_{\mu\nu}\big(\xi\cdot\partial\bar\xi^\mu - \bar\xi\cdot\partial\xi^\mu\big)\xi^\nu - 2\bar\gamma_\mu h^\mu = \delta_\xi g, && (112)\\
\bar\gamma &= 2\bar\xi\cdot\partial g_{\mu\nu}\bar\xi^\mu\xi^\nu + 2g_{\mu\nu}\big(\xi\cdot\partial\bar\xi^\mu - \bar\xi\cdot\partial\xi^\mu\big)\bar\xi^\nu - 2\gamma_\mu h^\mu = \delta_{\bar\xi}g, && (113)\\
G &= 2b^\mu\partial_\mu g + 2b^\mu g_\mu - 2\bar\xi\cdot\partial\gamma - 2\bar\xi\cdot\partial\bar\xi^\mu f_\mu - 2b\cdot\partial\bar\xi^\rho\bar\gamma_\rho - 2\bar\xi\cdot\partial b^\rho\bar\gamma_\rho = \delta_\xi\bar\gamma \;=\; -\delta_{\bar\xi}\gamma. && (114)
\end{aligned}$$

This completes the verification of the horizontality condition. As expected, it leads to identifying the $\bar\vartheta$-and ϑ-superpartners of the metric as BRST and anti-BRST transforms.

4.5. Inverse of $G_{\mu\nu}(X)$

A fundamental ingredient of Riemannian geometry is the inverse metric. Therefore, in order to see whether a super-Riemannian geometry can be introduced in the supermanifold, we have to verify whether an inverse supermetric exists. We start by the inverse of $G_{\mu\nu}(X)$, which is defined by first writing it as follows:

$$G_{\mu\nu}(X) \;=\; g_{\mu\lambda}(x)\Big[\delta^\lambda_\nu + g^{\lambda\rho}\big(\vartheta\,\bar\Gamma_{\rho\nu}(x) + \bar\vartheta\Gamma_{\rho\nu}(x) + \vartheta\bar\vartheta V_{\rho\nu}(x)\big)\Big] \equiv g_{\mu\lambda}(x)(1+X)^\lambda{}_\nu,$$

then in matrix terms as follows:

$$\widehat G^{-1} = \Big(1 - X + X^2\Big)\hat g^{-1},$$

where $\hat g^{-1}$ is the inverse of g, i.e., the following:

$$\begin{aligned}
\widehat G^{\mu\nu} &= \Big(1 - X + X^2\Big)^\mu{}_\lambda\,\hat g^{\lambda\nu} && (115)\\
&= \big(\delta^\mu_\lambda - \hat g^{\mu\rho}\big(\vartheta\,\bar\Gamma_{\rho\lambda}(x) + \bar\vartheta\Gamma_{\rho\lambda}(x) + \vartheta\bar\vartheta V_{\rho\lambda}(x)\big) + \vartheta\bar\vartheta\,\hat g^{\mu\rho}\big(\Gamma_{\rho\sigma}g^{\sigma\tau}\bar\Gamma_{\tau\lambda} - \bar\Gamma_{\rho\sigma}g^{\sigma\tau}\Gamma_{\tau\lambda}\big)\big)\hat g^{\lambda\nu}\\
&\equiv \hat g^{\mu\nu}(x) + \vartheta\,\widehat{\bar\Gamma}^{\mu\nu}(x) + \bar\vartheta\widehat\Gamma^{\mu\nu}(x) + \vartheta\bar\vartheta\widehat V^{\mu\nu}(x),
\end{aligned}$$

and $\hat{g}^{\mu\nu}$ is the ordinary metric inverse. Moreover,

$$\begin{aligned}
\widehat{\Gamma}^{\mu\nu} &= -\hat{g}^{\mu\lambda}\Gamma_{\lambda\rho}\hat{g}^{\rho\nu}, &(116)\\
\widehat{\bar{\Gamma}}^{\mu\nu} &= -\hat{g}^{\mu\lambda}\bar{\Gamma}_{\lambda\rho}\hat{g}^{\rho\nu}, &(117)\\
\widehat{V}^{\mu\nu} &= \hat{g}^{\mu\lambda}\left(-V_{\lambda\rho}+\Gamma_{\lambda\sigma}\hat{g}^{\sigma\tau}\bar{\Gamma}_{\tau\rho}-\bar{\Gamma}_{\lambda\sigma}\hat{g}^{\sigma\tau}\Gamma_{\tau\rho}\right)\hat{g}^{\rho\nu}. &(118)
\end{aligned}$$

This contains the correct BRST transformation properties. For instance, we have the following:

$$\begin{aligned}
\widehat{\Gamma}^{\mu\nu} &= -\hat{g}^{\mu\lambda}\left(\xi\cdot\partial g_{\lambda\rho}+\partial_\lambda\xi^\tau g_{\tau\rho}+\partial_\rho\xi^\tau g_{\lambda\tau}\right)\hat{g}^{\rho\nu} &(119)\\
&= \xi\cdot\partial\hat{g}^{\mu\nu}-\partial_\lambda\xi^\mu\hat{g}^{\lambda\nu}-\partial_\lambda\xi^\nu\hat{g}^{\nu\lambda}=\delta_\xi\hat{g}^{\mu\nu}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\widehat{\bar{\Gamma}}^{\mu\nu} &= \delta_{\bar{\xi}}\hat{g}^{\mu\nu}, &(120)\\
\widehat{V}^{\mu\nu} &= \delta_\xi\delta_{\bar{\xi}}\hat{g}^{\mu\nu}. &(121)
\end{aligned}$$

The simplest way to obtain (121) is to proceed as follows:

$$\begin{aligned}
\delta_\xi\delta_{\bar{\xi}}\hat{g}^{\mu\nu} &= \delta_\xi\delta_{\bar{\xi}}\left(\hat{g}^{\mu\lambda}g_{\lambda\rho}\hat{g}^{\rho\nu}\right) &(122)\\
&= \delta_\xi\left(-\hat{g}^{\mu\lambda}\delta_{\bar{\xi}}g_{\lambda\rho}\,\hat{g}^{\rho\nu}\right)\\
&= -\hat{g}^{\mu\lambda}\delta_\xi\delta_{\bar{\xi}}g_{\lambda\rho}\,\hat{g}^{\rho\nu}+\hat{g}^{\mu\lambda}\delta_\xi g_{\lambda\rho}\,\hat{g}^{\rho\sigma}\delta_{\bar{\xi}}g_{\sigma\tau}\,\hat{g}^{\tau\nu}-\hat{g}^{\mu\lambda}\delta_{\bar{\xi}}g_{\lambda\rho}\,\hat{g}^{\rho\sigma}\delta_\xi g_{\sigma\tau}\,\hat{g}^{\tau\nu}\\
&= -\hat{g}^{\mu\lambda}V_{\lambda\rho}\,\hat{g}^{\rho\nu}+\hat{g}^{\mu\lambda}\Gamma_{\lambda\rho}\,\hat{g}^{\rho\sigma}\bar{\Gamma}_{\sigma\tau}\,\hat{g}^{\tau\nu}-\hat{g}^{\mu\lambda}\bar{\Gamma}_{\lambda\rho}\,\hat{g}^{\rho\sigma}\Gamma_{\sigma\tau}\,\hat{g}^{\tau\nu}.
\end{aligned}$$

4.6. $\widehat{G}^{MN}$

Now, we are ready to tackle the problem of the supermetric inverse. In ordinary Riemannian geometry, the inverse $\hat{g}^{\mu\nu}$ of the metric is defined by the following: $\hat{g}^{\mu\lambda}g_{\lambda\nu}=\delta^\mu_\nu$. However, $\hat{g}^{\mu\nu}$ can also be considered as a bi-vector such that

$$\hat{\mathfrak{g}}=\hat{g}^{\mu\nu}\frac{\partial}{\partial x^\mu}\vee\frac{\partial}{\partial x^\nu}, \tag{123}$$

is invariant under diffeomorphisms.

We can try to define the analog of (123) in the superspace, i.e.,

$$\widehat{\mathbb{G}}=\widehat{G}^{MN}(X)\frac{\tilde{\partial}}{\tilde{\partial}X^M}\vee\frac{\tilde{\partial}}{\tilde{\partial}X^N}, \tag{124}$$

where $\frac{\tilde{\partial}}{\tilde{\partial}X^M}=\left(\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial\vartheta},\frac{\partial}{\partial\bar{\vartheta}}\right)$, and

$$\begin{aligned}
\widehat{G}^{\mu\nu}(X) &= \hat{g}^{\mu\nu}(x)+\vartheta\,\widehat{\bar{\Gamma}}^{\mu\nu}(x)+\bar{\vartheta}\widehat{\Gamma}^{\mu\nu}(x)+\vartheta\bar{\vartheta}\widehat{V}^{\mu\nu}(x),\\
\widehat{G}^{\mu\vartheta}(X) &= \hat{\gamma}^\mu(x)+\vartheta\,\hat{\bar{g}}^\mu(x)+\bar{\vartheta}\hat{g}^\mu(x)+\vartheta\bar{\vartheta}\widehat{\Gamma}^\mu(x)=\widehat{G}^{\vartheta\mu}(X),\\
\widehat{G}^{\mu\bar{\vartheta}}(X) &= \hat{\bar{\gamma}}^\mu(x)+\vartheta\,\hat{\bar{f}}^\mu(x)+\bar{\vartheta}\hat{f}^\mu(x)+\vartheta\bar{\vartheta}\widehat{\bar{\Gamma}}^\mu(x)=\widehat{G}^{\bar{\vartheta}\mu}(X),\\
\frac{1}{2}\widehat{G}^{\vartheta\bar{\vartheta}}(X) &= \hat{g}(x)+\vartheta\,\hat{\bar{\gamma}}(x)+\bar{\vartheta}\hat{\gamma}(x)+\vartheta\bar{\vartheta}\widehat{G}(x)=-\frac{1}{2}\widehat{G}^{\bar{\vartheta}\vartheta}(X). &(125)
\end{aligned}$$

This suggests immediately the horizontality condition $\widehat{\mathfrak{G}}=\hat{\mathfrak{g}}$, i.e.,

$$\widehat{\tilde{G}}^{MN}(\tilde{X})\frac{\partial}{\tilde{\partial}\tilde{X}^M}\vee\frac{\tilde{\partial}}{\tilde{\partial}\tilde{X}^N}=\hat{g}^{\mu\nu}(x)\frac{\partial}{\partial x^\mu}\vee\frac{\partial}{\partial x^\nu}. \tag{126}$$

The partial derivative $\frac{\tilde\partial}{\tilde\partial \tilde X^M}$ can be derived from $\tilde d\tilde X$ by inverting the relation as follows:

$$\tilde d\tilde X^M = \begin{pmatrix} \delta^\mu_\nu - \vartheta\partial_\nu\bar\xi^\mu - \bar\vartheta\partial_\nu\xi^\mu + \vartheta\bar\vartheta\partial_\nu h^\mu & -\bar\xi^\mu + \bar\vartheta h^\mu & -\xi^\mu - \vartheta h^\mu \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^\nu \\ d\vartheta \\ d\bar\vartheta \end{pmatrix}. \tag{127}$$

The matrix has the structure $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, D are commuting square matrices, while B, C are anticommuting rectangular ones (in this case $C = 0$ and $D = 1$). Its inverse is $\begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & 1 \end{pmatrix}$. Therefore, we have the following:

$$\begin{aligned}
\frac{\tilde\partial}{\tilde\partial\tilde x^\mu} &= \frac{\partial}{\partial x^\mu} + \Big(\vartheta\partial_\mu\bar\xi^\nu + \bar\vartheta\partial_\mu\xi^\nu - \vartheta\bar\vartheta\Big(\partial_\mu h^\nu + \partial_\mu\bar\xi^\lambda\partial_\lambda\xi^\nu - \partial_\mu\xi^\lambda\partial_\lambda\bar\xi^\nu\Big)\Big)\frac{\partial}{\partial x^\nu}, \\
\frac{\tilde\partial}{\tilde\partial\tilde\vartheta} &= \frac{\partial}{\partial\vartheta} + \big(-\bar\xi^\nu + \vartheta\bar\xi\cdot\partial\bar\xi^\nu + \bar\vartheta\big(-h^\nu + \bar\xi\cdot\partial\xi^\nu\big) + \vartheta\bar\vartheta\big(h\cdot\partial\bar\xi^\nu - \bar\xi\cdot\partial h^\nu - \bar\xi\cdot\partial\bar\xi\cdot\partial\xi^\nu + \bar\xi\cdot\partial\xi\cdot\partial\bar\xi^\nu\big)\big)\frac{\partial}{\partial x^\nu}, \\
\frac{\tilde\partial}{\tilde\partial\tilde{\bar\vartheta}} &= \frac{\partial}{\partial\bar\vartheta} + \big(-\xi^\nu + \bar\vartheta\xi\cdot\partial\xi^\nu + \vartheta\big(h^\nu + \xi\cdot\partial\bar\xi^\nu\big) + \vartheta\bar\vartheta\big(h\cdot\partial\xi^\nu - \xi\cdot\partial h^\nu - \xi\cdot\partial\bar\xi\cdot\partial\xi^\nu + \xi\cdot\partial\xi\cdot\partial\bar\xi^\nu\big)\big)\frac{\partial}{\partial x^\nu}.
\end{aligned} \tag{128}$$

The explicit form of the RHS can be found on Appendix B (see (A23)) from which we can now proceed to identify the various fields in (125).

From the $\frac{\partial}{\partial\vartheta} \vee \frac{\partial}{\partial\bar\vartheta}$ term, we obtain the following equation:

$$\hat g - \big(\vartheta\bar\xi + \bar\vartheta\xi - \vartheta\bar\vartheta h\big)\cdot\partial\hat g + \vartheta\bar\vartheta\,\xi\bar\xi\cdot\partial^2\hat g + \vartheta\big(\hat{\bar\gamma} - \bar\vartheta\xi\cdot\partial\hat{\bar\gamma}\big) + \bar\vartheta\big(\hat\gamma - \vartheta\xi\cdot\hat\gamma\big) + \vartheta\bar\vartheta\widehat G = 0. \tag{129}$$

from which we deduce the following:

$$\hat g = 0, \quad \hat\gamma = 0, \quad \hat{\bar\gamma} = 0, \quad \widehat G = 0. \tag{130}$$

Similarly, from the $\frac{\partial}{\partial x^\mu} \vee \frac{\partial}{\partial\vartheta}$ term we deduce the following:

$$\hat\gamma^\mu = 0, \quad \hat g^\mu = 0, \quad \hat{\bar g}^\mu = 0, \quad \widehat\Gamma^\mu = 0, \tag{131}$$

and from $\frac{\partial}{\partial x^\mu} \vee \frac{\partial}{\partial\bar\vartheta}$

$$\hat{\bar\gamma}^\mu = 0, \quad \hat f^\mu = 0, \quad \hat{\bar f}^\mu = 0, \quad \widehat{\bar\Gamma}^\mu = 0. \tag{132}$$

Therefore, only the components of $\widehat G^{\mu\nu}(X)$ do not vanish. Equation (A23) becomes the following:

$$\begin{aligned}
&\hat g^{\mu\nu}(x)\frac{\partial}{\partial x^\mu} \vee \frac{\partial}{\partial x^\nu} = \widetilde{\widehat G}^{MN}(\tilde X)\frac{\tilde\partial}{\tilde\partial\tilde X^M} \vee \frac{\tilde\partial}{\tilde\partial\tilde X^N} \\
= &\Big(\hat g^{\mu\nu} - \big(\vartheta\bar\xi + \bar\vartheta\xi - \vartheta\bar\vartheta h\big)\cdot\partial\hat g^{\mu\nu} + \vartheta\bar\vartheta\,\xi\bar\xi\cdot\partial^2\hat g^{\mu\nu} + \vartheta\Big(\widehat{\bar\Gamma}^{\mu\nu} - \bar\vartheta\xi\cdot\partial\widehat{\bar\Gamma}^{\mu\nu}\Big) \\
&\quad + \bar\vartheta\Big(\widehat\Gamma^{\mu\nu} - \vartheta\bar\xi\cdot\partial\widehat\Gamma^{\mu\nu}\Big) + \vartheta\bar\vartheta\,\widehat V^{\mu\nu}(x)\Big) \\
&\cdot\bigg(\frac{\partial}{\partial x^\mu} + \Big(\vartheta\partial_\mu\bar\xi^\lambda + \bar\vartheta\partial_\mu\xi^\lambda - \vartheta\bar\vartheta\Big(\partial_\mu h^\lambda + \partial_\mu\bar\xi^\sigma\partial_\sigma\xi^\lambda - \partial_\mu\xi^\sigma\partial_\sigma\bar\xi^\lambda\Big)\Big)\frac{\partial}{\partial x^\lambda}\bigg) \\
&\vee\bigg(\frac{\partial}{\partial x^\nu} + \big(\vartheta\partial_\nu\bar\xi^\rho + \bar\vartheta\partial_\nu\xi^\rho - \vartheta\bar\vartheta\big(\partial_\nu h^\rho + \partial_\nu\bar\xi^\tau\partial_\tau\xi^\rho - \partial_\nu\xi^\tau\partial_\tau\bar\xi^\rho\big)\big)\frac{\partial}{\partial x^\rho}\bigg).
\end{aligned} \tag{133}$$

This implies the following:

$$\begin{aligned}
\widehat\Gamma^{\mu\nu} &= \xi\cdot\partial\hat g^{\mu\nu} - \partial_\lambda\xi^\mu\hat g^{\lambda\nu} - \partial_\lambda\xi^\nu\hat g^{\mu\lambda} = \delta_\xi\hat g^{\mu\nu}, \\
\widehat{\bar\Gamma}^{\mu\nu} &= \bar\xi\cdot\partial\hat g^{\mu\nu} - \partial_\lambda\bar\xi^\mu\hat g^{\lambda\nu} - \partial_\lambda\bar\xi^\nu\hat g^{\mu\lambda} = \delta_{\bar\xi}\hat g^{\mu\nu}, \\
\widehat V^{\mu\nu} &= \delta_\xi\delta_{\bar\xi}\hat g^{\mu\nu}.
\end{aligned} \tag{134}$$

If we impose $\hat{g}^{\mu\nu}(x)$ to be the inverse of $g_{\mu\nu}(x)$, these are identical to Equations (116)–(118).

4.7. Super-Christoffel Symbols and Super-Riemann Tensor

From the previous results and from Appendix B.5, it is clear that we cannot define an inverse of $G_{MN}(X)$; therefore, we must give up the idea of mimicking Riemannian geometry in the superspace. However, no obstacles exist if we limit ourselves to $G_{\mu\nu}(X)$. We have seen that its inverse exists. Therefore, we can introduce a horizontal Riemannian geometry in the superspace, that is, a Riemannian geometry where the involved tensors are horizontal, i.e., they do not have components in the anticommuting directions. To start with, we can define the super-Christoffel symbol as follows:

$$\begin{aligned} \mathbf{\Gamma}^{\lambda}_{\mu\nu} &= \frac{1}{2}\widehat{G}^{\lambda\kappa}\left(\partial_\mu G_{\nu\kappa} + \partial_\nu G_{\mu\kappa} - \partial_\kappa G_{\mu\nu}\right) && (135)\\ &= \Gamma^{\lambda}_{\mu\nu} + \vartheta \overline{K}^{\lambda}_{\mu\nu} + \bar{\vartheta} K^{\lambda}_{\mu\nu} + \vartheta\bar{\vartheta} H^{\lambda}_{\mu\nu}, && (136) \end{aligned}$$

where

$$\begin{aligned} \overline{K}^{\lambda}_{\mu\nu} &= \frac{1}{2}\left(\widehat{\overline{\Gamma}}^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) + \hat{g}^{\lambda\rho}\left(\partial_\mu \overline{\Gamma}_{\rho\nu} + \partial_\nu \overline{\Gamma}_{\mu\rho} - \partial_\rho \overline{\Gamma}_{\mu\nu}\right)\right) && (137)\\ &= \frac{1}{2}\left(\delta_{\bar{\xi}}\hat{g}^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) + \hat{g}^{\lambda\rho}\left(\partial_\mu \delta_{\bar{\xi}} g_{\rho\nu} + \partial_\nu \delta_{\bar{\xi}} g_{\mu\rho} - \partial_\rho \delta_{\bar{\xi}} g_{\mu\nu}\right)\right)\\ &= \delta_{\bar{\xi}}\Gamma^{\lambda}_{\mu\nu}. \end{aligned}$$

Similarly, we note the following:

$$K^{\lambda}_{\mu\nu} = \delta_{\xi}\Gamma^{\lambda}_{\mu\nu}, \tag{138}$$

and

$$\begin{aligned} H^{\lambda}_{\mu\nu} &= \frac{1}{2}\Big(\widehat{V}^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) + \hat{g}^{\lambda\rho}\left(\partial_\mu V_{\rho\nu} + \partial_\nu V_{\mu\rho} - \partial_\rho V_{\mu\nu}\right) && (139)\\ &\quad + \widehat{\Gamma}^{\lambda\rho}\left(\partial_\mu \overline{\Gamma}_{\rho\nu} + \partial_\nu \overline{\Gamma}_{\mu\rho} - \partial_\rho \overline{\Gamma}_{\mu\nu}\right) - \widehat{\overline{\Gamma}}^{\lambda\rho}\left(\partial_\mu \Gamma_{\rho\nu} + \partial_\nu \Gamma_{\mu\rho} - \partial_\rho \Gamma_{\mu\nu}\right)\Big)\\ &= \frac{1}{2}\Big(\delta_\xi\delta_{\bar{\xi}}\hat{g}^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) + \hat{g}^{\lambda\rho}\left(\partial_\mu \delta_\xi\delta_{\bar{\xi}} g_{\rho\nu} + \partial_\nu \delta_\xi\delta_{\bar{\xi}} g_{\mu\rho} - \partial_\rho \delta_\xi\delta_{\bar{\xi}} g_{\mu\nu}\right)\\ &\quad + \delta_\xi \hat{g}^{\lambda\rho}\left(\partial_\mu \delta_{\bar{\xi}} g_{\rho\nu} + \partial_\nu \delta_{\bar{\xi}} g_{\mu\rho} - \partial_\rho \delta_{\bar{\xi}} g_{\mu\nu}\right) - \delta_{\bar{\xi}}\hat{g}^{\lambda\rho}\left(\partial_\mu \delta_\xi g_{\rho\nu} + \partial_\nu \delta_\xi g_{\mu\rho} - \partial_\rho \delta_\xi g_{\mu\nu}\right)\Big)\\ &= \delta_\xi\delta_{\bar{\xi}}\Gamma^{\lambda}_{\mu\nu}, \end{aligned}$$

in agreement with (136).

The super-Riemann curvature is

$$\begin{aligned} \mathbf{R}_{\mu\nu\lambda}{}^{\rho} &= -\partial_\mu \mathbf{\Gamma}^{\rho}_{\nu\lambda} + \partial_\nu \mathbf{\Gamma}^{\rho}_{\mu\lambda} - \mathbf{\Gamma}^{\rho}_{\mu\sigma}\mathbf{\Gamma}^{\sigma}_{\nu\lambda} + \mathbf{\Gamma}^{\rho}_{\nu\sigma}\mathbf{\Gamma}^{\sigma}_{\mu\lambda}\\ &= R_{\mu\nu\lambda}{}^{\rho} + \vartheta\,\overline{\Omega}_{\mu\nu\lambda}{}^{\rho} + \bar{\vartheta}\,\Omega_{\mu\nu\lambda}{}^{\rho} + \vartheta\bar{\vartheta}\,S_{\mu\nu\lambda}{}^{\rho}, && (140) \end{aligned}$$

where

$$\begin{aligned} \overline{\Omega}_{\mu\nu\lambda}{}^{\rho} &= -\partial_\mu \overline{K}^{\rho}_{\nu\lambda} + \partial_\nu \overline{K}^{\rho}_{\mu\lambda} - \Gamma^{\rho}_{\mu\sigma}\overline{K}^{\sigma}_{\nu\lambda} + \Gamma^{\rho}_{\nu\sigma}\overline{K}^{\sigma}_{\mu\lambda} - \overline{K}^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} + \overline{K}^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\mu\lambda}\\ &= -\partial_\mu \delta_{\bar{\xi}}\Gamma^{\rho}_{\nu\lambda} + \partial_\nu \delta_{\bar{\xi}}\Gamma^{\rho}_{\mu\lambda} - \Gamma^{\rho}_{\mu\sigma}\delta_{\bar{\xi}}\Gamma^{\sigma}_{\nu\lambda} + \Gamma^{\rho}_{\nu\sigma}\delta_{\bar{\xi}}\Gamma^{\sigma}_{\mu\lambda} - \delta_{\bar{\xi}}\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} + \delta_{\bar{\xi}}\Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\mu\lambda}\\ &= \delta_{\bar{\xi}}\,R_{\mu\nu\lambda}{}^{\rho}. && (141) \end{aligned}$$

Likewise,

$$\Omega_{\mu\nu\lambda}{}^{\rho} = \delta_\xi\,R_{\mu\nu\lambda}{}^{\rho}, \tag{142}$$

and

$$\begin{aligned} S_{\mu\nu\lambda}{}^{\rho} &= \partial_\mu H^\rho_{\nu\lambda} + \partial_\nu H^\rho_{\mu\lambda} - H^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} + H^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda} - \Gamma^\rho_{\mu\sigma}H^\sigma_{\nu\lambda} + \Gamma^\rho_{\nu\sigma}H^\sigma_{\mu\lambda} \\ &\quad + \overline{K}^\rho_{\mu\sigma}K^\sigma_{\nu\lambda} - K^\rho_{\mu\sigma}\overline{K}^\sigma_{\nu\lambda} - \overline{K}^\rho_{\nu\sigma}K^\sigma_{\mu\lambda} + K^\rho_{\nu\sigma}\overline{K}^\sigma_{\mu\lambda} \\ &= -\partial_\mu\delta_\xi\delta_{\bar\xi}\Gamma^\rho_{\nu\lambda} + \partial_\nu\delta_\xi\delta_{\bar\xi}\Gamma^\rho_{\mu\lambda} - \delta_\xi\delta_{\bar\xi}\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} + \delta_\xi\delta_{\bar\xi}\Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda} - \Gamma^\rho_{\mu\sigma}\delta_\xi\delta_{\bar\xi}\Gamma^\sigma_{\nu\lambda} + \Gamma^\rho_{\nu\sigma}\delta_\xi\delta_{\bar\xi}\Gamma^\sigma_{\mu\lambda} \\ &\quad + \delta_{\bar\xi}\Gamma^\rho_{\mu\sigma}\delta_\xi\Gamma^\sigma_{\nu\lambda} - \delta_\xi\Gamma^\rho_{\mu\sigma}\delta_{\bar\xi}\Gamma^\sigma_{\nu\lambda} - \delta_{\bar\xi}\Gamma^\rho_{\nu\sigma}\delta_\xi\Gamma^\sigma_{\mu\lambda} + \delta_\xi\Gamma^\rho_{\nu\sigma}\delta_{\bar\xi}\Gamma^\sigma_{\mu\lambda} \\ &= \delta_\xi\delta_{\bar\xi}\, R_{\mu\nu\lambda}{}^{\rho}. \end{aligned} \tag{143}$$

This gives immediately the super-Ricci tensor

$$\mathbf{R}_{\mu\lambda} \equiv \mathbf{R}_{\mu\nu\lambda}{}^{\nu} = R_{\mu\lambda} + \vartheta\,\overline{\Omega}_{\mu\lambda} + \bar\vartheta\,\Omega_{\mu\lambda} + \vartheta\bar\vartheta\,S_{\mu\lambda}, \tag{144}$$

with $\overline{\Omega}_{\mu\lambda} = \overline{\Omega}_{\mu\nu\lambda}{}^{\nu}$, $\Omega_{\mu\lambda} = \Omega_{\mu\nu\lambda}{}^{\nu}$ and $S_{\mu\lambda} = S_{\mu\nu\lambda}{}^{\nu}$. Of course,

$$\overline{\Omega}_{\mu\lambda} = \delta_{\bar\xi}R_{\mu\lambda}, \qquad \Omega_{\mu\lambda} = \delta_\xi R_{\mu\lambda}, \qquad \text{and} \qquad S_{\mu\lambda} = \delta_\xi\delta_{\bar\xi}R_{\mu\lambda}. \tag{145}$$

The super-Ricci scalar is the following:

$$\mathbf{R} \equiv \widehat{G}^{\mu\nu}\,\mathbf{R}_{\mu\nu} = R + \vartheta\,\overline{\Omega} + \bar\vartheta\,\Omega + \vartheta\bar\vartheta\,S. \tag{146}$$

It is easy to show the following:

$$\overline{\Omega} = \delta_{\bar\xi}R, \qquad \Omega = \delta_\xi R, \qquad S = \delta_\xi\delta_{\bar\xi}R. \tag{147}$$

5. The Vielbein

If we want to include fermions in a theory in curved spacetime, we need frame fields. This section is devoted to introducing vielbein in the superspace. We define the supervierbein as the following d-vector 1-form:

$$\mathbb{E}^a = E^a_M(X)\tilde{d}X^M, \tag{148}$$

where

$$\begin{aligned} E^a_\mu(X) &= e^a_\mu(x) + \vartheta\bar\phi^a_\mu(x) + \bar\vartheta\phi^a_\mu(x) + \vartheta\bar\vartheta f^a_\mu(x), \\ E^a_\vartheta(X) &= \chi^a(x) + \vartheta\bar C^a(x) + \bar\vartheta C^a(x) + \vartheta\bar\vartheta\psi^a(x), \\ E^a_{\bar\vartheta}(X) &= \lambda^a(x) + \vartheta\bar D^a(x) + \bar\vartheta D^a(x) + \vartheta\bar\vartheta\rho^a(x). \end{aligned} \tag{149}$$

The natural horizontality condition is the following:

$$\widetilde{E}^a_M(\widetilde{X})\tilde{d}\widetilde{X}^M = e^a_\mu(x)dx^\mu. \tag{150}$$

This is the same condition as for a vector field. So, we immediately obtain the following results:

$$\phi^a_\mu = \xi\cdot\partial e^a_\mu + \partial_\mu\xi^\lambda e^a_\lambda = \delta_\xi e^a_\mu, \tag{151}$$

$$\bar\phi^a_\mu = \bar\xi\cdot\partial e^a_\mu + \partial_\mu\bar\xi^\lambda e^a_\lambda = \delta_{\bar\xi} e^a_\mu, \tag{152}$$

$$\begin{aligned} f^a_\mu &= \xi\cdot\partial\bar\phi^a_\mu - \bar\xi\cdot\partial\phi^a_\mu - \xi\bar\xi\cdot\partial^2 e^a_\mu + \partial_\mu\bar\xi^\lambda\xi\cdot\partial e^a_\lambda - \partial_\mu\xi^\lambda\bar\xi\cdot\partial e^a_\lambda \\ &\quad -\partial_\mu\bar\xi^\lambda\phi^a_\lambda + \partial_\mu\xi^\lambda\bar\phi^a_\lambda - h\cdot\partial e^a_\mu - \partial_\mu h\cdot e^a = -\delta_{\bar\xi}\phi^a_\mu = \delta_\xi\bar\phi^a_\mu, \end{aligned} \tag{153}$$

$$\begin{aligned}
\chi^a &= e^a_\mu \bar{\xi}^\mu, & (154)\\
C^a &= -\xi\cdot\partial e^a_\mu \bar{\xi}^\mu + \phi^a_\mu \bar{\xi}^\mu + \xi\cdot\partial\chi^a - h\cdot e^a = \delta_\xi \chi^a, & (155)\\
\bar{C}^a &= -\bar{\xi}\cdot\partial e^a_\mu \bar{\xi}^\mu + \bar{\phi}^a_\mu \bar{\xi}^\mu + \bar{\xi}\cdot\partial\chi^a = \delta_{\bar{\xi}} \chi^a, & (156)\\
\psi^a &= \xi\bar{\xi}\cdot\partial^2 e^a_\mu \bar{\xi}^\mu - \xi\cdot\partial\bar{\phi}^a_\mu \bar{\xi}^\mu + \bar{\xi}\cdot\partial\phi^a_\mu \bar{\xi}^\mu + f^a_\mu \bar{\xi}^\mu - \xi\bar{\xi}\cdot\partial^2\chi^a + \xi\cdot\partial\bar{C}^a - \bar{\xi}\cdot\partial C^a & (157)\\
&\quad -\bar{\xi}\cdot\partial e^a\cdot h + \bar{\phi}^a\cdot h - h\cdot\partial\chi^a + h\cdot\partial e^a_\mu \bar{\xi}^\mu = \delta_\xi \bar{C}^a = -\delta_{\bar{\xi}} C^a,
\end{aligned}$$

and

$$\begin{aligned}
\lambda^a &= e^a_\mu \xi^\mu, & (158)\\
D^a &= -\xi\cdot\partial e^a_\mu \xi^\mu + \phi^a_\mu \xi^\mu + \xi\cdot\partial\lambda^a = \delta_\xi \lambda^a, & (159)\\
\bar{D}^a &= -\bar{\xi}\cdot\partial e^a_\mu \xi^\mu + \bar{\phi}^a_\mu \xi^\mu + \bar{\xi}\cdot\partial\lambda^a + h\cdot e^a = \delta_{\bar{\xi}} \lambda^a, & (160)\\
\rho^a &= \xi\bar{\xi}\cdot\partial^2 e^a_\mu \xi^\mu - \xi\cdot\partial\bar{\phi}^a_\mu \xi^\mu + \bar{\xi}\cdot\partial\phi^a_\mu \xi^\mu + f^a_\mu \xi^\mu - \xi\bar{\xi}\cdot\partial^2\lambda^a + \xi\cdot\partial\bar{D}^a - \bar{\xi}\cdot\partial D^a & (161)\\
&\quad -\xi\cdot\partial e^a\cdot h + \phi^a\cdot h - h\cdot\partial\lambda^a + h\cdot\partial e^a_\mu \xi^\mu = \delta_\xi \bar{D}^a = -\delta_{\bar{\xi}} D^a.
\end{aligned}$$

5.1. *The Inverse Vielbein* $\widehat{E}^\mu_a$

Here, we introduce the inverse vielbein $\widehat{E}^\mu_a$. Let us write it as the following:

$$E^a_\mu(X) = e^a_\lambda(x)\left(\delta^\lambda_\mu + \vartheta\bar{\phi}^\lambda_\mu(x) + \bar{\vartheta}\phi^\lambda_\mu(x) + \vartheta\bar{\vartheta} f^\lambda_\mu(x)\right) = e^a_\lambda (1+X)^\lambda_\mu, \quad (162)$$

where $\phi^\lambda_\mu = e^\lambda_a \phi^a_\mu$, $\bar{\phi}^\lambda_\mu = e^\lambda_a \bar{\phi}^a_\mu$ and $f^\lambda_\mu = e^\lambda_a f^a_\lambda$, and $e^\lambda_a(x) e^b_\lambda(x) = \delta^b_a$. Then we define

$$\widehat{E}^\mu_a = (1 - X + X^2)^\mu_\lambda \, e^\lambda_a. \quad (163)$$

The following is evident:

$$\widehat{E}^\mu_a \, E^b_\mu = \delta^b_a. \quad (164)$$

In terms of components, we have the following:

$$\begin{aligned}
\hat{\phi}^\mu_a &= -\hat{e}^\mu_b \phi^b_\lambda \hat{e}^\lambda_a = -\hat{e}^\mu_b \delta_\xi e^b_\lambda \hat{e}^\lambda_a = \delta_\xi \hat{e}^\mu_a, & (165)\\
\hat{\bar{\phi}}^\mu_a &= -\hat{e}^\mu_b \bar{\phi}^b_\lambda \hat{e}^\lambda_a = -\hat{e}^\mu_b \delta_{\bar{\xi}} e^b_\lambda \hat{e}^\lambda_a = \delta_{\bar{\xi}} \hat{e}^\mu_a, & (166)
\end{aligned}$$

and

$$\hat{f}^\mu_a = \hat{e}^\mu_b \left(-f^b_\lambda - \bar{\phi}^b_\rho e^\rho_c \, \phi^c_\lambda + \phi^b_\rho \, e^\rho_c \, \bar{\phi}^c_\lambda\right) \hat{e}^\lambda_a = \delta_\xi \delta_{\bar{\xi}} \hat{e}^\mu_a. \quad (167)$$

A simple way to prove the last step is as follows:

$$\begin{aligned}
\delta_\xi \delta_{\bar{\xi}} \hat{e}^\mu_a &= \delta_\xi \left(-\hat{e}^\mu_b \delta_{\bar{\xi}} e^b_\lambda \hat{e}^\lambda_a\right) = -\delta_\xi \hat{e}^\mu_b \delta_{\bar{\xi}} e^b_\lambda \hat{e}^\lambda_a - \hat{e}^\mu_b \delta_\xi \delta_{\bar{\xi}} e^b_\lambda \hat{e}^\lambda_a + \hat{e}^\mu_b \delta_{\bar{\xi}} e^b_\lambda \delta_{\bar{\xi}} \hat{e}^\lambda_a \\
&= \hat{e}^\mu_b \left(-f^b_\lambda - \bar{\phi}^b_\rho e^\rho_c \, \phi^c_\lambda + \phi^b_\rho \, e^\rho_c \, \bar{\phi}^c_\lambda\right) \hat{e}^\lambda_a = \hat{f}^\mu_a. & (168)
\end{aligned}$$

5.2. *The Inverse Supervielbein* $\widehat{E}^M_a$

In this subsection, as we did for the supermetric, we try to define the inverse of the supervielbein. Analogous with what we did for the metric, we define the following:

$$\widehat{\mathbb{E}}_a = \widehat{E}^M_a(X) \frac{\partial}{\partial X^M}, \quad (169)$$

where

$$\begin{aligned}\widehat{E}^{\mu}_{a}(X) &= \hat{e}^{\mu}_{a}(x)+\vartheta\hat{\bar{\phi}}^{\mu}_{a}(x)+\bar{\vartheta}\hat{\phi}^{\mu}_{a}(x)+\vartheta\bar{\vartheta}\hat{f}^{\mu}_{a}(x),\\ \widehat{E}^{\vartheta}_{a}(X) &= \hat{\chi}_{a}(x)+\vartheta\hat{\bar{C}}_{a}(x)+\bar{\vartheta}\hat{C}_{a}(x)+\vartheta\bar{\vartheta}\hat{\psi}_{a}(x),\\ \widehat{E}^{\bar{\vartheta}}_{a}(X) &= \hat{\lambda}_{a}(x)+\vartheta\hat{\bar{D}}_{a}(x)+\bar{\vartheta}\hat{D}_{a}(x)+\vartheta\bar{\vartheta}\hat{\rho}_{a}(x),\end{aligned} \tag{170}$$

and impose the following horizontality condition:

$$\widetilde{\widehat{E}}^{M}_{a}(X)\frac{\partial}{\partial\widetilde{X}^{M}}=\hat{e}^{\mu}_{a}\frac{\partial}{\partial x^{\mu}}. \tag{171}$$

The explicit form of the LHS of this equation can be found in Appendix B, see Equation (A24). In the latter, the coefficient of $\frac{\partial}{\partial\vartheta}$ leads to the following:

$$\hat{\chi}_{a}=\hat{\bar{C}}_{a}=\widehat{C}_{a}=\hat{\psi}_{a}=0, \tag{172}$$

and the equation proportional to $\frac{\partial}{\partial\bar{\vartheta}}$ to the following:

$$\hat{\lambda}_{a}=\hat{\bar{D}}_{a}=\widehat{D}_{a}=\hat{\rho}_{a}=0. \tag{173}$$

Therefore, only the components of $\widehat{E}^{\mu}_{a}(X)$ are nonvanishing. The remaining equations give the following:

$$\begin{aligned}\hat{\phi}^{\mu}_{a} &= \xi\cdot\hat{e}^{\mu}_{a}-\partial_{\lambda}\xi^{\mu}\hat{e}^{\lambda}_{a}=\delta_{\xi}\hat{e}^{\mu}_{a},\\ \hat{\bar{\phi}}^{\mu}_{a} &= \bar{\xi}\cdot\hat{e}^{\mu}_{a}-\partial_{\lambda}\bar{\xi}^{\mu}\hat{e}^{\lambda}_{a}=\delta_{\bar{\xi}}\hat{e}^{\mu}_{a},\\ \hat{f}^{\mu}_{a} &= -\xi\bar{\xi}\cdot\partial^{2}\hat{e}^{\mu}_{a}-h\cdot\partial e^{\mu}_{a}+\partial_{\lambda}h^{\mu}e^{\lambda}_{a}+\xi\cdot\partial\hat{\bar{\phi}}^{\mu}_{a}+\partial_{\lambda}\xi^{\mu}\hat{\bar{\phi}}^{\lambda}_{a}-\bar{\xi}\cdot\partial\hat{\phi}^{\mu}_{a}-\partial_{\lambda}\bar{\xi}^{\mu}\hat{\phi}^{\lambda}_{a},\\ &\quad+\hat{e}^{\lambda}_{a}\partial_{\lambda}\bar{\xi}\cdot\partial\xi^{\mu}+\xi\cdot\partial\hat{e}^{\lambda}_{a}\partial_{\lambda}\bar{\xi}^{\mu}-\hat{e}^{\lambda}_{a}\partial_{\lambda}\xi\cdot\partial\bar{\xi}^{\mu}-\bar{\xi}\cdot\partial\hat{e}^{\lambda}_{a}\partial_{\lambda}\xi^{\mu}=\delta_{\xi}\delta_{\bar{\xi}}\hat{e}^{\mu}_{a}.\end{aligned} \tag{174}$$

If $\hat{e}^{\mu}_{a}$ is the inverse of e^{a}_{μ}, these formulas coincide with those of the previous subsection.

The results of this section confirm what was found in the previous subsection. In the superspace, it makes sense to consider only horizontal tensors, i.e., tensors whose components in the anticommuting directions vanish. We continue, therefore, to define a frame geometry with this characteristic.

5.3. The Spin Superconnection

The spin superconnection is defined as follows:

$$\begin{aligned}\mathbf{\Omega}^{ab}_{\mu} &= \frac{1}{2}\Big[\widehat{E}^{a\nu}\Big(\partial_{\mu}E^{b}_{\nu}-\partial_{\nu}E^{b}_{\mu}\Big)-\widehat{E}^{b\nu}\Big(\partial_{\mu}E^{a}_{\nu}-\partial_{\nu}E^{a}_{\mu}\Big)-\widehat{E}^{a\nu}\widehat{E}^{b\lambda}(\partial_{\nu}E^{c}_{\lambda}-\partial_{\lambda}E^{c}_{\nu})E_{c\mu}\Big]\\ &= \omega^{ab}_{\mu}+\vartheta\overline{P}^{ab}_{\mu}+\bar{\vartheta}P^{ab}_{\mu}+\vartheta\bar{\vartheta}Q^{ab}_{\mu},\end{aligned} \tag{175}$$

where ω^{ab}_{μ} is the usual spin connection and the following holds:

$$\begin{aligned}P^{ab}_{\mu} &= \frac{1}{2}\Big[\hat{e}^{a\nu}\partial_{\mu}\phi^{b}_{\nu}+\hat{\phi}^{a\nu}\partial_{\mu}e^{b}_{\nu}-\hat{e}^{b\nu}\partial_{\mu}\phi^{a}_{\nu}-\hat{\phi}^{b\nu}\partial_{\mu}e^{a}_{\nu}-\Big(\hat{e}^{a\nu}\hat{\phi}^{b\lambda}+\hat{\phi}^{a\nu}\hat{e}^{b\lambda}\Big)(\partial_{\nu}e^{c}_{\lambda}-\partial_{\lambda}e^{c}_{\nu})e_{c\mu}\\ &\quad-\hat{e}^{a\nu}e^{b\lambda}(\partial_{\nu}\phi^{c}_{\lambda}-\partial_{\lambda}\phi^{c}_{\nu})e_{c\mu}-\hat{e}^{a\nu}e^{b\lambda}(\partial_{\nu}e^{c}_{\lambda}-\partial_{\lambda}e^{c}_{\nu})\phi_{c\mu}\Big]\\ &= \frac{1}{2}\Big[\hat{e}^{a\nu}\partial_{\mu}\delta_{\xi}e^{b}_{\nu}+\delta_{\xi}\hat{e}^{a\nu}\partial_{\mu}e^{b}_{\nu}-\hat{e}^{b\nu}\partial_{\mu}\delta_{\xi}e^{a}_{\nu}-\delta_{\xi}\hat{e}^{b\nu}\partial_{\mu}e^{a}_{\nu}-\Big(\hat{e}^{a\nu}\delta_{\xi}\hat{e}^{b\lambda}+\delta_{\xi}\hat{e}^{a\nu}\hat{e}^{b\lambda}\Big)(\partial_{\nu}e^{c}_{\lambda}-\partial_{\lambda}e^{c}_{\nu})e_{c\mu}\\ &\quad-\hat{e}^{a\nu}e^{b\lambda}(\partial_{\nu}\delta_{\xi}e^{c}_{\lambda}-\partial_{\lambda}\delta_{\xi}e^{c}_{\nu})e_{c\mu}-\hat{e}^{a\nu}e^{b\lambda}(\partial_{\nu}e^{c}_{\lambda}-\partial_{\lambda}e^{c}_{\nu})\delta_{\xi}e_{c\mu}\Big]=\delta_{\xi}\omega^{ab}_{\mu}.\end{aligned} \tag{176}$$

where $e^{a}_{\mu},\phi^{a}_{\mu},\hat{e}^{a\mu},\hat{\phi}^{a\mu}$ were explained earlier in Sections 5.1 and 5.2.

Similarly, we have the following:

$$\overline{P}^{ab}_{\mu} = \delta_{\bar{\xi}}\omega^{ab}_{\mu}, \tag{177}$$
$$\overline{Q}^{ab}_{\mu} = \delta_{\xi}\delta_{\bar{\xi}}\omega^{ab}_{\mu}. \tag{178}$$

Thus, we have recovered the complete set of (anti)BRST transformations for the spin superconnection.

5.4. The Curvature

The 2-form supercurvature is the following:

$$\mathbb{R}^{ab} = \mathbb{R}^{ab}_{\mu\nu} dx^{\mu} \wedge dx^{\nu}, \tag{179}$$

where

$$\begin{aligned} \mathbb{R}^{ab}_{\mu\nu} &= \partial_{\mu}\boldsymbol{\Omega}^{ab}_{\nu} - \partial_{\nu}\boldsymbol{\Omega}^{ab}_{\mu} + \boldsymbol{\Omega}^{a}_{\mu c}\boldsymbol{\Omega}^{cb}_{\nu} - \boldsymbol{\Omega}^{a}_{\nu c}\boldsymbol{\Omega}^{cb}_{\mu} \\ &= R^{ab}_{\mu\nu} + \vartheta\overline{\Sigma}^{ab}_{\mu\nu} + \bar{\vartheta}\Sigma^{ab}_{\mu\nu} + \vartheta\bar{\vartheta}S^{ab}_{\mu\nu}. \end{aligned} \tag{180}$$

$R^{ab}_{\mu\nu}$ is the usual spin connection curvature. Next, we have the following:

$$\begin{aligned} \Sigma^{ab}_{\mu\nu} &= \partial_{\mu}P^{ab}_{\nu} - \partial_{\nu}P^{ab}_{\mu} + P^{ac}_{\mu}\,\omega_{\nu c}{}^{b} + \omega^{ac}_{\mu}\,P_{\nu c}{}^{b} - P^{ac}_{\nu}\,\omega_{\mu c}{}^{b} - \omega^{ac}_{\nu}\,P_{\mu c}{}^{b} \\ &= \partial_{\mu}\delta_{\xi}\omega^{ab}_{\nu} - \partial_{\nu}\delta_{\xi}\omega^{ab}_{\mu} + \delta_{\xi}\omega^{ac}_{\mu}\,\omega_{\nu c}{}^{b} + \omega^{ac}_{\mu}\,\delta_{\xi}\omega_{\nu c}{}^{b} - \delta_{\xi}\omega^{ac}_{\nu}\,\omega_{\mu c}{}^{b} - \omega^{ac}_{\nu}\,\delta_{\xi}\omega_{\mu c}{}^{b} \\ &= \delta_{\xi}R^{ab}_{\mu\nu}. \end{aligned} \tag{181}$$

At the same time, we have the following identifications:

$$\overline{\Sigma}^{ab}_{\mu\nu} = \delta_{\bar{\xi}}R^{ab}_{\mu\nu}, \tag{182}$$
$$S^{ab}_{\mu\nu} = \delta_{\xi}\delta_{\bar{\xi}}R^{ab}_{\mu\nu}. \tag{183}$$

5.5. Fermions

Fermion fields, under diffeomorphisms, behave like scalars. A Dirac fermion superfield has the following expansion:

$$\boldsymbol{\Psi}(X) = \psi(x) + \vartheta\overline{F}(x) + \bar{\vartheta}F(x) + \vartheta\bar{\vartheta}\Theta(x, \tag{184}$$

$\psi, F, \overline{F}$ and Θ are four-component complex column vector fields. The horizontality condition is the following:

$$\widetilde{\boldsymbol{\Psi}}(\widetilde{X}) = \psi(x). \tag{185}$$

Repeating the analysis of the scalar superfield, we obtain the following:

$$F(x) = \delta_{\xi}\psi(x), \qquad \overline{F}(x) = \delta_{\bar{\xi}}\psi(x), \qquad \Theta(x) = \delta_{\xi}\delta_{\bar{\xi}}\psi(x). \tag{186}$$

The covariant derivative of a vector superfield is as follows:

$$\mathbf{D}_{\mu}\boldsymbol{\Psi} = \left(\partial_{\mu} + \frac{1}{2}\boldsymbol{\Omega}_{\mu}\right)\boldsymbol{\Psi},$$

where $\boldsymbol{\Omega}_{\mu} = \boldsymbol{\Omega}^{ab}_{\mu}\Sigma_{ab}$, and $\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$ are the Lorentz generators.

The Lagrangian density for a Dirac superfield is the following:

$$L = \sqrt{g}\, i\, \overline{\boldsymbol{\Psi}}\hat{\gamma}^{\mu}\mathbf{D}_{\mu}\boldsymbol{\Psi}, \tag{187}$$

with $\hat{\gamma}^{\mu} = \widehat{E}^{\mu}_{a}\gamma_a$. The Lagrangian density L is invariant under Lorentz transformations and, up to total derivatives, under diffeomorphisms.

5.6. The Super-LORENTZ Transformations

The fermion superfield transforms under local Lorentz transformations as follows:

$$\delta_\Lambda \mathbf{\Psi} = \frac{1}{2}\mathbf{\Lambda}\Psi, \qquad \mathbf{\Lambda} = \mathbf{\Lambda}^{ab}(X)\Sigma_{ab}, \tag{188}$$

where $\Lambda^{ab}(X)$ is an infinitesimal antisymmetric supermatrix with arbitrary entries:

$$\mathbf{\Lambda}^{ab}(X) = \lambda^{ab}(x) + \vartheta\bar{h}^{ab}(x) + \bar{\vartheta}h^{ab}(x) + \vartheta\bar{\vartheta}\Lambda^{ab}(x). \tag{189}$$

Local Lorentz transformations act on the supervielbein as follows:

$$\delta_\Lambda E^a_\mu = E^b_\mu \mathbf{\Lambda}_b{}^a, \qquad \delta_\Lambda \widehat{E}^\mu_a = \widehat{E}^\mu_b \mathbf{\Lambda}^b{}_a. \tag{190}$$

Using the definition (175), one finds the following:

$$\delta_\Lambda \mathbf{\Omega}_\mu = \partial_\mu \mathbf{\Lambda} + \frac{1}{2}[\mathbf{\Omega}_\mu, \mathbf{\Lambda}], \tag{191}$$

or

$$\delta_\Lambda \mathbf{\Omega}^{ab}_\mu = \partial_\mu \mathbf{\Lambda}^{ab} + \mathbf{\Omega}^{cb}_\mu \mathbf{\Lambda}_c{}^a + \mathbf{\Omega}^{ac}_\mu \mathbf{\Lambda}_c{}^b. \tag{192}$$

With this and

$$[\hat{\gamma}^\mu, \mathbf{\Lambda}] = 2E^\mu_a \mathbf{\Lambda}^{ab}\gamma_b, \tag{193}$$

one can easily prove that (187) is invariant under local Lorentz transformations.

6. Superfield Formalism and Consistent Anomalies

The superfield formalism nicely applies to the description of consistent anomalies. In this section, we first summarize the definitions and properties of gauge anomalies and then we apply to them the superfield description. Basic material for the following algebraic approach to anomalies can be found in [11,38,39,57–59].

Here, formulas refer to a d-dimensional spacetime M without boundary.

6.1. BRST, Descent Equations and Consistent Gauge Anomalies

The BRST operation $\mathfrak{s}$ in (3) is nilpotent. We represent with the same symbol $\mathfrak{s}$ the corresponding functional operator, i.e.,

$$\mathfrak{s} = \int d^d x \left(\mathfrak{s}A^a_\mu(x)\frac{\partial}{\partial A^a_\mu(x)} + \mathfrak{s}c^a(x)\frac{\partial}{\partial c^a(x)} \right). \tag{194}$$

To construct the descent equations, we start from a symmetric polynomial in the Lie algebra of degree n, i.e., $P_n(T^{a_1}, ..., T^{a_n})$, invariant under the adjoint transformations:

$$P_n([X, T^{a_1}], ..., T^{a_n}) + \ldots + P_n(T^{a_1}, ..., [X, T^{a_n}]) = 0, \tag{195}$$

for any element X of the Lie algebra $\mathfrak{g}$. In many cases, these polynomials are symmetric traces of the generators in the corresponding representation as follows:

$$P_n(T^{a_1}, ..., T^{a_n}) = Str(T^{a_1}...T^{a_n}), \tag{196}$$

(Str denotes the symmetric trace). With this, one can construct the $2n$-form

$$\Delta_{2n}(A) = P_n(F, F, \ldots F), \tag{197}$$

where $F = dA + \frac{1}{2}[A, A]$. It is easy to prove the following:

$$P_n(F, F, \dots F) = d\left(n \int_0^1 dt\, P_n(A, F_t, \dots, F_t)\right) \equiv d\Delta^{(0)}_{2n-1}(A), \tag{198}$$

where we have introduced the symbols $A_t = tA$ and its curvature $F_t = dA_t + \frac{1}{2}[A_t, A_t]$, where $0 \le t \le 1$. In the above expressions, the product of the forms is understood to be the exterior product. It is important to recall that in order to prove Equation (198), one uses in an essential way the symmetry of P_n and the graded commutativity of the exterior product of forms.

$\Delta^{(0)}_{2n-1}(A)$ is often denoted also as $TP_n(A)$. The following is known as the *transgression formula*:

$$TP_n(A) = n \int_0^1 dt\, P_n(A, F_t, \dots, F_t), \tag{199}$$

Equation (198) is the first of a sequence of equations that can be proven:

$$\Delta_{2n}(A) - d\Delta^{(0)}_{2n-1}(A) = 0, \tag{200}$$
$$s\Delta^{(0)}_{2n-1}(A) + d\Delta^{(1)}_{2n-2}(A, c) = 0, \tag{201}$$
$$s\Delta^{(1)}_{2n-2}(A, c) + d\Delta^{(2)}_{2n-3}(A, c) = 0, \tag{202}$$
$$\dots\dots$$
$$s\Delta^{(2n-1)}_0(c) = 0. \tag{203}$$

All the expressions $\Delta^{(p)}_k(A, c)$ are polynomials of and A, c, cdA, cdc and their commutators. The lower index k is the form degree, and the upper one p is the ghost number, i.e., the number of c factors. The last polynomial $\Delta^{(2n-1)}_0(c)$ is a 0-form and clearly a function only of c. All these polynomials have an explicit compact form. For instance, the next interesting case after Equation (200) is the following:

$$s\,\Delta_{2n-1}(A) = -d\left(n(n-1)\int_0^1 dt(1-t)P_n(dc, A, F_t, \dots F_t)\right). \tag{204}$$

This means, in particular, that integrating $\Delta_{2n-1}(A)$ over spacetime in $\mathsf{d} = 2n-1$ dimensions, we obtain an invariant local expression. This gives the gauge CS action in any odd dimension. What matters here is that the RHS contains the general expression of the consistent gauge anomaly in $\mathsf{d} = 2n-2$ dimension. Integrating (202) over spacetime, one obtains the following:

$$s\,\mathcal{A}[c, A] \;=\; 0, \tag{205}$$
$$\mathcal{A}[c, A] \;=\; \int d^d x\, \Delta^{(1)}_d(c, A), \qquad \text{where}$$
$$\Delta^1_d(c, A) \;=\; n(n-1)\int_0^1 dt(t-1)P_n(dc, A, F_t, \dots F_t), \tag{206}$$

where $\mathcal{A}[c, A]$ identifies the anomaly up to an overall numerical coefficient.

Thus, the existence of chiral gauge anomalies relies on the existence of the adjoint–invariant polynomials P_n. One can prove that the so-obtained cocycles are non-trivial.

Although the above formulas are formally correct, one should remark that, in order to describe a consistent anomaly in a $\mathsf{d} = 2n-2$ dimensional spacetime, we need two forms, $P_n(F, \dots, F)$ and $\Delta^{(0)}_{2n-1}(A)$, which are identically vanishing. This is an unsatisfactory aspect of the previous purely algebraic approach. The superfield formalism overcomes this difficulty and gives automatically the anomaly as well as its descendants [60].

6.2. Superfield Formalism, BRST Transformations and Anomalies

For simplicity, we introduce only one anticommuting variable ϑ and consider the superconnection (where $\tilde{d} = d + \frac{\partial}{\partial\vartheta} d\vartheta$) as follows:

$$\mathcal{A} = e^{-\vartheta c}(\tilde{d} + A)e^{\vartheta c} = A + \vartheta(dc + [A, c]) + \left(c - \vartheta \frac{1}{2}[c, c]\right) d\vartheta \equiv \phi + \eta\, d\vartheta. \tag{207}$$

The supercurvature is the following:

$$\mathcal{F} = e^{-\vartheta c} F e^{\vartheta c} = F + \vartheta[F, c]. \tag{208}$$

From these formulas, it is immediately visible that the derivative with respect to ϑ corresponds to the BRST transformation:

$$\begin{aligned}
\frac{\partial}{\partial\vartheta}\phi &= dc + [A, c] = D_A c = \mathfrak{s}A, \\
\frac{\partial}{\partial\vartheta}\eta &= -\frac{1}{2}[c, c] = \mathfrak{s}c, \\
\frac{\partial}{\partial\vartheta}\mathcal{F} &= [F, c] = \mathfrak{s}F.
\end{aligned}$$

Let us now consider the transgression formula as follows:

$$TP_n(\mathcal{A}) = n \int_0^1 dt\, P_n(\mathcal{A}, \mathcal{F}_t, \ldots, \mathcal{F}_t), \tag{209}$$

in which we have replaced A everywhere with $\mathcal{A}$, and F_t with $\mathcal{F}_t = t\,\tilde{d}\mathcal{A} + \frac{t^2}{2}[\mathcal{A}, \mathcal{A}]$ (t is an auxiliary parameter varying from 0 to 1).

The claim is that (209) contains all the information about the gauge anomaly, including the explicit form of all its descendants. To see this, it is enough to expand the polyform $TP_n(\mathcal{A})$ in component forms as follows:

$$TP_n(\mathcal{A}) = \sum_{i=0}^{2n-1} \tilde{\Delta}^{(i)}_{2n-i-1}(\phi, \eta) \underbrace{d\vartheta \wedge \ldots \wedge d\vartheta}_{i \text{ factors}}, \tag{210}$$

where $2n - i - 1$ is the spacetime form degree. Notice that the wedge product is commutative for the $d\vartheta$ factors. Of course, both $\tilde{\Delta}^{(0)}_{2n-1}(\phi)|_{\vartheta=0} = TP_n(A)$ and $\tilde{\Delta}^{(0)}_{2n-1}(\phi)$ vanish in dimension $\mathsf{d} = 2n - 2$. However, the remaining forms are nonvanishing.

Let us extract the term $\tilde{\Delta}^{(1)}_{2n-2}(A, c)$:

$$\tilde{\Delta}^{(1)}_{2n-2}(\phi, \eta) \;=\; n \int_0^1 dt \left(P_n(\eta, \tilde{F}_t, \ldots, \tilde{F}_t) + (n-1) P_n(\phi, t(d\eta - \partial_\vartheta \phi) + t^2[\phi, \eta], \tilde{F}_t, \ldots, \tilde{F}_t) \right), \tag{211}$$

where $\tilde{F}_t = t d\phi + \frac{t^2}{2}[\phi, \phi]$. Let us take the $\vartheta = 0$ part of this.

$$\tilde{\Delta}^{(1)}_{2n-2}(A, c) = n \int_0^1 dt \left(P_n(c, F_t, \ldots, F_t) + (n-1) P_n(A, (t^2 - t)[A, c], F_t, \ldots, F_t) \right). \tag{212}$$

Using $\int_0^1 dt(1-t)\frac{d}{dt} f(t) = \int_0^1 dt f(t)$ when $f(0) = 0$, we can rewrite this as follows:

$$\tilde{\Delta}^{(1)}_{2n-2}(A, c) = n(n-1) \int_0^1 dt\, (1-t) \left(P_n(c, \frac{dF_t}{dt}, \ldots, F_t) - P_n(A, [tA, c], F_t, \ldots, F_t) \right). \tag{213}$$

Using $\frac{dF_t}{dt} = D_{tA}A$ and the ad invariance of P_n, we obtain finally the following:

$$\begin{aligned}\tilde{\Delta}^{(1)}_{2n-2}(A,c) &= d\left(n(n-1)\int_0^1 dt\,(1-t)\Big(P_n(c,A,\ldots,F_t)\Big)\right.\\ &\quad -n(n-1)\int_0^1 dt\,(1-t)P_n(dc,A,F_t,\ldots,F_t).\end{aligned} \tag{214}$$

Therefore $\tilde{\Delta}^{(1)}_{2n-2}(A,c)$ coincides with the opposite of $\Delta^{(1)}_{2n-2}(A,c)$ up to a total spacetime derivative, which is irrelevant for the integrated anomaly.

The ϑ derivative of $\tilde{\Delta}^{(1)}_{2n-2}(\phi,\eta)$ is the BRST transformation of $\tilde{\Delta}^{(1)}_{2n-2}(A,c)$, and turns out to be a total spacetime derivative. This can be checked with an explicit calculation.

Remark 2. *The cocycles Δ and $\tilde{\Delta}$ may differ. For instance, in the case $\mathsf{d}=2, n=2$, we obtain the following:*

$$\tilde{\Delta}^{(1)}_2(A,c) = P_2(c,dA), \qquad \tilde{\Delta}^{(2)}_1(A,c) = \frac{1}{2}P_2(A,[c,c]), \qquad \tilde{\Delta}^{(3)}_0(c) = \frac{1}{6}P_2(c,[c,c]), \tag{215}$$

while

$$\Delta^{(1)}_2(A,c) = P_2(dc,A), \qquad \Delta^{(2)}_1(c) = P_2(dc,c), \qquad \Delta^{(3)}_0(c) = \frac{1}{6}P_2(c,[c,c]). \tag{216}$$

This originates from the difference of a total derivative between $\tilde{\Delta}^{(1)}_2(A,c)$ and $\Delta^{(1)}_2(A,c)$.

6.3. Anomalies with Background Connection

The expressions of anomalies introduced so far are generally well defined in a local patch of spacetime, but they may not be globally well defined on the whole spacetime (they may not be basic forms in the language of fiber bundles, i.e., they may be well-defined forms in the total space but not in the base spacetime). To obtain globally well defined anomalies, we need to introduce a background connection A_0, which is invariant under BRST transformations: $\mathfrak{s}A_0 = 0$. The dynamical connection transforms, instead, in the usual way; see (3). We call F and F_0 the curvatures of A and A_0. Since the space of connections is affine, also $\widehat{A}_t = tA + (1-t)A_0$, with $0 \le t \le 1$ is a connection. We call $\widehat{F}_t$ its curvature, which takes the values F and F_0 for $t=1$ and $t=0$, respectively.

The relevant connection is now the following:

$$\begin{aligned}\widehat{\mathcal{A}}_t &= te^{-\vartheta c}(\tilde{d}+A)e^{\vartheta c} + (1-t)A_0 = t\mathcal{A} + (1-t)A_0\\ &= t\left(A + \vartheta(dc+[A,c]) + \left(c - \vartheta\frac{1}{2}[c,c]\right)d\vartheta\right) + (1-t)A_0 \equiv \hat{\phi}_t + \hat{\eta}_t\,d\vartheta,\end{aligned} \tag{217}$$

where $\hat{\phi}_t = t\phi + (1-t)A_0$ and $\hat{\eta}_t = t\eta$. We call $\widehat{\mathcal{F}}_t$ the curvature of $\widehat{\mathcal{A}}_t$. Notice that $\widehat{\mathcal{F}}_1 = \mathcal{F}$ and $\widehat{\mathcal{F}}_0 = F_0$, which is straightforward to be checked.

We start again from the transgression formula as follows:

$$\mathcal{T}P_n(\mathcal{A},A_0) = n\int_0^1 dt\,P_n(\mathcal{A}-A_0,\widehat{\mathcal{F}}_t,\ldots,\widehat{\mathcal{F}}_t). \tag{218}$$

In the same way as before, one can prove that, if we assume the spacetime dimension is $\mathsf{d} = 2n-2$, we have the following:

$$\tilde{d}\,\mathcal{T}P_n(\mathcal{A},A_0) = P_n(\mathcal{F},\ldots,\mathcal{F}) - P_n(F_0,\ldots,F_0) = P_n(F,\ldots,F) - P_n(F_0,\ldots,F_0) = 0. \tag{219}$$

As before, we decompose the following:

$$\mathcal{T}P_n(\mathcal{A}, A_0) = \sum_{i=0}^{2n-1} \widehat{\Delta}^{(i)}_{2n-i-1}(\phi, \eta, A_0)\, \underbrace{d\vartheta \wedge \ldots \wedge d\vartheta}_{i \text{ factors}}. \tag{220}$$

The relevant term for the anomaly is $\Delta^{(1)}_{2n-2}(\phi, \eta, A_0)$, i.e.,

$$\begin{aligned}\widehat{\Delta}^{(1)}_{2n-2}(\phi, \eta, A_0) \quad &= \quad n\int_0^1 dt\, \Big[P_n\Big(\phi, \widehat{\Phi}_t, \ldots, \widehat{\Phi}_t\Big) + P_n\Big(\mathcal{A} - A_0, \Big(d_{\hat{\phi}_t}\hat{\eta}_t - \frac{\partial}{\partial\vartheta}\hat{\phi}_t\Big), \widehat{\Phi}_t, \ldots, \widehat{\Phi}_t\Big) \\ &\quad + \ldots + P_n\Big(\mathcal{A} - A_0, \widehat{\Phi}_t, \ldots, \widehat{\Phi}_t, \Big(d_{\hat{\phi}_t}\hat{\eta}_t - \frac{\partial}{\partial\vartheta}\hat{\phi}_t\Big)\Big)\Big],\end{aligned} \tag{221}$$

where $d_{\hat{\phi}_t} = d + [\hat{\phi}_t, \cdot]$, and $\widehat{\Phi}_t = d\hat{\phi}_t + \frac{1}{2}[\hat{\phi}_t, \hat{\phi}_t]$. We have to select the $\vartheta = 0$ part

$$\begin{aligned}\widehat{\Delta}^{(1)}_{2n-2}(\phi, \eta, A_0)\Big|_{\vartheta=0} \quad &= \quad n\int_0^1 dt\, \Big[P_n\Big(c, \widehat{F}_t, \ldots, \widehat{F}_t\Big) - t(1-t)P_n\Big(A - A_0, [A - A_0, c], \widehat{F}_t, \ldots, \widehat{F}_t\Big) \\ &\quad - \ldots - t(1-t)P_n\Big(A - A_0, \widehat{F}_t, \ldots, \widehat{F}_t, [A - A_0, c]\Big)\Big].\end{aligned} \tag{222}$$

This is the chiral anomaly with background connection. It can be written in a more familiar form by rewriting the first term on the RHS, using $\int_0^1 dt(1-t)\frac{d}{dt}f(t) + f(0) = \int_0^1 dt f(t)$:

$$\begin{aligned}\widehat{\Delta}^{(1)}_{2n-2}(\phi, \eta, A_0)\Big|_{\vartheta=0} \quad &= \quad nP_n(c, F_0, \ldots, F_0) - n(n-1)\int_0^1 dt\,(1-t)P_n(d_{A_0}c, A - A_0, \widehat{F}_t, \ldots, \widehat{F}_t) \\ &\quad + d\Big(n(n-1)\int_0^1 dt\,(1-t)P_n(c, A - A_0, \widehat{F}_t, \ldots, \widehat{F}_t)\Big).\end{aligned} \tag{223}$$

Integrating over the spacetime M, the last term drops out. If we set $A_0 = 0$, we recover the formula (214). We notice that, as expected, the RHS of (223) is a basic quantity.

6.4. Wess-Zumino Terms in Field Theories with the Superfield Method

In a gauge field theory with connection A, valued in a Lie algebra with generators T^a and structure constants f^{abc}, an anomaly $\mathcal{A}^a$ must satisfy the WZ consistency conditions [61]:

$$X^a(x)\mathcal{A}^b(y) - X^b(y)\mathcal{A}^a(x) + f^{abc}\mathcal{A}^c(x)\delta(x-y) = 0, \tag{224}$$

where

$$X^a(x) = \partial_\mu \frac{\delta}{\delta A^a_\mu(x)} + f^{abc}A^b_\mu(x)\frac{\delta}{\delta A^c_\mu(x)}. \tag{225}$$

Equation (224) is integrability conditions. This means that one can find a functional of the fields $\mathcal{B}_{WZ}$ such that the following holds:

$$X^a(x)\mathcal{B}_{WZ} = \mathcal{A}^a(x). \tag{226}$$

In this section, we show how to construct a term $\mathcal{B}_{WZ}$ which, upon BRST variation, generates the following anomaly:

$$\mathcal{A} = -\int_{\mathsf{M}} c^a(x)\mathcal{A}^a(x) = n(n-1)\int_{\mathsf{M}}\int_0^1 dt\,(1-t)P_n(dc, A, F_t, \ldots, F_t), \tag{227}$$

where M is the spacetime of dimension $\mathsf{d} = 2n-2$, and $F_t = tdA + \frac{t^2}{2}[A, A]$.

This is possible provided we enlarge the set of fields of the theory by adding new fields as follows. Let us introduce a set of auxiliary fields $\sigma(x) = \sigma^a(x)T^a$, which under a gauge transformations with parameters $\lambda(x) = \lambda^a(x)T^a$, transform as the following:

$$e^{\sigma(x)} \longrightarrow e^{\sigma'(x)} = e^{-\lambda(x)}e^{\sigma(x)}. \tag{228}$$

Using the Campbell–Hausdorff formula, this means $\delta\sigma(x) = -\lambda(x) - \frac{1}{2}[\lambda(x), \sigma(x)] + \ldots$. Next, we pass to the infinitesimal transformations and replace the infinitesimal parameter $\lambda(x)$ with anticommuting fields $c(x) = c^a(x)T^a$. We have the following BRST transformations:

$$\mathfrak{s}e^{\sigma(x)} = -c(x)e^{\sigma(x)}, \qquad \mathfrak{s}\sigma(x) = -c(x) + \frac{1}{2}[\sigma(x), c(x)] - \frac{1}{12}[\sigma(x), [\sigma(x), c(x)]] + \ldots \tag{229}$$

Now, we use a superspace technique by adding to the spacetime coordinates x^μ and the anticommuting one ϑ, but, simultaneously, we enlarge the spacetime with the addition of an auxiliary commuting parameter, s, $0 \le s \le 1$. So the local coordinates in the superspace are (x^μ, s, ϑ). In particular, we have the following:

$$e^{-s(\sigma+\vartheta\mathfrak{s}\sigma)} = e^{-s\sigma} + \vartheta\,\mathfrak{s}e^{-s\sigma}. \tag{230}$$

On this superspace, the superconnection is the following:

$$\begin{aligned} \widetilde{\mathcal{A}}(x,s,\vartheta) &= e^{-s(\sigma+\vartheta\mathfrak{s}\sigma)}e^{-\vartheta c}(\tilde{d} + A)e^{\vartheta c}e^{s(\sigma+\vartheta\mathfrak{s}\sigma)} \\ &= e^{-s(\sigma+\vartheta\mathfrak{s}\sigma)}\left(\mathcal{A} + d + \frac{\partial}{\partial s}ds\right)e^{s(\sigma+\vartheta\mathfrak{s}\sigma)}, \end{aligned} \tag{231}$$

where $\tilde{d} = d + \frac{\partial}{\partial s}ds + \frac{\partial}{\partial\vartheta}d\vartheta$, and $\mathcal{A} = A + \vartheta(dc + [A, c]) + (c - \vartheta cc)d\vartheta$. We decompose $\widetilde{\mathcal{A}}$ as follows:

$$\begin{aligned} \widetilde{\mathcal{A}}(x,s,\vartheta) &= \phi(x,s,\vartheta) + \phi_s(x,s,\vartheta)ds + \phi_\vartheta(x,s,\vartheta)d\vartheta, \\ \phi(x,s,\vartheta) &= A_s + \vartheta(dc_s + [A_s, c_s]), \\ \phi_s(x,s,\vartheta) &= \sigma + \vartheta\mathfrak{s}\sigma, \\ \phi_\vartheta(x,s,\vartheta) &= c_s - \vartheta c_s c_s, \end{aligned} \tag{232}$$

where

$$\begin{aligned} A_s &= e^{-s\sigma}A\,e^{s\sigma} + e^{-s\sigma}de^{s\sigma}, \\ c_s &= e^{-s\sigma}c\,e^{s\sigma} + e^{-s\sigma}\mathfrak{s}e^{s\sigma}. \end{aligned} \tag{233}$$

In particular, $A_0 = A$ and c_s interpolates between c and 0. Since the derivative with respect to ϑ corresponds to the BRST transformation, we deduce the following:

$$\mathfrak{s}A_s = dc_s + [A_s, c_s], \qquad \mathfrak{s}c_s = -c_s c_s. \tag{234}$$

Moreover, if we denote by $\widetilde{\mathcal{F}}, \mathcal{F}$ and F the curvatures of $\widetilde{\mathcal{A}}, \mathcal{A}$ and A, respectively, we have the following:

$$\widetilde{\mathcal{F}} = e^{-s(\sigma+\vartheta\mathfrak{s}\sigma)}\mathcal{F}\,e^{s(\sigma+\vartheta\mathfrak{s}\sigma)} = e^{-s(\sigma+\vartheta\mathfrak{s}\sigma)}e^{-\vartheta c}A\,e^{s(\sigma+\vartheta\mathfrak{s}\sigma)}e^{\vartheta c}. \tag{235}$$

Now, suppose the spacetime M has dimension d; choose any ad-invariant polynomial P_n with $n = \frac{\mathsf{d}}{2} - 1$. Then, the following holds:

$$P_n(\widetilde{\mathcal{F}}, \ldots, \widetilde{\mathcal{F}}) = P_n(\mathcal{F}, \ldots, \mathcal{F}) = P_n(F, \ldots, F) = 0, \tag{236}$$

where the last equality holds for dimensional reasons. Now, we can write the following:

$$P_n(\widetilde{\mathcal{F}},\ldots,\widetilde{\mathcal{F}}) = \tilde{d}\Big(n\int_0^1 dt\, P_n(\widetilde{\mathcal{A}},\widetilde{\mathcal{F}}_t,\ldots,\widetilde{\mathcal{F}}_t)\Big) \equiv \tilde{d}\left(TP_n(\widetilde{\mathcal{A}})\right). \tag{237}$$

For notational simplicity, let us set $\widetilde{\mathcal{Q}}(\widetilde{\mathcal{A}}) \equiv TP_n(\widetilde{\mathcal{A}})$ and decompose it in the various components according to the form degree as follows:

$$\widetilde{\mathcal{Q}} = \sum_{\substack{k,i,j \\ k+i+j=2n-1}} \widetilde{\mathcal{Q}}^j_{(k,i)}, \qquad \widetilde{\mathcal{Q}}^j_{(k,i)} = \left(\mathcal{Q}^j_{(k,i)} + \vartheta\hat{\mathcal{Q}}^j_{(k,i)}\right)(d\vartheta)^j(ds)^i, \tag{238}$$

where k denotes the form degree in spacetime, j is the ghost number, and i is either 0 or 1. Next, let us decompose the equation $\tilde{d}\widetilde{\mathcal{Q}} = 0$ into components, and select, in particular, the component $(2n-2,1,1)$, i.e., the following:

$$0 = d\widetilde{\mathcal{Q}}^1_{(2n-3,1)} + \frac{\partial}{\partial\vartheta}\widetilde{\mathcal{Q}}^0_{(2n-2,1)}d\vartheta + \frac{\partial}{\partial s}\widetilde{\mathcal{Q}}^1_{(2n-2,0)}ds, \tag{239}$$

and let us integrate it over M and s. We obtain the following:

$$0 = \int_{\mathsf{M}}\int_0^1 ds\, \mathfrak{s}\mathcal{Q}^0_{(2n-2,1)} + \int_{\mathsf{M}}\int_0^1 ds\, \frac{\partial}{\partial s}\mathcal{Q}^1_{(2n-2,0)}. \tag{240}$$

Since $\mathcal{Q}^1_{(2n-2,0})$ is linear in c_s and $c_0 = c, c_1 = 0$, we obtain finally the following:

$$\int_{\mathsf{M}} \mathcal{Q}^1_{(2n-2,0)} = \mathfrak{s}\int_{\mathsf{M}}\int_0^1 ds\, \mathcal{Q}^0_{(2n-2,1)}. \tag{241}$$

Now, we remark that $\int_{\mathsf{M}} \mathcal{Q}^1_{(2n-2,0})$ is linear in c and coincides precisely with the anomaly. On the other hand, $\mathcal{Q}^0_{(2n-2,1)}$ has the same expression as the anomaly with c replaced by σ and A by A_s, i.e., the following:

$$\mathcal{Q}^0_{(2n-2,1)}(\sigma, A_s) = n(n-1)\int_0^1 dt\,(1-t)\, P_n(d\sigma, A_s, F_{s,t},\ldots,F_{s,t}), \tag{242}$$

where $F_{s,t} = tdA_s + \frac{t^2}{2}[A_s, A_s]$. We call

$$\mathcal{B}_{\mathrm{WZ}} = \int_{\mathsf{M}}\int_0^1 ds\, \mathcal{Q}^0_{(2n-2,1)}, \tag{243}$$

the *Wess–Zumino term.*

The existence of $\mathcal{B}_{WZ}$ for any anomaly seems to contradict the non-triviality of anomalies. This is not so because the price we have to pay to construct the term (243) is the introduction of the new fields σ^a, which are not present in the initial theory. The proof of the non-triviality of anomalies is based on a definite differential space formed by c, A and their exterior derivatives and commutators, which constrains the anomaly to be a polynomial in these fields. Of course, in principle, it is not forbidden to enlarge the theory by adding new fields plus the WZ term. However, the resulting theory is different from the initial one. Moreover, the σ^a fields have zero canonical dimension. This means that, except in 2d, it is possible to construct new invariant action terms with more than two derivatives, which renders renormalization problematic.

7. HS-YM Models and Superfield Method

In this section, we apply the superfield method to higher-spin[5] Yang–Mills (HS-YM) models. These models are characterized by a local gauge symmetry, the higher spin symmetry, with infinite parameters, encompassing, in particular, both ordinary gauge transformations and diffeomorphisms. In a sense, they unify ordinary gauge and gravitational theories. This makes them interesting in themselves but particularly for the superfield method, to whose bases they seem to perfectly adhere. These models were only recently introduced in the literature, and they are largely unexplored. For this reason, we devote a rather long and hopefully sufficiently detailed introduction .

HS-YM models in Minkowski spacetime are formulated in terms of master fields $h_a(x,p)$, which are local in the phase space (x,p), with $[\hat{x}^\mu, \hat{p}_\nu] = i\hbar\delta^\mu_\nu$ ($\hbar$ will be set to the value 1), where $\hat{x}, \hat{p}$ are the operators whose classical symbols are x, p, according to the Weyl–Wignar quantization. The master field can be expanded in powers of p as follows:

$$\begin{aligned} h_a(x,p) &= \sum_{n=0}^{\infty} \frac{1}{n!} h_a^{\mu_1 \dots, \mu_n}(x) p_{\mu_1} \dots p_{\mu_n} \\ &= A_a(x) + \chi_a^\mu(x) p_\mu + \frac{1}{2} b_a^{\mu\nu}(x) p_\mu p_\nu + \frac{1}{6} c_a^{\mu\nu\lambda}(x) p_\mu p_\nu p_\lambda + \dots, \end{aligned} \tag{244}$$

where $h_a^{\mu_1 \dots \mu_n}(x)$ are ordinary tensor fields, symmetric in $\mu_1, \dots, \mu_n$. The indices $\mu_1, \dots, \mu_n$ are upper (contravariant) Lorentz indices, $\mu_i = 0, \dots, d-1$. The index a is also a vector index, but it is of a different nature. In fact, it will be interpreted as a flat index and h_a will be referred to as a *frame-like master field*. Of course, when the background metric is flat, all indices are on the same footing, but it is preferable to keep them distinct to facilitate the correct interpretation.

The master field $h^a(x,p)$ can undergo the following (HS) gauge transformations, whose infinitesimal parameter $\varepsilon(x,p)$ is itself a master field:

$$\delta_\varepsilon h_a(x,p) = \partial_a^x \varepsilon(x,p) - i[h_a(x,p) \overset{*}{,} \varepsilon(x,p)] \equiv \mathcal{D}_a^{*x} \varepsilon(x,p), \tag{245}$$

where we have introduced the following covariant derivative:

$$\mathcal{D}_a^{*x} = \partial_a^x - i[h_a(x,p) \overset{*}{,} \quad]. \tag{246}$$

The $*$ product is the Moyal product, defined by the following:

$$f(x,p) * g(x,p) = f(x,p) e^{\frac{i}{2}\left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right)} g(x,p), \tag{247}$$

between two regular phase-space functions $f(x,p)$ and $g(x,p)$.

Like in ordinary gauge theories, we use the compact notation $d = \partial_a\, dx^a, h = h_a dx^a$ and write (245) as the following:

$$\delta_\varepsilon h(x,p) = d\varepsilon(x,p) - i[h(x,p) \overset{*}{,} \varepsilon(x,p)] \equiv D\varepsilon(x,p), \tag{248}$$

where it is understood that $[h(x,p) \overset{*}{,} \varepsilon(x,p)] = [h_a(x,p) \overset{*}{,} \varepsilon(x,p)] dx^a$.

Next, we introduce the curvature notation as follows:

$$G = dh - \frac{i}{2}[h \overset{*}{,} h], \tag{249}$$

with the transformation property $\delta_\varepsilon G = -i[G \overset{*}{,} \varepsilon]$.

The action functionals we consider are integrated polynomials of G or of its components G_{ab}. To imitate ordinary non-Abelian gauge theories, we need a 'trace property', similar to the trace of polynomials of Lie algebra generators. In this framework, we have the following:

$$\langle\langle f * g \rangle\rangle \equiv \int d^d x \int \frac{d^d p}{(2\pi)^d} f(x,p) * g(x,p) = \int d^d x \int \frac{d^d p}{(2\pi)^d} f(x,p) g(x,p) = \langle\langle g * f \rangle\rangle. \tag{250}$$

From this, plus associativity, it follows that

$$\begin{aligned}&\langle\langle f_1 * f_2 * \ldots * f_n \rangle\rangle = \langle\langle f_1 * (f_2 * \ldots * f_n) \rangle\rangle \\ &= (-1)^{\epsilon_1(\epsilon_2+\ldots+\epsilon_n)} \langle\langle (f_2 * \ldots * f_n) * f_1 \rangle\rangle = (-1)^{\epsilon_1(\epsilon_2+\ldots+\epsilon_n)} \langle\langle f_2 * \ldots * f_n * f_1 \rangle\rangle,\end{aligned} \tag{251}$$

where ϵ_i is the Grassmann degree of f_i (this is usually referred to as *cyclicity* property).

This property holds also when the f_i is valued in a (finite dimensional) Lie algebra, provided that the symbol $\langle\langle \quad \rangle\rangle$ includes also the trace over the Lie algebra generators.

The HS Yang-Mills action. The curvature components, see (249), are as follows:

$$G_{ab} = \partial_a h_b - \partial_b h_a - i[h_a \overset{*}{,} h_b], \tag{252}$$

with the following transformation rule:

$$\delta_\varepsilon G_{ab} = -i[G_{ab} \overset{*}{,} \varepsilon]. \tag{253}$$

If we consider the functional $\langle\langle G^{ab} * G_{ab} \rangle\rangle$, it follows from the above that

$$\delta_\varepsilon \langle\langle G^{ab} * G_{ab} \rangle\rangle = -i \langle\langle G^{ab} * G_{ab} * \varepsilon - \varepsilon * G^{ab} * G_{ab} \rangle\rangle = 0. \tag{254}$$

Therefore,

$$\mathcal{YM}(h) = -\frac{1}{4g^2} \langle\langle G^{ab} * G_{ab} \rangle\rangle, \tag{255}$$

is invariant under HS gauge transformations and it is a well-defined functional. This is the HS-YM-like action.

From (255), we obtain the following eom:

$$\partial_b G^{ab} - i[h_b \overset{*}{,} G^{ab}] \equiv \mathcal{D}_b^* G^{ab} = 0, \tag{256}$$

which is covariant by construction under HS gauge transformations

$$\delta_\varepsilon \left(\mathcal{D}_b^* G^{ab} \right) = -i[\mathcal{D}_b^* G^{ab}, \varepsilon]. \tag{257}$$

We recall that $\mathcal{D}_b^*$ is the covariant $*$-derivative and ε, the HS gauge parameter.

All that has been said so far can be repeated for non-Abelian models with minor changes. For simplicity, here, we limit ourselves to the Abelian case.

Gravitational interpretation. The novel property of HS YM-like theories is that, nothwithstanding their evident similarity with ordinary YM theories, they can describe also gravity. To see this, let us expand the gauge master parameter $\varepsilon(x,p)$:

$$\varepsilon(x,p) \quad = \quad \epsilon(x) + \xi^\mu(x) p_\mu + \frac{1}{2} \Lambda^{\mu\nu}(x) p_\mu p_\nu + \frac{1}{3!} \Sigma^{\mu\nu\lambda}(x) p_\mu p_\nu p_\lambda + \ldots \tag{258}$$

In Appendix C, we show that the parameter $\epsilon(x)$ is the usual $U(1)$ gauge parameter for the field $A_a(x)$, while $\xi^\mu(x)$ is the parameter for general coordinate transformations, and $\chi_a^\mu(x)$ can be interpreted as the fluctuating inverse vielbein field.

Scalar and spinor master fields. To HS YM-like theories, we can couple matter-type fields of any spin, for instance, a complex multi-index boson field,

$$\Phi(x,p) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{\mu_1 \mu_2 \ldots \mu_n}(x) p_{\mu_1} p_{\mu_2} \ldots p_{\mu_n}, \tag{259}$$

which, under a master gauge transformation (245), transforms like $\delta_\varepsilon \Phi = i\varepsilon * \Phi$. The covariant derivative is $\mathcal{D}_a^* \Phi = \partial_a \Phi - i h_a * \Phi$ with the property $\delta_\varepsilon \mathcal{D}_a^* \Phi = i\, \varepsilon * \mathcal{D}_a^* \Phi$. With these properties, the kinetic action term $\frac{1}{2}\langle\langle(\mathcal{D}^a_*\Phi)^\dagger * \mathcal{D}^*_a\Phi\rangle\rangle$ and the potential terms $\langle\langle(\Phi^\dagger * \Phi)^n_*\rangle\rangle$ are HS-gauge invariant.

In a quite similar manner, we can introduce master spinor fields,

$$\Psi(x,p) = \sum_{n=0}^{\infty}\frac{1}{n!}\Psi^{\mu_1\ldots\mu_n}_{(n)}(x)p_{\mu_1}\ldots p_{\mu_n}, \tag{260}$$

where $\Psi_{(0)}$ is a Dirac field. The HS gauge transformations are $\delta_\varepsilon\Psi = i\varepsilon * \Psi$, and the covariant derivative is $\mathcal{D}^*_a\Psi = \partial_a\Psi - ih_a * \Psi$ with $\delta_\varepsilon(\mathcal{D}^*_a\Psi) = i\varepsilon * (\mathcal{D}^*_a\Psi)$.

With these properties, the following action integral is invariant.

$$S(\Psi,h) = \langle\langle\overline{\Psi}i\gamma^a\mathcal{D}_a\Psi\rangle\rangle = \langle\langle\overline{\Psi}\gamma^a(i\partial_a + h_a*)\Psi\rangle\rangle, \tag{261}$$

BRST quantization of HS Yang-Mills. Fixing the Lorenz gauge with parameter α and applying the standard Faddeev–Popov approach, the quantum action becomes the following:

$$\mathcal{YM}(h_a,c,B) = \frac{1}{g^2}\langle\langle -\frac{1}{4}G_{ab} * G^{ab} - h^a * \partial_a B - i\partial^a\bar{c} * \mathcal{D}^*_a c + \frac{\alpha}{2}B * B\rangle\rangle, \tag{262}$$

where $c,\bar{c}$ and B are the ghost, antighost and Nakanishi–Lautrup master fields, respectively. $c,\bar{c}$ are anticommuting fields, while B is commuting.

The action (262) is invariant under the following BRST transformations:

$$sh_a = \mathcal{D}^*_a c, \qquad sc = ic * c = \frac{i}{2}[c \overset{*}{,} c], \qquad s\bar{c} = iB, \qquad sB = 0, \tag{263}$$

which are nilpotent. In particular, $s(\mathcal{D}^*_a c) = 0$ and $s(c * c) = 0$.

7.1. Anomalies in HS Theories

The effective action of HS-YM theories is functional of the master field $h_a(x,p)$ defined as follows:

$$\begin{aligned}\mathcal{W}[h] &= \mathcal{W}[0] \\ &+ \sum_{n=1}^{\infty}\frac{i^{n-1}}{n!}\int\prod_{i=1}^{n}d^dx_i\,\frac{d^dp_i}{(2\pi)^d}\,\langle J_{a_1}(x_1,p_1)\ldots J_{a_n}(x_n,p_n)\rangle\, h^{a_1}(x_1,p_1)\ldots h^{a_n}(x_n,p_n),\end{aligned} \tag{264}$$

where $J_a(x,p)$ are fermion master currents coupled to $h_a(x,p)$. In a quantum theory, it may happen that the HS symmetry is not preserved (the Ward identity is violated):

$$\delta_\varepsilon\mathcal{W}[h] = \mathcal{A}[\varepsilon,h] \neq 0, \tag{265}$$

but in this case, we have a consistency condition. Since the following holds,

$$\begin{aligned}(\delta_{\varepsilon_2}\delta_{\varepsilon_1} - \delta_{\varepsilon_1}\delta_{\varepsilon_2})h_a(x,p) &= i(\partial_x[\varepsilon_1 \overset{*}{,} \varepsilon_2](x,p) - i[h_a(x,p) \overset{*}{,} [\varepsilon_1 \overset{*}{,} \varepsilon_2](x,p)]]) \\ &= i\,\mathcal{D}^{*x}_a[\varepsilon_1 \overset{*}{,} \varepsilon_2](x,p),\end{aligned} \tag{266}$$

we must have the following:

$$\delta_{\varepsilon_2}\mathcal{A}[\varepsilon_1,h] - \delta_{\varepsilon_1}\mathcal{A}[\varepsilon_2,h] = \mathcal{A}[[\varepsilon_1 \overset{*}{,} \varepsilon_2],h]. \tag{267}$$

If we exclude the possibility that $\mathcal{A}[\varepsilon,h]$ is trivial (i.e., $\mathcal{A}[\varepsilon,h] = \delta_\varepsilon\mathcal{C}[h]$, for an integrated local counterterm $\mathcal{C}[h]$), then we are faced with a true anomaly, which breaks the covariance of the effective action.

As illustrated above, in ordinary gauge theories, the form of chiral anomalies (and the CS action) is given by elegant formulas: the descent equations. It is natural to inquire whether in HS-YM theories there are similar formulas. We would be tempted to derive the relevant descent equations by mimicking the constructions of the previous sections. For instance, besides (249), we can introduce the standard (ordinary gauge theory) definitions as follows:

$$G_t = dh_t - \frac{i}{2}[h_t \stackrel{*}{,} h_t], \qquad h_t = th, \qquad h = h_a dx^a, \quad d = \partial_a dx^a. \tag{268}$$

The difference in the HS case is that, unlike in ordinary gauge theories, we cannot use graded commutativity. There is also another difficulty: the already signaled trace problem. We can use Equations (250) and (251), but we have to integrate over the full phase-space. Therefore, it is impossible to reproduce the unintegrated descent equations in the same way as in the ordinary gauge theories. The best we can do is try to reproduce each equation separately in integrated form. While it is rather simple, starting from the expression with n G entries,

$$\langle\langle G * G * \ldots * G \rangle\rangle, \tag{269}$$

to derive the Chern–Simons action in $2n-1$ dimension

$$\mathcal{CS}(h) = n \int_0^1 dt \langle\langle h * G_t * \ldots * G_t \rangle\rangle, \tag{270}$$

and prove that $\delta_\varepsilon \mathcal{CS}(h) = 0$, the remaining derivations are unfortunately lengthy and very cumbersome.

It is at this point that the superfield formulation of HS-YM theories comes to our rescue and gives us all these relations for free.

7.2. Superfield Formulation of HS YM

We introduce the master superfield 1-form as follows:

$$\mathbb{H} = \mathbf{h}_a(x,p,\vartheta,\bar{\vartheta})dx^a + \boldsymbol{\phi}(x,p,\vartheta,\bar{\vartheta})d\bar{\vartheta} + \overline{\boldsymbol{\phi}}(x,p,\vartheta,\bar{\vartheta})d\vartheta, \tag{271}$$

where all the component masterfields $\mathbf{h}_a, \boldsymbol{\phi}, \overline{\boldsymbol{\phi}}$ are valued in the Lie algebra with generators T^α. The component master fields are as follows:

$$\begin{aligned}
\mathbf{h}_a(x,p,\vartheta,\bar{\vartheta}) &= h_a(x,p) + \vartheta\bar{\zeta}_a(x,p) + \bar{\vartheta}\zeta_a(x,p) + \vartheta\bar{\vartheta}t_a(x,p), &(272)\\
\boldsymbol{\phi}(x,p,\vartheta,\bar{\vartheta}) &= c(x,p) + \vartheta\bar{B}(x,p) + \bar{\vartheta}R(x,p) + \vartheta\bar{\vartheta}\varsigma(x,p), &(273)\\
\overline{\boldsymbol{\phi}}(x,p,\vartheta,\bar{\vartheta}) &= \bar{c}(x,p) + \vartheta\bar{R}(x,p) + \bar{\vartheta}B(x,p) + \vartheta\bar{\vartheta}\bar{\varsigma}(x,p). &(274)
\end{aligned}$$

The supercurvature is as follows:

$$\mathbb{G} = \tilde{d}\mathbb{H} - \frac{i}{2}[\mathbb{H} \stackrel{*}{,} \mathbb{H}], \tag{275}$$

where $\tilde{d} = d + \frac{d}{d\vartheta}d\vartheta + \frac{d}{d\bar{\vartheta}}d\bar{\vartheta}$. Notice that the $*$ commutator on the RHS persists also in the Abelian case. Setting the following,

$$\mathbf{G} \equiv d\mathbf{h} - \frac{i}{2}[\mathbf{h} \stackrel{*}{,} \mathbf{h}], \qquad \mathbf{h} = \mathbf{h}_a(x,p,\vartheta,\bar{\vartheta})dx^a, \tag{276}$$

the horizontality condition in this case is as follows:

$$\mathbb{G} = \mathbf{G}. \tag{277}$$

The procedure is the same as in the ordinary YM case, except that the ordinary products are replaced by $*$ products. From the vanishing of the $dx^\mu \wedge d\vartheta$ and $dx^\mu \wedge d\bar{\vartheta}$ component of $\mathbb{G}$, we obtain the following:

$$\zeta_a(x,p) = D^*_a c(x,p), \qquad \overline{\zeta}_a(x,p) = D^*_a \bar{c}(x,p), \tag{278}$$
$$t_a(x,p) = D^*_a B(x,p) + i[D^*_a c(x,p) \stackrel{*}{,} \bar{c}(x,p)], \tag{279}$$

where $D^*_a = \partial^x_a - i[h_a(x,p) \stackrel{*}{,} \quad]$; see (246). From the $d\vartheta \vee d\bar{\vartheta}$, $d\vartheta \vee d\vartheta$ and $d\bar{\vartheta} \vee d\bar{\vartheta}$ components, we obtain the CF restriction as follows:

$$B(x,p) + \overline{B}(x,p) - i[c(x,p) \stackrel{*}{,} \bar{c}(x,p)] = 0, \tag{280}$$

and

$$R(x,p) = \frac{i}{2}[c(x,p) \stackrel{*}{,} c(x,p)], \qquad \overline{R}(x,p) = \frac{i}{2}[\bar{c}(x,p) \stackrel{*}{,} \bar{c}(x,p)], \tag{281}$$
$$\varsigma(x,p) = -i[\overline{B}(x,p) \stackrel{*}{,} c(x,p)], \qquad \bar{\varsigma}(x,p) = i[B(x,p) \stackrel{*}{,} \bar{c}(x,p)].$$

Let us recall again that the $*$-commutators are present also in the Abelian case we are considering. Expanding, for instance, the c ghost master field in ordinary field components according to Equation (258), we obtain the following:

$$c(x,p) \quad = \quad c(x) + c^\mu(x) p_\mu + \frac{1}{2} c^{\mu\nu}(x) p_\mu p_\nu + \frac{1}{3!} c^{\mu\nu\lambda}(x) p_\mu p_\nu p_\lambda + \ldots, \tag{282}$$

where now all the component fields are anticommuting. c is the gauge ghost field, c^μ is the diffeomorphism ghost, and the subsequent are the HS ghost fields. For instance, the BRST transformations of A_a and χ^μ_a are the following:

$$sA_a \quad = \quad \partial_a c - c\cdot\partial A_a - \partial_\lambda c^\lambda_a + \ldots, \tag{283}$$
$$s\chi^\mu_a \quad = \quad \partial_a c^\mu + c\cdot\chi^\mu_a - \partial_\rho c^\mu \chi^\rho_a + \partial^\rho A_a c_\rho{}^\mu - \partial_\rho c \, b_a{}^{\rho\mu} + \ldots \tag{284}$$

For later use, we introduce the following notations:

$$\frac{d}{dt}\mathbb{G}_t \quad = \quad \tilde{d}\mathbb{H} - it[\mathbb{H} \stackrel{*}{,} \mathbb{H}] = \tilde{d}_t \mathbb{H}, \qquad \tilde{d}_t = \tilde{d} - i[\mathbb{H}_t \stackrel{*}{,} \quad], \tag{285}$$
$$\tilde{d}\mathbb{G}_t \quad = \quad i[\mathbb{H}_t \stackrel{*}{,} \mathbb{G}_t], \qquad \delta\mathbb{G}_t = \tilde{d}\delta\mathbb{H}_t - i[\mathbb{H}_t \stackrel{*}{,} \delta\mathbb{H}_t] = \tilde{d}_t \delta\mathbb{H}_t, \tag{286}$$

and analogous formulas for $\mathbf{G}_t = t\big(\tilde{d}\,\mathbf{h} - i\frac{t}{2}[\mathbf{h} \stackrel{*}{,} \mathbf{h}]\big)$ and $G_t = t\big(\tilde{d}\,h - i\frac{t}{2}[h \stackrel{*}{,} h]\big)$.

7.3. Derivation of HS-YM Anomalies

In order to limit the size and complication of the formulas, we limit ourselves to only one anticommuting variable ϑ and consider the following superderivation ($\tilde{d} = d + \frac{\partial}{\partial\vartheta} d\vartheta$):

$$\mathbb{H} = e^{i\vartheta c} * (i\tilde{d} + h*) e^{-i\vartheta c} = h + \vartheta(dc - i[h \stackrel{*}{,} c]) + \left(c + i\vartheta\frac{1}{2}[c \stackrel{*}{,} c]\right) d\vartheta \equiv \mathbf{h} + \boldsymbol{\phi}\, d\vartheta, \tag{287}$$

where $c = c(x,p)$. It follows that the supercurvature is the following:

$$\mathbb{G} = e^{i\vartheta c} * G * e^{-i\vartheta c} = \mathbf{G} \equiv G - i\vartheta[F \stackrel{*}{,} c]. \tag{288}$$

From these formulas, it is immediately visible that the derivative with respect to ϑ corresponds to the BRST transformations:

$$
\begin{aligned}
\frac{\partial}{\partial\vartheta}\mathbf{h} &= dc - i[h \stackrel{*}{,} c] = D_h^* c = \mathfrak{s}h, \\
\frac{\partial}{\partial\vartheta}\boldsymbol{\phi} &= \frac{i}{2}[c \stackrel{*}{,} c] = \mathfrak{s}c, \\
\frac{\partial}{\partial\vartheta}\mathbf{G} &= -i[G \stackrel{*}{,} c] = \mathfrak{s}G.
\end{aligned}
$$

Let us start from the phase space integral with n $\mathbb{G}$ entries:

$$
\langle\!\langle \mathbb{G} * \mathbb{G} * \ldots * \mathbb{G} \rangle\!\rangle, \tag{289}
$$

where $\langle\!\langle \quad \rangle\!\rangle$ means integration over a phase space of dimension $< 4n$. Let us introduce the following notations:

$$
\frac{d}{dt}\mathbb{G}_t = \tilde{d}\,\mathbb{H} - it[\mathbb{H} \stackrel{*}{,} \mathbb{H}] = \tilde{d}_t\,\mathbb{H}, \qquad \tilde{d}_t = \tilde{d} - i[\mathbb{H}_t \stackrel{*}{,} \quad], \qquad \tilde{d}\,\mathbb{G}_t = i[\mathbb{H}_t \stackrel{*}{,} \mathbb{G}_t]. \tag{290}
$$

Then, consider the expression with $n-1$ $\mathbb{G}_t$ entries:

$$
\begin{aligned}
\int_0^1 dt \langle\!\langle \tilde{d}(\mathbb{H} * \mathbb{G}_t * \ldots * \mathbb{G}_t) \rangle\!\rangle &= \int_0^1 dt \langle\!\langle \tilde{d}\,\mathbb{H} * \mathbb{G}_t * \ldots * \mathbb{G}_t \rangle\!\rangle - \int_0^1 dt \langle\!\langle \mathbb{H} * \tilde{d}\,\mathbb{G}_t * \ldots * \mathbb{G}_t \rangle\!\rangle - \ldots \\
\cdots - \int_0^1 dt \langle\!\langle \mathbb{H} * \mathbb{G}_t * \ldots * \tilde{d}\,\mathbb{G}_t \rangle\!\rangle &= \int_0^1 dt \Big(\langle\!\langle \tilde{d}\,\mathbb{H} * \mathbb{G}_t * \ldots * \mathbb{G}_t \rangle\!\rangle - i \langle\!\langle \mathbb{H} * [\mathbb{H}_t \stackrel{*}{,} \mathbb{G}_t] * \ldots * \mathbb{G}_t \rangle\!\rangle - \ldots \\
&\quad \ldots - i \langle\!\langle \mathbb{H} * \mathbb{G}_t * \ldots * [\mathbb{H}_t \stackrel{*}{,} \mathbb{G}_t] \Big) \rangle\!\rangle,
\end{aligned} \tag{291}
$$

using the last of (290). Then, using the first of (290) together with (251), this becomes the following:

$$
\begin{aligned}
&= \int_0^1 dt \langle\!\langle (\tilde{d}\,\mathbb{H} - i[\mathbb{H}_t \stackrel{*}{,} \mathbb{H}]) * \mathbb{G}_t * \ldots * \mathbb{G}_t \rangle\!\rangle = \int_0^1 dt \langle\!\langle (\frac{d}{dt}\mathbb{G}_t * \mathbb{G}_t * \ldots * \mathbb{G}_t \rangle\!\rangle \\
&= \frac{1}{n}\int_0^1 dt\, \frac{d}{dt} \langle\!\langle \mathbb{G}_t * \mathbb{G}_t * \ldots * \mathbb{G}_t \rangle\!\rangle = \frac{1}{n} \langle\!\langle \mathbb{G} * \mathbb{G} * \ldots * \mathbb{G} \rangle\!\rangle = \frac{1}{n} \langle\!\langle \mathbf{G} * \mathbf{G} * \ldots * \mathbf{G} \rangle\!\rangle = 0.
\end{aligned} \tag{292}
$$

Since they are integrated over a spacetime of dimension $\mathsf{d} < 2n$, these expressions vanish. However, this is the way we identify the primitive functional action for HS CS (if the spacetime dimension is $\mathsf{d} = 2n-1$). In other words, the HS CS action is the following:

$$
\mathcal{CS}(h) = n \int_0^1 dt \langle\!\langle \mathbf{h} * \mathbf{G}_t * \ldots * \mathbf{G}_t \rangle\!\rangle \Big|_{\vartheta=0} = n \int_0^1 dt \langle\!\langle h * G_t * \ldots * G_t \rangle\!\rangle, \tag{293}
$$

where $\langle\!\langle \quad \rangle\!\rangle$ means now integration over a phase space of dimension $4n - 2$.

Expressions relevant to anomalies appear if the spacetime dimension is $\mathsf{d} = 2n - 2$ and the phase space one is $\mathsf{d} = 4n - 4$. In this case, the (unintegrated) expression

$$
\begin{aligned}
&\int_0^1 dt (\mathbb{H} * \mathbb{G}_t * \ldots * \mathbb{G}_t + \mathbb{G}_t * \mathbb{H} * \cdots * \mathbb{G}_t + \ldots + \mathbb{G}_t * \mathbb{G}_t * \ldots * \mathbb{H}) \\
&= \sum_{i=0}^{2n-1} Q^{(i)}_{2n-i-1}(\mathbf{h}, \boldsymbol{\phi})\, \underbrace{d\vartheta \wedge \ldots \wedge d\vartheta}_{i \text{ factors}},
\end{aligned} \tag{294}
$$

is a spacetime polyform of degree $2n-1, \ldots, 1, 0$: i represents the ghost number and $2n-i-1$ is the spacetime form degree. Of course, its components of degree $2n$ and $2n-1$ vanish for dimensional reasons. Then, excluding vanishing factors and recalling that $\frac{\partial}{\partial\vartheta}$

corresponds to the BRST transform $\mathfrak{s}$, the integrand of Equation (294) can be written as follows:

$$
\begin{aligned}
&\sum_{i=2}^{2n-1} dQ^{(i)}_{2n-i-1}(\mathbf{h},\boldsymbol{\phi})\underbrace{d\vartheta\wedge\ldots\wedge d\vartheta}_{i\text{ factors}}+\sum_{i=1}^{2n-1}\frac{\partial}{\partial\vartheta}Q^{(i)}_{2n-i-1}(\mathbf{h},\boldsymbol{\phi})\underbrace{d\vartheta\wedge\ldots\wedge d\vartheta}_{i+1\text{ factors}}\\
&=\left(\sum_{i=2}^{2n-1} dQ^{(i)}_{2n-i-1}(\mathbf{h},\boldsymbol{\phi})+\sum_{i=2}^{2n-2} sQ^{(i-1)}_{2n-i}(\mathbf{h},\boldsymbol{\phi})\right)\underbrace{d\vartheta\wedge\ldots\wedge d\vartheta}_{i\text{ factors}}.
\end{aligned}
\tag{295}
$$

Now, this decomposition must be inserted inside the integration symbol $\langle\langle\cdot\rangle\rangle$. This symbol needs a specification and must be interpreted as follows: any form $Q^{(i)}_{2n-i-1}\sim dx^{\mu_1}\wedge d^{\mu_{2n-i-1}}$ is understood to be multiplied by a trivial factor $dx^{\mu_{2n-i}}\wedge\ldots\wedge dx^{\mu_{2n-2}}$, so that integration over spacetime makes sense. In conclusion, the following equation

$$
\int_0^1 dt\langle\langle\tilde{d}(\mathbb{H}*\mathbb{G}_t*\ldots*\mathbb{G}_t)\rangle\rangle=0, \tag{296}
$$

means the following:

$$
\begin{aligned}
s\langle\langle Q^{(1)}_{2n-2}(\mathbf{h},\boldsymbol{\phi})\rangle\rangle &= 0, &(297)\\
s\langle\langle Q^{(2)}_{2n-3}(\mathbf{h},\boldsymbol{\phi})\rangle\rangle &= 0, &(298)\\
\ldots &= \ldots\\
s\langle\langle Q^{(2n-1)}_{0}(\boldsymbol{\phi})\rangle\rangle &= 0. &(299)
\end{aligned}
$$

The anomaly is given by $\langle\langle Q^{(1)}_{2n-2}(\mathbf{h},\boldsymbol{\phi})\rangle\rangle\Big|_{\vartheta=0}$, and the following:

$$
\begin{aligned}
\langle\langle Q^{(1)}_{2n-2}(\mathbf{h},\boldsymbol{\phi})\rangle\rangle &= n\int_0^1 dt\Big(\langle\langle\boldsymbol{\phi}*\mathbf{G}_t*\ldots*\mathbf{G}_t\rangle\rangle+\langle\langle\mathbf{h}*(d_t\boldsymbol{\phi}-\partial_\vartheta\mathbf{h})*\mathbf{G}_t*\ldots*\mathbf{G}_t\rangle\rangle\\
&\quad+\ldots+\langle\langle\mathbf{h}*\mathbf{G}_t*\ldots*(d_t\boldsymbol{\phi}-\partial_\vartheta\mathbf{h}\Big),
\end{aligned}
\tag{300}
$$

where $d_t=d-it[\mathbf{h},]$. It follows that

$$
\begin{aligned}
\langle\langle Q^{(1)}_{2n-2}(\mathbf{h},\boldsymbol{\phi})\rangle\rangle|_{\vartheta=0} &= n\int_0^1 dt\Big(\langle\langle c*G_t*\ldots*G_t\rangle\rangle-it(1-t)\langle\langle h*[h\stackrel{*}{,}c]*G_t*\ldots*G_t\rangle\rangle\\
&\quad+\ldots-it(1-t)\langle\langle h*G_t*\ldots*G_t*[h\stackrel{*}{,}c]\Big).
\end{aligned}
\tag{301}
$$

Now, using $\frac{d}{dt}G_t=d_{th}h=dh-it[h\stackrel{*}{,}h]$ and $dG_t=it[h,G_t]$, one can easily find the more familiar expression of the anomaly as follows:

$$
\langle\langle Q^{(1)}_{2n-2}(\mathbf{h},\boldsymbol{\phi})\rangle\rangle|_{\vartheta=0} = -n\int_0^1 dt\Big(\langle\langle dc*h*G_t*\ldots*G_t\rangle\rangle+\ldots+\langle\langle dc*G_t*\ldots*G_t*h\rangle\rangle\Big). \tag{302}
$$

8. The Superfield Formalism in Supersymmetric Gauge Theories

To conclude this review, we present the 'BRST supergeometry' of an N = 1 supersymmetric gauge theory formulated in the superspace. Let us start with a summary of the superspace presentation of this theory.

8.1. The Supermanifold Formulation of SYM

From ch. XIII of [68], a supersymmetric gauge theory can be introduced as follows. One starts from a torsionful (but flat) superspace with supercoordinates $z^M=(z^m,\theta^\mu,\bar{\theta}^{\dot{\mu}})$ and introduces a supervielbein basis as follows:

$$
e^A(z)=dz^M\,e_M{}^A(z)
$$

where $A = (a, \alpha, \dot{\alpha})$ are flat indices. The vielbein satisfy the following:

$$e_A{}^M e_M{}^B = \delta_A{}^B, \qquad e_M{}^A e_A{}^N = \delta_M{}^N, \qquad \delta_M{}^N = \begin{pmatrix} \delta_m{}^n & 0 & 0 \\ 0 & \delta_\mu{}^\nu & 0 \\ 0 & 0 & \delta^{\dot{\mu}}{}_{\dot{\nu}} \end{pmatrix}.$$

The vielbein are *chosen* to be the following:

$$e_A{}^M = \begin{pmatrix} \delta_a{}^m & 0 & 0 \\ i\sigma^m_{\alpha\dot{\alpha}}\theta^{\dot{\alpha}} & \delta_\alpha{}^\mu & 0 \\ i\bar{\theta}^\alpha \sigma^m_{\alpha\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}} & 0 & \delta^{\dot{\alpha}}{}_{\dot{\mu}} \end{pmatrix}, \qquad e_M{}^A = \begin{pmatrix} \delta_m{}^a & 0 & 0 \\ -i\sigma^a_{\mu\dot{\mu}}\bar{\theta}^{\dot{\mu}} & \delta_\mu{}^\alpha & 0 \\ -i\theta^\nu \sigma^a_{\nu\dot{\nu}} \epsilon^{\dot{\nu}\dot{\mu}} & 0 & \delta^{\dot{\mu}}{}_{\dot{\alpha}} \end{pmatrix}.$$

In such a type of supergeometry, one has the following:

$$\begin{aligned} de^A &= dz^M dz^N \frac{\partial}{\partial z^N} e_M{}^A(z), \qquad \text{i.e.,} \\ de^a &= -2ie^\alpha \sigma^a{}_{\alpha\dot{\alpha}} e^{\dot{\alpha}}, \\ de^\alpha &= 0, \\ de^{\dot{\alpha}} &= 0. \end{aligned} \tag{303}$$

The flat indices derivatives $D_A = e_A{}^M \partial_M$ correspond to the following:

$$D_a = e^m_a \partial_m = \partial_a, \qquad D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m{}_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_m, \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \sigma^m{}_{\alpha\dot{\alpha}} \partial_m,$$

because in flat space, $e^m_a = \delta^m_a$. Moreover, we have the following:

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^m{}_{\alpha\dot{\alpha}} \partial_m, \qquad \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0.$$

The superconnection is defined by the following:

$$\phi = e^A \phi_A, \qquad \phi_A = iT^r \phi^r_A, \qquad \phi^r_m\Big|_{\theta=\bar{\theta}=0} = v^r_m, \tag{304}$$

where v^r_m is the *ordinary* non-Abelian potential and T^r are the Hermitean generators of the gauge Lie algebra.

The gauge curvature is given by the following superform:

$$F = d\phi - \phi\phi = \frac{1}{2} e^A e^B F_{BA}.$$

On the flat basis, this becomes the following:

$$F = de^A \phi_A + \frac{1}{2} e^A e^B \Big(D_B \phi_A - (-)^{ab} D_A \phi_B - \phi_B \phi_A + (-)^{ab} \phi_A \phi_B \Big), \tag{305}$$

where the torsion term is the first on the RHS. We have the following[6]:

$$F_{ab}\big|_{\theta=\bar{\theta}=0} = iT^r v^r_{ab}.$$

The dynamics are determined by the super-Bianchi identity, $\mathcal{D}F = dF - [\phi, F] = 0$. They are solved by the following conditions:

$$F_{\alpha\beta} = F_{\dot{\alpha}\dot{\beta}} = F_{\alpha\dot{\beta}} = 0, \tag{306}$$

with further restrictions stemming from the following:

$$\sigma^a{}_{\alpha\dot{\gamma}} F_{a\beta} + \sigma^a{}_{\beta\dot{\gamma}} F_{a\alpha} = 0, \qquad \sigma^a{}_{\gamma\dot{\beta}} F_{a\dot{\alpha}} + \sigma^a{}_{\gamma\dot{\alpha}} F_{a\dot{\beta}} = 0. \tag{307}$$

This allows us to write the following:

$$F_{a\alpha} = -i\sigma_{a\alpha\dot\beta}\bar W^{\dot\beta}, \qquad \bar W^{\dot\alpha} = -\frac{i}{4}\bar\sigma^{a\dot\alpha\alpha}F_{a\alpha}. \tag{308}$$

Similarly, we have the following:

$$F_{a\dot\alpha} = -iW^{\beta}\sigma_{a\beta\dot\alpha}, \qquad W^{\alpha} = -\frac{i}{4}F_{a\dot\alpha}\bar\sigma^{a\dot\alpha\alpha}. \tag{309}$$

Moreover the Ws must satisfy the following:

$$\bar{\mathcal D}\bar W - \mathcal D W = \bar{\mathcal D}_{\dot\alpha}\bar W^{\dot\alpha} - \mathcal D^{\alpha}W_{\alpha} = 0, \qquad \mathcal D_{\alpha}\bar W_{\dot\alpha} = 0, \qquad \bar{\mathcal D}_{\dot\alpha}W_{\alpha} = 0, \tag{310}$$

where we have introduced the covariant super-derivative $\mathcal D_A = D_A - [\phi_A, \ \ ]$.

8.2. The $\vartheta, \bar\vartheta$ Superfield Formalism

Now, we switch on two anticommuting coordinates, ϑ and $\bar\vartheta$, and call super-superfield (ss-field) a superfield that is a function *also* of these coordinates. In terms of $\tilde Z^{\tilde M} = (x^m, \theta^\mu, \theta^{\dot\mu}, \vartheta, \bar\vartheta) = (z^M, \vartheta, \bar\vartheta)$ we have the following:

$$\tilde f(\tilde Z) = \tilde f(z, \vartheta, \bar\vartheta) = f(z) + \vartheta\,\bar g(z) + \bar\vartheta g(z) + \vartheta\bar\vartheta h(z),$$

where $f(z), g(z), \bar g(z)$ and $h(z)$ are ordinary supersymmetric superfields. The BRST-anti-BRST interpretation is the following:

$$g = \mathfrak s f, \qquad \bar g = \bar{\mathfrak s} f, \qquad h = \bar{\mathfrak s} g = -\mathfrak s \bar g. \tag{311}$$

We introduce also the ss-exterior derivative $\tilde d = d\tilde Z^{\tilde M}\frac{\partial}{\partial \tilde Z^{\tilde M}} = d + d\vartheta\frac{\partial}{\partial\vartheta} + d\bar\vartheta\frac{\partial}{\partial\bar\vartheta}$. The super-super-connection (ss-connection) is the following:

$$\tilde\Phi = \tilde e^{\tilde A}\tilde\Phi_{\tilde A}. \tag{312}$$

We choose the following:

$$\tilde e^{\tilde A}(\tilde Z) = \begin{pmatrix} e^A(z) & 0 & 0 \\ 0 & d\vartheta & 0 \\ 0 & 0 & d\bar\vartheta \end{pmatrix}.$$

So we have the following:

$$\tilde\Phi = \tilde\phi + d\vartheta\tilde\phi_\vartheta + d\bar\vartheta\tilde\phi_{\bar\vartheta},$$

where

$$\begin{aligned} \tilde\phi &= e^A\tilde\phi_A = e^A\left(\phi_A + \vartheta\bar\psi_A + \bar\vartheta\psi_A + \vartheta\bar\vartheta\pi_A\right), \\ \tilde\phi_\vartheta &= \varphi_\vartheta + \vartheta\bar\psi_\vartheta + \bar\vartheta\psi_\vartheta + \vartheta\bar\vartheta\varpi_\vartheta, \\ \tilde\phi_{\bar\vartheta} &= \varphi_{\bar\vartheta} + \vartheta\bar\psi_{\bar\vartheta} + \bar\vartheta\psi_{\bar\vartheta} + \vartheta\bar\vartheta\varpi_{\bar\vartheta}. \end{aligned} \tag{313}$$

In the above, $\phi_A, \psi_A, \ldots, \varpi_{\bar\vartheta}$ are ordinary superfields valued in the gauge Lie algebra with generators T^r.

The ss-curvature can be written as follows:

$$\begin{aligned} \tilde{\mathcal F} &= \tilde d\tilde\Phi - \tilde\Phi\tilde\Phi \\ &= \tilde F + d\vartheta((\partial_\vartheta - \tilde\phi_\vartheta)\tilde\phi - (d - \tilde\phi)\tilde\phi_\vartheta) + d\bar\vartheta((\partial_{\bar\vartheta} - \tilde\phi_{\bar\vartheta})\tilde\phi - (d - \tilde\phi)\tilde\phi_{\bar\vartheta}) \\ &\quad + d\vartheta d\vartheta(\partial_\vartheta\tilde\phi_\vartheta - \phi_\vartheta\tilde\phi_\vartheta) + d\bar\vartheta d\bar\vartheta(\partial_{\bar\vartheta}\tilde\phi_{\bar\vartheta} - \phi_{\bar\vartheta}\tilde\phi_{\bar\vartheta}) \\ &\quad + d\vartheta d\bar\vartheta(\partial_\vartheta\tilde\phi_{\bar\vartheta} + \partial_{\bar\vartheta}\tilde\phi_\vartheta - \tilde\phi_\vartheta\tilde\phi_{\bar\vartheta} - \tilde\phi_{\bar\vartheta}\tilde\phi_\vartheta), \end{aligned} \tag{314}$$

where $\tilde{F}$ has nonzero components only in the $x^\mu, \theta^\mu, \theta^{\dot{\mu}}$ directions. The horizontality condition is the following:

$$\tilde{d}\tilde{\Phi} - \tilde{\Phi}\tilde{\Phi} = \tilde{F}. \tag{315}$$

It gives rise to the following set of equations:

$$\tilde{\mathcal{F}}\big|_{d\vartheta=0=d\bar{\vartheta}} = \tilde{F}, \tag{316}$$
$$(\partial_\vartheta - \tilde{\phi}_\vartheta)\tilde{\phi} - (d - \tilde{\phi})\tilde{\phi}_\vartheta = 0, \tag{317}$$
$$(\partial_{\bar{\vartheta}} - \tilde{\phi}_{\bar{\vartheta}})\tilde{\phi} - (d - \tilde{\phi})\tilde{\phi}_{\bar{\vartheta}} = 0, \tag{318}$$
$$\partial_\vartheta\tilde{\phi}_\vartheta - \phi_\vartheta\phi_\vartheta = 0, \tag{319}$$
$$\partial_{\bar{\vartheta}}\tilde{\phi}_{\bar{\vartheta}} - \phi_{\bar{\vartheta}}\phi_{\bar{\vartheta}} = 0, \tag{320}$$
$$\partial_\vartheta\tilde{\phi}_{\bar{\vartheta}} + \partial_{\bar{\vartheta}}\tilde{\phi}_\vartheta - \phi_\vartheta\phi_{\bar{\vartheta}} - \phi_{\bar{\vartheta}}\phi_\vartheta = 0. \tag{321}$$

Equations (319) and (320) yield the identifications[7]

$$[A, B] = AB - (-1)^{\epsilon(A)\epsilon B} BA. \tag{322}$$

The total Grassmannality ϵ includes both the one related to supersymmetry and to the BRST symmetry.

$$\bar{\psi}_\vartheta = \varphi_\vartheta\varphi_\vartheta, \qquad \psi_{\bar{\vartheta}} = \varphi_{\bar{\vartheta}}\varphi_{\bar{\vartheta}}, \tag{323}$$
$$\varpi_\vartheta = [\psi_\vartheta, \varphi_\vartheta], \qquad \varpi_{\bar{\vartheta}} = -[\bar{\psi}_{\bar{\vartheta}}, \varphi_{\bar{\vartheta}}]. \tag{324}$$
$$\tag{325}$$

The following remaining equations are identically satisfied:

$$\bar{\psi}_\vartheta\varphi_\vartheta = \varphi_\vartheta\bar{\psi}_\vartheta, \qquad \psi_{\bar{\vartheta}}\varphi_{\bar{\vartheta}} = \varphi_{\bar{\vartheta}}\psi_{\bar{\vartheta}},$$
$$\varpi_\vartheta\varphi_\vartheta + \varphi_\vartheta\varpi_\vartheta + \bar{\psi}_\vartheta\psi_\vartheta - \psi_\vartheta\bar{\psi}_\vartheta = 0, \qquad \varpi_{\bar{\vartheta}}\varphi_{\bar{\vartheta}} + \varphi_{\bar{\vartheta}}\varpi_{\bar{\vartheta}} + \bar{\psi}_{\bar{\vartheta}}\psi_{\bar{\vartheta}} - \psi_{\bar{\vartheta}}\bar{\psi}_{\bar{\vartheta}} = 0, \tag{326}$$

The lowest component of φ_ϑ is an anticommuting scalar valued in the gauge Lie algebra, and is to be identified with the ghost field $c = c^r(x)\,T^r$. Its BRST transform is $\bar{\psi}_\vartheta$, where φ_ϑ is the BRST transform parameter. The lowest component of $\varphi_{\bar{\vartheta}}$ is to be identified with the dual ghost field $\bar{c} = \bar{c}^r(x)\,T^r$. Its anti-BRST transform is $\psi_{\bar{\vartheta}}$.

Equation (321) gives the following relation:

$$\psi_\vartheta + \bar{\psi}_{\bar{\vartheta}} = \varphi_\vartheta\varphi_{\bar{\vartheta}} + \varphi_{\bar{\vartheta}}\varphi_\vartheta, \tag{327}$$

which is to be interpreted as the Curci–Ferrari relation, ref. [12], and $\psi_\vartheta, \bar{\psi}_{\bar{\vartheta}}$ are the Nakanishi–Lautrup superfields. Using (327), the following remaining relations turn out to be identically verified:

$$\varpi_\vartheta = [\varphi_\vartheta, \bar{\psi}_{\bar{\vartheta}}] + [\varphi_{\bar{\vartheta}}, \bar{\psi}_\vartheta],$$
$$\varpi_{\bar{\vartheta}} = -[\varphi_\vartheta, \psi_{\bar{\vartheta}}] - [\varphi_{\bar{\vartheta}}, \psi_\vartheta],$$
$$[\bar{\psi}_\vartheta, \psi_{\bar{\vartheta}}] + [\bar{\psi}_{\bar{\vartheta}}, \psi_\vartheta] + [\varpi_\vartheta, \varphi_{\bar{\vartheta}}] + [\varpi_{\bar{\vartheta}}, \varphi_\vartheta] = 0, \tag{328}$$

Let us come next to the constraint (317). It implies the following definitions:

$$\bar{\psi}_A = D_A\varphi_\vartheta - [\phi_A, \varphi_\vartheta] = \mathcal{D}_A\varphi_\vartheta, \tag{329}$$
$$\pi_A = \mathcal{D}_A\psi_\vartheta - [\psi_A, \varphi_\vartheta], \tag{330}$$

and the following identities:

$$\mathcal{D}_A \bar{\psi}_\vartheta - [\bar{\psi}_{A}, , \varphi_\vartheta] = 0,$$
$$\mathcal{D}_A \varpi_\vartheta - [\pi_A, \varphi_\vartheta] + [\bar{\psi}_{A}, , \psi_\vartheta] - [\psi_{A}, , \bar{\psi}_\vartheta] = 0, \tag{331}$$

while, from (318), we obtain the following definitions:

$$\psi_A = D_A \varphi_{\bar{\vartheta}} - [\phi_A, \varphi_{\bar{\vartheta}}] = \mathcal{D}_A \varphi_{\bar{\vartheta}}, \tag{332}$$
$$\pi_A = -\mathcal{D}_A \bar{\psi}_{\bar{\vartheta}} + [\bar{\psi}_A, \varphi_{\bar{\vartheta}}], \tag{333}$$

as well as the following identities:

$$\mathcal{D}_A \psi_{\bar{\vartheta}} - [\psi_A, \varphi_{\bar{\vartheta}}] = 0,$$
$$\mathcal{D}_A \varpi_{\bar{\vartheta}} - [\pi_A, \varphi_{\bar{\vartheta}}] + [\bar{\psi}_{A}, , \psi_{\bar{\vartheta}}] - [\psi_{A}, , \bar{\psi}_{\bar{\vartheta}}] = 0. \tag{334}$$

The superfield $\psi_A, \bar{\psi}_A, \pi_A$ are easily recognized as the (anti)BRST transform. The equivalence of (330) and (333) can be proven by means of the CF condition. Next, let us come to (316). In general, using (305), one can show the following:

$$\tilde{F}_{AB} = F_{AB} - \vartheta[F_{AB}, \varphi_\vartheta] - \bar{\vartheta}[F_{AB}, \varphi_{\bar{\vartheta}}] - \vartheta\bar{\vartheta}([F_{AB}, \psi_\vartheta] - [[F_{AB}, \varphi_{\bar{\vartheta}}], \varphi_\vartheta]). \tag{335}$$

In proving this, particular attention must be paid to the $(A, B) = (\alpha, \dot{\beta})$ case. The definition (305) includes in this case also a contribution from the supertorsion, but this contribution is exactly canceled by an analogous term coming from the first commutator (304).

From (335) it is evident that the constraints (306) can be covariantly implemented in the BRST formalism. Next, we have to consider the constraints (307). However, instead of solving the ss-Bianchi identity, we prefer to covariantize the constraints extracted from it in chapter XIII of [68]. In the same way as (306), we can covariantly implement also (307) in the BRST formalism. To this end, we introduce the BRST covariant definitions of $W^\alpha, \overline{W}^{\dot{\alpha}}$

$$\widetilde{W}^\alpha = -\frac{i}{4}\tilde{F}_{a\dot{\alpha}}\bar{\sigma}^{a\dot{\alpha}\alpha}, \qquad \widetilde{\overline{W}}^{\dot{\alpha}} == -\frac{i}{4}\bar{\sigma}^{a\dot{\alpha}\alpha}\tilde{F}_{a\alpha}. \tag{336}$$

Therefore, the ss-field expression for $\widetilde{W}_\alpha$ is the following:

$$\widetilde{W}_\alpha = W_\alpha - \vartheta[W_\alpha, \varphi_\vartheta] - \bar{\vartheta}[W_\alpha, \varphi_{\bar{\vartheta}}] - \vartheta\bar{\vartheta}([W_\alpha, \psi_\vartheta] - [[W_\alpha, \varphi_{\bar{\vartheta}}], \varphi_\vartheta]), \tag{337}$$

and an analogous one for $\widetilde{\overline{W}}_{\dot{\alpha}}$. The next issue is now to BRST-covariantize the constraints (310).

Let us use the compact notation $\underline{\alpha}$ to denote both α and $\dot{\alpha}$ and introduce the BRST super-covariant derivative as follows:

$$\widetilde{\mathcal{D}}_{\underline{\alpha}}\widetilde{W}_{\underline{\beta}} = D_{\underline{\alpha}}\widetilde{W}_{\underline{\beta}} - [\tilde{\phi}_{\underline{\alpha}}, \widetilde{W}_{\underline{\beta}}], \tag{338}$$

then, it is lengthy but straightforward to prove the following:

$$\widetilde{\mathcal{D}}_{\underline{\alpha}}\widetilde{W}_{\underline{\beta}} = \mathcal{D}_{\underline{\alpha}}W_{\underline{\beta}} - \vartheta[\mathcal{D}_{\underline{\alpha}}W_{\underline{\beta}}, \varphi_\vartheta] - \bar{\vartheta}[\mathcal{D}_{\underline{\alpha}}W_{\underline{\beta}}, \varphi_{\bar{\vartheta}}] - \vartheta\bar{\vartheta}\Big([\mathcal{D}_{\underline{\alpha}}W_{\underline{\beta}}, \psi_\vartheta] - [[\mathcal{D}_{\underline{\alpha}}W_{\underline{\beta}}, \varphi_{\bar{\vartheta}}], \varphi_\vartheta]\Big). \tag{339}$$

This allows us to write down the constraints (310) in a BRST covariant form. They combine perfectly with the BRST superfield formalism.

This section shows that the BRST formalism can be consistently embedded in a supermanifold that encompasses also the supersymmetric spinorial directions.

9. Conclusions and Comments

In this paper, we have reviewed (or proposed for the first time) several applications of the superfield method to represent the BRST and anti-BRST algebra in field theories with both gauge and diffeomorphism symmetries[8]. We have shown, in many examples, that it correctly reproduces the transformations and, in more complicated cases, it helps finding them. Beyond that, we have shown that it is instrumental in practical applications, such as in the subject of consistent gauge anomalies and their integration (Wess–Zumino terms). In such instances, it can be of invaluable help as an effective algorithmic method, as opposed to laborious alternative trial and error methods.

In Appendix A, we have reported a geometrical description of the BRST and anti-BRST algebras based on the geometry of principal fiber bundles and infinite dimensional groups of gauge transformations. Although elegant and with the appeal of classical geometry, this description can hardly be extended to the full quantum theory as defined by the perturbative expansion. The superfield description, which incorporates gauge, ghosts and auxiliary fields in a unique expression, seems instead to be more apt for this purpose, although a full attempt to exploit this possibility, to our best of knowledge, has never been made.

Elementary examples, in this sense, are the gauge-fixing action terms. Gauge fixing is a necessary step of quantization. There is freedom in choosing the gauge-fixing terms, except for a few obvious limitations: they must break completely the relevant gauge symmetry, have a zero ghost number, and then they must be real and of a canonical dimension not larger than 4 (in 4D) in order to guarantee unitarity and renormalizability. However, of course, they must be BRST (and, if possible, anti-BRST) invariant. Having at our disposal the superfield method, it is relatively easy to construct such terms. For instance, a well-known case is that of non-Abelian gauge theories. With reference to the notation in Section 2.1, one such term is the trace of $\frac{\partial}{\partial\vartheta}\frac{\partial}{\partial\bar\vartheta}(\Phi_\mu\Phi^\mu)$, which, being the coefficient of $\vartheta\bar\vartheta$, is automatically BRST and anti-BRST invariant. It gives rise to the action term $\mathrm{tr}(A_\mu\partial^\mu B - \partial^\mu\bar{c}D_\mu c)$. Another possibility is the trace of $\left(\frac{\partial\bar\eta}{\partial\vartheta}\right)^2$ and $\left(\frac{\partial\eta}{\partial\bar\vartheta}\right)^2$, which give rise to the action terms $\mathrm{tr}(\bar{B}^2)$ and $\mathrm{tr}(B^2)$, respectively, and so on.

Against the backdrop of these well known examples, we would like to produce a few analogous terms in the case of gravity. The Einstein–Hilbert action for gravity in the superfield formalism can be written as follows [54]:

$$S = \kappa\int d^4x d\bar\vartheta d\vartheta\, \vartheta\bar\vartheta\, \sqrt{G}\,\mathbf{R} = \kappa\int d^4x\, R. \tag{340}$$

On the same footing, we easily add matter fields and their interaction with gravity. Then, gauge-fixing terms invariant under BRST and anti-BRST transformations can be easily produced with the superfield formalism of Sections 4 and 5, for instance, the coefficients of $\vartheta\bar\vartheta$ in any local expression of the superfields of dimension 4. One such term is the following:

$$L^{(1)}_{g.f.} = (\partial_\vartheta\partial_{\bar\vartheta}G_{\mu\nu}(X))\,(\partial_\vartheta\partial_{\bar\vartheta}\widehat{G}^{\mu\nu}(X)) = V_{\mu\nu}(x)\widehat{V}^{\mu\nu}(x). \tag{341}$$

The dimensional counting is based on assigning to X^M dimension -1: $[x^\mu] = [\vartheta] = [\bar\vartheta] = -1$ and to $\tilde{d}X^M$ dimension 0. Of course, $[G_{MN}] = 0$. This fixes the dimensions of all the component fields. For instance $[h] = 1, [\xi^\mu] = 0, etc$. Another possible term is the following:

$$L^{(2)}_{g.f.} = (\partial_\vartheta\partial_{\bar\vartheta}G_{\vartheta\bar\vartheta}(X))^4 = G(x)^4, \tag{342}$$

which is, however, quadratic in h.

Another way to obtain the BRST invariant term is to consider the following:

$$\eta^{\mu\nu}(\partial_\vartheta\partial_{\bar\vartheta}G_{\mu\underline{\vartheta}}(X))\,(\partial_\vartheta\partial_{\bar\vartheta}G_{\nu\underline{\vartheta}}(X)),$$

where $\underline{\vartheta}$ stands either for ϑ or $\bar\vartheta$. The only nonvanishing term is the following:

$$L^{(3)}_{g.f.} = \eta^{\mu\nu}\Gamma_\mu(x)\overline{\Gamma}_\nu(x). \tag{343}$$

Using the supervierbein, one can construct the following:

$$L^{(4)}_{g.f.} = (\partial_\vartheta \partial_{\bar\vartheta} E^a_\mu(X))(\partial_\vartheta \partial_{\bar\vartheta} \widehat{E}^\mu_a(X)) = f^a_\mu(x) \hat{f}^\mu_a(x), \tag{344}$$

and

$$L^{(5)}_{g.f.} = \eta_{ab}(\partial_\vartheta \partial_{\bar\vartheta} E^a_\vartheta(X))(\partial_\vartheta \partial_{\bar\vartheta} E^b_{\bar\vartheta}(X)) = \eta_{ab} \psi^a(x) \rho^b(x). \tag{345}$$

It is remarkable that these gauge-fixing terms are generally more than quadratic in the fields, which implies, in particular, that not only linear gauge-fixing terms enjoy both BRST *and* anti-BRST symmetry. We see from this that aspects of BRST quantization of gauge and gravity field theories need further investigation and may reserve surprises. We plan to return to them.

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Appendix A. Evaluation Map and BRST

The purpose of this appendix is to present an interpretation of the BRST transformations within the framework of the geometry of a principal fiber bundle [11] so that one is in the condition to appreciate the remarkable similarity between this geometry and the superfield formalism.

Let $\mathsf{P}(\mathsf{M}, \mathsf{G})$ be a principal bundle with a d-dimensional manifold M as base, structure group G, which we suppose to be compact, and total space P. π will denote the projection $\pi : \mathsf{P} \to \mathsf{M}$. An automorphism is a diffeomorphism of P, $\psi : \mathsf{P} \to \mathsf{P}$, such that $\psi(pg) = \psi(p)g$, for any $p \in \mathsf{P}$ and any $g \in \mathsf{G}$. A vertical automorphism does not move the base point: $\pi(\psi(p)) = \pi(p)$. Vertical authomorphisms form a group denoted as $\mathsf{Aut_v}(\mathsf{P})$. The latter is to be identified with the group $\mathcal{G}$ of gauge transformations. The corresponding Lie algebra will be denoted by $\mathsf{aut_v}(\mathsf{P}) \equiv \mathsf{Lie}(\mathcal{G})$; it is a space of vector fields in P generated by one-parameter subgroups of $\mathsf{Aut_v}(\mathsf{P})$. The reason for this identification is clear from the way a connection transforms under vertical automorphisms. Let A be a connection with curvature $F = dA + \frac{1}{2}[A, A]$. In local coordinates, it takes the form $A = A^a_\mu T^a dx^\mu$, where T^a are the generators of $\mathsf{Lie}(\mathsf{G})$. Let ψ be a vertical automorphism: we can associate to it a map $\gamma :\ \mathsf{P} \to \mathsf{G}$ defined by $\psi(p) = p\,\gamma(p)$ satisfying $\gamma(pg) = g^{-1}\gamma(u)g$. Then, one can show that the following holds:

$$\psi^* A = \gamma^{-1} A \gamma + \gamma^{-1} d\gamma, \qquad \psi^* F = \gamma^{-1} F \gamma.$$

Next, we introduce the evaluation map:

$$ev : \mathsf{P} \times \mathcal{G} \to \mathsf{P}, \qquad \mathsf{ev}(\mathsf{p}, \text{-}) = \text{-}(\mathsf{p}). \tag{A1}$$

We suppose that $\mathsf{P} \times \mathcal{G}$ is a principal fiber bundle over $M \times \mathcal{G}$ with group G. This means that pulling back a connection A from P, we obtain a connection $\mathcal{A} = ev^* A$ in $P \times \mathcal{G}$.

This connection contains all information about FP-ghosts and BRST transformations. Let us see how this comes about.

We evaluate ev^*A over a couple (X, Y). Here $X \in T_pP$ and $\mathrm{Y} \in T_\psi\mathcal{G}$, where $T_pP, T_\psi\mathcal{G}$ denote the tangent space of P at p and of $\mathcal{G}$ at ψ, respectively. Since $\mathcal{G}$ is a Lie group, there exists a $\mathrm{Z} \in T_{id}\mathcal{G}$, such that $\psi_*\mathrm{Z} = \mathrm{Y}$. If ψ_t ($t \in \mathbb{R}$) with $\psi_0 = \psi$ generates Y, i.e., given $f \in C^\infty(\mathcal{G})$, we have the following:

$$\mathrm{Y}f = \frac{d}{dt}f(\psi_t)\Big|_{t=0}, \tag{A2}$$

then $\mathring{\psi}_t = \psi^{-1}\psi_t$ generates Z:

$$\mathrm{Z}f = \frac{d}{dt}f(\mathring{\psi}_t)\Big|_{t=0} = (\psi_*^{-1}\mathrm{Y})f. \tag{A3}$$

Now, it is useful to introduce two auxiliary maps:

$$\begin{aligned} ev_p &: \mathcal{G} \to \mathsf{P}, \qquad ev_p(\psi) = \psi(p), \\ ev_\psi &: \mathsf{P} \to \mathsf{P}, \qquad ev_\psi(p) = \psi(p). \end{aligned}$$

For any $h \in C^\infty(\mathsf{P})$, we have the following:

$$(ev_{p*}\mathrm{Y})h = \frac{d}{dt}h(ev_p \circ \psi_t)\Big|_{t=0} = \frac{d}{dt}h(\psi_t(p))\Big|_{t=0} = Y_{\psi(p)}h. \tag{A4}$$

It follows that

$$\begin{aligned} (ev^*A)_{p,\psi}(\mathrm{X},\mathrm{Y}) &= A_{\psi(p)}(ev_{\psi*}\mathrm{X}) + A_{\psi(p)}(ev_{p*}\mathrm{Y}) = A_{\psi(p)}(\psi_*\mathrm{X}) + A_{\psi(p)}\Big(\mathrm{Y}_{\psi(p)}\Big) \\ &= (\psi^*A)_p(\mathrm{X}) + (\psi^*A)_p(\mathrm{Z}_p) = (\psi^*A)_p(\mathrm{X}) + \Big(i_{\psi_*^{-1}(\mathrm{Y})}\psi^*A\Big)_{\psi(p)} \\ &= (\psi^*A)_p(\mathrm{X}) + \Big(i_{\psi_*^{-1}(\cdot)}\psi^*A\Big)_{\psi(p)}(\mathrm{Y}). \end{aligned} \tag{A5}$$

At $\psi = id$, the identity of $\mathcal{G}$, this formula can be written in the compact form

$$ev^*A = A + i_{(\cdot)}A, \tag{A6}$$

where i is the interior product, and $i_{(\cdot)}A$ denotes the map $\mathrm{Z} \to i_\mathrm{Z}A$ that associates to every $\mathrm{Z} \in \mathsf{Lie}(\mathcal{G})$ the map $\xi_\mathrm{Z} = A(\mathrm{Z}) : \mathsf{P} \to \mathsf{Lie}(\mathsf{G})$. Here, ξ_Z is an infinitesimal gauge transformation, for let us recall that the action of Z over the connection A is given by the Lie derivative L_Z, which takes the following form:

$$\begin{aligned} L_\mathrm{Z}A &= (di_\mathrm{Z} + i_\mathrm{Z}d)A = d(i_\mathrm{Z}A) + i_\mathrm{Z}(dA) = d(i_\mathrm{Z}A) + i_\mathrm{Z}\left(F - \frac{1}{2}[A,A]\right) \\ &= i_\mathrm{Z}F + d(i_\mathrm{Z}A) + [A, i_\mathrm{Z}A] = d\xi_\mathrm{Z} + [A, \xi_\mathrm{Z}], \end{aligned} \tag{A7}$$

because $i_\mathrm{Z}F = 0$, Z being a vertical vector, while F is a basic (i.e., horizontal) two-form.

The above means, in particular, that $i_{(\cdot)}A$ behaves like the Maurer–Cartan form on the group G. From now on, our purpose is to justify the fact that $i_{(\cdot)}A$ can play the role of the ghost field c.

If $Q(A)$ is a polynomial of A, dA and the exterior product $A \wedge A$, the formula (A6) is generalized to the following expression:

$$ev^*Q(A) = Q(A) + i_{(\cdot)}Q(A) - i_{(\cdot)}i_{(\cdot)}Q(A) - \ldots + (-1)^{\frac{k(k-1)}{2}}\underbrace{i_{(\cdot)}\ldots i_{(\cdot)}}_{k \text{ terms}}Q(A) + \ldots, \tag{A8}$$

where the interior products are understood with respect to vectors in $Lie(\mathcal{G})$. Of course, we have also the following:

$$ev^*Q(A) = Q(ev^*A) = Q(A + i_{(\cdot)}A), \tag{A9}$$

because the pull-back 'passes through' exterior product and differential. So the RHS of this equation equals the RHS of (A8). This allows us to read off the meaning of the expression $Q(A + i_{(\cdot)}A)$.

Let us consider an example. A remarkable consequence of (A6) is the following:

$$\mathcal{F} = ev^*F = F, \tag{A10}$$

because $i_Z F = 0$ for any $Z \in Lie(\mathcal{G})$. Let us write down the explicit form of ev^*F:

$$ev^*F \quad = \quad F + i_{(\cdot)}F + i_{(\cdot)}i_{(\cdot)}F = F + i_{(\cdot)}dA + [i_{(\cdot)}A, A] + i_{(\cdot)}i_{(\cdot)}dA + \frac{1}{2}[i_{(\cdot)}A, i_{(\cdot)}A]. \tag{A11}$$

On the other hand, if $\hat{\delta}$ is the exterior differential in $\mathcal{G}$, we have the following:

$$\begin{aligned} F(ev^*A) &= (d+\hat{\delta})\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \qquad \text{(A12)} \\ &= F + \hat{\delta}A + di_{(\cdot)}A + \frac{1}{2}[A, i_{(\cdot)}A] + \frac{1}{2}[i_{(\cdot)}A, A] + \hat{\delta}i_{(\cdot)}A + \frac{1}{2}[i_{(\cdot)}A, i_{(\cdot)}A], \end{aligned}$$

which means, splitting it according to the form degree,

$$\hat{\delta}A \quad = \quad -di_{(\cdot)}A - \frac{1}{2}[A, i_{(\cdot)}A] - \frac{1}{2}[i_{(\cdot)}A, A], \tag{A13}$$

$$\hat{\delta}i_{(\cdot)}A \quad = \quad -\frac{1}{2}[i_{(\cdot)}A, i_{(\cdot)}A]. \tag{A14}$$

The first equation must correspond with the term $i_{(\cdot)}F$ in Equation (A11). However,

$$i_{(\cdot)}F = i_{(\cdot)}dA + [i_{(\cdot)}A, A] = L_{(\cdot)}A - di_{(\cdot)}A - [A, i_{(\cdot)}A] = 0. \tag{A15}$$

Therefore, we must understand that $\hat{\delta} = (-1)^k \mathfrak{s}$, where k is the order of the form in P it acts on, and $\mathfrak{s}$ is the ordinary BRST variation. Moreover $[i_{(\cdot)}A, A] = [A, i_{(\cdot)}A]$, i.e., A and $i_{(\cdot)}A$ behave like one-forms (remember Equation (5)!). Finally, we have the following:

$$\mathfrak{s}A \quad = \quad di_{(\cdot)}A + [A, i_{(\cdot)}A], \tag{A16}$$

$$\mathfrak{s}i_{(\cdot)}A \quad = \quad -\frac{1}{2}[i_{(\cdot)}A, i_{(\cdot)}A]. \tag{A17}$$

On the other hand, Equation (A14) must correspond to the last two terms in Equation (A11). To see that this is the case, one must recall some basic formulas in differential geometry where, for any one-form ω and any two vector fields X, Y, we have the following:

$$d\omega(X, Y) = \frac{1}{2}\big(X\,!(Y) - Y\,!(X)\big) - !([X, Y]), \tag{A18}$$

and

$$L_X\omega(Y) = X\,\omega(Y) - \omega([X, Y]). \tag{A19}$$

The skew double interior product $i_{(\cdot)}i_{(\cdot)}dA$ in (A11) must be understood as follows:

$$i_{Z_2}i_{Z_1}dA = dA(Z_1, Z_2) = \frac{1}{2}\big(Z_1 A(Z_2) - Z_2\, A(Z_1)\big) - A([Z_1, Z_2]) = L_{Z_1}A(Z_2) - L_{Z_2}A(Z_1).$$

In other words, $i_{(\cdot)}i_{(\cdot)}dA$ is to be understood as the Lie derivative of $i_{(\cdot)}A$, or its BRST transform, once we interpret $i_{(\cdot)}A$ as the FP ghost, in agreement with (A17).

From the above formulas we see that the evaluation map provides a geometrical interpretation of the BRST transformations. Let us see now the relation with the superfield formalism. To start, let us remark that the geometrical formula (A6) corresponds to the superfield expression:

$$\tilde{A}|_{\vartheta=0} = A + c\, d\vartheta, \tag{A20}$$

where ϑ is our anticommuting variable (see Section 2). The expression $i_{(\cdot)}A$ is the component of $\mathcal{A}$ in the direction of $\mathcal{G}$ in the product $\mathsf{P} \times \mathcal{G}$. Therefore, we see that ϑ represents this direction, rather than the vertical direction in P. The ϑ partners of A, F and c are nothing but the Lie derivatives in this direction with respect to the vector fields $Z \in \mathsf{Lie}(\mathcal{G})$. Therefore, the superfield method captures the geometry of the principal fiber bundles: $\mathcal{A} \to \mathcal{A}/\mathcal{G}$ where $\mathcal{A}$ is the space of connections.

Appendix B. Auxiliary Formulae

In this appendix, we collect a few cumbersome formulas in expanded form with the aim of clarifying the main text of the paper.

Appendix B.1. Expansion of Equation (79)

We start by expanding the LHS of (79):

$$\begin{aligned}
\tilde{A}_M(\tilde{X})d\tilde{X}^M = &\Big(A_\mu - \big(\vartheta\bar{\xi} + \bar{\vartheta}\xi - \vartheta\bar{\vartheta}h\big)\cdot\partial A_\mu + \vartheta\bar{\vartheta}\,\xi\bar{\xi}\cdot\partial^2 A_\mu + \vartheta\big(\bar{\phi}_\mu - \bar{\vartheta}\xi\cdot\partial\bar{\phi}_\mu\big)\\
&+\bar{\vartheta}\big(\phi_\mu - \vartheta\bar{\xi}\cdot\partial\phi_\mu\big) + \vartheta\bar{\vartheta}\,B_\mu(x)\Big)\\
&\cdot\Big(dx^\mu - \vartheta\,\partial_\lambda\bar{\xi}^\mu dx^\lambda - \bar{\vartheta}\,\partial_\lambda\xi^\mu dx^\lambda + \vartheta\bar{\vartheta}\partial_\lambda h^\mu dx^\lambda - (\bar{\xi}^\mu - \bar{\vartheta}h^\mu)d\vartheta - (\xi^\mu + \vartheta h^\mu)d\bar{\vartheta}\Big)\\
&+\Big(\chi(x) - \big(\vartheta\bar{\xi} + \bar{\vartheta}\xi - \vartheta\bar{\vartheta}h\big)\cdot\partial\chi(x) + \vartheta\bar{\vartheta}\,\xi\bar{\xi}\cdot\partial^2\chi(x)\\
&+\vartheta\big(\bar{C}(x) - \bar{\vartheta}\xi\cdot\partial\bar{C}(x)\big) + \bar{\vartheta}\big(C(x) - \vartheta\bar{\xi}\cdot\partial C(x)\big) + \vartheta\bar{\vartheta}\psi(x)\Big)d\vartheta\\
&+\Big(\lambda(x) - \big(\vartheta\bar{\xi} + \bar{\vartheta}\xi - \vartheta\bar{\vartheta}h\big)\cdot\partial\lambda(x) + \vartheta\bar{\vartheta}\,\xi\bar{\xi}\cdot\partial^2\lambda(x)\\
&+\vartheta\big(\bar{D}(x) - \bar{\vartheta}\xi\cdot\partial\bar{D}(x)\big) + \bar{\vartheta}\big(D(x) - \vartheta\bar{\xi}\cdot\partial D(x)\big) + \vartheta\bar{\vartheta}\rho(x)\Big)d\bar{\vartheta} = A_\mu(x)dx^\mu,
\end{aligned} \tag{A21}$$

where $\xi\bar{\xi}\cdot\partial^2 = \xi^\mu\bar{\xi}^\nu\partial_\mu\partial_\nu$.

Appendix B.2. Expansion of Equation (99)

The next auxiliary formula is the explicit expression of the LHS of (99):

$$
\begin{aligned}
& g_{\mu\nu}(x)dx^\mu \vee dx^\nu = \widetilde{G}_{MN}(\tilde{X})\tilde{d}\tilde{X}^M \vee \tilde{d}\tilde{X}^N \qquad \text{(A22)} \\
= \; & \Big(g_{\mu\nu} - (\vartheta\bar{\xi} + \bar{\vartheta}\xi - \vartheta\bar{\vartheta}h)\cdot\partial g_{\mu\nu} + \vartheta\bar{\vartheta}\,\xi\bar{\xi}\cdot\partial^2 g_{\mu\nu} + \vartheta(\bar{\Gamma}_{\mu\nu} - \bar{\vartheta}\xi\cdot\partial\bar{\Gamma}_{\mu\nu}) \\
& \quad +\bar{\vartheta}(\Gamma_{\mu\nu} - \vartheta\bar{\xi}\cdot\partial\Gamma_{\mu\nu}) + \vartheta\bar{\vartheta}\,V_{\mu\nu}(x)\Big) \\
& \Big(dx^\mu - (\vartheta\,\partial_\lambda\bar{\xi}^\mu + \bar{\vartheta}\,\partial_\lambda\xi^\mu - \vartheta\bar{\vartheta}\partial_\lambda h^\mu)dx^\lambda - (\bar{\xi}^\mu - \bar{\vartheta}h^\mu)d\vartheta - (\xi^\mu + \vartheta h^\mu)d\bar{\vartheta}\Big) \\
& \vee\big(dx^\nu - (\vartheta\,\partial_\rho\bar{\xi}^\nu + \bar{\vartheta}\,\partial_\rho\xi^\nu - \vartheta\bar{\vartheta}\partial_\rho h^\nu)dx^\rho - (\bar{\xi}^\nu - \bar{\vartheta}h^\nu)d\vartheta - (\xi^\nu + \vartheta h^\nu)d\bar{\vartheta}\big) \\
& +2\Big(\gamma_\mu - (\vartheta\bar{\xi} + \bar{\vartheta}\xi - \vartheta\bar{\vartheta}h)\cdot\partial\gamma_\mu + \vartheta\bar{\vartheta}\,\xi\bar{\xi}\cdot\partial^2\gamma_\mu + \vartheta(\bar{g}_\mu - \bar{\vartheta}\xi\cdot\partial\bar{g}_\mu) \\
& \quad +\bar{\vartheta}(g_\mu - \vartheta\bar{\xi}\cdot\partial g_\mu) + \vartheta\bar{\vartheta}\,\Gamma_\mu(x)\Big) \\
& \Big(dx^\mu - (\vartheta\,\partial_\lambda\bar{\xi}^\mu + \bar{\vartheta}\,\partial_\lambda\xi^\mu - \vartheta\bar{\vartheta}\partial_\lambda h^\mu)dx^\lambda - (\bar{\xi}^\mu - \bar{\vartheta}h^\mu)d\vartheta - (\xi^\mu + \vartheta h^\mu)d\bar{\vartheta}\Big) \vee d\vartheta \\
& +2\Big(\bar{\gamma}_\mu - (\vartheta\bar{\xi} + \bar{\vartheta}\xi)\cdot\partial\bar{\gamma}_\mu + \vartheta\bar{\vartheta}\,\xi\bar{\xi}\cdot\partial^2\bar{\gamma}_\mu + \vartheta(\bar{f}_\mu - \bar{\vartheta}\xi\cdot\partial\bar{f}_\mu) \\
& +\bar{\vartheta}(f_\mu - \vartheta\bar{\xi}\cdot\partial f_\mu) + \vartheta\bar{\vartheta}\,\bar{\Gamma}_\mu(x)\Big) \\
& \Big(dx^\mu - (\vartheta\,\partial_\lambda\bar{\xi}^\mu + \bar{\vartheta}\,\partial_\lambda\xi^\mu - \vartheta\bar{\vartheta}\partial_\lambda h^\mu)dx^\lambda - (\bar{\xi}^\mu - \bar{\vartheta}h^\mu)d\vartheta - (\xi^\mu + \vartheta h^\mu)d\bar{\vartheta}\Big) \vee d\bar{\vartheta} \\
& +\Big(g - (\vartheta\bar{\xi} + \bar{\vartheta}\xi - \vartheta\bar{\vartheta}h)\cdot\partial g + \vartheta\bar{\vartheta}\,\xi\bar{\xi}\cdot\partial^2 g + \vartheta(\bar{\gamma} - \bar{\vartheta}\xi\cdot\partial\bar{\gamma}) + \bar{\vartheta}(\gamma - \vartheta\xi\cdot\gamma) + \vartheta\bar{\vartheta}G\Big)d\vartheta \vee d\bar{\vartheta},
\end{aligned}
$$

where all the fields on the RHS are function of x.

Appendix B.3. Expansion of Equation (126)

Here, we expand the horizontality condition for the inverse supermetric Equation (126):

$$
\begin{aligned}
&\hat{g}^{\mu\nu}(x)\frac{\partial}{\partial x^\mu}\vee\frac{\partial}{\partial x^\nu}=\widetilde{\widehat{G}}^{MN}(\tilde{X})\frac{\partial}{\tilde{\partial}\tilde{X}^M}\vee\frac{\tilde{\partial}}{\tilde{\partial}\tilde{X}^N} \qquad \text{(A23)}\\
=\;&\Big(\hat{g}^{\mu\nu}-(\vartheta\bar{\xi}+\bar{\vartheta}\xi-\vartheta\bar{\vartheta}h)\cdot\partial\hat{g}^{\mu\nu}+\vartheta\bar{\vartheta}\,\xi\bar{\xi}\cdot\partial^2\hat{g}^{\mu\nu}+\vartheta\Big(\widehat{\bar{\Gamma}}^{\mu\nu}-\bar{\vartheta}\xi\cdot\partial\widehat{\bar{\Gamma}}^{\mu\nu}\Big)\\
&\quad+\bar{\vartheta}\Big(\widehat{\Gamma}^{\mu\nu}-\vartheta\bar{\xi}\cdot\partial\widehat{\Gamma}^{\mu\nu}\Big)+\vartheta\bar{\vartheta}\,\widehat{V}^{\mu\nu}(x)\Big)\\
&\cdot\Big(\frac{\partial}{\partial x^\mu}+\Big(\vartheta\partial_\mu\bar{\xi}^\lambda+\bar{\vartheta}\partial_\mu\xi^\lambda-\vartheta\bar{\vartheta}\Big(\partial_\mu h^\lambda+\partial_\mu\bar{\xi}^\sigma\partial_\sigma\xi^\lambda-\partial_\mu\xi^\sigma\partial_\sigma\bar{\xi}^\lambda\Big)\Big)\frac{\partial}{\partial x^\lambda}\Big)\\
&\vee\Big(\frac{\partial}{\partial x^\nu}+\big(\vartheta\partial_\nu\bar{\xi}^\rho+\bar{\vartheta}\partial_\nu\xi^\rho-\vartheta\bar{\vartheta}(\partial_\nu h^\rho+\partial_\nu\bar{\xi}^\tau\partial_\tau\xi^\rho-\partial_\nu\xi^\tau\partial_\tau\bar{\xi}^\rho)\big)\frac{\partial}{\partial x^\rho}\Big)\\
&+2\Big(\hat{\gamma}^\mu-(\vartheta\bar{\xi}+\bar{\vartheta}\xi-\vartheta\bar{\vartheta}h)\cdot\partial\hat{\gamma}^\mu+\vartheta\bar{\vartheta}\,\xi\bar{\xi}\cdot\partial^2\hat{\gamma}^\mu+\vartheta(\hat{\bar{g}}^\mu-\bar{\vartheta}\xi\cdot\partial\hat{\bar{g}}^\mu)\\
&\quad+\bar{\vartheta}(\hat{g}^\mu-\vartheta\xi\cdot\partial\hat{g}^\mu)+\vartheta\bar{\vartheta}\,\widehat{\Gamma}_\mu(x)\Big)\\
&\cdot\Big(\frac{\partial}{\partial x^\mu}+\Big(\vartheta\partial_\mu\bar{\xi}^\lambda+\bar{\vartheta}\partial_\mu\xi^\lambda-\vartheta\bar{\vartheta}\Big(\partial_\mu h^\lambda+\partial_\mu\bar{\xi}^\sigma\partial_\sigma\xi^\lambda-\partial_\mu\xi^\sigma\partial_\sigma\bar{\xi}^\lambda\Big)\Big)\frac{\partial}{\partial x^\lambda}\Big)\\
&\vee\Big(\frac{\partial}{\partial\vartheta}+\Big(-\bar{\xi}^\rho+\vartheta\bar{\xi}\cdot\partial\bar{\xi}^\rho+\bar{\vartheta}\Big(-h^\rho+\bar{\xi}\cdot\partial\xi^\rho\Big)\\
&\quad+\vartheta\bar{\vartheta}\big(h\cdot\partial\bar{\xi}^\rho-\bar{\xi}\cdot\partial h^\rho-\bar{\xi}\cdot\partial\bar{\xi}\cdot\partial\xi^\rho+\bar{\xi}\cdot\partial\xi\cdot\partial\bar{\xi}^\rho\big)\Big)\frac{\partial}{\partial x^\rho}\Big)\\
&+2\Big(\hat{\bar{\gamma}}^\mu-(\vartheta\bar{\xi}+\bar{\vartheta}\xi)\cdot\partial\hat{\bar{\gamma}}^\mu+\vartheta\bar{\vartheta}\,\xi\bar{\xi}\cdot\partial^2\hat{\bar{\gamma}}^\mu+\vartheta\Big(\hat{\bar{f}}^\mu-\bar{\vartheta}\xi\cdot\partial\hat{\bar{f}}^\mu\Big)\\
&\quad+\bar{\vartheta}\Big(\hat{f}^\mu-\vartheta\xi\cdot\partial\hat{f}^\mu\Big)+\vartheta\bar{\vartheta}\,\widehat{\bar{\Gamma}}^\mu(x)\Big)\\
&\cdot\Big(\frac{\partial}{\partial x^\mu}+\Big(\vartheta\partial_\mu\bar{\xi}^\lambda+\bar{\vartheta}\partial_\mu\xi^\lambda-\vartheta\bar{\vartheta}\Big(\partial_\mu h^\lambda+\partial_\mu\bar{\xi}^\sigma\partial_\sigma\xi^\lambda-\partial_\mu\xi^\sigma\partial_\sigma\bar{\xi}^\lambda\Big)\Big)\frac{\partial}{\partial x^\lambda}\Big)\\
&\vee\Big(\frac{\partial}{\partial\bar{\vartheta}}+\Big(-\xi^\rho+\bar{\vartheta}\xi\cdot\partial\xi^\rho+\vartheta\big(h^\rho+\xi\cdot\partial\bar{\xi}^\rho\big)\\
&\quad+\vartheta\bar{\vartheta}\big(h\cdot\partial\xi^\rho-\xi\cdot\partial h^\rho-\xi\cdot\partial\bar{\xi}\cdot\partial\xi^\rho+\xi\cdot\partial\xi\cdot\partial\bar{\xi}^\rho\big)\Big)\frac{\partial}{\partial x^\rho}\Big)\\
&+\Big(\hat{g}-(\vartheta\bar{\xi}+\bar{\vartheta}\xi-\vartheta\bar{\vartheta}h)\cdot\partial\hat{g}+\vartheta\bar{\vartheta}\,\xi\bar{\xi}\cdot\partial^2\hat{g}+\vartheta(\hat{\bar{\gamma}}-\bar{\vartheta}\xi\cdot\partial\hat{\bar{\gamma}})+\bar{\vartheta}(\hat{\gamma}-\vartheta\xi\cdot\hat{\gamma})+\vartheta\bar{\vartheta}\widehat{G}\Big)\\
&\cdot\Big(\frac{\partial}{\partial\vartheta}+\Big(-\bar{\xi}^\lambda+\vartheta\bar{\xi}\cdot\partial\bar{\xi}^\lambda+\bar{\vartheta}\Big(-h^\lambda+\bar{\xi}\cdot\partial\xi^\lambda\Big)\\
&\quad+\vartheta\bar{\vartheta}\Big(h\cdot\partial\bar{\xi}^\lambda-\bar{\xi}\cdot\partial h^\lambda-\bar{\xi}\cdot\partial\bar{\xi}\cdot\partial\xi^\lambda+\bar{\xi}\cdot\partial\xi\cdot\partial\bar{\xi}^\lambda\Big)\Big)\frac{\partial}{\partial x^\lambda}\Big)\\
&\vee\Big(\frac{\partial}{\partial\bar{\vartheta}}+\Big(-\xi^\rho+\bar{\vartheta}\xi\cdot\partial\xi^\rho+\vartheta\big(h^\rho+\xi\cdot\partial\bar{\xi}^\rho\big)\\
&\quad+\vartheta\bar{\vartheta}\big(h\cdot\partial\xi^\rho-\xi\cdot\partial h^\rho-\xi\cdot\partial\bar{\xi}\cdot\partial\xi^\rho+\xi\cdot\partial\xi\cdot\partial\bar{\xi}^\rho\big)\Big)\frac{\partial}{\partial x^\rho}\Big).
\end{aligned}
$$

Appendix B.4. Expansion of Equation (171)

Finally, we consider the explicit form of the LHS of (171), which can be expanded as follows:

$$
\begin{aligned}
\hat{e}_a^\mu \frac{\partial}{\partial x^\mu} = &\Big(\hat{e}_a^\mu - (\vartheta\bar{\xi} + \bar{\vartheta}\xi - \vartheta\bar{\vartheta}h)\cdot\partial\hat{e}_a^\mu + \vartheta\bar{\vartheta}\,\xi\bar{\xi}\cdot\partial^2\hat{e}_a^\mu + \vartheta\Big(\hat{\bar{\phi}}_a^\mu - \bar{\vartheta}\xi\cdot\partial\hat{\bar{\phi}}_a^\mu\Big) \\
&\quad + \bar{\vartheta}\Big(\hat{\phi}_a^\mu - \vartheta\bar{\xi}\cdot\partial\hat{\phi}_a^\mu\Big) + \vartheta\bar{\vartheta}\,\hat{f}_a^\mu)\Big) \\
&\cdot\bigg(\frac{\partial}{\partial x^\mu} + \Big(\vartheta\partial_\mu\bar{\xi}^\lambda + \bar{\vartheta}\partial_\mu\xi^\lambda - \vartheta\bar{\vartheta}\Big(\partial_\mu h^\lambda + \partial_\mu\bar{\xi}^\sigma\partial_\sigma\xi^\lambda - \partial_\mu\xi^\sigma\partial_\sigma\bar{\xi}^\lambda\Big)\Big)\frac{\partial}{\partial x^\lambda}\bigg) \\
&+\Big(\hat{\chi}_a - (\vartheta\bar{\xi} + \bar{\vartheta}\xi - \vartheta\bar{\vartheta}h)\cdot\partial\hat{\chi}_a + \vartheta\bar{\vartheta}\,\xi\bar{\xi}\cdot\partial^2\hat{\chi}_a + \vartheta\Big(\widehat{\bar{C}}_a - \bar{\vartheta}\xi\cdot\partial\widehat{\bar{C}}_a\Big) \\
&\quad + \bar{\vartheta}\Big(\widehat{C}_a - \vartheta\bar{\xi}\cdot\partial\widehat{C}_a\Big) + \vartheta\bar{\vartheta}\,\hat{\psi}_a)\Big) \\
&\cdot\bigg(\frac{\partial}{\partial\vartheta} + \Big(-\bar{\xi}^\lambda + \vartheta\bar{\xi}\cdot\partial\bar{\xi}^\lambda + \bar{\vartheta}\Big(-h^\lambda + \bar{\xi}\cdot\partial\xi^\lambda\Big) \\
&\qquad + \vartheta\bar{\vartheta}\Big(h\cdot\partial\bar{\xi}^\lambda - \bar{\xi}\cdot\partial h^\lambda - \bar{\xi}\cdot\partial\bar{\xi}\cdot\partial\xi^\lambda + \bar{\xi}\cdot\partial\xi\cdot\partial\bar{\xi}^\lambda\Big)\Big)\frac{\partial}{\partial x^\lambda}\bigg) \\
&+\Big(\hat{\lambda}_a - (\vartheta\bar{\xi} + \bar{\vartheta}\xi - \vartheta\bar{\vartheta}h)\cdot\partial\hat{\lambda}_a + \vartheta\bar{\vartheta}\,\xi\bar{\xi}\cdot\partial^2\hat{\lambda}_a + \vartheta\Big(\widehat{\bar{D}}_a - \bar{\vartheta}\xi\cdot\partial\widehat{\bar{D}}_a\Big) \\
&\quad + \bar{\vartheta}\Big(\widehat{D}_a - \vartheta\bar{\xi}\cdot\partial\widehat{D}_a\Big) + \vartheta\bar{\vartheta}\,\hat{\rho}_a)\Big) \\
&\cdot\bigg(\frac{\partial}{\partial\bar{\vartheta}} + \Big(-\xi^\rho + \bar{\vartheta}\xi\cdot\partial\xi^\rho + \vartheta(h^\rho + \xi\cdot\partial\bar{\xi}^\rho) \\
&\qquad + \vartheta\bar{\vartheta}(h\cdot\partial\xi^\rho - \xi\cdot\partial h^\rho - \xi\cdot\partial\bar{\xi}\cdot\partial\xi^\rho + \xi\cdot\partial\xi\cdot\partial\bar{\xi}^\rho)\Big)\frac{\partial}{\partial x^\rho}\bigg).
\end{aligned}
\tag{A24}
$$

Appendix B.5. Inverse Supermetric

Here is an additional argument (in 4D) that shows that the inverse of G_{MN} does not exist. Suppose that the inverse $\widehat{G}^{MN}$ of G_{MN} exists. We have the expansions (125), which involve 76 component functions. The inversion condition is the following:

$$\widehat{G}^{ML}G_{LN} = \delta^M{}_N, \tag{A25}$$

where $\delta^\mu{}_\nu = \delta_\nu{}^\mu = \delta^\mu_\nu$ while $1 = \delta^\vartheta{}_{\bar{\vartheta}} = -\delta_{\bar{\vartheta}}{}^\vartheta$. Decomposing the condition (A25) into components, one realizes that it implies 88 quadratic equations. This is to be compared with the ordinary inverse metric $\hat{g}^{\mu\nu}$ in 4D, which has 10 independent components, while the independent inversion conditions are also 10. It is clear that (A25) cannot be satisfied without imposing constraints on the supermetric components. For instance, one of the equations is the following:

$$\hat{\gamma}^\mu\gamma_\lambda + \hat{\bar{\gamma}}^\mu\bar{\gamma}_\lambda = 0, \tag{A26}$$

which means that either $\hat{\gamma}^\mu$ and $\hat{\bar{\gamma}}^\mu$ vanish or they are constrained to one another (no such constraint exists for γ^μ and $\bar{\gamma}^\mu$).

Appendix C. Gauge Transformations in HS-YM

In this subsection, we examine in more detail the gauge transformation (245) and propose an interpretation of the lowest spin fields. Let us expand the master gauge parameter as in (258) and consider the first few terms in the transformation law of the lowest spin fields ordered in such a way that component fields and gauge parameters are infinitesimals of the same order. To the lowest ($\delta^{(0)}$) order, the transformation (245) reads as follows:

$$
\begin{aligned}
&\delta^{(0)} A_a = \partial_a \epsilon, \\
&\delta^{(0)} \chi_a^\nu = \partial_a \xi^\nu, \\
&\delta^{(0)} b_a{}^{\nu\lambda} = \partial_a \Lambda^{\nu\lambda}.
\end{aligned} \tag{A27}
$$

To the first ($\delta^{(1)}$) order, we have the following:

$$
\begin{aligned}
\delta^{(1)} A_a &= \xi \cdot \partial A_a - \partial_\rho \epsilon \, \chi_a^\rho, \\
\delta^{(1)} \chi_a^\nu &= \xi \cdot \partial \chi_a^\nu - \partial_\rho \xi^\nu \chi_a^\rho + \partial^\rho A_a \Lambda_\rho{}^\nu - \partial_\lambda \epsilon \, b_a{}^{\lambda\nu}, \\
\delta^{(1)} b_a^{\nu\lambda} &= \xi \cdot \partial b_a{}^{\nu\lambda} - \partial_\rho \xi^\nu b_a{}^{\rho\lambda} - \partial_\rho \xi^\lambda b_a{}^{\rho\nu} + \partial_\rho \chi_a^\nu \Lambda^{\rho\lambda} + \partial_\rho \chi_a^\lambda \Lambda^{\rho\nu} - \chi_a^\rho \partial_\rho \Lambda_{\nu\lambda}.
\end{aligned} \tag{A28}
$$

The next orders contain three and higher derivatives.

These transformation properties allow us to associate the first two component fields of h_a to an ordinary U(1) gauge field and to a vielbein. To see this, let us denote by $\tilde{A}_a$ and $\tilde{E}_a^\mu = \delta_a^\mu - \tilde{\chi}_a^\mu$ the standard gauge and vielbein fields. The standard gauge and diffeomorphism transformations are the following:

$$
\begin{aligned}
\delta \tilde{A}_a &\equiv \delta\left(\tilde{E}_a^\mu \tilde{A}_\mu\right) \equiv \delta\left((\delta_a^\mu - \tilde{\chi}_a^\mu)\tilde{A}_\mu\right) \\
&= \left(-\partial_a \xi^\mu - \xi \cdot \partial \tilde{\chi}_a^\mu + \partial_\lambda \xi^\mu \tilde{\chi}_a^\lambda\right)\tilde{A}_\mu + (\delta_a^\mu - \tilde{\chi}_a^\mu)\left(\partial_\mu \epsilon + \xi \cdot \partial \tilde{A}_\mu + \tilde{A}_\lambda \partial_\mu \xi^\lambda\right) \\
&= \partial_a \epsilon + \xi \cdot \partial \tilde{A}_a - \tilde{\chi}_a^\mu \partial_\mu \epsilon,
\end{aligned} \tag{A29}
$$

and

$$
\delta \tilde{E}_a^\mu \equiv \delta(\delta_a^\mu - \tilde{\chi}_a^\mu) = \xi \cdot \partial \tilde{e}_a^\mu - \partial_\lambda \xi^\mu \tilde{e}_a^\lambda = -\xi \cdot \partial \tilde{\chi}_a^\mu - \partial_a \xi^\mu + \partial_\lambda \xi^\mu \tilde{\chi}_a^\lambda, \tag{A30}
$$

so that

$$
\delta \tilde{\chi}_a^\mu = \xi \cdot \partial \tilde{\chi}_a^\mu + \partial_a \xi^\mu - \partial_\lambda \xi^\mu \tilde{\chi}_a^\lambda, \tag{A31}
$$

where we have retained only the terms at most linear in the fields.

Now, it is important to understand the derivative ∂_a in Equations (245) and (A27) in the appropriate way: the derivative ∂_a means $\partial_a = \delta_a^\mu \partial_\mu$, not $\partial_a = E_a^\mu \partial_\mu = \left(\delta_a^\mu - \chi_a^\mu\right)\partial_\mu$. In fact, the linear correction $-\chi_a^\mu \partial_\mu$ is contained in the term $-i[h_a(x,p) \overset{*}{,} \varepsilon(x,p)]$; see, for instance, the second term on the RHS of the first Equation (A28).

From the above, it is now immediate to make the following identifications:

$$
A_a = \tilde{A}_a, \qquad \chi_a^\mu = \tilde{\chi}_a^\mu. \tag{A32}
$$

The transformations (A27) and (A28) allow us to interpret χ_a^μ as the fluctuation of the inverse vielbein; therefore, the HS-YM action may accommodate gravity. However, a gravitational interpretation requires also that the frame field transforms under local Lorentz transformations. Therefore, we expect that the master field h_a transforms and the action be invariant under local Lorentz transformations. In [69], it was shown that this symmetry can actually be implemented.

Notes

1 But there seem to exist models where the anti-BRST symmetry is of particular interest, for instance the one of the vector supersymmetry (combining with BRST and anti-BRST symmetries) in certain topological field theories like Chern-Simons theory [18] for which this symmetry is at the origin of the finiteness of models [19]. More recently, the fundamental role of the anti-BRST symmetry in the construction of Hodge-type theories was pointed out in [20].

2 The symbol $[\cdot,\cdot]$ denotes an ordinary commutator when both entries are non-anticommuting, and an anticommutator when both entries are anticommuting.

3 There are also more general superdiffeomorphisms, which we ignore here.

4 Whenever possible we use Greek letters for anticommuting auxiliary fields and Latin letters for commuting ones.

5 Basic literature on higher spin theories can be found in [62–67].
6 A reviewer of this paper pointed out to us that this result holds in the gauge-real representation of SYM theory.
7 In this section the square bracket notation [,] denotes a *graded commutator*, with grading according to the total Grassmannality ϵ of the two entries
8 For recent applications of the superspace/supervariable approach to (non-)susy 1d and 2d diffeomorphism invariant theories, see [31].

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