universe

MDPI

Communication

New Anisotropic Exact Solution in Multifield Cosmology

Andronikos Paliathanasis [1,2]

[1] Institute of Systems Science, Durban University of Technology, P.O. Box 1334, Durban 4000, South Africa; anpaliat@phys.uoa.gr
[2] Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia 5090000, Chile

Abstract: In the case of two-scalar field cosmology, and specifically for the Chiral model, we determine an exact solution for the field equations with an anisotropic background space. The exact solution can describe anisotropic inflation with a Kantowski–Sachs geometry and can be seen as the anisotropic analogue of the hyperbolic inflation. Finally, we investigate the stability conditions for the exact solution.

Keywords: multifield cosmology; anisotropic spacetimes; Kantowski–Sachs; anisotropic inflation

check for updates

Citation: Paliathanasis, A. New Anisotropic Exact Solution in Multifield Cosmology. *Universe* **2021**, 7, 323. https://doi.org/10.3390/universe7090323

Academic Editor: Lorenzo Iorio

Received: 16 August 2021
Accepted: 27 August 2021
Published: 30 August 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2021 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

The early acceleration epoch of the universe is the inflationary era [1], to which the isotropy and homogeneity of the observed universe are due [2]. The origin of the inflation is unknown. However, the introduction of a minimally coupled scalar field, the inflation, into the cosmological dynamics of Einstein's General Relativity provides an acceleration when the scalar field potential dominates. Hence, the scalar field drives the spacetime towards a locally isotropic and homogeneous space form that leaves only very small residual anisotropies, which are left from the pre-inflationary era [3,4]. Therefore, anisotropies may have been important for the evolution of the universe. Thus, the investigation of exact solutions in anisotropic inflationary models is a subject of special interest.

Exact and analytic solutions are important for the study of the evolution and of the viability of a given cosmological model. In one scalar field cosmology, exact and analytic solutions in a homogeneous and isotropic background space can be found in [5–14]. On the other hand, there are few known anisotropic exact solutions with one scalar field [15–21].

Multiscalar field models have been proposed as alternative models for the description of the whole cosmological history [22,23]. In the multiscalar field model the additional degrees of freedom provide new dynamical behaviours in the cosmological dynamics [24–30]. Some anisotropic exact solutions in multifield cosmology can be found in [20,31,32].

A multiscalar field model that has drawn the attention of cosmologists in recent years is the Chiral model. The Lagrangian function of the Chiral model is inspired by the σ-model [33] and is composed of two scalar fields, and the kinetic energy is defined on a two-dimensional hyperbolic space [34]. The Chiral model with an exponential potential provides a new inflationary solution known as hyperbolic inflation [35,36]. Hyperinflation solves various problems of inflationary physics. In hyperbolic inflation, the dynamics are driven by all of the matter components of the field equations, that is, by the scalar field potential and the kinetic parts of the two scalar fields. Moreover, the initial conditions in the start and in the end of the inflation can be different in the Chiral model, which means that the curvature perturbations depend upon the number of the e-fold [37]. Furthermore, detectable non-Gaussianities in the power spectrum are supported by the multifield inflation [38].

In this study we investigate the existence of a new exact solution in Chiral cosmology with an anisotropic background space. As far as isotropic and homogeneous models are concerned, Chiral theory has been widely studied previously with many interesting results—see for instance [39–42]—while recently, extensions of Chiral cosmology were considered by assuming one of the two scalar fields to be a phantom field [43]. In our consideration for the background space we consider locally rotational spacetimes (LRS) with two scale factors that belong to the family of Bianchi I, Bianchi III and Kantowski–Sachs spacetimes. These anisotropic spacetimes have the property that they fall into the spatially flat, closed and open Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime when they reach isotropy. The plan of the paper is as follows.

In Section 2 we present the cosmological model of our consideration and we derive the gravitational field equations. In Section 3 we present the new solution of our analysis, which is that of anisotropic hyperbolic inflation. The analysis of homogeneous perturbations is presented in Section 4, where we discuss the stability properties of the new exact solution. Finally, in Section 5 we draw our conclusions.

2. Chiral Cosmology

We consider the gravitational Action Integral

$$S = \int \sqrt{-g} dx^4 \left(R + L_C\left(\phi, \nabla_\mu \phi, \psi, \nabla_\mu \psi\right) \right) \tag{1}$$

in which $R(x^\kappa)$ is the Ricci scalar of the metric tensor $g_{\mu\nu}(x^\kappa)$, and $L_C(\phi, \nabla_\mu\phi, \psi, \nabla_\mu\psi)$ is the Lagrangian function for the Chiral model, which describes the dynamics for the two scalar fields $\phi(x^\kappa)$ and $\psi(x^\kappa)$, that is:

$$L_C(\phi, \nabla_\mu\phi, \psi, \nabla_\mu\psi) = -\frac{1}{2} g^{\mu\nu}(x^\kappa) \left(\nabla_\mu \phi(x^\kappa) \nabla_\nu \phi(x^\kappa) + e^{-2\kappa\phi(x^\kappa)} \nabla_\mu \psi(x^\kappa) \nabla_\nu \psi(x^\kappa) \right) + V(\phi(x^\kappa)). \tag{2}$$

From the kinetic term of Equation (2) we observe that the scalar field lies on two geometries: The physical space with metric tensor $g_{\mu\nu}(x^\kappa)$ and the two-dimensional space of constant curvature with metric $h_{AB} = diag\left(1, e^{-2\kappa\phi}\right)$ and curvature $R_h \simeq -\kappa^2$, $A, B = 1,2$. The parameter κ is assumed to be a nonzero constant, otherwise the line element h_{AB} reduces to the two-dimensional flat space and the Lagrangian Equation (2) is reduced to that of multiquintessence theory. In general, the potential function (Equation (2)) has been assumed to also be a function of the second field $\psi\left(x^k\right)$. However, hyperbolic inflation in the case of FLRW space follows for the exponential potential [35] $V(\phi(x^\kappa)) = V_0 \exp(-\lambda\phi(x^\kappa))$, which we shall consider in this analysis.

Anisotropic Spacetime

In this study for the physical space we consider the LRS anisotropic line element in the Milne variables

$$ds^2 = -dt^2 + e^{2\alpha(t)} \left(e^{2\beta(t)} dx^2 + e^{-\beta(t)} \left(dy^2 + f^2(y) dz^2 \right) \right) \tag{3}$$

in which the function $f(y)$ has one of the following forms, $f_A(y) = 1$, and the line element describes a Bianchi I spacetime, $f_B(y) = \sinh\left(\sqrt{|K|}y\right)$, where $g_{\mu\nu}(x^\kappa)$ is that of Bianchi III spacetime and $f_C(y) = \sin\left(\sqrt{|K|}y\right)$, where $g_{\mu\nu}$ takes the form of Kantowski–Sachs space. Variable $\beta(t)$ indicates the existence of anisotropy. When $\dot{\beta}(t) = 0$, the background space is that of the FLRW universe.

For the line element (Equation (3)) and the Action Integral (Equation (1)) it follows that the equations of motions that drive the dynamics for the variables $\alpha(t)$, $\beta(t)$, $\phi(t)$ and $\psi(t)$ are

$$e^{3\alpha} \left(3\dot{\alpha}^2 - \frac{3}{4}\dot{\beta}^2 - \frac{1}{2}\left(\dot{\phi}^2 + e^{-2\kappa\phi}\dot{\psi}^2 \right) - V(\phi) \right) - e^{\alpha-\beta} K = 0, \tag{4}$$

$$2\ddot{\alpha} + 3\dot{\alpha}^2 + \frac{3}{4}\dot{\beta}^2 + \frac{1}{2}\left(\dot{\phi}^2 + e^{-2\kappa\phi}\dot{\psi} \right) - V(\phi) - \frac{1}{3} e^{-2\alpha-\beta} K = 0, \tag{5}$$

$$\ddot{\beta} + 3\dot{\alpha}\dot{\beta} + \frac{2}{3}e^{-2\alpha-\beta}K = 0, \tag{6}$$

$$\ddot{\phi} + \kappa e^{-2\kappa\phi}\dot{\psi}^2 + 3\dot{\alpha}\dot{\phi} + V_{,\phi} = 0, \tag{7}$$

$$\ddot{\psi} - 2\kappa\dot{\phi}\dot{\psi} + 3\dot{\alpha}\dot{\psi} = 0, \tag{8}$$

where $K = \frac{f(y)_{,yy}}{f(y)}$ is the spatial curvature of the three-dimensional hypersurface of Equation (3). For Bianchi I spacetime, $K = 0$, for Bianchi III space, $K > 0$, and for the Kantowski–Sachs spacetime, $K < 0$.

3. Exact Solution

We assume the exponential potential $V(\phi) = V_0 \exp(-\lambda\phi)$. Moreover, we observe that Equation (8) is total derivative, i.e., $\frac{d}{dt}\left(\dot{\psi}e^{3\alpha-2\kappa\phi}\right) = 0$. Hence, the conservation law for the field equations is

$$I_0 = \dot{\psi}e^{3\alpha-2\kappa\phi}. \tag{9}$$

Equation (5) can be seen as a second conservation law for the dynamical system. In the case of a spatially flat FLRW universe, i.e., $\dot{\beta} = 0$ and $K = 0$, the analytic solution of the field equation was presented recently in [28] using the Lie symmetry approach.

Hence, in order to investigate the existence of additional conservation laws, we apply the theory of Lie symmetries. For a review on applications of the Lie symmetry analysis in cosmology we refer the reader to [44]. We omit the presentation of the calculations and we directly present the results.

The dynamical system consisting of the second-order differential Equations (5)–(8) for the exponential potential admits the symmetry vectors

$$X_1 = \partial_\psi \ ,\ X_2 = 2t\partial_t + \frac{2}{3}(\partial_\alpha + \partial_\beta) + \frac{4}{\lambda}(\partial_\phi + \kappa\psi\partial_\psi) \ ,\ \text{for } \lambda \neq 0 \tag{10}$$

with the corresponding conservation laws I_0 and

$$I_1 = e^{3\alpha}\left(\dot{\beta} - 4\dot{\alpha} + \frac{4}{\lambda}\left(\dot{\phi} + \kappa e^{-2\kappa\phi}\dot{\psi}\right)\right). \tag{11}$$

For $\lambda = 0$, that is $V(\phi) = V_0$, the admitted symmetry vectors are the elements of the $so(3)$ algebra for the metric tensor h_{AB}. They are

$$Z_1 = \partial_\psi,\ Z_2 = (\partial_\phi + \kappa\psi\partial_\psi)\ , \tag{12}$$

$$Z_3 = \psi\partial_\phi + \kappa\left(\frac{\psi^2}{2} + \psi - \frac{1}{2\kappa}e^{2\kappa\phi}\right), \tag{13}$$

with conservation laws I_0 and

$$\bar{I}_2 = \left(\dot{\phi} + \kappa e^{-2\kappa\phi}\dot{\psi}\right) \tag{14}$$

and

$$\bar{I}_3 = \psi\dot{\phi} + \kappa\left(\left(\frac{\psi^2}{2} + \psi\right)e^{-2\kappa\phi} - \frac{1}{2\kappa}\right)\dot{\psi}. \tag{15}$$

We focus on the case for which $\lambda \neq 0$. We observe that the two conservation laws I_0, I_1 are not in involution, that is, $\{I_0, I_1\} \neq 0$, where $\{,\}$ is the Poisson Bracket. Consequently we cannot infer the Liouville integrability of the field equations. However, the existence of the symmetry vector X_2 indicates the existence of invariant functions. We follow [44] and we search for the exact solution of the form

$$a(t) = p_1 \ln t\ ,\ \beta(t) = p_2 \ln t\ ,\ \phi(t) = p_3 \ln t. \tag{16}$$

We substitute into the conservation law I_0, which gives $I_0 = t^{3p_1 - 2\kappa p_3}\dot{\psi}$, that is,

$$\psi(t) = \frac{I_0}{1 - 3p_1 + 2\kappa p_3} t^{1-3p_1+2\kappa p_3} \, , \; 3p_1 - 2\kappa p_3 \neq 1 \tag{17}$$

$$\psi(t) = I_0 \ln t \, , \; 3p_1 - 2\kappa p_3 = 1. \tag{18}$$

In addition, we assume that $I_0 \neq 0$, otherwise we reduce to the case of anisotropic spaces with a quintessence field [15].

Let us assume now that $3p_1 - 2\kappa p_3 \neq 1$, then by replacing Equations (16) and (17) in the field Equations (4)–(7) we arrive at the exact solution

$$\alpha(t) = \frac{1}{3}\left(1 + 2\frac{\kappa}{\lambda}\right)\ln t \, , \; \beta(t) = \frac{4}{3}\left(1 - \frac{\kappa}{\lambda}\right)\ln t \, , \; \phi(t) = \frac{2}{\lambda}\ln t \; , \tag{19}$$

$$\psi(t) = \frac{\lambda I_0}{2\kappa} t^{2\frac{\kappa}{\lambda}} \, , \; V_0 = \frac{\kappa}{\lambda}\left(4 + I_0^2\lambda^2\right) \, , \; K = 4\frac{\kappa}{\lambda}\left(1 - \frac{\kappa}{\lambda}\right) \tag{20}$$

with the constraint equation

$$\left(4(1 - \kappa\lambda) + \left(2 + I_0^2\right)\lambda^2\right) = 0. \tag{21}$$

Hence, $3p_1 - 2\kappa p_3 = \frac{2\kappa}{\lambda}$, which means that $2\kappa - \lambda \neq 0$.

This is a new anisotropic exact solution with two scalar fields. For $K = 0$, it follows that $\lambda = \kappa$. Thus, $\beta(t) = 0$, which means we end with the spatially flat FLRW spacetime. The background spacetime is that of Bianchi III spacetime when $\kappa(\lambda - \kappa) > 0$, while the Kantowski–Sachs metric is recovered when $\kappa(\lambda - \kappa) < 0$.

The anisotropic scale factor can be written as $\beta(t) = \frac{\lambda}{\kappa}K \ln t$, which means that the existence of a spatial curvature indicates the existence of anisotropy. Thus, the isotropic open or closed FLRW spacetimes, similarly to Kasner-like universes, are not provided by this exact solution.

Indeed, this solution is the analogue of the hyperbolic inflation in the anisotropic background space. The deceleration parameter is defined as $q = -1 - \frac{\ddot{a}}{\dot{a}^2}$, that is, $q(t) = -\frac{2(\kappa-\lambda)}{2\kappa+\lambda}$. Consequently, when the acceleration parameter is negative, $q(t) < 0$, the exact solution describes an accelerated solution. Thus, when $-\frac{2(\kappa-\lambda)}{2\kappa+\lambda} < 0$, we observe that $K < 0$. Hence acceleration exists only for the Kantowski–Sachs background space. This inflationary solution is an extension of the inflationary solution found before for the inflation field in Kantowski–Sachs geometry [45].

Finally, for the case $3p_1 - 2\kappa p_3 = 1$, by replacing Equations (16) and (18) in the field equations, from Equation (6) we find $p_2 = 2(1 - p_1)$ and $K = -3\left(1 - 4p_1 + 3p_1^2\right)$. Thus, from Equation (7) it follows that

$$2I_0^2\kappa^2 + (1 - 3p_1)t^{-1+3p_1} - 2t^{1+3p_1+\frac{\lambda - 3p_1\lambda}{2\kappa}} V_0\kappa\lambda = 0. \tag{22}$$

Because we are interested in solutions with two scalar fields, we study cases with $I_0 \neq 0$ and $p_1 \neq \frac{1}{3}$. Thus, the polynomial Equation (22) can not be solved, which means that there is no anisotropic solution of the form of Equation (16) for $2p_1 - 2\kappa p_3 = 1$.

4. Stability Analysis

We continue our analysis with the study of the stability properties for the new anisotropic inflationary solution. We define the new variable $H = \dot{a}$, and we substitute into Equations (4)–(7)

$$H = \frac{\left(1 + 2\frac{\kappa}{\lambda}\right)}{3t} + \delta H(t) \, , \;\; \beta(t) = \frac{4}{3}\left(1 - \frac{\kappa}{\lambda}\right)\ln t + \delta\beta(t) \, , \tag{23}$$

$$\phi(t) = \frac{2}{\lambda}\ln t \; + \delta\phi \, , \; \dot{\psi} = I_0 e^{-3\alpha + 2\kappa\phi} \, , \; V_0 = \frac{\kappa}{\lambda}\left(4 + I_0^2\lambda^2\right) \, , \; K = 4\frac{\kappa}{\lambda}\left(1 - \frac{\kappa}{\lambda}\right) \tag{24}$$

and we linearize. Moreover, we perform the new change of variable, this time for the dependent variable $t = e^s$. Hence we obtain the system of two linear second-order differential equations

$$0 = 3\lambda\Big(\lambda(2\kappa+\lambda)\delta\beta'' + \Big(8\kappa^2 - 6\kappa\lambda + 4\lambda^2\Big)\delta\beta' + 4(\lambda-\kappa)\delta\phi'\Big) \\ - 8\kappa(5\kappa - 2\lambda)(\kappa-\lambda)\delta\beta + 24\kappa(\kappa-\lambda)\delta\phi \tag{25}$$

and

$$0 = \lambda\big(6(\lambda-\kappa)\delta\beta' + \lambda(2\kappa+\lambda)\delta\phi'' + 2(\kappa(2\kappa+\lambda)+3)\delta\phi'\big) \\ + 12\kappa(\kappa-\lambda)\delta\beta + 2\kappa\Big((2\kappa+\lambda)\Big(4\kappa^2\lambda - 4\kappa - \lambda^3\Big) - 6\Big)\delta\phi \tag{26}$$

with the constraint

$$\delta H(s) = \frac{e^{-s}(\lambda((\lambda-\kappa)\delta\beta' + \delta\phi') + 2\kappa(\kappa-\lambda)\delta\beta - 2\kappa\delta\phi)}{\lambda(2\kappa+\lambda)} \tag{27}$$

and $\delta\beta' = \frac{d\delta\beta}{ds}$.

The solutions of the perturbations are expressed as

$$(\delta\beta, \delta\phi)^T = \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_3 & \zeta_4 \end{pmatrix} (\exp(\mu_1(\lambda,\kappa)s), \exp(\mu_2(\lambda,\kappa)s))^T, \tag{28}$$

where $\mu_1(\lambda,\kappa)$, $\mu_2(\lambda,\kappa)$ are the eigenvalues for the linearized system. The asymptotic solution is stable, when $\mathrm{Re}(\mu_1 < 0)$ and $\mathrm{Re}(\mu_2 < 0)$.

In Figure 1 we present the region plots for the parameters μ_1 and μ_2 is the space (λ, κ), where the perturbations decay.

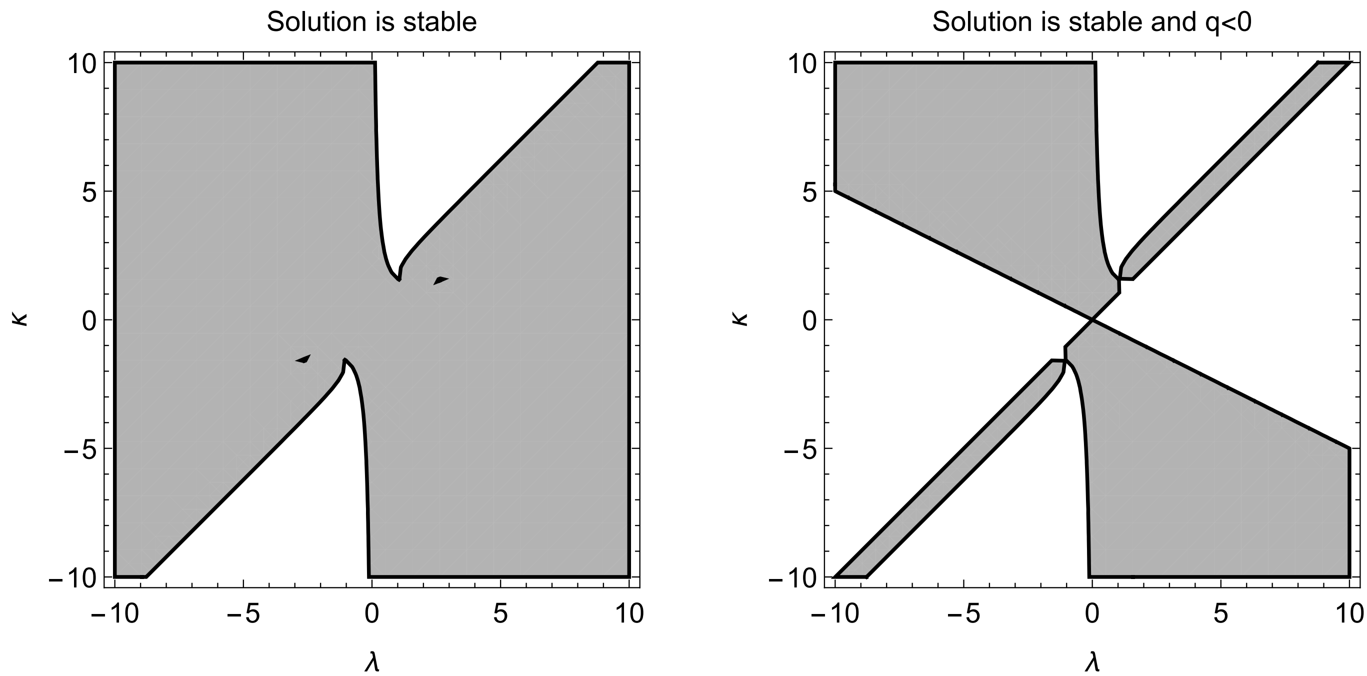

Figure 1. Region plot for the eigenvalues $\mu_1(\lambda,\kappa)$, $\mu_2(\lambda,\kappa)$, where the perturbations around the new anistropic solution decay (**left**), and the perturbations around the new anistropic solution decay and the new anisotropic solution describes an anisotopic inflationary universe (**right**).

5. Conclusions

In this work we investigated the existence of inflationary solutions on multifield cosmology with a homogeneous LRS anisotropic background space. In the context of Chiral cosmology and for the model that describes the hyperbolic inflation in an FLRW background space, we found an anisotropic exact solution that provides anisotropic inflation when the background spacetime has a negative spatial curvature, that is, the physical space is described by the Kantowski–Sachs spacetime.

For the exact solution, the anisotropic parameter and the spatial curvature are analogues. Therefore, when the curvature term vanishes, the physical space becomes isotropic. The method that we applied for the derivation of the exact solution is based upon the investigation of Lie invariant functions, by calculating the Lie symmetries for the cosmological field equations. Finally, the stability properties for these exact solutions were studied. We found that the inflationary anisotropic solution can be a stable solution.

In contrary to the slow-roll inflationary solution for the single scalar field [1], in which $\dot{\psi} = 0$, and $3\dot{\alpha}\dot{\phi} \simeq -V_{,\phi}$, in the hyperinflation the following expressions are true [35]:

$$\dot{\phi} \simeq \frac{6}{2\kappa + \lambda}\dot{\alpha}\,, \tag{29}$$

which means that the evolution of the scalar field is independent on the derivative of the potential. Hence, by replacing the new anisotropic solution in Equation (29) we find that it is true, while for the second field $\psi(t)$ it holds that

$$e^{-2\kappa\phi}\dot{\psi}^2 = 6\left(1 - \frac{6}{2\kappa + \lambda}\right)\dot{\alpha}^2 - \dot{\alpha} - \frac{3}{2}\dot{\beta}^2 - -2V(\phi) - 2e^{-2\alpha-\beta}K \tag{30}$$

where we conclude that we have derived the analogue for the hyperinflation in an anisotropic background space.

At this point it is important to mention that the exact solution that we found does not provide the limit of the cosmological constant [46]. Indeed, the declaration parameter is $q(t) = -\frac{2(\kappa-\lambda)}{2\kappa+\lambda}$ and the limit for the cosmological constant is recovered when $q(t) = 1$, that is $\lambda = 0$. However, in our analysis we considered $\lambda \neq 0$. For other forms of the scalar field potential, it is possible that there exist exact solutions that provide the limit of the cosmological constant. Such an analysis is outside the scope of this work, since we focused on the exponential potential. From this result we can infer that the Chiral model provides inflationary anisotropic solutions that can be used as a toy model for the study of the very early universe.

Let us assume now the new anisotropic exact solution in the limit where $\frac{\kappa}{\lambda} \simeq 1 + \varepsilon$, then $\alpha(t) = \left(1 + \frac{2}{3}\varepsilon\right)\ln t$ and $\beta(t) = \frac{4}{3}\varepsilon \ln t$. Hence, for $\varepsilon^2 = 0$ the anisotropies are small, and inflation can be described by the Hubble slow roll parameters [47] $\varepsilon_H = -\frac{\dot{H}}{H^2}$, $\eta_H = \frac{\dot{\varepsilon}_H}{H\varepsilon_H}$, from where we calculate $\varepsilon_H \simeq 1 - \frac{2}{3}\varepsilon$ and $\eta_H = 0$. However, these slow-roll parameters are similar to those of the exponential potential for the inflation field. However, because of the additional degrees of freedom, the solution may not be always stable, and thus the actual solution will be different from the exact solution.

In a future study we plan to investigate the stability properties for the general model and also to investigate the behaviour of the inflationary parameters with initial conditions near the region of the exact solution.

Funding: This work is based on research supported in part by the National Research Foundation ofSouth Africa (Grant Numbers 131604).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Guth, A. Inflationary universe: A possible solution to the horizon and flatness problems. *Phys. Rev. D* **1981**, *23*, 347–356. [CrossRef]
2. Saadeh, D.; Feeney, S.; Pontzen, A.; Peiris, H.; McEwen, J.D. How isotropic is the Universe? *Phys. Rev. Lett.* **2016**, *117*, 131302. [CrossRef]
3. Goldwirth, D.S. Inhomogeneous initial conditions for inflation. *Phys. Rev. D* **1991**, *43*, 3204–3213. [CrossRef]
4. Clough, K.; Lim, E.A.; Di Nunno, B.S.; Fischler, W.; Flauger, R.; Paban, S. Robustness of Inflation to Inhomogeneous Initial Conditions. *J. Cosmol. Astropart. Phys.* **2017**, *2017*, 025. [CrossRef]
5. Ellis, G.F.R.; Madsen, M.S. Exact scalar field cosmologies. *Class. Quantum Gravity* **1991**, *8*, 667–676. [CrossRef]
6. Ram, B. Infinitely many solutions of Einstein cosmology in 'slow roll'. *Phys. Lett. A* **1993**, *172*, 404–406. [CrossRef]
7. Barrow, J.D.; Saich, P. Scalar field cosmologies. *Class. Quantum Gravity* **1993**, *10*, 297. [CrossRef]
8. Maartens, R.; Taylor, D.R.; Roussos, N. Exact inflationary cosmologies with exit. *Phys. Rev. D* **1995**, *52*, 3358. [CrossRef]
9. Barrow, J.D.; Paliathanasis, A. Observational Constraints on New Exact Inflationary Scalar-field Solutions. *Phys. Rev. D* **2016**, *94*, 083518. [CrossRef]
10. Chimento, L.P.; Jakubi, A.S. Scalar field cosmologies with perfect fluid in robertson-walker metric. *Int. J. Mod. Phys. D* **1996**, *5*, 71. [CrossRef]
11. Banerjee, N.; Sen, S. Power law inflation and scalar field cosmology with a causal viscous fluid. *Phys. Rev. D* **1998**, *57*, 4614–4619. [CrossRef]
12. Chimento, L.P.; Mendez, V.; Zuccala, N. Cosmological models arising from generalized scalar field potentials. *Class. Quantum Gravity* **1999**, *16*, 3749–3763. [CrossRef]
13. Kruger, A.T.; Norburry, J.W. Another exact inflationary solution. *Phys. Rev. D* **2000**, *61*, 087303. [CrossRef]
14. Andrianov, A.A.; Cannata, F.; Kamenshchik, A.Y. General solution of scalar field cosmology with a (piecewise) exponential potential. *J. Cosmol. Astropart. Phys.* **2011**, *10*, 004. [CrossRef]
15. Demianski, M.; de Ritis, R.; Rubano, C.; Scudellaro, P. Scalar fields and anisotropy in cosmological models. *Phys. Rev. D* **1992**, *46*, 1391–1398. [CrossRef]
16. Chimento, L.P.; Cossarini, A.E.; Zuccala, N.A. Isotropic and anisotropic N-dimensional cosmologies with exponential potentials. *Class. Quantum Gravity* **1998**, *15*, 57–74. [CrossRef]
17. Fadragas, C.R.; Leon, G.; Saridakis, E.N. Dynamical analysis of anisotropic scalar-field cosmologies for a wide range of potentials. *Class. Quantum Gravity* **2014**, *31*, 075018. [CrossRef]
18. Tsamparlis, M.; Paliathanasis, A. The geometric nature of Lie and Noether symmetries. *Gen. Relativ. Gravit.* **2011**, *43*, 1861. [CrossRef]
19. Christodoulakis, T.; Grammenos, T.; Helias, C.; Kevrekidis, P.G.; Spanou, A. Decoupling of the general scalar field mode and the solution space for Bianchi type I and V cosmologies coupled to perfect fluid sources. *J. Math. Phys.* **2006**, *47*, 042505. [CrossRef]
20. Do, T.Q.; Kao, W.F. Anisotropic power-law inflation for a model of two scalar and two vector fields. *Eur. Phys. J. C* **2021**, *81*, 525. [CrossRef]
21. Faraoni, V.; Jose, S.; Dussault, S. Multi-fluid cosmology in Einstein gravity: Analytical solutions. *arXiv* **2021**, arXiv:2107.12488.
22. Di Valentino, E.; Mena, O.; Pan, S.; Visinelli, L.; Yang, W.; Melchiorri, A.; Mota, D.F.; Riess, A.G.; Silk, J. In the Realm of the Hubble tension—A Review of Solutions. *Class. Quantum Gravity* **2021**, *38*, 153001. [CrossRef]
23. Cai, Y.-F.; Saridakis, E.; Setare, M.R.; Xia, J.-Q. Quintom Cosmology: Theoretical implications and observations. *Phys. Rep.* **2010**, *493*, 1–60. [CrossRef]
24. Coley, A.A.; Hoogen, R.V.D. The Dynamics of Multi-Scalar Field Cosmological Models and Assisted Inflation. *Phys. Rev. D* **2000**, *92*, 023517. [CrossRef]
25. Calcagni, G.; Liddle, A.R. Stability of multifield cosmological solutions. *Phys. Rev. D* **2008**, *77*, 023522. [CrossRef]
26. Collinucci, A.; Nielsen, M.; Van Riet, T. Scalar cosmology with multi-exponential potentials. *Class. Quantum Gravity* **2005**, *22*, 1269–1287. [CrossRef]
27. Paliathanasis, A. Dynamics of Chiral Cosmology. *Class. Quantum Gravity* **2020** *37*, 195014. [CrossRef]
28. Christodoulidis, P.; Paliathanasis, A. N-field cosmology in hyperbolic field space: Stability and general solutions. *J. Cosmol. Astropart. Phys.* **2021**, *5*, 038. [CrossRef]
29. Christodoulidis, P. General solutions to N-field cosmology with exponential potentials. *Eur. Phys. J. C* **2021**, *81*, 471. [CrossRef]
30. Socorro, J.; Pérez-Payán, S.; Hernández, R.; Espinoza-García, A.; Díaz-Barrón, L.R. Classical and quantum exact solutions for a FRW in chiral like cosmology. *Class. Quantum Gravity* **2021**, *38*, 135027. [CrossRef]
31. Socorro, J.; Nuñez, O.E. Scalar potentials with multi-scalar fields from quantum cosmology and supersymmetric quantum mechanics. *Eur. Phys. J. Plus* **2017**, *132*, 168. [CrossRef]
32. Díaz-Barrón, L.R.; Espinoza-García, A.; Pérez-Payán, S.; Socorro, J. Anisotropic chiral cosmology: Exact solutions. *Int. J. Mod. Phys. D.* **2021**, in press. [CrossRef]
33. Ketov, S.V. *Quantum Non-linear Sigma Models*; Springer: Berlin, Germany, 2000.
34. Chervon, S.V. Chiral Cosmological Models: Dark Sector Fields Description. *Quantum Matter* **2013**, *2*, 71–82. [CrossRef]
35. Brown, A.R. Hyperbolic Inflation. *Phys. Rev. Lett.* **2018**, *121*, 251601. [CrossRef]

36. Mizuno, S.; Mukohyama, S. Primordial perturbations from inflation with a hyperbolic field-space. *Phys. Rev. D* **2017**, *96*, 103533. [CrossRef]
37. Lyth, D.H. A numerical study of non-gaussianity in the curvaton scenario. *J. Cosmol. Astropart. Phys.* **2006**, *2006*, 008.
38. Langlois, D.; Renaux-Peterl, S. Perturbations in generalized multi-field inflation. *J. Cosmol. Astropart. Phys.* **2008**, *2008*, 017. [CrossRef]
39. Christodoulidis, P.; Roest, D.; Sfakianakis, E.I. Scaling attractors in multi-field inflation. *J. Cosmol. Astropart. Phys.* **2019**, *2019*, 059. [CrossRef]
40. Christodoulidis, P.; Roest, D.; Sfakianakis, E.I. Angular inflation in multi-field α-attractors. *J. Cosmol. Astropart. Phys.* **2019**, *2019*, 002. [CrossRef]
41. Paliathanasis, A.; Tsamparlis, M. Two scalar field cosmology: Conservation laws and exact solutions. *Phys. Rev. D* **2014**, *90*, 103524. [CrossRef]
42. Giacomini, A.; Gonzalez, E.; Leon, G.; Paliathanasis, A. Variational symmetries and superintegrability in multifield cosmology. *arXiv* **2021**, arXiv:2104.13649.
43. Paliathanasis, A.; Leon, G. Dynamics of a two scalar field cosmological model with phantom terms. *Class. Quantum Gravity* **2021**, *38*, 075013. [CrossRef]
44. Tsamparlis, M.; Paliathanasis, A. Symmetries of Differential Equations in Cosmology. *Symmetry* **2018**, *10*, 233. [CrossRef]
45. Mendes, L.E.; Henriques, A. Inflation in a simple Kantowski–Sachs model. *Phys. Lett. B* **1991**, *254*, 44–48. [CrossRef]
46. Gron, O. Expansion isotropization during the inflationary era. *Phys. Rev. D* **1985**, *32*, 2522–2527. [CrossRef] [PubMed]
47. Liddle, A.R.; Parson, P.; Barrow, J.D. Formalising the Slow-Roll Approximation in Inflation. *Phys. Rev. D* **1994**, *50*, 7222–7232. [CrossRef]