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GEMS Embeddings of Schwarzschild and RN Black Holes in Painlevé-Gullstrand Spacetimes

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Abstract: Making use of the higher dimensional global embedding Minkowski spacetime (GEMS), we embed (3 + 1)-dimensional Schwarzschild and Reissner-Nordström (RN) black holes written by the Painlevé-Gullstrand (PG) spacetimes, which have off-diagonal components in metrics, into (5 + 1)- and (5 + 2)-dimensional flat ones, respectively. As a result, we have shown the equivalence of the GEMS embeddings of the spacetimes with the diagonal and off-diagonal terms in metrics. Moreover, with the aid of their geodesic equations satisfying various boundary conditions in the flat embedded spacetimes, we directly obtain freely falling temperatures. We also show that freely falling temperatures in the PG spacetimes are well-defined beyond the event horizons, while they are equivalent to the Hawking temperatures, which are obtained in the original curved ones in the ranges between the horizon and the infinity. These will be helpful to study GEMS embeddings of more realistic Kerr, or rotating BTZ black holes.

Keywords: Schwarzschild; Reissner-Nordström; Painlevé-Gullstrand spacetimes; global flat embedding; Unruh effect

1. Introduction

Any low dimensional Riemannian manifold can be locally isometrically embedded in a higher dimensional flat one [1–3]. To be specific, by using isometric embedding, it is expected that it may be possible to find the higher dimensional flat manifold than the original curved one with singularity. Moreover, this global embedding Minkowski spacetime (GEMS) method can be used to show their equivalence of Hawking [4] and Unruh effects [5]. Deser and Levin [6–8] showed that the Hawking temperature for a fiducial observer in a curved spacetime can be considered as the Unruh one for a uniformly accelerated observer in a higher-dimensional flat spacetime. Since then, there have been a lot of works on the GEMS approach to confirm these ideas in various other spacetimes [9–24] and an interesting extension to embedding gravity [25–28]. Later, Brynjolfsson and Thorlacius (BT) [29] introduced a local temperature measured by a freely falling observer in the GEMS method. The local free fall temperature they obtained remains finite at the event horizon and it approaches the Hawking temperature at spatial infinity. Here, a freely falling local temperature is defined at special turning points of radial geodesics where a freely falling observer is momentarily at rest with respect to a black hole. Following the BT's approach, we have extended the methods to various interesting curved spacetimes [30–35] to investigate local temperatures of corresponding spacetimes and their equivalence to Hawking ones. Meanwhile, radial geodesics can be categorized into drip, rain, and hail frames according to which initial states observers or objects are following [36]. Observers are said to be in a drip frame when they are freely falling from rest at a finite initial radius r_0. In particular, when observers are freely falling at rest from infinitely far away or $r_0 \to \infty$, they are said

to be in a rain frame. Observers in a hail frame are hurled inward with initial velocity v_∞ toward black holes at infinity. In this classification scheme, freely falling local temperature of the BT's method is the one seen by an observer in a drip frame. It is interesting to extend the BT's work to the other two frames.

On the other hand, the PG coordinates [37,38] have been newly recognized as an alternative to the Schwarzschild coordinates to avoid the singularity at the event horizon. The PG spacetime remains regular across the horizon. It describes stationary, but not static spacetimes. Observers falling into a black hole in the PG spacetime use their own proper times with spatial flat hypersurfaces flowing radially inward. The importance of the PG spacetime was newly received attention related to the Hawking radiation as a tunnelling process [39–41]. In this scenario, when self-gravitation of particles is taken into account, positive energy virtual particles triggered by vacuum fluctuations just inside an event horizon can tunnel out across the horizon. It was extended to more general spherical spacetimes with interesting physical interpretations [42–45]. Laboratory analogues of gravity [46–49] can also be recast in the PG spacetime coordinates by conformal transformations. Moreover, the PG spacetime used to describe gravitational collapse dynamically both inside and outside the horizon in a single coordinate patch [50], compared to the standard model of Oppenheimer and Snyder [51,52] which uses two different spacetimes corresponding to the interior and exterior regions of the collapsing body. From a modern point of view, several of the subtle features of the PG spacetime are under investigation [53–56].

The main goal of this paper is the construction and analysis of the GEMS embeddings of spacetimes with off-diagonal components in metric, whose embedding is extremely difficult to carry out, and actually there are no such GEMS embedding models as far as we know. Since the more physically realistic Kerr, or rotating BTZ holes have the off-diagonal terms in metrics, it would be interesting enough in itself if their GEMS embeddings can be found. If then, one can further find various thermodynamic functions including freely falling temperatures seen by a freely falling observer beyond event horizons. As a preliminary to such a road, in this paper, we will study first GEMS embeddings of the Schwarzschild and the Reissner-Nordström (RN) black holes described by the PG spacetimes, which are also known to be nontrivial due to the existence of off-diagonal terms in metrics. We will also show that their Hawking and Unruh temperatures are the exactly same with the ones in the original GEMS approach. Moreover, by following BT's method and full geodesic equations satisfying various boundary conditions, we will find freely falling temperatures seen by observers in the PG-embedded Schwarzschild and RN black holes, which can be extended smoothly through the future event horizon. The remainder of the paper is organized as follows. In Section 2, we will construct GEMS embeddings and various temperatures of the Schwarzschild black hole in the PG spacetime and compare them with the ones of the Schwarzschild black hole in the original spherically symmetric spacetime. In Section 3, we also present GEMS embeddings of the RN black hole in the PG spacetime and study various temperatures in drip, rain and hail frames. Conclusions are drawn in Section 4.

2. GEMS Embedding of the Schwarzschild Black Hole in the PG Spacetime
2.1. GEMS Embedding of the Schwarzschild Black Hole in the Spherically Symmetric Spacetime

In this subsection, we briefly recapitulate the GEMS embeddings [1–3] and their way of finding Hawking, Unruh, freely falling temperatures at rest [29] of the Schwarzschild black hole in the spherically symmetric spacetime. The spherically symmetric Schwarzschild black hole is described by the metric

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

with

$$f(r) = 1 - \frac{2m}{r}. \tag{2}$$

From the metric, one can find the surface gravity [57] as

$$k_H = \left. \sqrt{-\frac{1}{2}(\nabla^\mu \xi^\nu)(\nabla_\mu \xi_\nu)} \right|_{r=r_H} = \frac{1}{2r_H}, \tag{3}$$

where ξ^μ is a Killing vector and the event horizon is given by $r_H = 2m$. Then, the Hawking temperature T_H seen by an asymptotic observer is given by

$$T_H = \frac{k_H}{2\pi} = \frac{1}{4\pi r_H}. \tag{4}$$

Moreover, a local fiducial temperature measured by an observer who rests at a distance from the black hole is given by

$$T_{\text{FID}}(r) = \frac{T_H}{\sqrt{f(r)}} = \frac{r^{1/2}}{4\pi r_H(r - r_H)^{1/2}}. \tag{5}$$

Note that the fiducial temperature T_{FID} diverges at the event horizon, while it becomes the Hawking temperature to an asymptotic observer.

Now, by following the GEMS approach [1–3], the (3 + 1)-dimensional spherically symmetric Schwarzschild spacetime (1) can be embedded in a (5 + 1)-dimensional flat one as

$$ds^2 = \eta_{IJ} dz^I dz^J, \text{ with } \eta_{IJ} = \text{diag}(-1, 1, 1, 1, 1, 1), \tag{6}$$

where embedding coordinates are given by

$$
\begin{aligned}
z^0 &= k_H^{-1} f^{1/2}(r) \sinh k_H t, \\
z^1 &= k_H^{-1} f^{1/2}(r) \cosh k_H t, \\
z^2 &= r \sin\theta \cos\phi, \\
z^3 &= r \sin\theta \sin\phi, \\
z^4 &= r \cos\theta, \\
z^5 &= \int dr \left(\frac{r_H(r^2 + rr_H + r_H^2)}{r^3} \right)^{1/2}.
\end{aligned} \tag{7}
$$

Note the analyticity of z^5 in $r > 0$. This helps the embedding coordinates (7) to cover the region $0 < r < r_H$ by analytic extension.

In static detectors (r, θ, ϕ = constant) described by a fixed point in the (z^2, z^3, z^4, z^5) plane on the GEMS embedded spacetime, an observer who is uniformly accelerated in the (5 + 1)-dimensional flat spacetime, follows a hyperbolic trajectory in (z^0, z^1) described by

$$a_6^{-2} = (z^1)^2 - (z^0)^2 = \frac{f(r)}{k_H^2}. \tag{8}$$

Thus, one can arrive at the Unruh temperature for the uniformly accelerated observer in the (5 + 1)-dimensional flat spacetime

$$T_U = \frac{a_6}{2\pi} = \frac{r^{1/2}}{4\pi r_H(r - r_H)^{1/2}}. \tag{9}$$

This corresponds to the fiducial temperature (5) for the observer located at a distance from the Schwarzschild black hole. The Hawking temperature T_H seen by an asymptotic observer can be obtained as

$$T_H = \sqrt{-g_{00}} T_U = \frac{k_H}{2\pi}. \tag{10}$$

As a result, one can see that the Hawking effect for a fiducial observer in a black hole spacetime is equal to the Unruh effect for a uniformly accelerated observer in a higher-dimensional flat spacetime.

It is appropriate to comment that the static detectors following a timelike Killing vector $\xi = \partial_t$ with the same condition (r, θ, ϕ = constant) in the original (3 + 1)-dimensional spacetime have constant 4-acceleration as

$$a_4 = \frac{r_H}{2r^{3/2}(r - r_H)^{1/2}},$$ (11)

where the constant 4-acceleration is given by $a_4^2 = a_\mu a^\mu$ with $a_\mu = \xi_{\nu;\mu}\xi^\nu / |\xi|^2$. Comparing this with the acceleration (9) in the (5 + 1)-dimensional embedded spacetime, one can have

$$a_6 = \frac{r^2}{r_H^2}a_4.$$ (12)

On the other hand, before finding freely falling accelerations in the embedded spacetime, let us first consider the geodesic Equation [57]

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0,$$ (13)

where $x^\mu = (t, r, \theta, \phi)$. Here, the Christoffel symbol is given by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}).$$ (14)

From the metric (1), the geodesic equation gives us explicitly as follows

$$\frac{dv^0}{d\tau} + \frac{2m}{r^2 - 2mr}v^0v^1 = 0,$$ (15)

$$\frac{dv^1}{d\tau} + \frac{m}{r^3}(r - 2m)(v^0)^2 - \frac{m}{r^2 - 2mr}(v^1)^2 - (r - 2m)\left[(v^2)^2 + \sin^2\theta(v^3)^2\right] = 0,$$ (16)

$$\frac{dv^2}{d\tau} + \frac{2}{r}v^1v^2 - \sin\theta\cos\theta(v^3)^2 = 0,$$ (17)

$$\frac{dv^3}{d\tau} + \frac{2}{r}v^1v^3 + 2\cot\theta v^2v^3 = 0,$$ (18)

where $v^\mu = dx^\mu/d\tau$ denotes the four velocity vector. Without loss of generality, one can consider the geodesics on the equatorial plane given by $\theta = \pi/2$. Then, one has $v^2 = d\theta/d\tau = 0$ and the geodesic equations are reduced to

$$\frac{dv^0}{d\tau} + \frac{2m}{r^2 - 2mr}v^0v^1 = 0,$$ (19)

$$\frac{dv^1}{d\tau} + \frac{m}{r^3}(r - 2m)(v^0)^2 - \frac{m}{r^2 - 2mr}(v^1)^2 - (r - 2m)(v^3)^2 = 0,$$ (20)

$$\frac{dv^3}{d\tau} + \frac{2}{r}v^1v^3 = 0.$$ (21)

One can easily integrate Equations (19) and (21) as

$$v^0 = \frac{dt}{d\tau} = \frac{c_1 r}{r - 2m},$$ (22)

$$v^3 = \frac{d\phi}{d\tau} = \frac{c_2}{r^2},$$ (23)

respectively, where c_1 and c_2 are integration constants [58].

Note that for the Killing vectors $\xi^\mu = (1,0,0,0)$ and $\psi^\mu = (0,0,0,1)$, two conserved quantities of E and L are given by

$$E = -g_{\mu\nu}\xi^\mu v^\nu = \left(1 - \frac{2m}{r}\right)v^0, \tag{24}$$

$$L = g_{\mu\nu}\psi^\mu v^\nu = r^2 v^3. \tag{25}$$

Comparing these relations with Equations (22) and (23), one can fix the integration constants

$$c_1 = E, \quad c_2 = L \tag{26}$$

in terms of the conserved quantities E and L. Finally, by letting $ds^2 = -kd\tau^2$ in Equation (1) and making use of Equations (22)–(26), one can obtain

$$v^1 = \frac{dr}{d\tau} = \pm\left[E^2 - \left(k + \frac{L^2}{r^2}\right)\left(1 - \frac{2m}{r}\right)\right]^{1/2}, \tag{27}$$

where the $-(+)$ sign is for inward (outward) motion and $k = 1(0)$ is for a timelike (nulllike) geodesic.

Now, let us find a freely falling acceleration and corresponding temperature in the $(5 + 1)$-dimensional embedded flat spacetime, which is described by Equation (7). For simplicity, we assume that an observer is moving along a timelike geodesic with zero angular momentum. Thus, we choose $k = 1$ and $L = 0$ in the geodesic equations, which read as follows

$$\frac{dt}{d\tau} = \frac{E}{f(r)}, \tag{28}$$

$$\frac{dr}{d\tau} = -\left[E^2 - f(r)\right]^{1/2}. \tag{29}$$

(a) A drip frame: for an observer who is freely falling from rest $r = r_0$ at $\tau = 0$, one has $E = \pm\sqrt{f(r_0)}$. Since this case was already obtained, let us briefly summarize it [29–33]. The equations of motion are reduced to

$$\frac{dt}{d\tau} = \frac{\sqrt{f(r_0)}}{f(r)},$$

$$\frac{dr}{d\tau} = -[f(r_0) - f(r)]^{1/2}, \tag{30}$$

where we have chosen the $(+)$ sign in E. Then, making use of the embedding coordinates in Equation (7) and the geodesic equations in Equation (30), one can explicitly find a freely falling acceleration $\bar{a}_6$ as

$$\bar{a}_6^2 = \sum_{I=0}^{5} \eta_{IJ}\frac{dz^I}{d\tau}\frac{dz^J}{d\tau}\bigg|_{r=r_0} = \frac{r^3 + r_H r^2 + r_H^2 r + r_H^3}{4r_H^2 r^3}. \tag{31}$$

This gives us the freely falling temperature measured by the freely falling observer in the drip frame as

$$T_{\mathrm{DF}} = \frac{\bar{a}_6}{2\pi} = \frac{1}{4\pi r_H}\sqrt{1 + \frac{r_H}{r} + \frac{r_H^2}{r^2} + \frac{r_H^3}{r^3}}. \tag{32}$$

Note that r_0 is replaced with r in Equation (31). As $r \to \infty$, the freely falling temperature T_{DF} is reduced to the Hawking temperature (4). Note also that at the event horizon the freely falling temperature (32) is finite, while the fiducial temperature (5) diverges [29,30].

(b) A rain frame: for an observer who is freely falling from rest at the asymptotic infinity, one has $E = 1$. Then, the equations of motion are reduced to

$$\frac{dt}{d\tau} = \frac{r}{r - 2m},$$
$$\frac{dr}{d\tau} = -\sqrt{\frac{2m}{r}}. \tag{33}$$

Then, in this case, making use of the embedding coordinates in Equation (7) and the geodesic equations in Equation (30), one can obtain a freely falling acceleration $\bar{a}_6$

$$\bar{a}_6^2 = \sum_{I=0}^{5} \eta_{IJ} \frac{dz^I}{d\tau} \frac{dz^J}{d\tau} \bigg|_{r=\infty} = \frac{1}{4r_H^2}, \tag{34}$$

and thus the freely falling temperature as

$$T_{\mathrm{RF}} = \frac{\bar{a}_6}{2\pi} = \frac{1}{4\pi r_H}. \tag{35}$$

Note that the acceleration $\bar{a}_6$ is only defined at spatial infinity. Thus, T_{RF} is the freely falling temperature T_{DF} in Equation (32) with $r \to \infty$. Therefore, one can see that the Hawking temperature in a curved spacetime is equal to the Unruh temperature in a higher-dimensional flat spacetime having a freely falling acceleration starting from rest at the asymptotic infinity.

(c) A hail frame: finally, if an observer starts to fall with an inward non-zero velocity at the asymptotic infinity as

$$\frac{dr}{dt} = -v_\infty \tag{36}$$

where the $(-)$ sign is for the inward direction, one has $E = (1 - v_\infty^2)^{-1/2}$. Then, the equations of motion are reduced to

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v_\infty^2 \left(1 - \frac{2m}{r}\right)}},$$
$$\frac{dr}{d\tau} = -\left(\frac{v_\infty^2}{1 - v_\infty^2} + \frac{2m}{r}\right)^{1/2}. \tag{37}$$

It seems to appropriate to comment that the equations of motion (37) give

$$\frac{dr}{dt}\bigg|_{r=r_H} = -1 \tag{38}$$

at the event horizon, which shows that the velocity of the observer does not exceed the speed of light [36], independent of how fast the observer is hurled at the asymptotic infinity given by (36).

Now, by making use of the embedding coordinates in Equation (7) and the geodesic equations in Equation (37), we can obtain an acceleration $\bar{a}_6$ as

$$\bar{a}_6^2 = \sum_{I=0}^{5} \eta_{IJ} \frac{dz^I}{d\tau} \frac{dz^J}{d\tau} \bigg|_{r=\infty} = \frac{1}{4r_H^2 (1 - v_\infty^2)^2}. \tag{39}$$

Thus, the freely falling temperature in the hail frame is obtained as

$$T_{\mathrm{HF}} = \frac{\bar{a}_6}{2\pi} = \frac{1}{4\pi r_H (1 - v_\infty^2)}. \tag{40}$$

Note that when $v_\infty = 0$, this temperature in the hail frame is reduced to the freely falling one in the rain frame as expected. The freely falling temperature in the hail frame can also be rewritten as

$$T_{\mathrm{HF}} = \frac{E^2}{4\pi r_H} \tag{41}$$

in terms of the energy per unit mass. It is appropriate to comment that even though we may have the freely falling temperature in the hail frame, it would not be actually well-defined in the hail frame since the observer is not at rest even momentarily.

2.2. GEMS of the Schwarzschild Black Hole in the PG Spacetime

In this subsection, we newly find GEMS embeddings of the (3 + 1)-dimensional Schwarzschild black hole in the PG spacetime, comparing with the previous results of the Schwarzschild black hole in the spherically symmetric spacetime. In the PG spacetime, GEMS embedding is obtained by redefining the PG time [42] as

$$t = t_S - F_S(r), \tag{42}$$

where t_S is the Schwarzschild time written in the metric (1) and

$$F_S(r) = -\int \sqrt{\frac{2m}{r}} \left(1 - \frac{2m}{r}\right)^{-1} dr. \tag{43}$$

This integral can be easily performed and one has

$$F_S(r) = -4m \left(y_s - \frac{1}{2}\ln\frac{y_s + 1}{y_s - 1}\right) \tag{44}$$

with the definition

$$y_s \equiv (2m/r)^{-1/2}. \tag{45}$$

Thus, the Schwarzschild black hole in the PG spacetime is described by the metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + 2\sqrt{\frac{2m}{r}}dtdr + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{46}$$

which has an off-diagonal term. This PG spacetime is stationary, but not static. Note also that there is no coordinate singularity at $r_H = 2m$, which is the Schwarzschild radius. Unlike the Schwarzschild coordinate, it is possible to define an effective vacuum state of a quantum field with this well-behaved coordinate system at the horizon by requiring that it annihilates negative frequency modes with respect to the PG time t. Moreover, positive energy particles can tunnel out through a barrier set by the energies of outgoing particles themselves [39–41]. Observers in this PG spacetime, who are freely falling from rest at infinity and see nothing abnormal at the horizon, carry their own proper times. Each spatial slice of the metric with $dt = 0$ corresponds to the flat metric in the spherical coordinates. Thus, the spacetime curvature information is contained in the off-diagonal component of the metric (46), which structure we will study in the following GEMS method in detail.

Inspired by the previous GEMS scheme of the Schwarzschild spacetime, the (3 + 1)-dimensional Schwarzschild black hole in the PG spacetime can be embedded into a (5 + 1)-dimensional flat one described as

$$ds^2 = \eta_{IJ}dz^I dz^J, \text{ with } \eta_{IJ} = \mathrm{diag}(-1, 1, 1, 1, 1, 1), \tag{47}$$

with

$$
\begin{aligned}
z^0 &= k_H^{-1} f^{1/2}(r) \sinh k_H(t + F_S(r)), \\
z^1 &= k_H^{-1} f^{1/2}(r) \cosh k_H(t + F_S(r)), \\
z^2 &= r \sin\theta \cos\phi, \\
z^3 &= r \sin\theta \sin\phi, \\
z^4 &= r \cos\theta, \\
z^5 &= \int dr \left(\frac{r_H(r^2 + rr_H + r_H^2)}{r^3} \right)^{1/2},
\end{aligned} \tag{48}
$$

where $f(r)$ is denoted in Equation (2) and $k_H = 1/4m$ can be calculated directly from the metric (46) with the definition of (3). Note that compared with the GEMS embeddings in Equation (7), the coordinates z^0 and z^1 are transformed to include the PG time t, which partly lead to the off-diagonal term in the PG metric (46). On the other hand, the coordinates z^2, z^3, and z^4 constitute the spatial hypersurface, which is the same with the constant time spacial slice of the metric (46).

Now, in static detectors (r, θ, ϕ =constant) described by a fixed point in the (z^2, z^3, z^4, z^5) plane, a uniformly accelerated observer in the (5 + 1)-dimensional flat spacetime, follows a hyperbolic trajectory in (z^0,z^1) described by a proper acceleration a_6 as follows

$$
a_6^{-2} = (z^1)^2 - (z^0)^2 = \frac{f(r)}{k_H^2}. \tag{49}
$$

Thus, we arrive at the Unruh temperature for the uniformly accelerated observer in the (5 + 1)-dimensional flat spacetime

$$
T_U = \frac{a_6}{2\pi} = \frac{k_H}{2\pi\sqrt{f(r)}}. \tag{50}
$$

It is also appropriate to comment that the static detectors having the same condition (r, θ, ϕ = constant) in this PG spacetime following timelike Killing vector $\xi = \partial_t$ have the same constant 4-acceleration (11) and thus satisfy the same relation (12).

Now, let us consider the geodesic Equation (13) in the Schwarzschild black hole in the PG spacetime. From the metric (46), one can obtain the geodesic equations explicitly as

$$
\frac{dv^0}{d\tau} + \frac{m}{r^2}\sqrt{\frac{2m}{r}}(v^0)^2 + \frac{2m}{r^2}v^0v^1 + \frac{1}{r}\sqrt{\frac{m}{2r}}(v^1)^2 - \sqrt{2mr}[(v^2)^2 + \sin^2\theta(v^3)^2] = 0, \tag{51}
$$

$$
\frac{dv^1}{d\tau} + \frac{m}{r^3}(r - 2m)(v^0)^2 - \frac{2m}{r^2}\sqrt{\frac{2m}{r}}v^0v^1 - \frac{m}{r^2}(v^1)^2 - (r - 2m)\left[(v^2)^2 + \sin^2\theta(v^3)^2\right] = 0, \tag{52}
$$

$$
\frac{dv^2}{d\tau} + \frac{2}{r}v^1v^2 - \sin\theta\cos\theta(v^3)^2 = 0, \tag{53}
$$

$$
\frac{dv^3}{d\tau} + \frac{2}{r}v^1v^3 + 2\cot\theta v^2v^3 = 0. \tag{54}
$$

As before, for simplicity, one can consider the geodesics on the equatorial plane $\theta = \pi/2$. Then, the geodesic equations are reduced to

$$
\frac{dv^0}{d\tau} + \frac{m}{r^2}\sqrt{\frac{2m}{r}}(v^0)^2 + \frac{2m}{r^2}v^0v^1 + \frac{1}{r}\sqrt{\frac{m}{2r}}(v^1)^2 - \sqrt{2mr}(v^3)^2 = 0, \tag{55}
$$

$$
\frac{dv^1}{d\tau} + \frac{m}{r^3}(r - 2m)(v^0)^2 - \frac{2m}{r^2}\sqrt{\frac{2m}{r}}v^0v^1 - \frac{m}{r^2}(v^1)^2 - (r - 2m)(v^3)^2 = 0, \tag{56}
$$

$$
\frac{dv^3}{d\tau} + \frac{2}{r}v^1v^3 = 0. \tag{57}
$$

First of all, one can easily integrate (57) as

$$v^3 = \frac{d\phi}{d\tau} = \frac{c_3}{r^2},\qquad(58)$$

where c_3 is an integration constant. Note that as before, for the Killing vectors $\xi^\mu = (1,0,0,0)$ and $\psi^\mu = (0,0,0,1)$, one can have conserved quantities of E and L given by

$$E = -g_{\mu\nu}\xi^\mu v^\nu = \left(1 - \frac{2m}{r}\right)v^0 - \sqrt{\frac{2m}{r}}v^1,\qquad(59)$$

$$L = g_{\mu\nu}\psi^\mu v^\nu = r^2 v^3.\qquad(60)$$

Comparing (60) with Equation (58), one can fix the integration constant as

$$c_3 = L\qquad(61)$$

in terms of the conserved quantity L. Also, by letting $ds^2 = -kd\tau^2$ in Equation (46) and making use of Equations (58) and (59), one can obtain

$$v^1 = \frac{dr}{d\tau} = \pm\left[E^2 - \left(k + \frac{L^2}{r^2}\right)\left(1 - \frac{2m}{r}\right)\right]^{1/2},\qquad(62)$$

where the $-(+)$ sign is for inward (outward) motion and $k = 1(0)$ is for a timelike (nulllike) geodesic. Finally, making use of Equation (62), one can obtain

$$v^0 = \frac{dt}{d\tau} = \frac{E - \sqrt{\frac{2m}{r}}\left[E^2 - \left(k + \frac{L^2}{r^2}\right)\left(1 - \frac{2m}{r}\right)\right]^{1/2}}{1 - \frac{2m}{r}}.\qquad(63)$$

Here, we have used the $(-)$ sign for v^1 describing the inward motion for later convenience.

Now, let us find a freely falling acceleration and corresponding temperature in the (5 + 1)-dimensional embedded flat spacetime. As in the previous subsection, for simplicity, we assume that an observer is moving along a timelike geodesic with zero angular momentum. Thus, we choose $k = 1$ and $L = 0$ in the geodesic equations, which read as follows

$$\frac{dt}{d\tau} = \frac{E - \sqrt{\frac{2m}{r}[E^2 - f(r)]}}{f(r)},\qquad(64)$$

$$\frac{dr}{d\tau} = -\left[E^2 - f(r)\right]^{1/2}.\qquad(65)$$

(a) A drip frame: for an observer who is freely falling from rest $r = r_0$ at $\tau = 0$ [29–33], one has $E = \pm\sqrt{f(r_0)}$. Then, the equations of motion are reduced to

$$\frac{dt}{d\tau} = \frac{\sqrt{f(r_0)} - \sqrt{\frac{2m}{r}[f(r_0) - f(r)]}}{f(r)},$$

$$\frac{dr}{d\tau} = -[f(r_0) - f(r)]^{1/2}.\qquad(66)$$

Note that we have chosen the $(+)$ sign in E.

Now, by exploiting the embedding coordinates in Equation (48) and the equations of motion in Equation (66), one can easily obtain a freely falling acceleration $\bar{a}_6$ in the (5 + 1)-dimensional GEMS embedded spacetime as

$$\bar{a}_6^2 = \sum_{I=0}^{5} \eta_{IJ} \frac{dz^I}{d\tau} \frac{dz^J}{d\tau}\bigg|_{r=r_0} = \frac{r^3 + r_H r^2 + r_H^2 r + r_H^3}{4 r_H^2 r^3}, \tag{67}$$

which is exactly the same with the freely falling acceleration (31) as expected. Thus, the freely falling temperature at rest measured by the freely falling observer can be written as

$$T_{\text{DF}} = \frac{\bar{a}_6}{2\pi} = \frac{1}{4\pi r_H} \sqrt{1 + \frac{r_H}{r} + \frac{r_H^2}{r^2} + \frac{r_H^3}{r^3}}. \tag{68}$$

In Figure 1, we have drawn the freely falling and fiducial temperatures in the unit of the Hawking temperature. For fiducial observers at rest from a distance in the PG coordinates, freely falling temperature is seen to diverge at the event horizon. However, for freely falling observers in the PG coordinates, freely falling temperature is well-defined at the horizon and moreover is continuously increased to the $r = 0$ singularity by passing through the horizon. It is appropriate to comment that if the observers are at rest from a distance in the usual Schwarzschild coordinates as in the previous subsection, this freely falling temperature would be ended at finite temperature at the event horizon which is denoted by the point p in Figure 1.
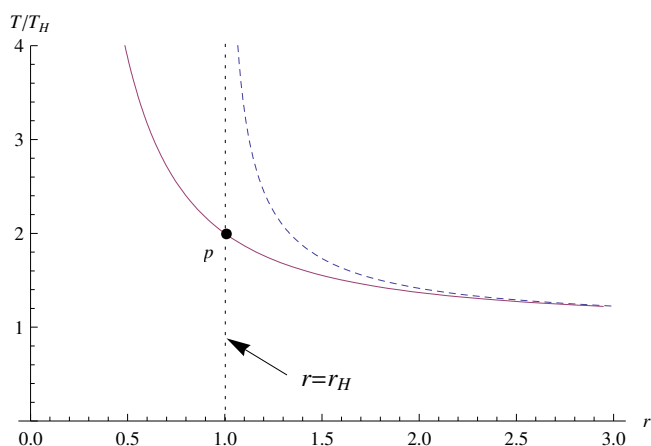

Figure 1. Freely falling temperature in a drip frame (solid line) and fiducial temperature (dashed line) in the PG coordinates in the unit of the Hawking temperature with $r_H = 1$. For fiducial observers, freely falling temperature is seen to diverge at the event horizon. However, for freely falling observers in a drip frame, freely falling temperature is well-defined at the horizon and increased to the $r = 0$ singularity.

(b) A rain frame: for an observer who is freely falling from rest at the asymptotic infinity, one has $E = 1$. Then, the equations of motion are reduced to

$$\begin{aligned} \frac{dt}{d\tau} &= 1, \\ \frac{dr}{d\tau} &= -\sqrt{\frac{2m}{r}}. \end{aligned} \tag{69}$$

Thus, one can obtain a freely falling acceleration $\bar{a}_6$ as

$$\bar{a}_6^2 = \sum_{I=0}^{5} \eta_{IJ} \frac{dz^I}{d\tau} \frac{dz^J}{d\tau}\bigg|_{r=\infty} = \frac{1}{4 r_H^2}, \tag{70}$$

and the freely falling temperature at rest at the asymptotic infinity as

$$T_{\mathrm{RF}} = \frac{1}{4\pi r_H},$$

(71)

which is exactly the same with the Hawking temperature (4) as before.

It is appropriate to comment that four velocity in the rain frame can be obtained from the drip frame by taking limit of $r_0 \to \infty$, which is given by

$$v^a = \left(1, -\sqrt{\frac{2m}{r}}, 0, 0\right),$$

(72)

which shows that the observer's time is given by the proper time. Then, the velocity seen by the observer in the asymptotic infinity becomes

$$\frac{dr}{dt} = -\sqrt{\frac{2m}{r}},$$

(73)

which is the escape velocity. Thus, one can see that the observer is freely falling with the flat space slicing which is flowing radially inwards at the escape velocity. Note that at the event horizon $r_H = 2m$, it becomes the speed of light.

(c) A hail frame: for an observer who has thrown with a velocity v_∞ at the asymptotic infinity as

$$\frac{dr}{dt} = -v_\infty,$$

(74)

one has $E = (1 - v_\infty^2)^{-1/2}$. Then, the equations of motion are reduced to

$$\begin{aligned}
\frac{dt}{d\tau} &= \frac{(1 - v_\infty^2)^{-1/2} - \sqrt{\frac{2m}{r}\left(\frac{v_\infty^2}{1 - v_\infty^2} + \frac{2m}{r}\right)}}{1 - \frac{2m}{r}}, \\
\frac{dr}{d\tau} &= -\left(\frac{v_\infty^2}{1 - v_\infty^2} + \frac{2m}{r}\right)^{1/2}.
\end{aligned}$$

(75)

In this case, one can obtain an acceleration $\bar{a}_6$ by making use of the embedding coordinates in Equation (48) as

$$\bar{a}_6^2 = \sum_{I=0}^{5} \eta_{IJ} \frac{dz^I}{d\tau} \frac{dz^J}{d\tau} \bigg|_{r=\infty} = \frac{1}{4r_H^2(1 - v_\infty^2)^2}.$$

(76)

Thus, the freely falling-like temperature in the hail frame is obtained as

$$T_{\mathrm{HF}} = \frac{1}{4\pi r_H(1 - v_\infty^2)}.$$

(77)

Note that for the case of $v_\infty = 0$, T_{HF} reduces to T_{RF} at infinity and as the throwing velocity v_∞ is increased, T_{HF} is also increased. As before, the freely falling temperature in the hail frame can also be rewritten as

$$T_{\mathrm{HF}} = \frac{E^2}{4\pi r_H}$$

(78)

in terms of the energy per unit mass.

3. GEMS of the RN Black Hole in the PG Spacetime

Let us consider the RN black hole in the spherically symmetric spacetime

$$ds^2 = -f(r)dt_{\mathrm{RN}}^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(79)

with

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}. \tag{80}$$

It was well-known that this RN spacetime can be embedded into the $(5+2)$-dimensional flat spacetime [8,10,14,22,29,30]. So, we do not repeat it. Instead, in the following, we will concentrate on GEMS embedding of the RN black hole in the PG spacetime.

Now, by defining a new time in the PG coordinates as

$$t = t_{RN} - F_{RN}(r), \tag{81}$$

where t_{RN} is the time in the usual RN spacetime and

$$F_{RN}(r) = -\int \sqrt{\frac{2m}{r} - \frac{q^2}{r^2}} \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr, \tag{82}$$

one can obtain the RN black hole in the PG spacetime as

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + 2\sqrt{\frac{2m}{r} - \frac{q^2}{r^2}} \, dt dr + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{83}$$

which also has an off-diagonal term as before.

First of all, it seems appropriate to comment on the integral (82). By redefining

$$y_q \equiv \left(\frac{2m}{r} - \frac{q^2}{r^2}\right)^{-1/2}, \tag{84}$$

let us rewrite $F_{RN}(r)$ as

$$F_{RN}(r) = -4m \int dy_q \frac{y_q^2}{y_q^2 - 1} (1 - \alpha)^{-1}, \tag{85}$$

with $\alpha = (qy_q/r)^4$. Then, through tedious calculations, we have newly obtained the exact solution of the nontrivial form

$$F_{RN}(r) = F_S^q(r) - 8m \sum_{k=1}^{\infty} \frac{(k-1)(2k-3)!}{k!(k-2)!} \left(\frac{q}{2m}\right)^{2k} \left(\sum_{n=0}^{k-2} \frac{y_q^{-(2n+1)}}{2n+1} - \frac{1}{2} \ln\frac{y_q+1}{y_q-1}\right), \tag{86}$$

where

$$F_S^q(r) = -4m\left(y_q - \frac{1}{2}\ln\frac{y_q+1}{y_q-1}\right). \tag{87}$$

Note that in y_q one should require the condition of $2mr > q^2$. Otherwise, there is an obstruction to implement spatially flat PG hypersurfaces [59]. In the limit of $q \to 0$, $F_{RN}(r)$ becomes $F_S(r)$ in Equation (44) as expected.

Now, exploiting the GEMS approach, we can embed the $(3+1)$-dimensional RN black hole in the PG spacetime into one in a $(5+2)$-dimensional flat spacetime as

$$ds^2 = \eta_{IJ} dz^I dz^J, \text{ with } \eta_{IJ} = \text{diag}(-1,1,1,1,1,1,-1), \tag{88}$$

where the embedding coordinates are explicitly given by

$$
\begin{aligned}
z^0 &= k_H^{-1} f^{1/2}(r) \sinh k_H(t + F_{RN}(r)), \\
z^1 &= k_H^{-1} f^{1/2}(r) \cosh k_H(t + F_{RN}(r)), \\
z^2 &= r \sin\theta \cos\phi, \\
z^3 &= r \sin\theta \sin\phi, \\
z^4 &= r \cos\theta, \\
z^5 &= \int dr \left(\frac{r^2(r_+ + r_-) + r_+^2(r_+ + r_-)}{r^2(r - r_-)} \right)^{1/2}, \\
z^6 &= \int dr \left(\frac{4r_+^5 r_-}{r^4(r_+ - r_-)^2} \right)^{1/2}.
\end{aligned}
\tag{89}
$$

Here, k_H is given by

$$
k_H = \frac{r_+ - r_-}{2r_+^2}
\tag{90}
$$

with $r_\pm = m \pm \sqrt{m^2 - q^2}$. Again, compared with the previous GEMS embeddings of the RN black hole in the spherically symmetric spacetime [8,10,14,22,29,30], the coordinates z^0 and z^1 include the new PG time t, which partly lead to the off-diagonal term in the PG metric (83). And the coordinates z^2, z^3, and z^4 constitute the flat spatial hypersurfaces as far as we keep the condition of $2mr > q^2$.

In static detectors (r, θ, ϕ = constant) described by a fixed point in the (z^2, z^3, z^4, z^5, z^6) plane, a uniformly accelerated observer in the (5 + 2)-dimensional flat spacetime, follows a hyperbolic trajectory in (z^0, z^1) described by

$$
a_7^{-2} = (z^1)^2 - (z^0)^2 = \frac{4r_+^4(r - r_+)(r - r_-)}{r^2(r_+ - r_-)^2}.
\tag{91}
$$

Thus, we arrive at the Unruh temperature for a uniformly accelerated observer in the (5 + 2)-dimensional flat spacetime

$$
T_U = \frac{a_7}{2\pi} = \frac{r(r_+ - r_-)}{4\pi r_+^2 (r - r_+)^{1/2}(r - r_-)^{1/2}}.
\tag{92}
$$

It is also appropriate to comment that the static detectors having the same condition (r, θ, ϕ = constant) in the original (3 + 1)-dimensional spacetime following the timelike Killing vector $\xi = \partial_t$ have the constant 4-acceleration as

$$
a_4 = \frac{(r_+ + r_-)r - 2r_+ r_-}{2r^2[(r - r_+)(r - r_-)]^{1/2}}.
\tag{93}
$$

Comparing this with the acceleration (91) in the (5 + 2)-dimensional embedded spacetime, one can have

$$
a_7 = \frac{r^3(r_+ - r_-)}{r_+^2[(r_+ + r_-)r - 2r_+ r_-]} a_4.
\tag{94}
$$

In the Schwarzschild limit of $r_+ = r_H$ and $r_- = 0$, the acceleration a_7 becomes a_6, while a_4 becomes (11) so that the relation (12) is recovered as expected.

Now, as in the previous section, let us consider the geodesic equation (13) in the RN black hole in the PG spacetime. First of all, non-vanishing independent components of the Christoffel symbols are given by

$$
\begin{aligned}
\Gamma^0_{00} &= -\Gamma^1_{01} = \frac{1}{r}\left(\frac{m}{r}-\frac{q^2}{r^2}\right)\left(\frac{2m}{r}-\frac{q^2}{r^2}\right)^{1/2}, & \Gamma^0_{01} &= -\Gamma^1_{11} = \frac{1}{r}\left(\frac{m}{r}-\frac{q^2}{r^2}\right), \\
\Gamma^0_{11} &= \frac{1}{r}\left(\frac{m}{r}-\frac{q^2}{r^2}\right)\left(\frac{2m}{r}-\frac{q^2}{r^2}\right)^{-1/2}, & \Gamma^0_{22} &= -r\left(\frac{2m}{r}-\frac{q^2}{r^2}\right)^{1/2}, \\
\Gamma^0_{33} &= -r\left(\frac{2m}{r}-\frac{q^2}{r^2}\right)^{1/2}\sin^2\theta, & \Gamma^1_{00} &= \frac{1}{r}\left(\frac{m}{r}-\frac{q^2}{r^2}\right)\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right), \\
\Gamma^1_{22} &= -r\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right), & \Gamma^1_{33} &= -r\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)\sin^2\theta, \\
\Gamma^2_{12} &= \Gamma^3_{13} = \frac{1}{r}, & \Gamma^2_{33} &= -\sin\theta\cos\theta, \\
\Gamma^3_{23} &= \cot\theta.
\end{aligned}
\tag{95}
$$

Then, from the metric (83), one can obtain the geodesic equations explicitly as

$$
\frac{dv^0}{d\tau} + \frac{\sqrt{2mr-q^2}(mr-q^2)}{r^4}(v^0)^2 + \frac{2(mr-q^2)}{r^3}v^0 v^1 + \frac{mr-q^2}{r^2\sqrt{2mr-q^2}}(v^1)^2
$$
$$
-\sqrt{2mr-q^2}[(v^2)^2 + \sin^2\theta(v^3)^2] = 0,
\tag{96}
$$

$$
\frac{dv^1}{d\tau} + \frac{[(r-2m)r+q^2](mr-q^2)}{r^5}(v^0)^2 - \frac{2\sqrt{2mr-q^2}(mr-q^2)}{r^4}v^0 v^1 - \frac{mr-q^2}{r^3}(v^1)^2
$$
$$
-\frac{r^2-2mr+q^2}{r}\left[(v^2)^2 + \sin^2\theta(v^3)^2\right] = 0,
\tag{97}
$$

$$
\frac{dv^2}{d\tau} + \frac{2}{r}v^1 v^2 - \sin\theta\cos\theta(v^3)^2 = 0,
\tag{98}
$$

$$
\frac{dv^3}{d\tau} + \frac{2}{r}v^1 v^3 + 2\cot\theta v^2 v^3 = 0.
\tag{99}
$$

As before, for simplicity, one can consider the geodesics on the equatorial plane $\theta = \pi/2$. Then, the geodesic equations are reduced to

$$
\frac{dv^0}{d\tau} + \frac{\sqrt{2mr-q^2}(mr-q^2)}{r^4}(v^0)^2 + \frac{2(mr-q^2)}{r^3}v^0 v^1 + \frac{mr-q^2}{r^2\sqrt{2mr-q^2}}(v^1)^2
$$
$$
-\sqrt{2mr-q^2}(v^3)^2 = 0,
\tag{100}
$$

$$
\frac{dv^1}{d\tau} + \frac{[(r-2m)r+q^2](mr-q^2)}{r^5}(v^0)^2 - \frac{2\sqrt{2mr-q^2}(mr-q^2)}{r^4}v^0 v^1 - \frac{mr-q^2}{r^3}(v^1)^2
$$
$$
-\frac{r^2-2mr+q^2}{r}(v^3)^2 = 0,
\tag{101}
$$

$$
\frac{dv^3}{d\tau} + \frac{2}{r}v^1 v^3 = 0.
\tag{102}
$$

Firstly, one can easily integrate (102) as

$$
v^3 = \frac{d\phi}{d\tau} = \frac{c_4}{r^2},
\tag{103}
$$

where c_4 is an integration constant. Now, for the Killing vectors $\xi^\mu = (1,0,0,0)$ and $\psi^\mu = (0,0,0,1)$, one can have conserved quantities of E and L given by

$$
E = -g_{\mu\nu}\xi^\mu v^\nu = \left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)v^0 - \sqrt{\frac{2m}{r}-\frac{q^2}{r^2}}v^1,
\tag{104}
$$

$$
L = g_{\mu\nu}\psi^\mu v^\nu = r^2 v^3.
\tag{105}
$$

Comparing (105) with Equation (103), one can fix the integration constant as

$$
c_4 = L
\tag{106}
$$

in terms of the conserved quantity L. Also, by letting $ds^2 = -kd\tau^2$ in Equation (79) and making use of Equations (103) and (104), one can obtain

$$v^1 = \frac{dr}{d\tau} = \pm \left[E^2 - \left(k + \frac{L^2}{r^2}\right)\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) \right]^{1/2}, \tag{107}$$

where the $-(+)$ sign is for inward (outward) motion and $k = 1(0)$ is for a timelike (nulllike) geodesic. Finally, making use of (107), one can obtain

$$v^0 = \frac{dt}{d\tau} = \frac{E - \sqrt{\frac{2m}{r} - \frac{q^2}{r^2}}\left[E^2 - \left(k + \frac{L^2}{r^2}\right)\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)\right]^{1/2}}{1 - \frac{2m}{r} + \frac{q^2}{r^2}}. \tag{108}$$

As before, we have here used the $(-)$ sign for v^1 describing the inward motion for later convenience.

Now, let us find a freely falling acceleration and corresponding temperature in the $(5 + 2)$-dimensional embedded flat spacetime. As in the previous section, for simplicity, we assume that an observer is moving along a timelike geodesic with zero angular momentum. Thus, we choose $k = 1$ and $L = 0$ in the geodesic equations, which read as follows

$$\frac{dt}{d\tau} = \frac{E - \sqrt{\left(\frac{2m}{r} - \frac{q^2}{r^2}\right)[E^2 - f(r)]}}{f(r)}, \tag{109}$$

$$\frac{dr}{d\tau} = -\left[E^2 - f(r)\right]^{1/2}. \tag{110}$$

(a) A drip frame: for an observer who is freely falling from rest $r = r_0$ at $\tau = 0$ [29–33], one has $E = \pm\sqrt{f(r_0)}$. Then, the equations of motion are reduced to

$$\frac{dt}{d\tau} = \frac{\sqrt{f(r_0)} - \sqrt{\left(\frac{2m}{r} - \frac{q^2}{r^2}\right)[f(r_0) - f(r)]}}{f(r)},$$

$$\frac{dr}{d\tau} = -[f(r_0) - f(r)]^{1/2}. \tag{111}$$

Note that we have chosen the $(+)$ sign in E.

Now, by exploiting the embedding coordinates in Equation (89) and the equations of motion in Equation (111), one can easily obtain a freely falling acceleration $\bar{a}_7$ in the $(5 + 2)$-dimensional GEMS embedded spacetime as

$$\bar{a}_7^2 = \sum_{I=0}^{6} \eta_{IJ} \frac{dz^I}{d\tau}\frac{dz^J}{d\tau}\bigg|_{r=r_0} = \frac{(r_+ - r_-)^2(r^3 + r_+r^2 + r_+^2 r + r_+^3)r^2 - 4r_+^5 r_-(r - r_-)}{4(r - r_-)r_+^4 r^4}. \tag{112}$$

In the last expression, we have replaced r_0 with r. Thus, the freely falling temperature at rest measured by the freely falling observer can be written as

$$T_{\mathrm{DF}} = \frac{\bar{a}_7}{2\pi} = \frac{(r_+ - r_-)}{4\pi r_+^2}\sqrt{\frac{r^3 + r_+r^2 + r_+^2 r + r_+^3}{(r - r_-)r^2} - \frac{4r_+^5 r_-}{(r_+ - r_-)^2 r^4}}. \tag{113}$$

In the limit of $r \to \infty$, the above T_{DF} is reduced to the Hawking temperature

$$T_H = \frac{r_+ - r_-}{4\pi r_+^2}. \tag{114}$$

For the case of $q = 0$ ($r_+ = r_H$, $r_- = 0$), T_{DF} becomes the Schwarzschild one in Equation (68). It also seems appropriate to comment that when the observer approaches to the event horizon, T_{DF} does not diverge but is finite as

$$T_{DF} = \frac{\sqrt{r_+(r_+ - 2r_-)}}{2\pi r_+^2}, \tag{115}$$

while the fiducial temperature (92) diverges.

In Figure 2, we have drawn squared freely falling temperatures in a drip frame, which can be rewritten in terms of a dimensionless variable $x = r_+/r$ and a parameter $b = r_-/r_+$ as

$$T_{DF} = \frac{1}{4\pi r_+^2} \sqrt{\frac{(1-b)^2(1 + x + x^2 + x^3) - 4bx^4(1 - bx)}{1 - bx}}. \tag{116}$$

Here, the event horizon of $r = r_+$ ($r = \infty$) corresponds to $x = 1$ ($x = 0$) and the parameter b lies between $0 \le b \le 1$. Note that at the horizon ($x = 1$), this is reduced to

$$T_{DF} = \frac{\sqrt{1 - 2b}}{2\pi r_+^2}. \tag{117}$$

Thus, as seen in Figure 2, when $b \le 0.5$, squared freely falling temperatures are finitely well-defined in the whole range of x. On the other hand, when $b > 0.5$, there is a region with no thermal radiation surrounding the horizon, where T_{DF} becomes imaginary. These are exactly the same with the ones in the BT's work [29]. However, as stated before, the freely falling temperatures in the PG coordinates are defined beyond the horizon, while they are not in the original RN spacetime coordinates. To understand this situation better, we have drawn freely falling temperatures in the unit of the Hawking temperature in a drip frame in Figure 3. Then, one can see that the freely falling temperatures in a drip frame, which are obtained from the PG spacetimes, are increased, reached at a peak, and then decreased to zero as the observers fall into the singularity after passing through the horizon. Note that if they are obtained from the original RN coordinates having the spherical symmetric spacetime (79), freely falling temperatures are ended at finite values, such as p and q, at the horizon in Figures 2 and 3. Finally, in Figure 3, the regions of no thermal radiation are in between the $r = 0$ singularity and the positions where the freely falling temperatures vanish.
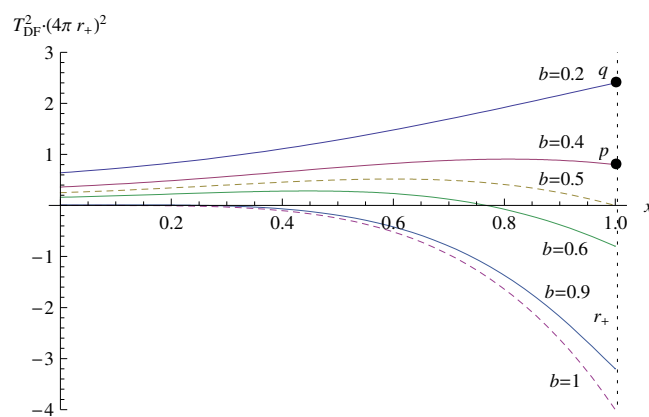

Figure 2. Squared freely falling temperatures in a drip frame, $T_{DF}^2 \cdot (4\pi r_+^2)^2$ along $x = r_+/r$ for $b = 0.2, 0.4, 0.5, 0.6, 0.9,$ and 1, where $b = r_-/r_+$. The dotted line at $x = 1$ is for the horizon r_+.
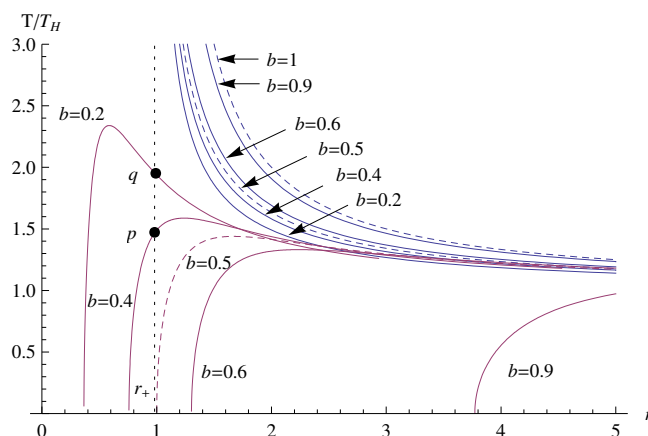
Figure 3. Freely falling temperatures T_{DF}/T_H in the unit of the Hawking temperature in a drip frame (lower curves) and fiducial temperatures T_{FID}/T_H (upper curves) along r for $b = 0.2, 0.4, 0.5, 0.6, 0.9$, and 1. The dotted line at $r = 1$ is for the horizon r_+. Note that there is no freely falling temperature for $b = 1$, which is imaginary for all x as easily seen in Equation (116). Freely falling temperatures for $b < 0.5$ can be drawn beyond the horizon (over p and q in the figure) by freely falling observers who are described by the PG coordinates, while fiducial temperatures described by the original RN coordinates cannot be drawn due to divergences at the horizon.

(b) A rain frame: for an observer who is freely falling from rest at the asymptotic infinity, one has $E = 1$. Then, the equations of motion are reduced to

$$
\begin{aligned}
\frac{dt}{d\tau} &= 1, \\
\frac{dr}{d\tau} &= -\left(\frac{2m}{r} - \frac{q^2}{r^2}\right)^{1/2}.
\end{aligned}
\tag{118}
$$

Again, by exploiting the embedding coordinates in Equation (89), one can obtain a freely falling acceleration $\bar{a}_7$

$$
\bar{a}_7^2 = \frac{(r_+ - r_-)^2}{4r_+^4}.
\tag{119}
$$

Thus, the freely falling temperature at rest at the asymptotic infinity is

$$
T_{\mathrm{RF}} = \frac{r_+ - r_-}{4\pi r_+^2},
\tag{120}
$$

which is exactly the same with the Hawking temperature.

(c) A hail frame: if an observer starts to fall with a velocity at the asymptotic infinity as

$$
\frac{dr}{dt} = -v_\infty
\tag{121}
$$

one has $E = (1 - v_\infty^2)^{-1/2}$. Then, the equations of motion are reduced to

$$
\begin{aligned}
\frac{dt}{d\tau} &= \frac{(1 - v_\infty^2)^{-1/2} - \sqrt{\left(\frac{2m}{r} - \frac{q^2}{r^2}\right)\left(\frac{v_\infty^2}{1 - v_\infty^2} + \frac{2m}{r} - \frac{q^2}{r^2}\right)}}{1 - \frac{2m}{r} + \frac{q^2}{r^2}}, \\
\frac{dr}{d\tau} &= -\left(\frac{v_\infty^2}{1 - v_\infty^2} + \frac{2m}{r} - \frac{q^2}{r^2}\right)^{1/2}.
\end{aligned}
\tag{122}
$$

In this case, one can obtain an acceleration in the hail frame $\bar{a}_7$ by making use of the embedding coordinates in Equation (89) as

$$\bar{a}_7^2 = \frac{(r_+ - r_-)^2}{4r_+^4(1 - v_\infty^2)^2}. \tag{123}$$

Thus, the hail frame temperature is obtained as

$$T_{\mathrm{HF}} = \frac{r_+ - r_-}{4\pi r_+^2(1 - v_\infty^2)}. \tag{124}$$

As in the case of the Schwarzschild black hole in the PG spacetime, when $v_\infty = 0$, T_{HF} reduces to T_{RF}. Finally, the freely falling temperature in the hail frame can also be rewritten as

$$T_{\mathrm{HF}} = \frac{E^2(r_+ - r_-)}{4\pi r_H} \tag{125}$$

in terms of the energy per unit mass.

4. Discussion

In summary, we have globally embedded the (3 + 1)-dimensional Schwarzschild and RN black holes in the PG spacetimes into (5 + 1)- and (5 + 2)-dimensional flat ones, respectively, which were made by the introduction of the PG time. The advantage of introducing the PG time in the GEMS embeddings is that hypersurfaces constructed by z^2, z^3 and z^4 either for the Schwarzschild and the RN black holes remain flat spacelike as in the original spacetimes. As a result, making use of the embedding coordinates, we have directly obtained the Hawking, Unruh, and freely falling temperatures in the drip, rain, and hail frames. Moreover, we have shown that the Hawking effect for a fiducial observer in a curved spacetime is exactly equal to the Unruh effect for a uniformly accelerated observer in higher-dimensionally embedded flat spacetimes.

On the other hand, the Hawking, Unruh, and freely falling temperatures in the drip frame in the Schwarzschild and RN black holes in the PG spacetime have been shown to be well-defined to the $r = 0$ singularity by passing through the event horizon, while to the event horizon they are exactly the same as the ones obtained from the previous GEMS embeddings of the spherically symmetric Schwarzschild and RN spacetimes. We have also shown that the freely falling temperatures in the rain frame, which are seen by an observer who is released freely from the asymptotic infinity, are nothing but the Hawking ones.

Finally, it seems appropriate to comment that we have shown the equivalence of the GEMS embeddings of the spacetimes with the diagonal and off-diagonal terms in metrics. Therefore, through further investigation, it would be very interesting to construct and analyze the GEMS embeddings of the more realistic Kerr, or rotating BTZ holes by using inverse PG type transformations.

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