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# On Maxwell Electrodynamics in Multi-Dimensional Spaces

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**Abstract:** The governing equations of Maxwell electrodynamics in multi-dimensional spaces are derived from the variational principle of least action, which is applied to the action function of the electromagnetic field. The Hamiltonian approach for the electromagnetic field in multi-dimensional pseudo-Euclidean (flat) spaces has also been developed and investigated. Based on the two arising first-class constraints, we have generalized to multi-dimensional spaces a number of different gauges known for the three-dimensional electromagnetic field. For multi-dimensional spaces of non-zero curvature the governing equations for the multi-dimensional electromagnetic field are written in a manifestly covariant form. Multi-dimensional Einstein's equations of metric gravity in the presence of an electromagnetic field have been re-written in the true tensor form. Methods of scalar electrodynamics are applied to analyze Maxwell equations in the two and one-dimensional spaces.

**Keywords:** electromagnetic; covariant; field; constraints; curved space

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**Citation:** Frolov, A.M. On Maxwell Electrodynamics in Multi-Dimensional Spaces. *Universe* **2022**, *8*, 20. https://doi.org/10.3390/universe8010020

Academic Editors: Panayiotis Stavrinos, Emmanuel N. Saridakis and Xue-Mei Deng

Received: 22 November 2021
Accepted: 24 December 2021
Published: 30 December 2021



## 1. Introduction

The main goal of this communication is to develop the logically closed and non-contradictory version of electrodynamics in the multi-dimensional (or $n$-dimensional) space. Right now, such a development can be considered as a pure theoretical (or model) task, but originally, our plan was to include the multi-dimensional electromagnetic fields in our Hamiltonian analysis of the metric gravity [1]. Note that all Hamiltonian approaches that are based on the $\Gamma - \Gamma$ Lagrangian (see, e.g., [1] and earlier references therein) have been derived in the manifestly covariant form and can be applied to multi-dimensional (or $n$-dimensional, where $n$ ($\geq$3) is an arbitrary integer) Riemannian spaces without any modification. On the other hand, our current Maxwell theory of electromagnetic fields and corresponding Hamiltonian approach can be used only for three-dimensional (geometrical) spaces. This contradiction creates numerous problems for the development of any united theory of the coupled electromagnetic and gravitational fields. Furthermore, it is hard to believe that in reality one can smoothly combine two theories that have different properties with respect to their extensions on multi-dimensional spaces.

After our investigations began, it did not take long to understand that such a theory of the free electromagnetic fields in multi-dimensions simply does not exist even in the first-order approximation (in contrast with metric gravity). There are quite a few reasons why a similar generalization of the classical electrodynamics to multi-dimensional spaces has not been developed earlier. For instance, the explicit expression for the action integral and therefore for the Lagrangian of the electromagnetic field in multi-dimensions is unknown. However, if we do not know the Lagrangian of the multi-dimensional electromagnetic field, then it is impossible to construct any valuable Hamiltonian. There have been a number of smaller problems which have substantially complicated any direct generalization of Maxwell theory to $n$-dimensional spaces. One of them is the lack of a reliable and practically valuable definition of a $curl-$operator (or $rot$-operator) in multi-dimensional spaces, where $n \geq 4$. In general, it is difficult to develop multi-dimensional electrodynamics without such an operator. Finally, we have decided to investigate this problem and derive some useful results which are of great interest for the Hamiltonian formulation of the metric gravity combined with electromagnetic field(s) in multi-dimensional spaces.

First, let us briefly discuss the classical Maxwell equations known for the three-dimensional electromagnetic fields. The Maxwell equations were first written by J.C. Maxwell in 1862 (published in 1865 [2] (see also [3,4])) for the intensities of electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields (or for the electric and magnetic field strengths):

$$\begin{aligned} div\mathbf{E} &= 4\pi\rho,\ curl\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \\ div\mathbf{H} &= 0,\ curl\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}, \end{aligned} \tag{1}$$

where $\rho$ and $\mathbf{j} = \rho\mathbf{v}$ are the electric charge density (scalar) and electric current density (vector), respectively. In this study, the charge density and current are defined exactly as in \$ 29 from [5]. Later, it was noticed by Hertz and others that these four equations from Equation (1) can be re-written in a simple form if we can introduce the four-dimensional potential $\bar{A} = (\varphi, \mathbf{A})$, where $\varphi$ is the scalar potential and $\mathbf{A}$ is the vector potential of the electromagnetic field. Note that the scalar potential $\varphi$ can equally be considered as the 0-component ($A_0$) of the four-dimensional vector potential $\bar{A}$ of the electromagnetic field. The $\varphi$ and $\mathbf{A}$ potentials are simply related to the intensities of electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields: $\mathbf{H} = curl\mathbf{A}$ and $\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - grad\varphi$. By using these relations between the potentials $(\varphi, \mathbf{A})$ and intensities $(\mathbf{E}, \mathbf{H})$ of electromagnetic field, one finds that the second equation in the first line and first equation in the second line of Equation (1) hold identically. The two remaining equations from Equation (1) lead to the following non-homogeneous equations:

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \Delta\mathbf{A} + grad\Big(div\mathbf{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t}\Big) = \frac{4\pi}{c}\mathbf{j} \tag{2}$$

$$-\ \Delta\varphi - \frac{1}{c}div\Big(\frac{\partial\mathbf{A}}{\partial t}\Big) = 4\pi\rho \tag{3}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the three-dimensional Laplace operator. By applying the "gauge condition" $\frac{\partial\varphi}{\partial t} + div\mathbf{A} = 0$ for the four-dimensional potential, one reduces the two last equations to the form

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \Delta\mathbf{A} = \frac{4\pi}{c}\rho\mathbf{v}, \tag{4}$$

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi = 4\pi\rho, \tag{5}$$

where the operator $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$ is the four-dimensional Laplace operator in pseudo-Euclidean space, which is often called the d'Alembertian operator.

It is interesting that all equations mentioned above can be derived by varying the action functional $S$ which is written for a system of particles and electromagnetic fields interacting with these particles. In Gauss units, the explicit form of this action function (or action, for short) $S$ is

$$S = S_p + S_{fp} + S_f = -\sum_k \int m_k c ds_k - \sum_k \int \frac{e_k}{c} A_\alpha(k) dx^\alpha - \frac{1}{16\pi}\int F_{\alpha\beta}F^{\alpha\beta}d\Omega, \tag{6}$$

where the two sums are taken over particles, $s = \sqrt{x_\mu x^\mu} = \sqrt{g_{\mu\nu}x^\mu x^\mu}$ is the interval, $S_p$ is the action for the particles ($k = 1, 2, \ldots$), and $S_{fp}$ is the action which describes the interaction between particles and electromagnetic field, while $S_f$ is the action for the electromagnetic field itself. The notation $e_k$ stands for the electric charge of the $k$-th particle, while $m_k$ means the mass of the same particle, and $A_\alpha$ is the covariant component of the four-dimensional vector potential $\bar{A}$ of the electromagnetic field. This formula, Equation (6), is written for the four-dimensional pseudo-Euclidean (flat) space-time. This fact drastically simplifies the analysis and derivation of the Maxwell and other equations in classical three-dimensional electrodynamics.

In this study, we discuss a possibility to generalize the usual (or three-dimensional) Maxwell equations to spaces of larger dimensions. In respect to this, below, we shall consider $n$-dimensional, pure geometrical spaces and $(n+1)$-dimensional space-time manifolds. Our main goal is to derive the correct form of multi-dimensional Maxwell equations and investigate their basic properties. In particular, we want to understand how many and what kind of changes we can expect in the multi-dimensional Hamiltonian of the free electromagnetic field and in a number of arising first-class constraints. A separate but closely related problem is the gauge invariance of the free electromagnetic field. Another interesting problem is to investigate the explicit form of multi-dimensional Maxwell equations in the presence of multi-dimensional gravitational fields. A brief discussion of scalar electrodynamics can be found in Appendix A. All new results obtained in the course of our current analysis will be used later to develop the modern united theory of electromagnetic and gravitational fields.

## 2. Scalar and Vector Potentials of the Electromagnetic Field

Let us derive the closed system of Maxwell equations for the $n$-dimensional (geometrical) space, where $n \geq 3$. The time $t$ is always considered as an independent scalar and special $(n+1)$-st variable. This means that we are dealing with manifolds of variables defined in $(n+1)$-dimensional space-time. First, we need to define the vector potential $\bar{A}$ in this $(n+1)$-dimensional space-time. Based on experimental facts known for actual electromagnetic systems considered in one, two, and three dimensions, below, we assume that the interaction of a point particle with the electromagnetic field is determined by a single, scalar parameter $e$, which is the electric charge of this particle. The parameter $e$ can be positive, negative, or equal to zero. The properties of the electromagnetic field are described by the $(n+1)$-dimensional vector potential $\bar{A}$. The notation $A_\mu$ (or $\bar{A}_\mu$) stands for the covariant $\mu-$component of this $(n+1)$-dimensional vector potential $\bar{A}$. In this study, we also deal with the $n$-dimensional space-like vector potential $\mathbf{A}$. Co- and contravariant components of this vector are designated by Latin indexes; e.g., $A_k$ and $A^k$, where $k = 1, 2, \ldots, n$. The same rule is applied to all vectors and tensors mentioned in this study: components of $(n+1)$-vectors are labeled by Greek indices (each of which varies between 0 and $n$), while spatial components of these $n$-dimensional vectors (each varying between 1 and $n$) are denoted by Latin indices. The generalization of this rule to the tensors of arbitrary ranks is straightforward and simple. Note also that in all formulas below, the following "summation rule" is applied: a repeated suffix (or index) in any formula means summations over all values of this suffix (or index).

In general, the vector potential $\bar{A}$ can be written in the form $\bar{A} = (\varphi, \mathbf{A})$, which includes the scalar potential $\varphi(= A_0)$ and $n$-dimensional vector potential $\mathbf{A} = (A_1, A_2, \ldots, A_n)$. For arbitrary scalar $\Phi$ and vector $\mathbf{V}$ functions in $n$-dimensional space, we can determine the first-order differential operators: the (a) gradient operator $\nabla$ (or *grad*) and (b) divergence operator *div*. They are defined as follows:

$$\nabla\Phi = grad\ \Phi = \Big(\frac{\partial\Phi}{\partial x_1}, \frac{\partial\Phi}{\partial x_2}, \ldots, \frac{\partial\Phi}{\partial x_n}\Big) \text{ and } div\ \mathbf{V} = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \ldots + \frac{\partial V_n}{\partial x_1} \tag{7}$$

Analogous definitions of these two operators can easily be generalized and applied to the scalar and vector functions defined in $(n+1)$-dimensional space. By using these definitions, we can discuss the gradient of the scalar potential $\nabla\varphi(= \nabla A_0)$ (vector) and divergence of vector potential $div\mathbf{A}$ (scalar) in the $n$-dimensional space.

The $(n+1)$-dimensional vector potential $\bar{A} = (A_0, A_1, \ldots, A_n)$ allows us to define the truly antisymmetric $(n+1) \times (n+1)$ electromagnetic field tensor $F_{\alpha\beta}(= -F_{\beta\alpha})$ by using the relation

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} = -F_{\beta\alpha}, \quad \text{and} \quad F^{\alpha\beta} = \frac{\partial A^\beta}{\partial x_\alpha} - \frac{\partial A^\alpha}{\partial x_\beta} = -F^{\beta\alpha}, \tag{8}$$

which formally coincides with the analogous definition of this tensor known in the four-dimensional space-time. For the $(n+1)$-dimensional space-time manifold, this tensor has zero-diagonal matrix elements (or components); i.e., $F_{\alpha\alpha} = 0$. Therefore, in $n$-dimensional space, each of the antisymmetric $F^{\alpha\beta}$ and $F_{\alpha\beta}$ tensors have $\frac{n(n-1)}{2}$ different and independent components. The double sum $F_{\alpha\beta}F^{\alpha\beta}$ is the first (or main) invariant of the electromagnetic field defined in the $(n+1)$-dimensional space. Now, let us write the following explicit formula for the action $S$ for the system, which includes the particles and electromagnetic field itself. This action takes the following form (see, e.g., [5]):

$$S = S_p + S_{fp} + S_f = -\sum_k \int m_k c ds - \sum_k \int \frac{e_k}{c} A_\alpha(k) dx^\alpha - a \int F_{\alpha\beta}F^{\alpha\beta} d\Omega, \tag{9}$$

where $s = \sqrt{x_\mu x^\mu} = \sqrt{g_{\mu\nu} x^\mu x^\mu}$ is the interval, $S_p$ is the action function for the particles, $S_{fp}$ is the action function which describes the interaction between particles and the electromagnetic field, and $S_f$ is the action function for the electromagnetic field itself. In this equation, the summation is performed over all particles (index $k$). The notation $A_\alpha(k)$ shows that the $\alpha-$component of the vector potential must be determined at the point of location of $k-$th particle. Note that the formula, Equation (9), is applicable in the flat pseudo-Euclidean and/or Euclidean spaces only. Its generalization to multi-dimensional Riemannian spaces (spaces of non-zero curvature) is considered below. In the next step, we need to determine the constant $a$ in Equation (9). This can be achieved by considering Coulomb's law in multi-dimensions (see the next section).

As a conclusion of this section, we want to emphasize the fact that our action function, which is chosen in the form of Equation (9), allows one to derive the equations of motion for a system of electrically charged, point particles which move in the electromagnetic field. For instance, for one electrically charged particle, by varying the coordinates of this particle (i.e., the $x^\mu$ and $x^\alpha$ variables) in the action function, Equation (9), one finds the following equation of motion for one electrically charged, point particle which moves in the non-flat multi-dimensional space:

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} - \frac{e}{c}F^{\alpha\beta}g_{\beta\gamma}\frac{dx^\gamma}{ds} = 0, \text{ or } \frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} - \frac{e}{mc^2}F^\alpha_\beta\frac{dx^\beta}{ds} = 0, \tag{10}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Cristoffel symbols of the second kind [6,7] which equal zero identically in any flat space. It is clear that the last term in the action function $S$ is not varied, and we do not know the exact numerical value of the constant $a$ in Equation (9). In addition, for the non-flat spaces, in the last term, we have to replace $d\Omega \to \sqrt{-g}d\Omega$.

### 3. Coulomb's Law in Multi-Dimensions

The explicit form of the Coulomb interaction between two point, electrically charged particles is of crucial importance for our present purposes. In Gauss units, which are used almost everywhere in this study, the Coulomb's law for three-dimensional space has a very simple form: $V(r_{21}) = \frac{q_1q_2}{r_{21}}$, where $V(r_{21})$ is the Coulomb potential, $q_1$ and $q_2$ are the electric charges of the two point particles (1 and 2), and $r_{21}$ is the interparticle distance, which equals $r = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$, where $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are the Cartesian coordinates of the two interacting particles. Note that the Coulomb interaction potential does not contain the factor $4\pi$. Furthermore, the Coulomb potential essentially coincides with the singular part of the Green's function for the three-dimensional Laplace operator; i.e., $V(r_{21}) = q_1 q_2 G(\mathbf{r}_1, \mathbf{r}_2) = q_1 q_2 G(\mid \mathbf{r}_1 - \mathbf{r}_2 \mid) = \frac{q_1 q_2}{|\mathbf{r}_2 - \mathbf{r}_1|}$ and $\Delta\Big(\frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|}\Big) = \nabla^2\Big(\frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|}\Big) = \nabla\Big(\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3}\Big) = -4\pi\delta(\mathbf{r}_2 - \mathbf{r}_1)$. The last equation can also be re-written for the intensity of electric field **E**, which is the negative gradient of the potential $\varphi$. This equation takes the familiar form $div\mathbf{E} = -\nabla\Big[\Big(\frac{q_1 q_2}{r_2 1}\Big)\Big] = q_1 q_2 \nabla\Big(\frac{\mathbf{r}_{21}}{r_{21}^3}\Big) = 4\pi\rho(\mathbf{r}_{21})$, where $\rho(\mathbf{x})$ is a continuous charge density. The derived expression coincides with the well-known differential form of Gauss's law of electrostatic and one of the Maxwell equations. These two properties (or two criteria) of three-dimensional Coulomb potential plays a crucial role in our definition of the multi-dimensional Coulomb potential (see below).

Now, we need to define the Coulomb potential in multi-dimensional (or $n$-dimensional) space. This is a crucial moment for the Maxwell electrodynamics in multi-dimensional spaces which we try to develop in this study. Any mistake in such a definition will cost too much for our present purposes. In this sense, this section was the most difficult part of our analysis and it was re-written quite a few times. Indeed, we cannot send someone to the four-dimensional (geometrical) space to repeat the well known Coulomb and Cavendish experiments; therefore, we need to find a way to make an analytical generalization of the Coulomb potential to multi-dimensional spaces. In respect to our first criterion formulated above, the Coulomb potential in the $n$-dimensional space must coincide with the singular part of the Green's function defined for the multi-dimensional (or $n$-dimensional) Laplace operator $\Delta = \Delta_n = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$. This leads [8] to the following general expression for the Coulomb potential in $n$-dimensional space: $V(r) = b\frac{q_1 q_2}{r_{21}^{n-2}} = b\frac{q_1 q_2}{r^{n-2}}$, where $b$ is some numerical factor, $n \geq 3$, and the explicit expression for the interparticle distance $r_{21} = r$ takes the multi-dimensional form $r = \sqrt{[x_2^{(1)} - x_1^{(1)}]^2 + [x_2^{(2)} - x_1^{(2)}]^2 + \ldots + [x_2^{(n)} - x_1^{(n)}]^2}$. Here $(x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(n)})$ and $(x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(n)})$ are the Cartesian coordinates of the two interacting particles in $n$-dimensional Euclidean space. The $n$-dimensional radius $r = \sqrt{[x^{(1)}]^2 + [x^{(2)}]^2 + \ldots + [x^{(n)}]^2}$ is, in fact, the hyper-radius of this point particle. To derive the explicit formula for the Coulomb potential in $n$-dimensional space, we have applied the method developed by A. Sokolov (see, e.g., [8,9] and earlier references therein) which allows one to determine the Green's functions for an arbitrary linear differential operator.

In order to determine the factor $b(n)$, we apply the second criterion (see above), which states that Gauss's law must be written in the form $\nabla\mathbf{E} = f(n)q_1 q_2$, where $f(n)$ is a pure angular (or hyper-angular for $n \geq 4$) factor. From here, one finds that $b = \frac{1}{n-2}$ and the explicit formula for Coulomb's law in $n$-dimensional space takes the final form $V(r) = \frac{q_1 q_2}{(n-2) r_{21}^{n-2}}$. Now, let us consider a slightly different problem. Suppose that we have to determine the static multi-dimensional Coulomb potential $\varphi(r)$ and the corresponding intensity of electric field **E**, which are generated by a point particle with the electric charge $Q$. For this problem, we write the following formulas for the potential $\varphi$ and for the field strength **E**: $\varphi = \frac{Q}{(n-2)r^{n-2}}$ and $\mathbf{E} = -\nabla\varphi = \frac{Q\mathbf{n_r}}{r^{n-1}}$, where $\mathbf{n}_r$ is the unit vector $\mathbf{n}_r = \frac{\mathbf{r}}{r}$ which is directed from the electric charge $Q$ to an observation point. To write Gauss's law in multi-dimensional space, let us assume that a point electrical charge $Q$ is located inside (and outside) of a closed $(n-1)$ dimensional hyper-surface. In this case, $r$ is the distance from the charge to a point on the hyper-surface, $\mathbf{n}$ is outwardly directed normal $\mathbf{n} = \frac{\mathbf{r}}{r}$ to the surface at that point, and $da$ is the element of the surface area. Then, for the normal component of **E** times the area element, we can write

$$(\mathbf{E} \cdot \mathbf{n})da = Q\ \frac{cos\Theta}{r^{n-1}}\ da = Q\ \frac{r^{n-1}d\Omega}{r^{n-1}} = Qd\Omega, \tag{11}$$

where $d\Omega$ is the element of the solid hyper-angle (in $n$-dimensional space) subtended by $da$ at the position of the charge. It is important here that the **E** is directed along the line from the hyper-surface element to the charge $Q$. This means that we have found no contradiction here between out two criteria and and Equation (11), since the following hyper-angular integration over $\Omega$ produces only an additional pure hyper-angular factor $f(n)$.

Now, by integrating the normal component of $\mathbf{E}$ over the whole hyper-surface, it is easy to find that

$$\oint (\mathbf{E} \cdot \mathbf{n}) da = Q \oint d\Omega = Q \ \frac{n\pi^{\left(\frac{n}{2}\right)}}{\Gamma\left(1 + \frac{n}{2}\right)} = f(n)Q, \tag{12}$$

where $f(n) = \frac{n\pi^{\left(\frac{n}{2}\right)}}{\Gamma\left(1+\frac{n}{2}\right)}$ is the geometrical (or hyper-angular) factor. In this equation, the symbol $\Gamma(x)$ stands for the Euler's gamma function (or Euler's integral of the second kind). It can be shown (see, e.g., [10]) that $\Gamma(1 + x) = x\Gamma(x)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. The formula, Equation (12), is true if the charge $Q$ lies inside of the $n$-dimensional hyper-surface. However, if this charge lies outside of this hyper-surface, the expression on the right-hand side of Equation (12) equals zero identically. Thus, we have reproduced Gauss's law in multi-dimensional spaces for a single point charge $Q$. For a discrete set of point charges and for a continuous charge density $\rho(\mathbf{r})$, Gauss's law becomes

$$\oint (\mathbf{E} \cdot \mathbf{n}) da = \frac{n\pi^{\left(\frac{n}{2}\right)}}{\Gamma\left(1 + \frac{n}{2}\right)} \ \sum_{k=1}^{K} Q_k \ = f(n) \ \sum_{k=1}^{K} Q_k \tag{13}$$

and

$$\oint (\mathbf{E} \cdot \mathbf{n}) da = \frac{n\pi^{\left(\frac{n}{2}\right)}}{\Gamma\left(1 + \frac{n}{2}\right)} \ \int_V \rho(\mathbf{r}) d^n\mathbf{r} \ = f(n) \ \int_V \rho(\mathbf{r}) d^n\mathbf{r} \tag{14}$$

respectively. In Equation (13), the sum is over only those charges inside of the hyper-surface $S$, while in Equation (14), the sum is over the volume (or hyper-volume) enclosed by $S$.

The differential form of these equations in $n$-dimensional Euclidean space is

$$div\mathbf{E} = -div\left(grad\ \varphi\right) = -\Delta\varphi = \frac{n\pi^{\left(\frac{n}{2}\right)}}{\Gamma\left(1 + \frac{n}{2}\right)} \ \rho(\mathbf{r}) = f(n)\rho(\mathbf{r}), \tag{15}$$

where $f(n) = \frac{n\pi^{\left(\frac{n}{2}\right)}}{\Gamma\left(1+\frac{n}{2}\right)}$ is the geometrical (or hyper-angular) factor, which is the volume $V_n$ of the $n$-dimensional unit ball times the dimension $n$ of geometrical space. In other words, the factor $f(n)$ is the surface area $S_n$ of the $n$-dimensional unit ball, since the equality $S_n = nV_n$ is always obeyed for the $n$-dimensional unit ball [11] and $n$ is an integer positive number. The physical sense of this factor $f(n)$ is simple: it is the total hyper-angle defined for a single point (central) particle located in the $n$-dimensional space. For a system of a few discrete charges, one has to replace $\rho(\mathbf{r}) \rightarrow \sum_{k=1}^{K} Q_k$, etc.

The $n$-dimensional hyper-angular factor $f(n)$ from Equation (12) plays a central role in our development of Maxwell electrodynamics in multi-dimensional spaces. In particular, the knowledge of this factor allows one to write the explicit formula for the action function (or action integral) of the electrically charged particles that move in the multi-dimensional (or $n$-dimensional) electromagnetic field. This problem is considered below.

## 4. Action Function and Maxwell Equations in Multi-Dimensional Flat Spaces

In this section, we consider Maxwell's equation in multi-dimensional flat spaces; e.g., in pseudo-Euclidean spaces. The results derived below are extensively used in the following sections of this study. First of all, by using the factor $f(n)$ obtained in Equation (12), we can write the final expression for the action function $S$ in Gauss units:

$$S = S_p + S_{fp} + S_f = -\sum_k \int m_k c ds - \frac{1}{c^2} \int A_\alpha j^\alpha dx^\alpha - \frac{1}{4cf(n)} \int F_{\alpha\beta} F^{\alpha\beta} d\Omega, \tag{16}$$

where $\frac{1}{4}$ (or $-\frac{1}{4}$) is the Heaviside constant, $c$ is the speed of light in a vacuum, while $j^\alpha$ is the electric current (or simply, current) in $(n+1)$-dimensional space. By varying all components of the $\bar{A}$ vector in this action integral, Equation (16), we derive the second group of Maxwell's equations, Equation (19), which contains, in the general case, the non-homogeneous differential equations. By omitting some obvious details, we can write the complete set of Maxwell's equations in the following tensor form:

$$\frac{\partial F_{\gamma\lambda}}{\partial x^\beta} + \frac{\partial F_{\lambda\beta}}{\partial x^\gamma} + \frac{\partial F_{\beta\gamma}}{\partial x^\lambda} = 0 \tag{17}$$

and

$$\frac{\partial F^{\alpha\beta}}{\partial x^\beta} = -\frac{n\pi^{\left(\frac{n}{2}\right)}}{c\Gamma\left(1+\frac{n}{2}\right)} j^\alpha = -\frac{f(n)}{c} j^\alpha, \tag{18}$$

where $j^\alpha$ is the $(n+1)$-dimensional current-vector (or current, for short) defined above. All equations from the both groups of these equations, Equations (17)–(19), are the first-order differential equations upon spatial coordinates and time $t$ (or temporal coordinate). From Equation (17) one finds the following condition for the current:

$$\frac{\partial^2 F^{\alpha\beta}}{\partial x^\alpha \partial x^\beta} = -\frac{f(n)}{c} \frac{\partial j^\alpha}{\partial x^\alpha} = 0\ . \tag{19}$$

This result is obvious, since the application of any symmetric operator (upon $\alpha \leftrightarrow \beta$ permutation)—e.g., the $\frac{\partial^2}{\partial x^\alpha \partial x^\beta}$ operator—to the truly antisymmetric $F^{\alpha\beta}$ tensor always gives zero. Thus, the equality $\frac{\partial j^\alpha}{\partial x^\alpha} = 0$ derived here is a necessary condition for any actual electric current. Note also that this equation is written in the form of $(n+1)$-dimensional divergence. In respect to Noether's second theorem, this equation $\frac{\partial j^\alpha}{\partial x^\alpha} = 0$ means some conservation law. It is easy to understand that this law describes the conservation of the total electric charge.

A very close similarity between the Maxwell equations derived for multi-dimensional spaces, where $n \geq 3$, and analogous Maxwell equations known in three-dimensional space is obvious. However, in some cases, this leads to fundamental mistakes, and most of such mistakes originate from Equation (17). Note here that in $n$-dimensional geometrical space, we have exactly $n$ components of the intensity of electric field **E** and $\frac{n(n+1)}{2}$ intensities of magnetic field **H**. For $n = 3$ (and only in this case), we have equal numbers of components in both **E** and **H** vectors. This leads to the well-known vector form of Maxwell electrodynamics. However, even for $n = 4$, the electric field has four components, while the magnetic field has six components. When $n$ increases, then the total number of components of the magnetic field grows rapidly (quadratically) and significantly exceeds the analogous number of components of the electric field. This fact substantially complicates the derivation of Maxwell equations written in terms of the intensities of electric and magnetic fields in multi-dimensional spaces. Plus, we have a certain problem with the general definition of the *curl* (or *rot*) operator in such cases.

Another interesting result follows from the analysis of tensor equations, Equation (17). If one of the indexes in this equation equals zero, then this group of equations gives us Faraday's law in multi-dimensional space, which describes the time-evolution of the magnetic field and is written in the form of $n$ equations. This is good, but what about other $\frac{n(n-1)(n-2)}{6}$ equations that are also included in tensor equations Equation (17)? After some transformations, one finds that these additional equations are written in a form whereby three-dimensional divergences of some three-dimensional pure-magnetic vectors equal zero. By pure magnetic vectors, we mean vectors assembled from the space-like components of the field tensor $F^{pq}$ (or $F_{pq}$) only (for flat spaces, it is always possible). Based on ideas by Dirac [12], we can formulate this result in the following form: *the magnetic field can have sources neither in our three-dimensional space nor in any three-dimensional subspace of multi-dimensional spaces*. This fundamental statement is directly and very closely related to the discrete nature of electric charge. Furthermore, the correctness of Maxwell electrodynamics (in any space) is essentially based on this statement. By taking into account arguments from [13], we can re-formulate our statement in the followingform: *the existence of magnetic monopoles in our three-dimensional space and, in general, in any three-dimensional subspace of multi-dimensional spaces is strictly prohibited*. Otherwise, the Maxwell electrodynamics will not be correct and must be replaced by a different approach.

To conclude this section, let us present the explicit formula for the energy momentum tensor in multi-dimensional space. The definition of this tensor and all details of its calculations are well described in [5]. Therefore, we can only present a few basic formulas here, which will be used below in Section 6. The explicit formula for the non-symmetrized energy momentum tensor is

$$T_{\alpha}^{\beta} = \frac{1}{f(n)}\Big(\frac{\partial A_{\gamma}}{\partial x_{\alpha}}F^{\gamma\beta} + \frac{1}{4}g_{\alpha}^{\beta}F_{\gamma\rho}F^{\gamma\rho}\Big), \tag{20}$$

where the factor $f(n)$ is the hyper-angular (or geometrical) factor mentioned above. After symmetrization, this tensor takes the form

$$T_{\alpha}^{\beta} = \frac{1}{f(n)}\Big(F_{\alpha\gamma}F^{\beta\gamma} + \frac{1}{4}g_{\alpha}^{\beta}F_{\gamma\rho}F^{\gamma\rho}\Big), \tag{21}$$

where $g_{\alpha}^{\beta} = \delta_{\alpha}^{\beta}$ is the substitution tensor [6]. The corresponding co- and contravariant tensors are

$$T_{\alpha\beta} = \frac{1}{f(n)}\Big(F_{\alpha\gamma}F_{\beta}^{\gamma} + \frac{1}{4}g_{\alpha\beta}F_{\gamma\rho}F^{\gamma\rho}\Big) \quad \text{and} \quad T^{\alpha\beta} = \frac{1}{f(n)}\Big(g_{\gamma}^{\alpha}F_{\beta\gamma}F^{\beta\gamma} + \frac{1}{4}g^{\alpha\beta}F_{\gamma\rho}F^{\gamma\rho}\Big), \tag{22}$$

where $f(n) = \frac{n\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)}$ is the geometrical (or hyper-angular) factor.

## 5. Hamiltonian of the Electromagnetic Field in Multi-Dimensional Flat Spaces

The second goal of this study is to develop the Hamiltonian formulation of the multi-dimensional electrodynamics. First, let us obtain the explicit formula for the Hamiltonian $H$ of the free electromagnetic field in multi-dimensional flat spaces. By using the formula, Equation (16), for the action integral, we can write the Lagrangian $L$ of the free electromagnetic field in multi-dimensional pseudo-Euclidean space (in Heaviside units)

$$L = -\frac{1}{4}\int F_{\alpha\beta}F^{\alpha\beta}\, d^n\mathbf{x} = -\frac{1}{4}\int F^{\alpha\beta}F_{\alpha\beta}\, d^n\mathbf{x}\,, \tag{23}$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the electromagnetic field tensor which is antisymmetric $F_{\mu\nu} = -F_{\nu\mu}$. From here, one finds the following equality $A_{\mu,\nu} = -F_{\mu\nu} + A_{\nu,\mu} = F_{\nu\mu} + A_{\nu,\mu}$. Variations of this Lagrangian are written in the following general form:

$$\delta L = -\frac{1}{2} \int F_{\alpha\beta} \delta F^{\alpha\beta} d^n \mathbf{x} = -\frac{1}{2} \int F^{\alpha\beta} \delta F_{\alpha\beta} d^n \mathbf{x}, \tag{24}$$

where $d^n\mathbf{x}$ means $dx^1 dx^2 \ldots dx^n$ and the integration is over $n$-dimensional space. Note that all integrals considered in this section are the spatial integrals that contain no integration over the temporal (or time) variable. Furthermore, in this section, we shall apply only the Heaviside units. The use of Gauss units complicates all formulas below, including the expressions for the momenta.

In order to develop the Hamiltonian approach for the electromagnetic field, we need to consider all variations of the velocities for each component of the $(n+1)$-dimensional vector potential $\bar{A}$. In other words, below, we deal with variations of the $A_{\mu,0}$ derivatives only, where $\mu = 0, 1, \ldots, n$. In other words, in our Hamiltonian formulation, all components of the $(n+1)$-dimensional vector potential $\bar{A}$—i.e., $A_0, A_1, \ldots, A_n$ components—are the generalized coordinates of our problem. For variations of the velocities $A_{\mu,0}$, our formula, Equation (24), for $\delta L$ is written in the form

$$\delta L = \int F^{\alpha 0} \delta A_{\alpha,0} \, d^n \mathbf{x} = \int B^{\alpha} \delta A_{\alpha,0} \, d^n \mathbf{x}, \tag{25}$$

where $B^{\alpha} = F^{\alpha 0}$ are the contravariant components of the $(n+1)$-dimensional vector momenta $\bar{B}$. In fact, this equation must be considered as the explicit definition of momenta. However, from this definition and the antisymmetry of the electromagnetic field tensor, one finds $B^0 = F^{00} = 0$. This means that the 0-component of momenta $\bar{B}$ of the electromagnetic field—i.e., $B^0$—must be equal to zero at all times. According to Dirac [14], all similar equations derived at this stage of the Hamiltonian procedure are the primary constraints. In our current case, this constraint is better to write in the form of a weak identity $B^0 \approx 0$.

By using our momenta $B^{\alpha}$, we can introduce the Hamiltonian of the free electromagnetic field in multi-dimensional pseudo-Euclidean (flat) space:

$$\begin{aligned} H &= \int B^{\alpha} A_{\alpha,0} \, d^n \mathbf{x} - L = \int \Big( F^{q0} A_{q,0} + \frac{1}{4} F^{pq} F_{pq} + \frac{1}{4} F^{p0} F_{p0} + \frac{1}{4} F^{0p} F_{0p} \Big) d^n \mathbf{x} \\ &= \int \Big( F^{q0} A_{q,0} + \frac{1}{4} F^{pq} F_{pq} + \frac{1}{2} F^{p0} F_{p0} \Big) d^n \mathbf{x} = \int \mathcal{H} d^n \mathbf{x}, \end{aligned} \tag{26}$$

where $\mathcal{H}$ is the Hamiltonian space-like density (scalar), which is

$$\mathcal{H} = F^{q0} A_{q,0} + \frac{1}{4} F^{pq} F_{pq} + \frac{1}{2} F^{p0} F_{p0} \tag{27}$$

For the $A_{q,0}$ derivative, we substitute its equivalent expression $A_{q,0} = -F_{q0} + A_{0,q}$ (see above) and obtain

$$H = \int \Big( \frac{1}{4} F^{pq} F_{pq} - \frac{1}{2} F^{p0} F_{p0} + F^{q0} A_{0,q} \Big) d^n \mathbf{x} = \int \Big( \frac{1}{4} F^{pq} F_{pq} + \frac{1}{2} B^p B^p + B^q A_{0,q} \Big) d^n \mathbf{x}. \tag{28}$$

In the last term of this Hamiltonian, we can perform a partial integration, which actually leads to the following replacement: $F^{q0} A_{0,q} \rightarrow -A_0 \frac{\partial F^{q0}}{\partial x_q} = -A_0 (B^q)_q = -A_0 B^p_{,p}$. This reduces our Hamiltonian, Equation (28), to the form

$$H = \int \Big( \frac{1}{4} F^{pq} F_{pq} - \frac{1}{2} F^{p0} F_{p0} + F^{q0} A_{0,q} \Big) d^n \mathbf{x} = \int \Big( \frac{1}{4} F^{pq} F_{pq} + \frac{1}{2} B^p B^p - A_0 B^p_{,p} \Big) d^n \mathbf{x} \ . \tag{29}$$

This is the Hamiltonian of the free electromagnetic field written in the closed analytical form. The corresponding Hamiltonian space-like density takes the form

$$\mathcal{H} = \frac{1}{4}F^{pq}F_{pq} + \frac{1}{2}B^{p}B^{p} - A_0 B^{p}_{,p}. \tag{30}$$

Note that by performing these transformations and deriving the Hamiltonian, Equation (29), we have gained even more than we wanted at the beginning of our procedure. In fact, the development of any Hamiltonian approach means that we have a simplectic structure, which is defined by the Poisson brackets between basic dynamical (Hamiltonian) variables: $(n+1)$ coordinates $A_\mu$ and $(n+1)$ momenta $B^\mu$. These Poisson brackets are defined as follows:

$$[A_\mu(\bar{x}_1), B^\nu(\bar{x}_2)] = g^\nu_\mu \delta^{(n)}(\mathbf{x}_1 - \mathbf{x}_2),\ [A_\mu(\mathbf{x}_1), A_\nu(\mathbf{x}_2)] = 0,\ [B^\mu(\mathbf{x}_1), B^\nu(\mathbf{x}_2)] = 0, \tag{31}$$

where $g^\nu_\mu = \delta^\nu_\mu$ is the Kronecker delta-function, while $\mu = 0, 1, \ldots, n$, and $\nu = 0, 1, \ldots, n$.

In general, the Poisson brackets are used as the main working tool in any Hamiltonian approach developed for a given physical system. Moreover, these brackets allow one to introduce a simplectic $(2n+2)$-dimensional phase space of the Hamiltonian variables $\{A_\alpha, B^\beta\}$ which are defined in each point $\bar{x}$ of the $(n+1)$-dimensional space-time manifold. The original configuration space of this problem is the direct sum of the $(n+1)$-dimensional subspace of $A_\mu-$coordinates and $(n+1)$-dimensional subspace of $A_{\mu,0}-$velocities. In turn, this allows one to consider and apply various canonical transformations of the Hamiltonian canonical variables. Furthermore, by using the Poisson brackets in Equation (31), we can complete our Hamiltonian approach for the classical electrodynamics and perform its quantization.

To illustrate this fact, let us go back to the primary constraint $B^0 \approx 0$ mentioned above. This constraint must remain satisfied at all times. This means that its time derivative $\frac{dB^0}{dt}$, which in our Hamiltonian approach equals the Poisson bracket $[B^0, H]$, must be zero at all times. This Poisson bracket is easily determined, since in the Hamiltonian, Equation (29), there is only one term (the last term) that does not commute with the momentum (or primary constraint) $B^0$:

$$[B^0, \frac{1}{4}F^{pq}F_{pq} - \frac{1}{2}F^{p0}F_{p0} - A_0 B^q_q] = -[B^0, A_0]B^p_{,p} = [A_0, B^0]B^p_{,p} = B^p_{,p} \tag{32}$$

In other words, we have found another weak equality $B^p_{,p} \approx 0$ that must be obeyed at all times. According to Dirac [14,15], this condition is the secondary constraint of our Hamiltonian formulation of the multi-dimensional Maxwell theory of radiation. The next Poisson bracket $[B^p_{,p}, H]$ (or $[B^p_{,p}, \mathcal{H}]$) equals zero identically, which indicates clearly that the chain of first-class constraints is closed, and our Hamiltonian formulation does not lead to any tertiary and/or other constraints of higher order. Briefly, this means the complete closure (or Dirac closure) of the Hamiltonian procedure for the free electromagnetic field in multi-dimensional space.

### 5.1. *Further Transformations of the Hamiltonian*

The first term in the Hamiltonian of the free electromagnetic field in multi-dimensional space, Equation (29), includes a number of different terms, but it does not contain any of the canonical variables. It is difficult to use such a Hamiltonian for the analysis and solution of actual problems in classical and/or quantum electrodynamics. Therefore, we have to transform this Hamiltonian to a form that explicitly contains canonical variables in each term. Then, our newly derived Hamiltonian $H$ and/or the corresponding Hamiltonian density $\mathcal{H}$ can be applied for the solution of many actual problems. For convenience, below, we shall deal with the Hamiltonian density $\mathcal{H}$. The partial integration of the first term in the Hamiltonian, Equation (29), leads to the following expression for the Hamiltonian density Equation (30):

$$\mathcal{H} = \Big(F^{qp}\Big)_{q} A_p + \frac{1}{2} B^p B^p - A_0 B^p_{,p} = \Big(\frac{\partial^2 A^p}{\partial x_q \partial x^q} - \frac{\partial^2 A^q}{\partial x_q \partial x_p}\Big) A_p + \frac{1}{2} B^p B^p - A_0 B^p_{,p}, \quad (33)$$

where $p = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, n$. For this Hamiltonian density, we can write the following system of canonical equations:

$$\frac{dA_p}{dt} = [A_p, \mathcal{H}] = \frac{1}{2}\,(2B^p) = B^p \quad (34)$$

and

$$\frac{dB^p}{dt} = [B^p, \mathcal{H}] = -\Big(\frac{\partial^2 A^p}{\partial x_q \partial x^q} - \frac{\partial^2 A^q}{\partial x_q \partial x_p}\Big) = \frac{\partial^2 A_p}{\partial x_q \partial x^q} - \frac{\partial^2 A_q}{\partial x_q \partial x_p}. \quad (35)$$

Combining these two equations, one finds

$$\frac{d^2 A_p}{dt^2} = \frac{\partial^2 A_p}{\partial x_q \partial x^q} - \frac{\partial^2 A_q}{\partial x_q \partial x_p}. \quad (36)$$

Taking into account the gauge condition $\frac{\partial A_q}{\partial x_q} = 0$ (see below), we reduce the last equation to the form

$$\frac{\partial^2 A_p}{\partial t^2} - \frac{\partial^2 A_p}{\partial x_q \partial x^q} = 0, \text{ or } \frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} = 0, \quad (37)$$

which is the wave equations written in the $(n+1)$-dimensional space-time. The $n$-dimensional Laplace operator $\Delta$ in this equation is

$$\Delta = \frac{\partial^2}{\partial x_q \partial x^q} = g^{qr} \frac{\partial^2}{\partial x^q \partial x^r} = g_{qr} \frac{\partial^2}{\partial x_q \partial x_r}. \quad (38)$$

Thus, in our Hamiltonian approach, the multi-dimensional wave equation for the free electromagnetic field is derived as a direct consequence of the canonical Hamilton equations obtained for this field. Such a derivation of the wave equation for a free electromagnetic field described here is, probably, the most direct, fast, and logically clear of all known (alternative) methods. In addition to this, we have rigorously derived the two additional conditions for the momenta of the free electromagnetic filed: $B^0 \approx 0$ and $B^p_{,p} \approx 0$. In our Hamiltonian formulation, these two weak equations are called the primary and secondary constraints, respectively. It is easy to show that these two constraints are first-class [14]. In the four-dimensional case, Dirac has suggested [14] that these two constraints are the generators (or generating functions) for infinitesimal contact transformations which do not change the actual physical state of the free electromagnetic field; i.e., they are two independent generators of internal symmetry. Twenty years later, this statement has rigorously been proven by L. Castellani [16]. All these results are the great and obvious advantages of the Dirac's (Hamiltonian) formulation of the Maxwell theory. Now, by using all first-class constraints that have been derived during the Hamiltonian formulation, one can determine the true symmetry of any given physical field. For the free electromagnetic field, such a symmetry group coincides with the Lorentz $SO(3,1)$-group. In general, by operating with the first-class constraints only, it is impossible to restore the so-called hidden (or additional) symmetries of the free electromagnetic field. For instance, for the free electromagnetic field considered in three-dimensional space, the complete group of point symmetry is the $SO(4,2)$-group, which has 15 generators [17], while the Lorentz $SO(3,1)$-group has only 6 generators. The powerful method of Bessel–Hagen [17] is based on applications of Noether's second theorem, which is applied to the Lagrangian of the free electromagnetic field. In this short paper, we cannot discuss all details of this interesting problem.

### 5.2. First-Class Constraints and Gauge Invariance

In this section, we consider a different symmetry (or invariance) of Maxwell equations that is directly and closely related to the primary and secondary first-class constraints. This invariance is the well-known gauge invariance (or symmetry) of the Maxwell equations. The gauge invariance of three-dimensional Maxwell equations has been studied by many famous authors, including Heitler [18], Jackson [19,20], Gelfand and Fomin [21], and others (see, e.g., [22]). Briefly, the gauge invariance means that we can impose some additional conditions upon the physical fields, or some of their components, and these additional conditions do not change solutions of the original problem (but they can change equations). The gauge conditions are often used to simplify the Hamiltonian equations of motion, either by reducing the total number of variable fields or by vanishing some terms (or combinations of terms) in these equations. Let us discuss the gauge invariance of the free electromagnetic field (or "pure radiation field" [18]) by using the two first-class constraints which we have derived above: $B^0 \approx 0$ and $B^p_{,p} \approx 0$. By re-writing these two constraints in terms of the components of the $(d+1)$-dimensional vector potential $\bar{A} = (\varphi, \mathbf{A})$ and their temporal derivatives, one finds

$$B^0 \approx 0 \Rightarrow \frac{\partial \varphi}{\partial t} = 0 \text{ and } B^p_{,p} \approx 0 \Rightarrow \frac{\partial}{\partial t}\Big(div\mathbf{A}\Big) = 0 \tag{39}$$

where we gave used the traditional sign of actual equality "=" instead of the weak equality "≈", which has been used above in the Dirac's Hamiltonian approach. The two equalities in the right-hand side of Equation (39) lead us to the two following equations: $\varphi = \varphi(\mathbf{r})$ and $div\mathbf{A} = C(\mathbf{r})$, where the scalars $\varphi(\mathbf{r})$ and $C(\mathbf{r})$ are the functions of $n$ spatial coordinates only, and they do not change with time; i.e., they are time-independent scalar functions. It is clear that these two time-independent scalars are not related in any way to the Hamiltonian formulation of the Maxwell theory of electromagnetic fields. Indeed, the Hamiltonian approaches describe only the time-evolution of the Hamiltonian dynamical variables. For static problems, there are other different methods. Therefore, without loss of generality, we can assume that these time-independent scalars $\varphi(\mathbf{r})$ and $C(\mathbf{r})$ equal zero identically at all times.

Based on these arguments, we can write the four following equations for the field dynamical variables (or Hamiltonian variables):

$$\varphi = 0, \;\; \frac{\partial \varphi}{\partial t} = 0, \;\; div\mathbf{A} = 0 \text{ and } \frac{\partial}{\partial t}\Big(div\mathbf{A}\Big) = 0, \tag{40}$$

which can be considered as the four independent "basis vectors". In general, the set of $N_g$ gauge conditions $\psi_i$ is represented as a linear combination of the four basis vectors from Equation (40):

$$\psi_i = \alpha_i \varphi + \beta_i \frac{\partial \varphi}{\partial t} + \gamma_i \, div\mathbf{A} + \delta_i \frac{\partial}{\partial t}\Big(div\mathbf{A}\Big) = 0, \tag{41}$$

where $i = 1, 2, 3, 4$, while $\alpha_i, \beta_i, \gamma_i$, and $\delta_i$ are some numerical constants. Let us discuss the principal question about the number $N_g$, which is the number of sufficient (or essential) gauge equations. For the free electromagnetic field, $N_g$ equals two, since exactly this number of conditions has been found in the Hamiltonian formulation of electrodynamics developed by Dirac (see above). The two equations $\frac{\partial \varphi}{\partial t} = 0$ and $\frac{\partial}{\partial t}\Big(div\mathbf{A}\Big) = 0$ define the so-called Dirac gauge, which is discussed above. Formally, for the Dirac gauge, we can introduce the third gauge condition $\varphi = 0$ and completely exclude the pair of variables $\Big(\varphi, \frac{\partial \varphi}{\partial t}\Big)$ from the list of our dynamical variables. However, this follows not from some general principle but from the explicit form of Dirac's Hamiltonian density, Equation (30), for the pure radiation field (see above), where the only term that includes the scalar potential $\varphi$ is written as a product of $\varphi$ (or $A_0$) and secondary constraint $B^p_{,p}$. This term equals zero on the shell of the first-class constraints.

An alternative choice of two gauge equations $\frac{\partial\varphi}{\partial t} = 0$ and $div\mathbf{A} = 0$ corresponds to the famous Coulomb gauge, which provides the best choice for many three-dimensional QED problems in atomic and molecular physics. In the Coulomb gauge, the scalar potential $\varphi(= A_0)$ is always a static potential, while the $n$-dimensional vector potential $\mathbf{A}$ is always transverse. The Coulomb gauge and other gauges discussed here are easily generalized for $n$-dimensional spaces. Another choice of the basic gauge equations defines the Lorentz gauge. Formally, this gauge is defined by one (Fermi's) equation $\frac{\partial\varphi}{\partial t} + div\mathbf{A} = 0$. In respect to the Dirac theory, this set of gauge conditions is not complete and a second gauge equation can be added. For instance, one can choose the second condition in the form $\frac{\partial\varphi}{\partial t} - div\mathbf{A} = 0$, which is a relativistic invariant for the electromagnetic wave that propagates from the present to the past. A different choice of the second equation for the Lorentz gauge corresponds to the so-called Heitler's gauge, which is based on the two equations $\frac{\partial\varphi}{\partial t} + div\mathbf{A} = 0$ and $\frac{\partial}{\partial t}\Big(div\mathbf{A}\Big) = 0$ for the free electromagnetic field [18]. The advantage of this useful gauge is obvious: if these equations hold at $t = 0$, then the equation $\frac{\partial\varphi}{\partial t} + div\mathbf{A} = 0$ is always satisfied. These simple examples of different gauges are mentioned here only to illustrate the ultimate power of Dirac's approach, which simplifies the internal analysis of various gauges.

Let us discuss the general source of gauges which often arise in different field theories; e.g., in Maxwell theory of radiation, metric gravity, tetrad gravity, etc. Here, we want to investigate this problem from the Hamiltonian point of view. First, let us assume that we have imposed all four conditions from Equation (40) on our dynamical variables. What does this mean for these variables? The first two equations $\varphi = 0$ and $\frac{\partial\varphi}{\partial t} = 0$ mean that the variable $\varphi$ and corresponding momentum $B^0$ (or velocity $\frac{\partial\varphi}{\partial t}$) are not dynamic (Lagrange) variables of our problem. In other words, we have to exclude these two variables before the application of our Hamiltonian procedure. The same statement is true for the two equations $div\mathbf{A} = 0$ and $\frac{\partial}{\partial t}\Big(div\mathbf{A}\Big) = 0$, but $div\mathbf{A}$ is not a regular dynamic variable of the original problem. In reality, the function $div\mathbf{A}$ appears in the secondary constraint in Dirac's Hamiltonian formulation developed for the pure radiation filed. This function is a linear combination of the first-order derivatives of covariant components of the multi-dimensional vector potential $\mathbf{A}$. The Hamiltonian canonical variables do not include any sum of the space-like derivatives of this potential. Therefore, it is not clear how we can exclude the scalar $div\mathbf{A}$ and its time-derivative from the list of our canonical variables. However, the main obstacle to the exclusion of the four variables, Equation (40), follows from the fact that we have only two gauge equations (not four). This means that we cannot correctly exclude all four variables and have to keep them in our procedure. These "extra" variables survive our Hamiltonian procedure only in the form of additional equations for the Hamiltonian dynamical variables. In other words, the gauge conditions are the integral parts of any Hamiltonian approach developed for an arbitrary physical field. This is the general principle that explains why different field theories with first-class constraints always have some number of non-trivial gauge conditions (or equations).

However, this is not the end of the story. Let us look at the constraints in multi-dimensional electrodynamics from a different point of view. Consider the following two-parametric $(\alpha, \beta)$-family of the Hamiltonian densities:

$$\mathcal{H}(\alpha,\beta) = \frac{1}{4}F^{pq}F_{pq} + \frac{1}{2}B^pB^p - A_0B^p_{,p} + \Big(\alpha B^0 + \beta B^p_{,p}\Big)^2. \tag{42}$$

where $B^0$ and $B^p_{,p}$ are the functions of the canonical variables of the problem. At this moment, we cannot assume that there are some restrictions on these two quantities. In other words, for now, the $B^0$ and $B^p_{,p}$ values are not yet constraints.

In general, to operate with the two-parametric family of Hamiltonian densities $\mathcal{H}(\alpha, \beta)$ in some constructive way, we have to formulate the following variational principle: the actual (or true) Hamiltonian density coincides with the minimal Hamiltonian density $\mathcal{H}(\alpha, \beta)$, Equation (42), in respect to possible variations of the two numerical parameters $\alpha$ and $\beta$. This principle immediately leads to the two following weak identities:

$$\left(\alpha B^0 + \beta B^p_{,p}\right) B^0 \approx 0 \quad \text{and} \quad \left(\alpha B^0 + \beta B^p_{,p}\right) B^p_{,p} \approx 0. \tag{43}$$

One obvious solution of this system gives us the two Dirac's constraints $B^0 \approx 0$ and $B^p_{,p} \approx 0$ which have been derived above. In general, there are other solutions of the system Equation (43), and one of them can be written in the form

$$\alpha_1 B^0 + \beta_1 B^p_{,p} \approx 0 \quad \text{and} \quad \alpha_2 B^0 + \beta_2 B^p_{,p} \approx 0. \tag{44}$$

where the coefficients $\alpha_1, \beta_1, \alpha_2$ and $\beta_2$ form a regular (i.e., invertible) $2 \times 2$ matrix. The principle formulated above is called the optimal principle for the constrained motions, since in actual physical systems, the motion along first-class constraints is optimal, or it can be considered as optimal.

## 6. Multi-Dimensional Maxwell Equations in Non-Flat Spaces

The Maxwell equations can be written in the covariant form, which is more appropriate in applications to the metric gravity (or general relativity) in multi-dimensional Riemannian spaces. In this and the next sections, we deal with the multi-dimensional Riemannian spaces only. These spaces are not flat, and they are often called the spaces of non-zero curvature. Indeed, the corresponding equations, Equations (17) and (19), for the flat multi-dimensional spaces have already been written in the tensor (or covariant) form. Furthermore, the electromagnetic field tensor $F_{\alpha\beta}$, which has been defined by Equation (8), is truly skew-symmetric with respect to permutations of its indexes; i.e., $F_{\alpha\beta} = -F_{\beta\alpha}$ and $F^{\alpha\beta} = -F^{\beta\alpha}$. These two facts simplify the process of derivation of the Maxwell equations in the covariant form. In fact, to derive the covariant form of Maxwell equations, one needs to replace all usual derivatives written in Cartesian coordinates by the tensor derivatives. After such a replacement, the first group of Maxwell equations in multi-dimensional Riemannian spaces takes the form

$$\nabla_\beta F_{\gamma\lambda} + \nabla_\lambda F_{\beta\gamma} + \nabla_\gamma F_{\lambda\beta} = 0 \quad (\text{or} \;\; \nabla_\beta F_{\gamma\lambda} = \nabla_\gamma F_{\beta\lambda} - \nabla_\lambda F_{\beta\gamma}), \tag{45}$$

where $\nabla_\beta$ is the tensor (or covariant) derivative; i.e.,

$$\nabla_\beta F_{\gamma\lambda} = \frac{\partial F_{\gamma\lambda}}{\partial x^\beta} - \Gamma^\mu_{\gamma\beta} F_{\mu\lambda} - \Gamma^\mu_{\lambda\beta} F_{\gamma\mu} \tag{46}$$

where $\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\gamma\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\gamma}}{\partial x^\gamma} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right) = \Gamma^\gamma_{\beta\alpha}$ are the Cristoffel symbols of the second kind. It is interesting to note that the form of Equation (45) does not depend explicitly upon the parameter $n$, which defines the dimension of Riemann space. By performing a few simple transformations, we can reduce the formula, Equation (46), to a form that exactly coincides with Equation (17). This has been noticed in many textbooks on three-dimensional electrodynamics (see, e.g., [5]).

The second group of Maxwell equations for multi-dimensional spaces of non-zero curvature is written in the form (in Gauss units)

$$\nabla_\beta F^{\alpha\beta} = \frac{1}{\sqrt{-g}} \frac{\left(\partial\sqrt{-g}F^{\alpha\beta}\right)}{\partial x^\beta} = -\frac{n\pi^{\left(\frac{n}{2}\right)}}{c\Gamma\left(1 + \frac{n}{2}\right)} j^\alpha = -\frac{f(n)}{c} j^\alpha, \tag{47}$$

since the tensor $F^{\alpha\beta}$ is antisymmetric. In this equation, $g$ is the determinant of the fundamental tensor, which is always negative in the metric gravity. By applying the operator $\nabla_\alpha$ to the last formula, one finds

$$\nabla_\alpha \nabla_\beta F^{\alpha\beta} = -\frac{f(n)}{c}\nabla_\alpha j^\alpha \implies -\frac{f(n)}{c}\nabla_\beta j^\beta = \nabla_\beta \nabla_\alpha F^{\beta\alpha} = -\nabla_\beta \nabla_\alpha F^{\alpha\beta}. \tag{48}$$

In other words, the expression on the left-hand side of these equations can be re-written in the following form:

$$\frac{1}{2}\Big(\nabla_\alpha \nabla_\beta + \nabla_\beta \nabla_\alpha\Big) F^{\alpha\beta}. \tag{49}$$

which equals zero identically, since here the truly symmetric tensor operator (upon $\alpha \leftrightarrow \beta$ permutation) is applied to an antisymmetric tensor (upon the same permutations). Finally, one finds that $\nabla_\alpha j^\alpha = 0$; i.e., the conservation law for electric charge written in the $(n+1)$-dimensional Riemannian space.

In many books and textbooks, the electrodynamic derivation of Maxwell equations in the manifestly covariant form is traditionally considered as the final step. A similar approach, however, ignores an additional group of governing equations that is obeyed for the electromagnetic field in the presence of actual gravitational fields. These additional equations determine the general properties, time evolution, and propagation of electromagnetic fields in the metric gravitational fields. The explicit derivation of these additional governing equations for the electromagnetic field tensor is straightforward. Indeed, if the electromagnetic field tensor $F_{\alpha\beta}$ is considered in the metric gravity, then the following equations must be obeyed:

$$\nabla_\lambda \nabla_\sigma F^\beta_\alpha - \nabla_\sigma \nabla_\lambda F^\beta_\alpha = F^\mu_\alpha R^\beta_{\sigma\lambda\mu} - F^\beta_\mu R^\mu_{\sigma\lambda\alpha}, \tag{50}$$

or in a slightly different form:

$$\nabla_\lambda \nabla_\sigma F_{\alpha\beta} - \nabla_\sigma \nabla_\lambda F_{\alpha\beta} = -F_{\mu\beta} R^\mu_{\sigma\lambda\alpha} - F_{\alpha\mu} R^\mu_{\sigma\lambda\beta} = F_{\alpha\mu} R^\mu_{\lambda\sigma\beta} + F_{\mu\beta} R^\mu_{\lambda\sigma\alpha}, \tag{51}$$

where the notation $R^\sigma_{\alpha\beta\gamma} = g^{\sigma\mu} R_{\alpha\beta\gamma\mu}$ is the Riemann-.Cristoffel tensor of the fourth rank, which is three times covariant and once contravariant (see, e.g., [6,7]). In turn, the $R_{\alpha\beta\gamma\sigma}$ is the Riemann curvature tensor (or Riemann–Cristoffel tensor):

$$R_{\alpha\beta\gamma\sigma} = \frac{1}{2}\Big[\frac{\partial^2 g_{\alpha\sigma}}{\partial x^\beta \partial x^\gamma} + \frac{\partial^2 g_{\beta\gamma}}{\partial x^\alpha \partial x^\sigma} - \frac{\partial^2 g_{\alpha\gamma}}{\partial x^\beta \partial x^\sigma} - \frac{\partial^2 g_{\beta\sigma}}{\partial x^\alpha \partial x^\gamma}\Big] + \Gamma_{\rho,\alpha\sigma}\Gamma^\rho_{\beta\gamma} - \Gamma_{\rho,\beta\sigma}\Gamma^\rho_{\alpha\gamma}\,, \tag{52}$$

where $\Gamma_{\gamma,\mu\nu} = \frac{1}{2}\Big(\frac{\partial g_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\Big)$ are the Cristoffel symbols of the first kind. The Riemann–Cristoffel tensor defined in Equation (52) is a covariant tensor of the fourth rank. Note that similar problems have been extensively studied since the 1920s in numerous papers and books on General Relativity (see, e.g., [23,24] and references therein). As follows from these equations, Equations (50) and (51), the propagation and other properties of the "free" electromagnetic fields in multi-dimensional spaces of non-zero curvature (or in non-flat spaces) are always affected by the gravitational fields. For relatively small gravitational fields, Equations (50) and (51) can be considered as small perturbations to the Maxwell equations. However, in strong gravitational fields, where some of the $|\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}|$ derivatives are very large, the laws of propagation and other properties of the electromagnetic fields can significantly be changed by the gravity. Briefly, we can say say that in similar non-flat spaces, the actual properties of electromagnetic fields cannot be described by the Maxwell equations only. Furthermore, in more complex "combined" theories of gravity and radiation—e.g., in the well known Born–Infeld theory (see, e.g., [25])—the total fundamental tensor is represented as a function—e.g., as a sum—of the gravitational $g_{\alpha\beta}$ and electromagnetic $F_{\alpha\beta}$

tensors, while the time-evolution and propagation of electromagnetic fields is described by the non-linear, well-coupled equations.

*6.1. Multi-Dimensional Electromagnetic Field in Metric Gravity*

Now, we are ready to vary the sum of action integrals for the gravitational $S_g$ and electromagnetic $S_f$ fields; i.e., to vary the $\delta(S_g + S_f)$ action. The both fields are considered as free; i.e., there are no masses, no free electric charges and no electric currents in the area of our interest. Our goal in this section is to derive (variationally) the governing Einstein equations (in multi-dimensions) in the presence of electromagnetic fields. To achieve this goal, we have to vary the gravitational field only; i.e., the components of the metric tensor $g_{\alpha\beta}$ (or $g^{\alpha\beta}$). The variation of the gravitational action $S_g$ is written in the form (see, e.g., [5,24]):

$$\delta S_g = -\frac{c}{f(n)\mathcal{K}} \int \Big( R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \Big) \delta g^{\alpha\beta} \sqrt{-g} d\Omega, \tag{53}$$

where $R_{\alpha\beta}$ is the Ricci tensor. In older works [6], authors used the the Einstein tensor, which is $G_{\alpha\beta} = -R_{\alpha\beta}$. The explicit form of the Ricci tensor is

$$R_{\alpha\beta} = \frac{\partial \Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\gamma}_{\alpha\gamma}}{\partial x^{\beta}} + \Gamma^{\gamma}_{\alpha\beta}\Gamma^{\lambda}_{\gamma\lambda} - \Gamma^{\lambda}_{\alpha\gamma}\Gamma^{\gamma}_{\beta\lambda}, \text{ or } R_{\alpha\beta} = g^{\mu\nu} R_{\mu\alpha\beta\nu} = g^{\nu\mu} R_{\nu\beta\alpha\mu} = R_{\beta\alpha} \tag{54}$$

and $R = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar (or Gauss) curvature of space. In addition, in this equation, the notation $\mathcal{K} = \frac{k}{c^2} = 7.4259155 \times 10^{-29}$ cm $\cdot$ s$^{-1}$ denotes the universal (or $n$-independent) gravitational constant. A similar variation of the electromagnetic action $S_f$ is

$$\delta S_f = \frac{2}{c} \int T_{\alpha\beta} \delta g^{\alpha\beta} \sqrt{-g} d\Omega = \frac{2}{c f(n)} \int \Big( F_{\alpha\gamma} F^{\gamma}_{\beta} + \frac{1}{4} g_{\alpha\beta} F_{\gamma\rho} F^{\gamma\rho} \Big) \delta g^{\alpha\beta} \sqrt{-g} d\Omega. \tag{55}$$

Therefore, for the variation of the sum of these two actions, we can write

$$\delta(S_g + S_f) = \frac{c}{f(n)\mathcal{K}} \int \Big( -R_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta} R + \frac{2 f(n)\mathcal{K}}{c^2} T_{\alpha\beta} \Big) \delta g^{\alpha\beta} \sqrt{-g} d\Omega. \tag{56}$$

Since variations of the gravitational field are arbitrary, then from this equation, one finds

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{2\mathcal{K}}{c^2} \Big( F_{\alpha\gamma} F^{\gamma}_{\beta} + \frac{1}{4} g_{\alpha\beta} F_{\gamma\rho} F^{\gamma\rho} \Big) = \frac{2\mathcal{K}}{c^2} \tilde{T}_{\alpha\beta}, \tag{57}$$

where $\tilde{T}_{\alpha\beta} = F_{\alpha\gamma} F^{\gamma}_{\beta} + \frac{1}{4} g_{\alpha\beta} F_{\gamma\rho} F^{\gamma\rho}$ is the reduced (or universal) energy–momentum tensor of the electromagnetic field, which does not include the hyper-angular $f(n)$ factor. The last equation, Equation (57), is the well known Einstein equation for the gravitational and electromagnetic field. This equation is a true tensor equation, since both parts of this equation do not include the geometrical (or hyper-angular) factor $f(n)$. In other words, by looking at this equation, one cannot say what the actual dimension of our working space is. For this reason, Flanders [11] and others have criticized classical tensor analysis: "In classical tensor analysis, one never knows what the range of applicability is simply because one is never told what the space is". However, for the purposes of this study, this fact is an obvious advantage. Any of the true tensor equations that appear in fundamental physics cannot include factors that explicitly depend upon the dimension $n$ (or $n + 1$) of the working Riemann space. Moreover, this is a simple criterion that can be used to separate the true (also universal, or absolute) tensor equations from similar tensor-like equations that can be correct only for some selected Riemannian spaces. As follows from the arguments presented above, both Einstein equations of the metric gravity for the free gravitational field, when $\tilde{T}_{\alpha\beta} = 0$ in Equation (57), and Einstein equations of metric gravity in the presence of electromagnetic field, Equation (57), are the true tensor equations.

*6.2. Radiation from a Rapidly Moving Electric Charge*

As is well known (see, e.g., [5,19]), any electric charge that accelerates in the electromagnetic field always emits EM radiation. Nowadays, this statement is repeated so often that a large number of students and researchers sincerely believe that EM radiation can only be emitted in the presence of an electromagnetic field. In general, this is not an absolute truth, and the emission of EM radiation is also possible in the presence of a strong (or rapidly varying) gravitational field. Below, we want to prove this statement and, for simplicity, here we restrict our analysis to the three-dimensional space only. However, all our formulas are written in the explicitly covariant form. This means that all these formulas can be generalized to describe the actual situation in multi-dimensional spaces as well. In General Relativity, the formula for the radiated four-momentum $dP^{\kappa}$ is written in the form (see, e.g., [5]):

$$dP^{\kappa} = -\frac{2e^2}{3c} g_{\alpha\mu} \Big(\frac{d^2x^{\alpha}}{ds^2}\Big)\Big(\frac{d^2x^{\mu}}{ds^2}\Big) dx^{\kappa} = -\frac{2e^2}{3c} g_{\alpha\mu} \Big(\frac{du^{\alpha}}{ds}\Big)\Big(\frac{du^{\mu}}{ds}\Big) u^{\kappa} ds, \quad (58)$$

where $u^{\beta} = \frac{dx^{\beta}}{ds}$ is the corresponding "velocity". Now, by taking the expression for the acceleration from Equation (10), one finds

$$\begin{aligned} dP^{\kappa} &= -\frac{2e^2}{3c} g_{\alpha\mu} \Big(\Gamma^{\alpha}_{\beta\gamma} u^{\beta} u^{\gamma} - \frac{e}{mc^2} F^{\alpha}_{\beta} u^{\beta}\Big)\Big(\Gamma^{\mu}_{\lambda\sigma} u^{\lambda} u^{\sigma} - \frac{e}{mc^2} F^{\mu}_{\sigma} u^{\sigma}\Big) u^{\kappa} ds = -\frac{2e^2}{3c} \times \\ &\Big(g_{\alpha\mu} \Gamma^{\alpha}_{\beta\gamma} \Gamma^{\mu}_{\lambda\sigma} u^{\beta} u^{\gamma} u^{\lambda} u^{\sigma} u^{\kappa} - \frac{2e}{mc^2} g_{\alpha\mu} \Gamma^{\alpha}_{\beta\gamma} F^{\mu}_{\sigma} u^{\beta} u^{\gamma} u^{\sigma} u^{\kappa} + \frac{e^2}{m^2c^4} g_{\alpha\mu} F^{\alpha}_{\beta} F^{\mu}_{\sigma} u^{\beta} u^{\sigma} u^{\kappa}\Big) \quad (59) \\ &= T^{\kappa}_{2} + T^{\kappa}_{2} + T^{\kappa}_{3} = -\frac{2e^2}{3c} \Gamma^{\alpha}_{\beta\gamma} \Gamma_{\alpha,\lambda\sigma} u^{\beta} u^{\gamma} u^{\lambda} u^{\sigma} u^{\kappa} + \frac{4e^3}{3mc^3} \Gamma^{\alpha}_{\beta\gamma} F_{\alpha\sigma} u^{\beta} u^{\gamma} u^{\sigma} u^{\kappa} \\ &- \frac{2e^4}{3m^2c^5} F^{\alpha}_{\beta} F_{\alpha\sigma} u^{\beta} u^{\sigma} u^{\kappa}, \end{aligned}$$

where the last term (vector) $T^{\kappa}_{3} = -\frac{2e^4}{3m^2c^5} F^{\alpha}_{\beta} F_{\alpha\sigma} u^{\beta} u^{\sigma} u^{\kappa}$. This term describes the emission of EM radiation by a single electrical charge that is rapidly moving in some electromagnetic field. This was extensively discussed in numerous books on classical electrodynamics (see, e.g., [5,19]), and we do not want to repeat these discussions below. The first term in Equation (59) $T^{\kappa}_{1} = -\frac{2e^2}{3c} \Gamma^{\alpha}_{\beta\gamma} \Gamma_{\alpha,\lambda\sigma} u^{\beta} u^{\gamma} u^{\lambda} u^{\sigma} u^{\kappa}$ is also a vector. This vector represents the emission of EM radiation by a point electric charge that rapidly moves in the gravitational field. The second term (vector) in Equation (59) describes the interference between gravitational and electromagnetic emissions of the EM field. The explicit formula for this term is $T^{\kappa}_{2} = \frac{4e^3}{3mc^3} \Gamma^{\alpha}_{\beta\gamma} F_{\alpha\sigma} u^{\beta} u^{\gamma} u^{\sigma} u^{\kappa}$.

There are a number of interesting observations that directly follow from the three formulas for the $T^{\kappa}_{1}$, $T^{\kappa}_{2}$, and $T^{\kappa}_{3}$ terms in Equation (59). First, let us note that the $T^{\kappa}_{1}$ term does not contain any particle mass. This means that one fast electron and/or one fast proton, which move with the equal velocities in a pure gravitational field, will always emit an equal amount of radiation. This the main distinguishing feature of the gravitation emission of EM radiation. Second, this term is a fifth-order power function of the velocities. Therefore, it is clear that overall contribution of this term will rapidly increase for relativistic particles that move with velocities close to the speed of light in a vacuum $c$. It is also clear that usually in Equation (59), the third term $T^{\kappa}_{3}$ is substantially larger than two other terms. In other words, the gravitational emission of EM radiation is hard to observe at "normal" gravitational conditions. However, in strong gravitational fields, where the absolute values of Cristoffel symbols are very large (or the $|\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}|$ derivatives are very large), the situation can be different. The second condition is simple: the rapidly moving particle must be truly relativistic; i.e., it must move with a velocity which is close to the speed of light $v \geq 0.9\ c$ with respect to the system where the rapidly changing gravitational field originated. If these two conditions are obeyed, then one can see a relatively intense gravitational EM radiation which is emitted by a single relativistic particle which has non-zero electric charge $e$.

## 7. Conclusions

We have generalized the three-dimensional Maxwell theory of radiation to multi-dimensional flat and curved spaces. Some equations derived in three-dimensional Maxwell electrodynamics do not change their form in multi-dimensional space. In other equations, we have to make a number of changes. In fact, all properties of the electromagnetic field are described by the $(n+1)$-dimensional vector potential $\bar{A} = (\phi, \mathbf{A})$, while the interaction between any particle and electromagnetic field is described by one experimental parameter, which is the electric charge $e$ of this particle. The governing Maxwell equations for the multi-dimensional electromagnetic field have been derived and written in the covariant (or tensor) form. These equations include the geometrical (or hyper-angular) factor $f(n) = \frac{n\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)}$, which explicitly depend upon the dimension of space $n$.

The Hamiltonian formulation of the Maxwell radiation field in multi-dimensional spaces is developed and investigated. We have found that the total number of first-class constraints in this Hamiltonian formulation equals two (one primary and one secondary constraints). This number exactly coincides with the number of first-class constraints in the analogous Hamiltonian formulation developed earlier by Dirac [14] for the pure radiation field in three-dimensional space. In other words, the total number of first-class constraints in any Hamiltonian formulation developed for the free radiation field does not depend upon the dimension of space $n$. To understand how lucky we are with the Hamiltonian formulations of electrodynamics, let us note that in the $(n+1)$-dimensional metric gravity, we always have $(n+1)$ primary and $(n+1)$ secondary first-class constraints. In addition to this, in many sets of canonical variables, the explicit form of all arising secondary constraints are very cumbersome (see, e.g., [26–29]), and this substantially complicates all operations with these values. By using these primary and secondary first-class constraints, we have investigated the gauge conditions in multi-dimensional electrodynamics.

In addition, in the last section, the Maxwell equations in multi-dimensional non-flat spaces are written in the manifestly covariant form. It is shown that any gravitation field changes the actual properties, time-evolution and space-time propagation of electromagnetic fields. For gravitation fields with large and very large connectivity coefficients $\Gamma^{\alpha}_{\beta\gamma}$, the "pure" radiation field cannot be described by the Maxwell equations only. Additional equations for the antisymmetric tensor of the electromagnetic field $F_{\alpha\beta}$ (and $F^{\beta}_{\alpha}$) have been derived in this study (see Equations (50) and (51)). An analogous equation for the reduced energy-momentum tensor of electromagnetic field is now written in the true tensor form (see Equation (57)), which does not contain any $n$-dependent factor.

In conclusion, we wish to note that the investigation of multi-dimensional Maxwell equations is not a purely academic problem. In fact, there are a number of advantages that one can gain by performing such an investigation. First, it helps to clarify additional and interesting features of Maxwell's equations in the usual three-dimensional space (or in four-dimensional space-time). By working only with the three-dimensional Maxwell equations in our everyday life, we simply do not pay attention to some fundamental and amazing facts. Second, if we have a complete and correct formulation for Maxwell's electrodynamics in multi-dimensional spaces, then it possible to develop the so-called unified theories of various fields, which include the electromagnetic field. In particular, the correct unified theory of the gravitational and electromagnetic fields in multi-dimensional spaces is of great interest in modern theoretical physics. Third, in experiments in high-energy physics, it has recently been noted that at very high collision energies, many results can be represented to very good numerical accuracy and with higher symmetry if we introduce multi-dimensional spaces at the intermediate stages of calculations. This fact is not completely unexpected, but we need to understand the internal nature of such a phenomenon. If multi-dimensional spaces do play a significant role during such processes, then this can change a great deal in modern physics and natural philosophy. Note that some of the problems mentioned in this study have been considered earlier (see, e.g., [30–33]).

**Funding:** No funding has been used for this research.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The author declares no conflict of interest.

## Appendix A. Scalar Electrodynamics

In this study, our analysis of electrodynamics in multi-dimensional spaces was restricted to spaces which have geometrical dimension $n \geq 3$. For the sake of completeness, we now want to consider the one and two-dimensional spaces. To investigate these small-dimensional cases, we shall apply one effective method which is based on the so-called scalar electrodynamics. This "pre-Maxwell" method was described and briefly discussed in [4]. Scalar electrodynamics can be introduced in three-dimensional space, where one can compare the arising equations with the usual Maxwell equations. The foundation of scalar electrodynamics is the well-known theorem from vector calculus (see, e.g., [6]) which states that an arbitrary vector **B** in three-dimensional space can be represented in the following $two - gradient$ form:

$$\mathbf{B} = \Psi_1 \nabla \Psi_2 + \nabla \Psi_3, \tag{A1}$$

where $\Psi_1, \Psi_2$, and $\Psi_3$ are three arbitrary analytical functions of three spatial coordinates and one temporal coordinate. In general, each of these functions can be real or complex. This formula can directly be applied to the vector potential of the electromagnetic field **A**. The four-dimensional vector potential $(\varphi, \mathbf{A})$ and intensities of electric **E** and magnetic **H** field are also represented in terms of the four $\Psi_1, \Psi_2, \Psi_3$, and $\varphi$ scalar functions. For two and one-dimensional (geometrical) spaces, the total numbers of such scalar functions equal three and two, respectively.

To derive the explicit expressions and obtain the governing equations of electrodynamics, one needs to use the two following formulas which play a central role in scalar electrodynamics:

$$curl\mathbf{A} = \nabla \Psi_1 \times \nabla \Psi_2 \quad \text{and} \quad div\mathbf{A} = \Psi_1 \Delta \Psi_2 + \nabla \Psi_1 \cdot \nabla \Psi_2 + \Delta \Psi_3 \tag{A2}$$

As follows from Equation (A2), in scalar electrodynamics, there are a number of advantages to choose some of the $\Psi_1, \Psi_2$ and $\Psi_3$ functions (where it is possible) as harmonic functions for which $\Delta \Psi_k = 0$, where $k$ = 1, 2, 3. Such a choice of functions reduces the total number of terms in Maxwell equations and gauge conditions. In turn, this simplifies the analysis and solutions of many problems in scalar electrodynamics. In fact, in three-dimensional spaces, the scalar electrodynamics cannot compete with the traditional vector approach. The main reason is obvious, since the regular Maxwell equations are linear for all components of the electromagnetic field. However, some selected three-dimensional problems can be solved (completely and accurately) if we apply the method of scalar electrodynamics.

The equation for two-dimensional spaces, Equation (A1), takes the form $\mathbf{A} = \Psi_1 \nabla \Psi_2$, since in this case we can assume that $\Psi_3 = 0$. The equality $\mathbf{A} \cdot curl\mathbf{A} = 0$ is a necessary and sufficient condition to represent the vector **A** in such a form [6] (it does obey in this case). This leads to the following equations:

$$\mathbf{H} = curl\mathbf{A} = \nabla \Psi_1 \times \nabla \Psi_2 \quad \text{and} \quad div\mathbf{A} = \Psi_1 \Delta \Psi_2 + \nabla \Psi_1 \cdot \nabla \Psi_2. \tag{A3}$$

We also need the explicit expression for the $curl\mathbf{H}$

$$curl\mathbf{H} = \nabla\Psi_1\Delta\Psi_2 - \nabla\Psi_2\Delta\Psi_1 + (\nabla\Psi_1 \cdot \nabla)\Psi_2 - (\nabla\Psi_2 \cdot \nabla)\Psi_1 = \nabla\Psi_1\Delta\Psi_2 - \nabla\Psi_2\Delta\Psi_1$$

One should also note that if $\Psi_2$ is chosen as a harmonic function—i.e., $\Delta\Psi_2 = 0$, and $\nabla\Psi_1 \perp \nabla\Psi_2$—then the gauge condition is obeyed automatically, and solutions of a large number of problems known in two-dimensional electrodynamics are simplified significantly. In general, it can be shown that the both two-dimensional electrodynamics and two-dimensional electrostatics include a number of operations with the harmonic functions (see, e.g., [34–36]. In turn, this leads to numerous successful applications of conformal mapping methods to describe the two-dimensional electromagnetic waves and determine solutions of numerous problems in two-dimensional electrostatics.

In the equation for the one-dimensional case, Equation (A1), one finds $\mathbf{A} = \nabla\Psi_2 = \nabla\Psi$. Therefore, the *curl* of the vector potential equals zero identically. This means that there is no classical magnetic field in one-dimensional space. Moreover, any time-variations of the electric field cannot generate any magnetic field; i.e., we have no Faraday's law in one-dimensional (geometrical) space. In other words, the one-dimensional electrodynamics does not exist. On the other hand, many one-dimensional electrostatic problems that include the potential and intensity of the electric field only can still be formulated and solved correctly.

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