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The Orbital and Epicyclic Frequencies in Axially Symmetric and Stationary Spacetime

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Abstract: Motivated by observational evidence of the electromagnetic signal from the X-ray binary system known as quasi-periodic oscillations in the light curves of astrophysical black holes or neutron stars, we examined the general relativity and alternative theory of gravity in the strong gravity regime. The orbital and epicyclic motion of test particles in general axially symmetric spacetime was investigated. We provide a general description to derive the exact analytical expressions for the fundamental frequencies, namely, Keplerian epicyclic (radial and vertical) frequencies of test particles in an arbitrary axisymmetric and stationary spacetime. The detailed derivation of the expressions for the orbital and epicyclic frequencies of test particles orbiting around the Kerr–Newman-NUT black hole is also shown.

Keywords: fundamental frequencies; orbital and epicyclic motions; axially symmetric spacetime

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1. Introduction

The observational evidence of the electromagnetic signals from an accretion disk surrounding a gravitational compact object, in particular, so-called quasi-periodic oscillations (QPOs) in the X-ray light curves of the astrophysical black holes [1,2] and neutron stars [3,4], is an important tool to test spacetime in the strong-field regime. One of the interesting and important topics in astrophysics is the oscillation of accretion disks around black holes, in particular, after the discovery of the QPOs by the Rossi X-ray Timing Explorer (RXTE) in neutron star X-ray binaries (see, e.g., [5]). Such observations allow estimation of the radius of the innermost stable circular orbit, orbital, and epicyclic frequencies of particles orbiting the compact objects. It has been discussed that the fundamental frequencies of the corresponding kilohertz QPOs are in rational ratios of small integers, such as 3:2 [6–9].

It is widely believed that the QPO occurs due to particle motion or the collective motion of matter in the close vicinity of the black hole; however, its physical origin has long been debated. From the observational point of view, QPOs in the light curve of the astrophysical objects can be observed in the spectrum of the Fourier transform of the power intensity:

$$\tilde{L}(\nu) = \frac{1}{2\pi}\int dte^{-2\pi\nu t}L(t)\,, \tag{1}$$

where $L(t)$ is the time-dependent luminosity of the X-ray source, and ν is the frequency of the electromagnetic signal. Several QPO models suggest that it arises either from some instability in the accretion disk, in particular, oscillations of the mass accretion rate or pressure [10–12], or a geometric oscillation that is related to relativistic precession due to general relativity or theories of gravity [13,14]. The geometric meaning of the QPO in the vicinity of the compact objects was explicitly studied in Ref. [15]. The practical formulae for the epicyclic frequencies of test particles in an arbitrary spacetime were derived in [16,17]. Based on the orbital resonance model for twin peaks kHz, QPO in the quasars and micro-quasars was discussed in [18,19]. The astrophysical application of the relativistic Keplerian orbital frequency and epicyclic frequencies near the extreme and naked Kerr black holes was studied in [20,21]. The quasi-harmonically oscillatory motion of the charged particle around magnetized (non)-rotating black holes and it application on QPO oscillations was investigated in [22–24].

The fundamental frequencies, namely, Keplerian and epicyclic frequencies, of test particles orbiting around the gravitational object of mass M in the framework of the Newtonian theory are found to be $\Omega = \Omega_r = \Omega_\theta = \sqrt{M/r^3}$, while in the Schwarzschild spacetime they can be reduced to $\Omega_r = \Omega\sqrt{1-6M/r}$ and $\Omega_\theta = \Omega = \sqrt{M/r^3}$, while in the Kerr spacetime, the above expressions read [25]

$$\Omega_r = \Omega\sqrt{1-\frac{6M}{r}+\frac{8a}{r}\sqrt{\frac{M}{r}}-\frac{3a^2}{r^2}}\,, \tag{2}$$

$$\Omega_\theta = \Omega\sqrt{1-\frac{4a}{r}\sqrt{\frac{M}{r}}+\frac{3a^2}{r^2}}\,, \tag{3}$$

$$\Omega = \frac{1}{r\sqrt{\frac{r}{M}}+a}\,, \tag{4}$$

where a is the spin parameter of the central object, i.e., $|a| \leq M$.

In this paper, we are interested in investigating the fundamental frequencies, namely, the Keplerian and epicyclic frequencies, of a test particle orbiting around Kerr-like black holes. The paper is organized as follows. In Section 2, we show the detailed derivation of the equations necessary for the orbital and oscillatory motion of the test particle in arbitrary spacetime. In Section 3, we present the exact analytical expressions for the fundamental frequencies in the Kerr-like spacetime. Finally, we summarize the obtained results in Section 4.

Throughout the paper, we use a space-like signature $(-,+,+,+)$, a system of units in which $G = c = 1$ and restore the Newtonian gravitational constant and speed of light when we need to compare the results with the observational data. Greek indices are taken to run from 0 to 3, and Latin indices from 1 to 3.

2. Formulations

The Lagrangian for a test particle is given by

$$L = \frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\,, \qquad \dot{x}^\alpha = \frac{dx^\alpha}{ds}\,, \tag{5}$$

where $\dot{x}^\alpha$ is the four-velocity of a particle normalized as $\dot{x}_\alpha\dot{x}^\alpha = -1$. From the Lagrangian (5), the geodesic equation is easily derived as

$$\ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0\,, \tag{6}$$

where $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols defined as $\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right)$, and ∂_α denotes a partial derivative with respect to x^α. There are two conserved quantities associated with the coordinates t and ϕ, the specific energy and specific angular momentum are found as

$$P_t = \frac{dL}{d\dot{t}} = g_{tt}\dot{t} + g_{t\phi}\dot{\phi} = -\mathcal{E}\,, \qquad P_\phi = \frac{dL}{d\dot{\phi}} = g_{\phi\phi}\dot{\phi} + g_{t\phi}\dot{t} = \mathcal{L}\,, \tag{7}$$

which allows finding $\dot{t}$ and $\dot{\phi}$ as

$$\dot{t} = -\frac{\mathcal{L}g_{t\phi} + \mathcal{E}g_{\phi\phi}}{g_{tt}g_{\phi\phi} - g_{t\phi}^2}\,, \qquad \dot{\phi} = \frac{\mathcal{L}g_{tt} + \mathcal{E}g_{t\phi}}{g_{tt}g_{\phi\phi} - g_{t\phi}^2}\,. \tag{8}$$

Now, using the normalization of the four-velocity, one can obtain the following equation:

$$g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + V(r,\theta) = 0\,, \tag{9}$$

where $V(r,\theta)$ is defined as

$$\begin{aligned} V(r,\theta) &= 1 + \frac{\mathcal{E}^2 g_{\phi\phi} + 2\mathcal{E}\mathcal{L}g_{t\phi} + \mathcal{L}^2 g_{tt}}{g_{tt}g_{\phi\phi} - g_{t\phi}^2} \\ &= 1 + \mathcal{E}^2 g^{tt} - 2\mathcal{E}\mathcal{L}g^{t\phi} + \mathcal{L}^2 g^{\phi\phi}\,. \end{aligned} \tag{10}$$

The constants of motion, $\mathcal{E}$ and $\mathcal{L}$, are limited by the following condition, $V(r_0,\theta_0) = 0$, where r_0 and θ_0 are, respectively, the stationary point of the function at which the conditions $\partial_r V(r,\theta) = 0$ and $\partial_\theta V(r,\theta) = 0$ can be found. Note that at the stationary point, test particles orbit around the black hole in a circular trajectory with four-velocity of $\dot{x}^\alpha = (\dot{t}, 0, 0, \dot{\phi})$. In this case, using the normalization of the four-velocity once again, one can obtain the following expression: $\dot{t} = 1/\sqrt{-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi}}$ and $\dot{\phi} = \Omega\dot{t}$. Consequently, the specific energy and specific angular momentum in (7) can be rewritten as

$$\mathcal{E} = -(g_{tt} + \Omega g_{t\phi})\dot{t} = -\frac{g_{tt} + \Omega g_{t\phi}}{\sqrt{-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi}}}\,, \tag{11}$$

$$\mathcal{L} = (g_{t\phi} + \Omega g_{\phi\phi})\dot{t} = \frac{g_{t\phi} + \Omega g_{\phi\phi}}{\sqrt{-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi}}}\,, \tag{12}$$

where $\Omega = d\phi/dt$ is an angular velocity measured by a distant observer, the so-called Keplerian frequency of test particle.

2.1. Keplerian Frequency

To derive the explicit expression for the Keplerian frequency of a test particle, one uses the geodesic Equation (6). Considering circular motion with four-velocity $\dot{x}^\alpha = \dot{t}(1,0,0,\Omega)$, the radial equation (i.e., $\alpha = r$) of (6) takes the form: $\partial_r g_{tt} + 2\Omega\partial_r g_{t\phi} + \Omega^2\partial_r g_{\phi\phi} = 0$, and the solution of the above equation with respect to the Keplerian frequency, Ω, can be expressed as

$$\Omega = -\frac{\partial_r g_{t\phi}}{\partial g_{\phi\phi}} \pm \sqrt{\left(\frac{\partial_r g_{t\phi}}{\partial_r g_{\phi\phi}}\right)^2 - \frac{\partial_r g_{tt}}{\partial_r g_{\phi\phi}}}\,, \tag{13}$$

where "$+$" and "$-$" are responsible for the retrograde and regrade rotations of the test particles. It is more common to express the Keplerian frequency in arbitrary axially symmetric and stationary spacetime. A very similar expression can be found in [26]. The sign in front of the square root indicates the retrograde and prograde motion.

One has to emphasize that there is another angular velocity that arises due to the angular equation (i.e., $\alpha = \theta$) of (6), which can be expressed as $\partial_\theta g_{tt} + 2\tilde{\Omega}\partial_\theta g_{t\phi} + \tilde{\Omega}^2\partial_\theta g_{\phi\phi} = 0$.

The solution of the equation is responsible for the frequency due to rotation or Gyroscope frequency. The explicit expression for the Gyroscope frequency reads

$$\tilde{\Omega} = -\frac{\partial_\theta g_{t\phi}}{\partial_\theta g_{\phi\phi}} \pm \sqrt{\left(\frac{\partial_\theta g_{t\phi}}{\partial_\theta g_{\phi\phi}}\right)^2 - \frac{\partial_\theta g_{tt}}{\partial_\theta g_{\phi\phi}}} \, . \tag{14}$$

2.2. Epicyclic Frequencies: 2D Oscillator Problem

Now, we focus on the epicyclic motion of the test particles along the circular orbit. In this case, the particles can oscillate with radial and vertical frequencies (Ω_r, Ω_θ) around the stationary points (r_0, θ_0), so-called epicyclic frequency. In the present paper, we provide the detailed derivation of the expressions for the epicyclic frequencies of the test particles orbiting around the rotating black hole. The epicyclic frequencies can be calculated by considering a small perturbation around the stable circular orbit of particles along the radial $r \to r_0 + \delta r$ and tangential $\theta \to \theta_0 + \delta\theta$ directions. Then, the function $V(r,\theta)$ in Equation (10) can be expanded as [27]

$$\begin{aligned} V(r,\theta) &= V(r_0,\theta_0) + \partial_r V(r,\theta)\Big|_{x_0} \delta r + \partial_\theta V(r,\theta)\Big|_{x_0} \delta\theta + \partial_r \partial_\theta V(r,\theta)\Big|_{x_0} \delta r \delta\theta \\ &\quad + \frac{1}{2}\partial_r^2 V(r,\theta)\Big|_{x_0} \delta r^2 + \frac{1}{2}\partial_\theta^2 V(r,\theta)\Big|_{x_0} \delta\theta^2 + \mathcal{O}\left(\delta r^3, \delta\theta^3\right) \\ &\simeq \frac{1}{2}\partial_r^2 V(r,\theta)\Big|_{x_0} \delta r^2 + \frac{1}{2}\partial_\theta^2 V(r,\theta)\Big|_{x_0} \delta\theta^2 \, , \end{aligned} \tag{15}$$

where $x_0 = (r_0, \theta_0)$. Here, we have used the following facts: (i) the first term of (15) vanishes due to the limiting condition of the constants of motion (i.e., $V(r_0,\theta_0) = 0$), (ii) on the other hand, the first order derivatives $\partial_r V(r,\theta) = \partial_\theta V(r,\theta) = 0$ can be removed from the condition of of the stability of $V(r,\theta)$, and (iii) finally, the terms proportional to the second order derivatives remain as shown in Equation (15). Before moving further, one has to replace the derivative with the affine parameter s to time t (i.e., $\frac{d}{ds} = \dot{t}\frac{d}{dt}$) in Equation (9), in order to obtain the observable frequencies in the radial and vertical directions. Taking into account all the facts mentioned above and after performing simple algebraic manipulations, the harmonic oscillator equations for the displacements δr and $\delta\theta$ can be obtained as

$$\frac{d^2}{dt^2}\delta r + \Omega_r^2 \delta r = 0 \, , \qquad \frac{d^2}{dt^2}\delta\theta + \Omega_\theta^2 \delta\theta = 0 \, , \tag{16}$$

where Ω_r and Ω_θ are, respectively, the radial and vertical angular frequencies of particle measured by a distant observer, defined as

$$\Omega_r^2 = \frac{1}{2 g_{rr} \dot{t}^2}\partial_r^2 V(r,\theta) \, , \qquad \Omega_\theta^2 = \frac{1}{2 g_{\theta\theta} \dot{t}^2}\partial_\theta^2 V(r,\theta) \, . \tag{17}$$

Finally, the explicit expressions for the epicyclic frequencies in the arbitrary axially symmetric spacetime of a test particle take the form:

$$\begin{aligned} \Omega_i^2 &= \frac{1}{2 g_{ii}}\left(\partial_i^2 g_{tt} + 2\Omega \partial_i^2 g_{t\phi} + \Omega^2 \partial_i^2 g_{\phi\phi}\right) + \frac{2 g^{t\phi}}{g_{ii}}(\partial_i g_{tt} + \Omega \partial_i g_{t\phi})(\partial_i g_{t\phi} + \Omega \partial_i g_{\phi\phi}) \\ &\quad + \frac{g^{tt}}{g_{ii}}(\partial_i g_{tt} + \Omega \partial_i g_{t\phi})^2 + \frac{g^{\phi\phi}}{g_{ii}}(\partial_i g_{t\phi} + \Omega \partial_i g_{\phi\phi})^2 \, , \qquad i = (r,\theta) \, , \end{aligned} \tag{18}$$

and alternatively, they can be rewritten as

$$\Omega_i^2 = \frac{1}{2g_{ii}}(g_{tt}+\Omega g_{t\phi})^2\partial_i^2 g^{tt} + \frac{1}{2g_{ii}}(g_{t\phi}+\Omega g_{\phi\phi})^2\partial_i^2 g^{\phi\phi} + \frac{1}{g_{ii}}(g_{tt}+\Omega g_{t\phi})(g_{t\phi}+\Omega g_{\phi\phi})\partial_i^2 g^{t\phi}\,. \tag{19}$$

For the static spherically symmetric spacetime (i.e., $g_{t\phi} = 0$), the epicyclic frequencies can be easily found as

$$\Omega^2 = -\frac{\partial_r g_{tt}}{\partial_r g_{\phi\phi}}\,, \tag{20}$$

$$\begin{aligned}\Omega_r^2 &= \frac{1}{2g_{rr}}\left(g_{tt}^2\partial_r^2 g^{tt} + \Omega^2 g_{\phi\phi}^2\partial_r^2 g^{\phi\phi}\right)\\ &= \frac{1}{g_{rr}}\left[\frac{1}{2}\left(\partial_r^2 g_{tt} + \Omega^2\partial_r^2 g_{\phi\phi}\right) + g^{tt}(\partial_r g_{tt})^2 + g^{\phi\phi}(\Omega\partial_r g_{\phi\phi})^2\right],\end{aligned} \tag{21}$$

$$\begin{aligned}\Omega_\theta^2 &= \frac{1}{2g_{\theta\theta}}\left(g_{tt}^2\partial_\theta^2 g^{tt} + \Omega^2 g_{\phi\phi}^2\partial_\theta^2 g^{\phi\phi}\right)\\ &= \frac{1}{g_{\theta\theta}}\left[\frac{1}{2}\left(\partial_\theta^2 g_{tt} + \Omega^2\partial_\theta^2 g_{\phi\phi}\right) + g^{tt}(\partial_\theta g_{tt})^2 + g^{\phi\phi}(\Omega\partial_\theta g_{\phi\phi})^2\right].\end{aligned} \tag{22}$$

3. The Fundamental Frequencies in Kerr-like Spacetime

Now, we apply the obtained expressions (18) or (19) for given spacetimes. In Boyer–Lindquist coordinates $x^\alpha = (t, r, \theta, \phi)$, the Kerr-like spacetime (Appendix A.1) is given as

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\phi - adt\right]^2, \tag{23}$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2m(r)r + a^2$, a is a spin of the black hole, and $m(r)$ is the mass function included the black hole parameters, and for the Kerr black hole, $m(r)$ should be equal to $m(r) = M$. Without losing generality, our detailed analyses showed that the stationary point of the function $V(r,\theta)$ was located at the equatorial plane, $\theta_0 = \pi/2$. Using the explicit expressions (18) or (19) and (13), the epicyclic and orbital frequencies of the test particles in the Kerr-like spacetime can be determined as

$$\Omega_r = \Omega\sqrt{1 - \frac{6m}{r} - \frac{3a^2}{r^2} + 4m' - \frac{(r^2 - 2mr + a^2)m''}{m - rm'} + \frac{8a\sqrt{m - rm'}}{r^{3/2}}}\,, \tag{24}$$

$$\Omega_\theta = \Omega\sqrt{1 - \frac{4am}{r^{3/2}\sqrt{m - rm'}} + \frac{a^2}{r^2}\left(1 + \frac{2m}{m - rm'}\right)}\,, \tag{25}$$

$$\Omega = \frac{1}{r\sqrt{\frac{r}{m - rm'}} + a}\,. \tag{26}$$

Using these expressions, one can determine the fundamental frequencies of test particles orbiting around the rotating black holes characterized by the mass function m. In Table 1, the mass functions for some of the well-known rotating black holes are listed; however, they might be extended to any other Kerr-like solutions (Figure 1).

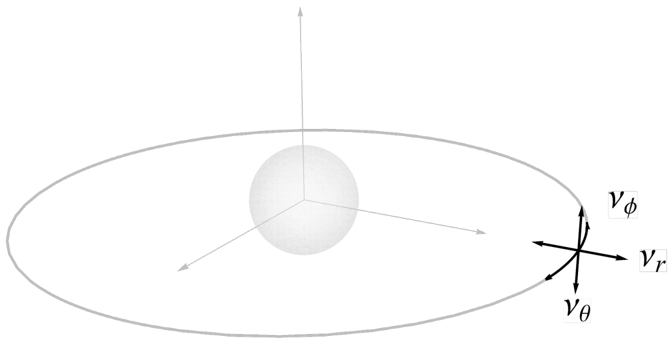

Figure 1. The schematic picture of the orbital and epicyclic motion of a particle around a black hole. The quantities ν_r, ν_θ, and ν_ϕ represent the radial, vertical, and orbital frequencies of the test particle, respectively. Here, the definitions of ν_i are given as $\nu_i = \Omega_i/2\pi$ with $i = (r, \theta, \phi)$.

Table 1. The mass function for some of the rotating black holes.

Spacetime	$m(r)$
Kerr	M
Kerr–Newman	$M - Q^2/2r$
Kerr-brane [28,29]	$M + Q_*/2r$
Kerr-Quintessential [30]	$M - cr^{-3\omega}$
Kerr-MOG [31,32]	$(1+\alpha)M + \alpha(1+\alpha)M^2/2r$

The Novel Feature of Kerr–Taub-NUT Spacetime

The Kerr–Taub-NUT spacetime is one of the vacuum solutions of the Einstein field equations for the rotating black hole parameterized by the gravitomagnetic monopole moment n in addition to the mass and spin of the black hole. It is described by the metric

$$ds^2 = -\frac{\Delta_{\rm KNT}}{\Sigma_{\rm KNT}}(dt - \chi d\phi)^2 + \frac{\Sigma_{\rm KNT}}{\Delta_{\rm KNT}}dr^2 + \Sigma_{\rm KNT}d\theta^2 + \frac{\sin^2\theta}{\Sigma_{\rm KNT}}[(\Sigma_{\rm KNT} + a\chi)d\phi - adt]^2\,, \quad (27)$$

with $\Delta_{\rm KNT} = r^2 - 2Mr + a^2 - n^2$, $\Sigma_{\rm KNT} = r^2 + (n + a\cos\theta)^2$, and $\chi = a\sin^2\theta - 2n\cos\theta$. It is worth noting that there are three main differences between the Kerr–Taub-NUT and Kerr spacetime. (i) The Kerr–Taub-NUT spacetime is asymptotically not flat unlike the Kerr spacetime; in particular, the non-diagonal component of the metric tensor tends to infinity at a large distance; (ii) it is a regular vacuum solution for the rotating black hole, which can be checked by determining the Kretschmann scalar at the origin for a non zero gravitomagnetic monopole moment, i.e., $K = 48(n^2 - M^2)(n + a\cos\theta)^{-6}$; and (iii) another interesting feature of the Kerr-NUT-Taub spacetime is that a test particle cannot orbit at the equatorial plane unlike that in the Kerr spacetime. Therefore, considering the oscillatory motion of test particles at the equatorial plane is meaningless. However, such cases have been considered by several authors, for instance, [33,34]. Indeed, one can consider particle motion at the equatorial plane in the Kerr–Taub-NUT space, and the angular frequency can be determined as [33]

$$\Omega = \frac{1}{\frac{r^2+n^2}{\sqrt{Mr+n^2(2-M/r)}} + a}\,. \quad (28)$$

However, one needs to keep in mind that the angular frequency $\tilde{\Omega}$ in Equation (14) takes a form:

$$\tilde{\Omega} = \frac{a}{a^2 + (n^2+r^2)\left(1 - \sqrt{\frac{\Delta}{\Delta - a^2}}\right)}\,, \quad (29)$$

which is always real. It turns out that the test particles orbiting around a Kerr–Taub-NUT black hole have an additional frequency along the Keplerian frequency, and the total frequency can be considered as $\Omega + \tilde{\Omega}$. Note that the frequency $\tilde{\Omega}$ is always a complex form in the Kerr-like spacetime if we calculate directly from Equation (14). This is another interesting feature of the Kerr–Taub-NUT spacetime, which differs from the Kerr one. Regarding the radial and vertical oscillation of a test particle, equation (9) reduces to

$$g_{rr}\dot{t}^2 \frac{d^2}{dt^2}\delta r + \frac{1}{2}\partial_r^2 V(r,\theta)\delta r + \partial_r\partial_\theta V(r,\theta)\delta\theta = -\partial_r V(r,\theta)\,, \tag{30}$$

$$g_{\theta\theta}\dot{t}^2 \frac{d^2}{dt^2}\delta\theta + \frac{1}{2}\partial_\theta^2 V(r,\theta)\delta\theta + \partial_r\partial_\theta V(r,\theta)\delta r = -\partial_\theta V(r,\theta)\,, \tag{31}$$

We already mentioned that particles cannot be located in the equatorial plane (i.e., $\theta_0 \neq \pi/2$) in the Kerr–Taub-NUT spacetime. However, to obtain the oscillatory motion of particles in the equatorial plane (i.e., $\theta_0 = \pi/2$), equations for the radial and vertical oscillations can be rewritten as

$$\frac{d^2}{dt^2}\delta r + \Omega_r^2\delta r + \gamma_r\delta\theta = F_r(r)\,, \tag{32}$$

$$\frac{d^2}{dt^2}\delta\theta + \Omega_\theta^2\delta\theta + \gamma_r\delta r = F_\theta(r)\,, \tag{33}$$

where

$$\begin{aligned}\Omega_r^2 = &\frac{(1-a\Omega)^2\left(a^2-4M^2+6Mr-n^2-3r^2\right)+4r^2\Omega^2\left(\Delta-a^2\right)-8ar\Omega(1-a\Omega)(r-M)}{\left(n^2+r^2\right)^2} \\ &+\frac{(1-a\Omega)\left[8r\left(\Delta-2a^2\right)(1-a\Omega)(r-M)-2\left(\Delta-a^2\right)\left(a\Omega\left(\Delta-8r^2\right)-\Delta\right)\right]}{2\left(n^2+r^2\right)^3} \\ &-\frac{4a^2r^2\left(a^2-\Delta\right)(a\Omega-1)^2}{\left(n^2+r^2\right)^4}-\frac{\Delta\Omega^2}{n^2+r^2}\,,\end{aligned} \tag{34}$$

$$\begin{aligned}\Omega_\theta^2 = &\frac{a^4(1-a\Omega)^2}{n^2+r^2}-\frac{\Delta(1-a\Omega)^2\left(a^2\left(5n^2+r^2\right)-4\Delta n^2\right)}{\left(n^2+r^2\right)^2} \\ &-a(1-a\Omega)\left(3a^2\Omega-a-2\Delta\Omega\right)+a\Omega(3a\Omega-2)\left(n^2+r^2\right)+\Omega^2\left(n^2+r^2\right)^2\,,\end{aligned} \tag{35}$$

$$\gamma_r = \frac{8\Delta nr\Omega\left(a^2-\Delta\right)(a\Omega-1)}{\left(n^2+r^2\right)^3}-\frac{4a\Delta n(a\Omega-1)^2\left(M\left(n^2-3r^2\right)-3n^2r+r^3\right)}{\left(n^2+r^2\right)^4} \tag{36}$$

$$\gamma_\theta = \frac{8nr\Omega\left(a^2-\Delta\right)(a\Omega-1)}{\left(n^2+r^2\right)^3}-\frac{4an(1-a\Omega)^2\left(M\left(n^2-3r^2\right)-3n^2r+r^3\right)}{\left(n^2+r^2\right)^4}\,, \tag{37}$$

$$F_r = \frac{2\Delta(a\Omega-1)^2\left(M\left(r^2-n^2\right)+2n^2r\right)}{\left(n^2+r^2\right)^3}-\frac{2\Delta r\Omega^2}{n^2+r^2}\,, \tag{38}$$

$$F_\theta = 2n(1-a\Omega)\left(\Delta-a^2\right)\left(a\frac{1-a\Omega}{n^2+r^2}-2\Omega\right)-2an\Omega^2\left(n^2+r^2\right)\,. \tag{39}$$

It is easy to check that, in the absence of the NUT parameter, one can obtain $F_r = F_\theta = \gamma_r = \gamma_\theta = 0$, while Ω_r and Ω_θ represent the epicyclic frequencies in the Kerr spacetime.

4. Conclusions and Future Prospects

In this work, we discussed the fundamental frequencies, namely, the orbital, radial, and vertical epicyclic frequencies of massive particles following circular orbits in the spacetime generated by generic stationary and axisymmetric black holes (as well as neutron stars) with arbitrary parameters. We derived the explicit expressions for the orbital, radial, and vertical frequencies of test particles around rotating and static black holes in terms of the arbitrary metric coefficients.

We showed that test particles orbit around Kerr-like black holes in the equatorial plane ($\theta_0 = \pi/2$), and the expressions for the fundamental frequencies can be expressed in terms of the mass function. This method allowed us to test the general relativity and alternative theory of gravity with the QPO of the disk surrounding the black holes and constraining the black hole's parameters. We also showed that this method was applicable for the complex spacetime metric with several parameters, for example, the Johannsen solution (A1), which describes the generalized Kerr spacetime.

We studied the fundamental frequencies of test particles in Kerr–Taub-NUT spacetime, and we showed that the test particles do not follow the oscillatory motion in the equatorial plane, unlike Kerr spacetime. As we mentioned before, there are three main differences between the Kerr–Taub-NUT spacetime and the Kerr one: (i) it is asymptotically not flat; (ii) it is a regular vacuum solution for the rotating black hole; and (iii) test particles cannot orbit at the equatorial plane, unlike in the Kerr spacetime. In addition, we showed that the test particles had an additional orbital frequency $\tilde{\Omega}$ that affected the Keplerian frequency.

It is well known that the QPO in neutron star X-ray binaries is practically well described and studied, in particular, after the detection of electromagnetic signals by the Rossi X-ray Timing Explorer (RXTE) telescope. Several papers have been published on the theoretical approach of the fundamental frequencies of particles in the Hartle–Thorne metric (A5) (Appendix A.2), which describes the spacetime around the rotating neutron star with quadruple moment (see, e.g., [35–37]). Indeed, the Hartle–Thorne spacetime contains complex terms, and it is difficult to derive the fundamental frequencies in this spacetime; therefore, researchers created the coordinate transformation to derive the approximated expressions for the fundamental frequencies. However, based on the results, in particular, using Equation (19), the exact expression for the epicyclic frequencies can be directly obtained.

The present paper mainly explored the practical expressions for the fundamental frequencies of particles orbiting in stationary and axisymmetric spacetime. In the future, we plan to constrain black holes and neutron star parameters with QPO in the X-ray binaries based on the results of this paper.

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Appendix A

Appendix A.1. The Johannsen Spacetime

Now, we focus on testing the Johannsen spacetime by determining the fundamental frequencies of the test particle. It is similar to the general form of the Kerr spacetime, as described in the following metric [38]:

$$\begin{aligned} ds^2 = &-\frac{\tilde{\Sigma}\Delta}{B^2}\Big[A_3(\theta)dt - aA_4(\theta)\sin^2\theta d\phi\Big]^2 + \frac{\tilde{\Sigma}}{\Delta A_5(r)}dr^2 + \frac{\tilde{\Sigma}}{A_6(\theta)}d\theta^2 \\ &+ \frac{\tilde{\Sigma}\sin^2\theta}{B^2}\Big[A_1(r)(r^2+a^2)d\phi - aA_2(r)dt\Big]^2 , \end{aligned} \tag{A1}$$

where $B = A_1(r)A_3(\theta)(r^2+a^2) - A_2(r)A_4(\theta)a^2\sin^2\theta$, and $\tilde{\Sigma} = \Sigma + f(r) + g(\theta)$. Notice that the Δ function is the same as in the Kerr spacetime. In general, it is impossible to find the stationary points of the function $V(r,\theta)$ in the background geometry (A1). However, for specific choice of the profile functions

$$A_1(r) = 1 + \sum_{n=3}^{\infty}\alpha_{1n}\left(\frac{M}{r}\right)^n , \qquad A_2(r) = 1 + \sum_{n=2}^{\infty}\alpha_{3n}\left(\frac{M}{r}\right)^n , \tag{A2}$$

$$A_5(r) = 1 + \sum_{n=2}^{\infty}\alpha_{5n}\left(\frac{M}{r}\right)^n , \qquad f(r) = r^2\sum_{n=3}^{\infty}\epsilon_n\left(\frac{M}{r}\right)^n , \tag{A3}$$

$$A_3(\theta) = A_4(\theta) = A_6(\theta) = 1 , \qquad g(\theta) = 0 , \tag{A4}$$

one can find that the stationary points of the function $V(r,\theta)$ are located at the equatorial plane $\theta_0 = \pi/2$. One has to emphasize that the Johannsen spacetime is characterized by a series of parameters, α_{1n}, α_{3n}, α_{5n}, and ϵ_n along the mass and spin of the black hole, and it is applicable to the phenomenological calculations in black hole astrophysics.

Appendix A.2. The Hartle–Thorne Metric

We also have checked that the test particles orbit around the equatorial plane in the Hartle–Thorne spacetime, which is given by the metric [39]:

$$\begin{aligned} ds^2 = &-\left(1-\frac{2M}{r}\right)\left[1+2k_1P_2(\cos\theta) - \frac{2J^2}{r^2}\left(1-\frac{2M}{r}\right)^{-1}(2\cos^2\theta - 1)\right]dt^2 \\ &+\left(1-\frac{2M}{r}\right)^{-1}\left[1-2\left(k_1-\frac{6J^2}{r^2}\right)P_2(\cos\theta) - \frac{2J^2}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\right]dr^2 \\ &+r^2[1-2k_2P_2(\cos\theta)](d\theta^2 + sin^2\theta d\phi^2) - \frac{4J}{r}\sin^2\theta dt d\phi , \end{aligned} \tag{A5}$$

where

$$k_1 = \frac{J^2}{Mr^3}\left(1+\frac{M}{r}\right) + \frac{5}{8}\left(\frac{Q}{M^3} - \frac{J^2}{M^4}\right)Q_2^2\left(\frac{r}{M}-1\right) , \tag{A6}$$

$$k_2 = k_1 + \frac{J^2}{r^4} + \frac{5}{4}\left(\frac{Q}{M^2r} - \frac{J^2}{M^3r}\right)\left(1-\frac{2M}{r}\right)^{-1/2}Q_2^1\left(\frac{r}{M}-1\right) , \tag{A7}$$

and $Q_2^1(x)$ and $Q_2^2(x)$ are the associated Legendre functions of the second kind. Here, M is the mass, J is the angular momentum, and Q is the quadruple moment of the neutron star. It shows that the the fundamental frequencies of the particle around a neutron star, described by the Hartle–Thorne spacetime, can be immediately determined using expressions (13) and (18).

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