universe

MDPI

Article

On the General Entangled State and Quantum Decoherence

Abasalt Rostami [1,2,*] and Javad T. Firouzjaee [1,3,4]

1 School of Physics, Institute for Research in Fundamental Sciences (IPM), Tehran P.O. Box 19395-5531, Iran
2 Department of Physics, Sharif University of Technology, Tehran P.O. Box 15875-4416, Iran
3 Department of Physics, K. N. Toosi University of Technology, Tehran P.O. Box 15875-4416, Iran
4 PDAT Laboratory, Department of Physics, K. N. Toosi University of Technology, Tehran P.O. Box 15875-4416, Iran
* Correspondence: aba-rostami@ipm.ir

Abstract: We study the primary entanglement effect on the decoherence of reduced-density matrices of scalar fields, which interact with other fields or independent mode functions. We study the (leading) tree-level evolution of the scalar bispectrum due to a coupling between two scalar fields. We show that the primary entanglement has a significant role in the decoherence of the given quantum state. We find that the existence of such an entanglement could couple dynamical equations coming from a Schrödinger equation. We show that if one wants to see no effect of the entanglement parameter in the decohering of the quantum system, then the ground state eigenvalues of the interaction terms in the Hamiltonian cannot be independent of each other Generally, including the primary entanglement destroys the independence of the interaction terms in the ground state. We show that the imaginary part of the entanglement parameter plays an important role in the decoherence process without posing any specific restriction to the interaction terms. Our results could be generalized to every scalar quantum field theory with a well-defined quantization of its fluctuations in a given curved space-time.

Keywords: quantum decoherence; Bunch-Davies vacuum; entanglement; Schrödinger field theory; inflation

check for updates

Citation: Rostami, A.; Firouzjaee, J.T. On the General Entangled State and Quantum Decoherence. *Universe* **2022**, *8*, 508. https://doi.org/10.3390/universe8100508

Academic Editor: Sigbjørn Hervik

Received: 22 August 2022
Accepted: 21 September 2022
Published: 26 September 2022

1. Introduction

Recent Cosmic Microwave Background (CMB) observations [1] of temperature fluctuations have conformity with the predictions made by inflation theory which states the universe was in an accelerated phase after the Planck era. According to the inflationary picture, not only do the primordial cosmological fluctuations (CMB temperature) have a quantum origin, but they are also created through a quantum state. These quantum fluctuations provide an elegant description of the advent of the large-scale structure in our universe, which explains how the density perturbations seeded structures in the cosmos.

The essential point is that no noted cosmological data would present the actual quantum state of primordial fluctuations because all known techniques of observation focus on a restricted set of properties of those fluctuations. Consequently, one has to consider not only the quantum aspects of the cosmological fluctuations but also the loss of quantum coherence prompted by the partial description of appropriate observation to study nontrivial quantum behavior.

To this end, main works on the quantum to the classical transition of inflation [2–5] have focused mainly on the squeezing of the quantum state for each mode. This implies that the inflaton effectively spans different values at widely separated points in space, which leads to inhomogeneity in the temperature after inflation. In other words, this means quantum expectation values of products of quantities in a highly squeezed state are identical to stochastic averages calculated from a stochastic distribution of classical quantity configurations.

However, this quantum to classical transition does not answer how we make a measurement. As is known, the inflaton field fluctuation is a quantum field represented by the Bunch–Davies vacuum state, which is entirely homogeneous and isotropic. The inflaton dynamics preserve the homogeneity and isotropy; thus, we cannot use the inflaton state to explain the observed inhomogeneous and anisotropic distribution of the primordial energy density in our universe. As a result, the homogeneous quantum state, which is a coherent superposition of all field configurations, collapses to a particular stochastic realization of classical inhomogeneities [6,7]. In this step, it needs to have a mechanism of quantum decoherence, which necessitates the presence of additional environment degrees of freedom that are coupled to quantum perturbations as a measuring device. Decoherence describes the transition from a pure state to the mixed one whenever the degrees of freedom of interest (quantum fluctuations) interact with an environment involving other degrees of freedom whose properties are not measured.

Decoherence is well-studied in the context of inflation [8,9]. Nelson [8] argued that the gravitational nonlinearities (from the coupling between long-wavelength fluctuations and an environmental sector with interaction terms) provide a minimal mechanism for generating classical stochastic perturbations from inflation via decoherence. The best-suited framework to discuss the decoherence of cosmological perturbations and the quantum to classical transition is Schrödinger field theory. This picture is the natural framework to study the entanglement between the fields [10] and can be used to study the entanglement effect on the curvature power spectrum [11] and bispectrum using the interaction picture [12]. There is no reason why one could not consider a more general initial state, such as an entangled one, especially with this hypothesis that inflation may be a low-energy effective theory of a fundamental theory, such as quantum gravity, or has multiple fields that, for example, arise in string theory [13–15].

The purpose of the present article is to study the primary entanglement effect on the decoherence of the reduced-density matrix of fields that interact with other fields or independent mode functions. In the usual inflationary scenarios, one can suppose inflaton as a system and tensor modes as an environment [16], or choose a massive isocurvature mode as an environment [17] (similar to all multi-field inflationary theory [13–15]); then, the interaction terms [8] are given by Maldacena's cubic terms. In fact, in all such theories, the action of the universe (including inflaton and other degrees of freedom in the inflation era) are effectively truncated up to the cubic terms. Although researchers in Ref. [11] have shown that inflation theories with an entanglement cloud can be regarded as a signal of an entanglement parameter in the power spectrum, we think it is necessary to show that such kinds of theories are consistent with the decoherence process in their dynamics. If such theories do not admit a decoherence process by themselves, they cannot describe the classicalization problem in the early universe. This work provides a simple way to study the decoherence process in such theories. This paper will show that the primary entanglement has a significant role in the decoherence of the quantum state (system). We discuss that the existence of entanglement could couple dynamical equations coming from a Schrödinger equation, and if someone wants to see no effect of the entanglement parameter in decoherence, then the eigenvalues of interaction terms at the ground state cannot be independent of each other. Generally, if we include the primary entanglement, the interaction terms cannot be independent.

The article is organized as follows: Section 2 reviews the decoherence in the quantum field theory. In Section 3, we present the entanglement of fields in the inflationary background and study the possible interactions in the third order. Section 4 is devoted to the study of the Schrödinger equation for the entangled state and how the decoherence happens in this picture. Finally, we conclude with a discussion in Section 5.

2. Decoherence in Quantum Physics

One of the most critical problems in quantum mechanics is the classicalization problem. Classicalization means a process that transforms a quantum system into a classical one.

The most straightforward (and, of course, the most difficult) process to unlock this problem is to replace the wave collapse assumption with a deterministic dynamical process that describes how the collapse happens. This idea entails a departure during measurement from the Born rule, for instance, collapse models [18]. One could try to find a non-fundamental solution for the case of statistical quantum systems. Although this method is not fundamental because it implicitly includes the Born rule, it can describe how we find a statistical quantum system in a classical statistical system. In other words, how quantum probabilities change to classical probabilities.

In recent years, what happens in the relationship between a system and its environment has emerged as a dramatic picture, which people like to call measurement. This has been widely due to the attention to the phenomenon of decoherence. In this section, we review the decoherence concept.

It is clear that the first requirement for the effect of the environment on the system under study is an evolution of the state vector in the Schrödinger picture, which creates a correlation between the system (like the inflaton at the early universe) and states of the environment (similar to other fields that affected the inflaton during inflation). Suppose that the system can be in various states labeled with an index s, while the environment can be in states labeled with an index e, such that the states of the total system in Hilbert space can be written in terms of a complete orthonormal basis of state vectors presented as Ψ_{se}. We assume that at $t = 0$, the environment is placed in an initial state denoted $e = 0$, with the system in a general superposition of its states (in a subspace of the total Hilbert space) so that the combined system would have an initial wave function, as in the following

$$\Psi(0) = \sum_s c_s \Psi_{s0}. \tag{1}$$

When we turn on an interaction between the system under study and its corresponding environment, the combined system evolves in a time t to $U\Psi(0)$, where U is the time evolution unitary operator $U = e^{-itH}$. To have ideal decoherence, we need to choose the Hamiltonian H to be in such a way that the basis states Ψ_{s0} should evolve into states $U\Psi_{s0} = \Psi_{se_s}$, with the index s unchanged, and with e_s labeling some definite state of the environment in a one-to-one correspondence with the state of the system under study, such that $e_s \neq e_{s'}$ if $s \neq s'$. For this, we just need

$$U_{s'e',s0} = \delta_{ss'}\delta_{e'e_s}. \tag{2}$$

It is always possible to choose the other elements of $U_{s'e',se}$ with $e \neq 0$ to make the whole transformation unitary. For instance, in the case of $e \neq 0$, we can take this transformation as in the following

$$U_{s'e',se} = \begin{cases} \delta_{ss'}\mathcal{U}^{(s)}_{ee'}, & \text{for} \quad e' \neq e_{s'} \\ 0, & \text{for} \quad e' = e_{s'} \end{cases} \tag{3}$$

where the matrix $\mathcal{U}^{(s)}$ has been constrained by the condition that, for all $e \neq 0$ and $e' \neq 0$,

$$\delta_{ee'} = \sum_{e'' \neq e_s} \mathcal{U}^{(s)*}_{e''e'}\mathcal{U}^{(s)}_{e''e}. \tag{4}$$

These conditions thus simply require that $\mathcal{U}^{(s)}$ are unitary matrices. Since they are not subject to any other constraints, one can establish any number of matrices that satisfy this condition.

After the system under study and the environment have interacted, the total system would be found in the following superposition

$$U\Psi(0) = \sum_s c_s \Psi_{se_s}, \tag{5}$$

which is an entangled state of the system and environment created by an interaction. We have no decoherence yet because the combined system is still in a pure state, and we just see a definite superposition of the basis. Based on the Born rule, the system must make a transition during the measurement to one or other of these states, with probabilities $|c_s|^2$. Here, by the classical state, we mean the favored states produced by measurement (the interaction between system and environment), to which the system under study goes. Zurek identified such states with the name "pointer states."

After this introduction, we are ready to ask why we see most systems around us as classical. The answer has to do with the phenomenon of decoherence. This happens because any specific environment will always be subjected to tiny noises, which could raise the environmental number of degrees of freedom. These perturbations could not by themselves change one classical state into another. We can investigate this issue in two equivalent looks. The decoherence converts Equation (5) as in the following

$$\sum_s c_s \Psi_{se_s} \longrightarrow \sum_s \exp(i\phi_s)\, c_s \Psi_{se_s}, \tag{6}$$

where the ϕ_s are randomly fluctuating phases. Consequently, when we take into account the expectation values, the interference between different terms in the above superposition would average to zero, and the expectation value of any observer operator A gives

$$\overline{\langle A \rangle} = \sum_s |c_s|^2 (\Psi_{se_s}, A\Psi_{se_s}), \tag{7}$$

with the bar over the expectation value indicating that it is averaged over the phases ϕ_s. Here, we see that the expectation value of A is just given by a classical distribution. One may note that this is not really a solution for the measurement problem because we have used the Born rule in (7).

One can also indicate this phenomenon in another way (equivalent to the former). To see it better, we go to the usual ket-bra notation. Suppose that $|E\rangle$ and $|S_i\rangle$ are states of the environment and system, respectively. Here, we have assumed that $|S_i\rangle$ states establish an orthonormal subspace. It is clear that the interaction defined in Equations (2) and (3) takes the combined system at t_0 to any later time as in the following

$$|E(t_0)\rangle |S_i(t_0)\rangle \longrightarrow |E_i(t)\rangle |S_i(t)\rangle. \tag{8}$$

Then, if we consider the effects of the environment during an ideal measurement, we will have

$$\langle E_i(t)|E_j(t)\rangle \approx \delta_{ij}. \tag{9}$$

Note that this would happen when there are many degrees of freedom for the environment. Now, an initially coherent superposition of the system goes to an entanglement state when time is past as (5)

$$|E(t_0)\rangle \left(\sum_i c_i |S_i(t_0)\rangle \right) \longrightarrow \sum_i c_i |E_i(t)\rangle |S_i(t)\rangle. \tag{10}$$

Therefore, to see how decoherence comes across, it is enough to find the reduced-density matrix of the system under study and notice that this matrix leads to a classical distribution when it is written down in basis $|S_i\rangle$. Using Equation (9), the components of the reduced-density matrix become

$$\rho_R(S_i, S_j) \approx |c_i|^2 \delta_{ij}. \tag{11}$$

Then, the effect of such interactions is to eliminate the off-diagonal components of a density matrix.

It would be useful to translate the above discussion to a scalar field theory in a Schrödinger picture. Against the Heisenberg picture for a field theory in which one works with a specific Fock space, in a Schrödinger picture, we use a wave functional to describe what is happening in a quantum system. To do that, suppose we have a scalar theory for $\phi(x)$. Then, in a Schrödinger picture, we have to find a basis for this theory. Thus, it is natural to choose the eigenstates of the operator $\hat{\phi}(x)$ as a suitable basis, which has been defined as follows

$$\hat{\phi}(x)|\phi(x)\rangle = \phi(x)|\phi(x)\rangle. \tag{12}$$

Now, an arbitrary state could be represented as a superposition of the field eigenstates,

$$|\Psi_\phi\rangle = \sum_{\phi(x)} \Psi[\phi(x)]\, |\phi(x)\rangle, \tag{13}$$

where $\Psi[\phi(x)]$ has the role of a wave functional. Note that the above summation is a functional integration, and we write it down formally. To establish a decohering system, it is enough to treat it similar to what we performed in Equation (10),

$$|\Psi_E\rangle|\phi(x)\rangle \longrightarrow \Big(|\Psi_E|_{\phi(x)}\rangle\Big)|\phi(x)\rangle. \tag{14}$$

Therefore, if one establishes a simple combined system at the initial state $|\Psi_E\rangle|\Psi_\phi\rangle$, one could easily show that the corresponding reduced-density matrix of $\phi(x)$ becomes

$$\rho_R[\phi(x), \phi'(x)] = \Psi_\phi[\phi(x)]\Psi^*_\phi[\phi'(x)] \sum_E \Big(\Psi_E[E]|_{\phi(x)}\Big)\Big(\Psi^*_E[E]|_{\phi'(x)}\Big). \tag{15}$$

Now, if the interaction leads the summation term in the above equality to zero, then we will have decoherence. To this end, we are tracking some interaction terms that satisfy this condition.

3. Schrödinger Equation for Entangled Fields

In this section, we expand the Schrödinger field theory during inflation for a combined system, including a scalar field $\phi(x)$ or any specific degree of freedom (as the main system) and another field such as $\chi(x)$ or the rest of the degrees of freedom (as the environment). What we are looking for is the wave function of the combined system. Once we find it, all information about the environment and the system under study will be obtained. Then, we will be able to see if the system can experience decoherence. Here, the field $\phi(x)$ has the role of fluctuating the inflaton, and suppose the field $\chi(x)$ is the quantum fluctuations of another field that could exist during inflation but has no role in the dynamics of inflation. Nevertheless, depending on the initial state, the entanglement between this field and inflaton could appear in the power spectrum and bispectrum of the inflationary universe [16]. We are interested in knowing what happens to the wave function when one starts from an entangled state of the system and environment. We would like to emphasize that the following is not just for two fields, namely inflation theory or different length modes, but could even be applied to the scalar and tensor modes interactions.

The total wave function would evolve according to the Schrödinger equation

$$i\frac{d}{dt}\Psi[E, S] = H[E, S; t]\Psi[E, S] \tag{16}$$

with the time-dependent Hamiltonian $H(t)$. We assume that the interaction between the system and environment could be treated perturbatively. To this end, one has to distinguish between the system and environment in the first step and then determine the interaction that affects the initial state of the total system (environment and system). In general, the environment may not be treated perturbatively, and one should try to find a non-perturbative method beyond the scope of this paper. In fact, the method we will use is suitable for standard inflationary theories and non-linear σ inflation models originating from the string theory in which the coupling between inflaton and the other given fields is weak. In the usual inflationary theories, one can suppose inflaton as a system and tensor modes as an environment [14–16], or choose a massive isocurvature mode as an environment [17] (similar to all multi-field inflationary theory [13]); then, the interaction terms [8] are given by Maldacena's cubic terms. In all such theories, the action of the universe (including inflaton and other degrees of freedom) in the inflation era is effectively truncated up to the cubic terms. The accuracy of such a calculation would be valid up to the third order of fluctuations. The Hamiltonians of such theories include the free terms of fields and the interaction part

$$H[E,S] = H_0[E,S] + H_{int}[E,S], \tag{17}$$

where H_{int} is the interaction between the system and environment[1]. Here, the free Hamiltonian H_0 includes a kinetic term with Fourier transformation

$$H_k[S] = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} f_s(\tau)\pi_p[S]\pi^*_p[S] \tag{18}$$

where the conjugate momentum is given by

$$\pi_p[S] = -i\frac{\delta}{\delta\phi_p}. \tag{19}$$

Here, f_s depends on the geometry of space-time and the kind of fields. For example, in the case of de Sitter space-time for a free scalar field action, one finds it as $f_s = \frac{1}{2a(\tau)}$, and it is independent of fields; however, in general, space-time could not be true [20]. To solve Equation (16) perturbatively, one needs to know the solution of the free part in Equation (17). The general solution $\Psi_{en}[E,S]$ for two independent free scalar fields should satisfy

$$i\frac{d}{dt}\Psi_{en}[E,S] = H_0[E,S;t]\Psi_{en}[E,S]. \tag{20}$$

One could perform it just by an anzats for the ground state

$$\Psi_{en} = N_{en}(\tau)\exp\left[-\int \frac{d^3k}{(2\pi)^3}(A_k(\tau)\phi_k\phi_{-k} + B_k(\tau)\chi_k\chi_{-k} + 2C_k(\tau)\phi_k\chi_{-k})\right], \tag{21}$$

and by finding some definite differential equations for A, B, and the entangled parameter C, can finally solve them with an initial condition suitable for an inflation theory. The entanglement is put in here when $C(\tau_0) \neq 0$ at the beginning. In fact, using a Schrödinger equation for the free part of a combined Hamiltonian, one could find the simple equation $\frac{C_k'}{C_k} = \frac{(A_k+B_k)}{a^2(\tau)}$. The non-vanishing value of the entangled parameter C_k at the start of inflation provides a non-zero value when inflation increases [16]. One notes that this form of the wave function is invariant under rotations and spatial translations. This solution would be Gaussian because of the quadratic form of the free Hamiltonian. Now, to solve the Hamiltonian, including the interactions of cubic terms, we suppose there is a solution as in the following

$$i\Psi_{en}\dot{\Psi}_{ng} = (H_k[S] + H_k[E] + H_{int}[E,S])\Psi_{en}\Psi_{ng}. \tag{22}$$

Here, we see that the effect of interactions appears as a non-Gaussian part Ψ_{ng} in the wave function at the ground state. Note that we have just used kinetic parts of a Hamiltonian. When one uses Equation (20), the potential terms in the free Hamiltonian would be canceled from the RHS of Equation (22). Using Equation (18), one can find the effective term of kinetic parts in Equation (22)

$$H_k[S]\Psi_{en}\Psi_{ng} \to \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} f_s(\tau)\{\frac{\delta\Psi_{en}}{\delta\phi_{-p}}\frac{\delta\Psi_{ng}}{\delta\phi_p}\} \tag{23}$$

and the same for the environment. In working out Equation (23), we have dropped out the terms in which H_k acts partly on Ψ_{ng}. This sector sources quartic and higher-order parts of a wave functional, which would be more suppressed by the interaction strength and not be captured correctly when the action is truncated at the third order in the fluctuations. One can again find an anzats for the non-Gaussian part of the wave function as

$$\Psi_{ng} = \exp \int_{k',k,p} \Big(\phi_p \chi_k \chi_{k'} F_{kk'p} + \phi_p \phi_k \phi_{k'} M_{kk'p} + \chi_p \phi_k \phi_{k'} N_{kk'p} + \chi_p \chi_k \chi_{k'} Q_{kk'p} \Big) \tag{24}$$

where the dynamical coefficients N, M, F, and Q must be found by a Schrödinger equation[2]. In the above integral, we use the usual convention $\int_{k,k',p} \equiv \int \frac{dk^3}{(2\pi)^3} \frac{dk'^3}{(2\pi)^3} \frac{dp^3}{(2\pi)^3} \delta(k + k' + p)$. Additionally, it is convenient to write down the action of the cubic interaction on the solution in Fourier space, such as

$$H_{int} \Psi_{en} \Psi_{ng} = \int_{k',k,p} \Big[\mathcal{H}^{(1)}_{kk'p} \phi_k \phi_{k'} \phi_p + \mathcal{H}^{(2)}_{kk'p} \chi_k \chi_{k'} \phi_p + \mathcal{H}^{(3)}_{kk'p} \phi_k \phi_{k'} \chi_p + \mathcal{H}^{(4)}_{kk'p} \chi_k \chi_{k'} \chi_p \Big] \Psi_{en} \Psi_{ng}, \tag{25}$$

where the integration is just an eigenvalue of the H_{int} operator. Substituting Equations (24), (25), and (21) in (22), and using the fact that all cubic multiplications of fields in a Schrödinger equation are independent, one would obtain four coupled first-order differential equations for unknown dynamical variables in non-Gaussian parts of a wave function (Equation (24)).[3]

$$\begin{gathered} -iH\tau \dot{F}_{kk'p} = \\ f_s(p,\tau) A_p(\tau) F_{kk'p} + 2 f_s(k,\tau) C_k(\tau) N_{kpk'} + 2 f_e(k',\tau) B_{k'}(\tau) F_{kk'p} + 3 f_e(p,\tau) C_p(\tau) Q_{kk'p} \\ + \mathcal{H}^{(2)}_{kk'p} \end{gathered} \tag{26}$$

which comes from $\chi^2 \phi$ coefficients in a Schrödinger equation,

$$-\, iH\tau \dot{M}_{kk'p} = 3 f_s(p,\tau) A_p(\tau) M_{kk'p} + f_e(p,\tau) C_p(\tau) N_{kk'p} + \mathcal{H}^{(1)}_{kk'p} \tag{27}$$

is the coefficient of ϕ^3,

$$-\, iH\tau \dot{Q}_{kk'p} = f_s(p,\tau) C_p(\tau) F_{kk'p} + 3 f_e(p,\tau) B_p(\tau) Q_{kk'p} + \mathcal{H}^{(4)}_{kk'p} \tag{28}$$

is related to the term of χ^3, and finally

$$\begin{gathered} -iH\tau \dot{N}_{kk'p} = \\ 3 f_s(p,\tau) C_p(\tau) M_{kk'p} + 2 f_s(k',\tau) A_{k'}(\tau) N_{kk'p} + 2 f_e(k,\tau) C_k(\tau) F_{kpk'} + f_e(p,\tau) B_p(\tau) N_{kk'p} \\ + \mathcal{H}^{(3)}_{kk'p}, \end{gathered} \tag{29}$$

which is the coefficient of $\phi^2 \chi$. In Equations (26)–(29), we have used dote as a conformal time derivation. After solving these equations, we can talk about all of the quantum effects on inflaton perturbations, such as decoherence. In the next section, we will show that only Equation (26) is related to decoherence and try to find some exact solutions for these equations. In general, one has to numerically solve these equations, which we have left for future studies.

Equations (26) and (29) imply that in the presence of entanglement, there would be a convolution between three different wavelengths, namely k, k', and p. In Equation (26), $N_{kpk'}$ appears, while the mode index of the other time-dependent factors appears in $kk'p$, and the same happens for $F_{kpk'}$ in Equation (29). Fortunately, this kind of convolution has not been seen in the bispectrums of standard inflationary theories (for example, see Ref. [19]). Consequently, this idea proposes that in the presence of the entanglement, if one demands that there is no convolution between dynamical equations, then either F or N should vanish at any time. Note that F cannot vanish because, in such a case, there would not be any decoherence in the system. Therefore, the only possible case is the case in which $N(\tau)$ vanishes.

We end this section with an understanding of this question: why is the entanglement parameter related to interaction parts? To answer this question, one may look at the path integral method. We know that one could relate the propagator to the wave function from the path integral method to the Schrödinger picture version of quantum mechanics

$$\Psi \leftrightarrow \int \mathcal{D}X \ \exp(iS). \tag{30}$$

Now, if we insert the interaction part to the exponent in RHS, then the LHS should be modified and vice versa. Because the RHS would be changed exponentially, the LHS would be modified similarly.

4. Decoherence from Entanglement

In this section, we want to find some exact solutions to Equations (26)–(29) and investigate the phenomenon of decoherence related to the form of the interactions. As mentioned before, it seems complicated to solve this coupled system of differential equations; however, one could try to obtain numerical solutions. Here, we shall check two special exact solutions.

4.1. Non-Entangled CASE

One of the compelling cases is non-entangled states. In this case, since $C = 0$ for all modes, the system and the environment are not correlated at an early time. Now, we analyze the above-coupled system of differential equations and understand more about decoherence in such theories. When we justify the entangled parameter to zero, this coupled system of equations is transformed into the decoupled one; therefore, the solution is easy. At first, we would like to focus on Equation (26)

$$iH\tau\dot{F}_{kk'p} + g(\tau;k,k',p)F_{kk'p} + \mathcal{H}^{(2)}_{kk'p} = 0 \tag{31}$$

where $g \equiv f_s A + 2f_e B$. To solve it, we need an initial condition. We are interested in theories in which interactions are active during inflation and have no effect at the early stages. With this assumption, it would be reasonable to assume $F(\tau_0) = 0$. Therefore, the solution is

$$F_{kk'p}(\tau) = i\int_{\tau_0}^{\tau} \frac{d\tau'}{H\tau'}\mathcal{H}^{(2)}_{kk'p}(\tau')\exp\left[i\int_{\tau'}^{\tau} d\tau'' g(\tau'';k,k',p)\right]. \tag{32}$$

This relation would be more simple in some cases. For example, in de Sitter space, suppose the theory in which $\mathcal{H}^{(2)}$ is proportional to a^n with $n > 0$ for scale factor [8]. If the coupling between the system and environment is weak enough, such that the density matrix remains close to Gaussian, then the real part of F does not grow at the late time. In other words, when τ is small and τ' is close to τ, the above integral gets its maximum value, and the exponent part in Equation (33) would be negligible. As a result, for the late time, we have just the imaginary part of this as the following

$$\lim_{\tau\to 0} ImF_{kk'p}(\tau) = \int_{\tau_0}^{\tau} \frac{d\tau'}{H\tau'}\mathcal{H}^{(2)}_{kk'p}(\tau'). \tag{33}$$

One can also reach this result by direct integration in de Sitter space-time. To manipulate Equation (32), we need the explicit form of $g(\tau; k, k', p)$. One can show that for a Gaussian wave function, $g(\tau; k, k', p)$ has the following form (see Ref. [8])

$$g(\tau; k, k', p) = -\tau^2 \frac{1 - \frac{i}{k\tau}}{1 + k^2\tau^2} k^3 + 2 \; perms, \tag{34}$$

which leads to

$$i \int^{\tau} d\tau' g(\tau; k, k', p) = (-ik\tau - \ln(1 - ik\tau)) + 2 \; perms. \tag{35}$$

Plugging this into Equation (32) with $\tau_0 \longrightarrow -\infty$, we have

$$F_{k,k',p}(\tau) = i \int_{-\infty}^{\tau} \frac{d\tau'}{H\tau'} \mathcal{H}^{(2)}_{kk'p}(\tau') e^{ik_t(\tau'-\tau)} \frac{1 - ik\tau'}{1 - ik\tau} \frac{1 - ik'\tau'}{1 - ik'\tau} \frac{1 - ip\tau'}{1 - ip\tau} \tag{36}$$

where $k_t \equiv k + k' + q$. When $\mathcal{H}^{(2)}_{kk'p}(\tau') \propto \frac{1}{\tau'^n}$ for $n \geq 1$, the result of this integral finds a pole of order $\frac{1}{\tau^n}$. To handle such an integral, we have to use the following integrals

$$-i \int_0^{\infty} dt \, e^{ikt} = \frac{1}{k + i\epsilon} \tag{37}$$

and

$$\int_{-\infty}^{\tau} \frac{dt}{t^{m+1}} e^{ikt} = -\frac{1}{\tau^m} \int_1^{\infty} \frac{dt}{t^{m+1}} e^{i\alpha t}, \qquad m > 0, \; \tau < 0; \; \alpha \equiv k\tau. \tag{38}$$

where τ goes to zero and the integral in the left-hand side of Equation (38) goes to $\frac{1}{m}$. Here, we have to mention that in deriving Equation (37), we have used the analytic continuation on k, equivalently changing the contour of the integral from the positive real line to $te^{i\epsilon}$. With the help of this information and using integration by parts, one can reach

$$F_{k,k',p}(\tau) = i \int_{-\infty}^{\tau} \frac{d\tau'}{H\tau'} \mathcal{H}^{(2)}_{kk'p}(\tau') e^{ik_t(\tau'-\tau)} \frac{1 - ik\tau'}{1 - ik\tau} \frac{1 - ik'\tau'}{1 - ik'\tau} \frac{1 - ip\tau'}{1 - ip\tau} = -\frac{i}{n\tau^n}\left(1 + \mathcal{G}_{k,k',p;\tau}\right), \tag{39}$$

where $\mathcal{G}_{k,k',p;\tau}$ is a regular function of τ and also goes to zero when $\tau \longrightarrow 0$. This result proves Equation (33).

Now, let us come back to the decoherence phenomenon. The summation in the reduced-density matrix Equation (15) is proportional to

$$\sum_E \left(\Psi_E[E]|_{\phi(x)}\right)\left(\Psi^*_E[E]|_{\phi'(x)}\right) \propto \int \mathcal{D}\chi \Psi_{en}[E,S]\Psi^*_{en}[E,S'] \exp\left[\int_{k,k',p} \chi_k \chi_{k'} \left(\phi_p F_{kk'p} + \phi'_p F^*_{kk'p}\right)\right] = \langle \exp\left[i \int_{k,k',p} \chi_k \chi_{k'} \Delta\phi_p Im(F_{kk'p})\right] \rangle \tag{40}$$

where $\Delta\phi_p = \phi_p - \phi'_p$, and this equation appears as an average value of the exponential on the environment's degrees of freedom. Thus, based on Riemann's integration theorem, if the imaginary part of F is large, then the off-diagonal elements of the density matrix go to zero, and decoherence occurs. From here, we can see that F is the most important term to which the decoherence phenomenon is related. This happens because $\Delta\phi$ has been coupled only with F.

Although the dynamics of Q, N, and M do not affect the decoherence, for a complete description, we shall solve them here. Because all these differential equations are decoupled and the same, the solutions are similar to the solution of F if we choose the same initial conditions:

$$M_{kk'p}(\tau) = i \int_{\tau_0}^{\tau} \frac{d\tau'}{H\tau'} \mathcal{H}^{(1)}_{kk'p}(\tau') \exp\left[i \int_{\tau'}^{\tau} d\tau'' m(\tau''; k, k', p)\right], \tag{41}$$

where $m \equiv 3f_sA$. For N, we have

$$N_{kk'p}(\tau) = i\int_{\tau_0}^{\tau} \frac{d\tau'}{H\tau'}\mathcal{H}^{(3)}_{kk'p}(\tau')\exp\left[i\int_{\tau'}^{\tau} d\tau'' n(\tau'';k,k',p)\right], \tag{42}$$

where $n \equiv 2f_sA + f_eB$, and finally

$$Q_{kk'p}(\tau) = i\int_{\tau_0}^{\tau} \frac{d\tau'}{H\tau'}\mathcal{H}^{(4)}_{kk'p}(\tau')\exp\left[i\int_{\tau'}^{\tau} d\tau'' q(\tau'';k,k',p)\right], \tag{43}$$

where $q \equiv 3f_eB$. These terms are significant in calculations related to the bispectrum but are not used in the decoherence of the density matrix or spectrum of CMB. In fact, one could establish any standard theory for inflation just by choosing $\mathcal{H}^{(1)} = \mathcal{H}^{(3)} = \mathcal{H}^{(4)} = 0$. In such theory, N, Q, and M are equal to zero, and F has a non-zero value that can contribute to the CMB bispectrum.

4.2. Entangled Case

In this part of the paper, we investigate a theory in which decoherence happens similarly to the previous case, with the difference being that the theory has an entangled initial state. To realize such a theory, we have to restrict the interaction terms. In the previous case, we saw that if there was no entanglement in the combined system, there would not be any correlation between other interaction terms. Here, we will find a solution in the presence of entanglement such that the eigenvalues of interaction terms in the ground state are related to the entanglement variable C.

To find such a solution, we have to choose $N(\tau) = Q(\tau) = 0$. Therefore, Equation (26), which is responsible for the decoherence phenomenon, would be the same as before, and the decoherence would happen similar to the case $C = 0$. One can easily see that the solution for M is the same as Equation (41); however, there are two consistency relations for $\mathcal{H}^{(3)}$ and $\mathcal{H}^{(4)}$, as in the following equations:

$$3if_s(p,\tau)C_p(\tau)\int_{\tau_0}^{\tau} \frac{d\tau'}{H\tau'}\mathcal{H}^{(1)}_{kk'p}(\tau')\exp\left[i\int_{\tau'}^{\tau} d\tau'' m(\tau'';k,k',p)\right] + \\ 2if_e(k,\tau)C_k(\tau)\int_{\tau_0}^{\tau} \frac{d\tau'}{H\tau'}\mathcal{H}^{(2)}_{kk'p}(\tau')\exp\left[i\int_{\tau'}^{\tau} d\tau'' g(\tau'';k,k',p)\right] = -\mathcal{H}^{(3)}_{kk'p} \tag{44}$$

and

$$if_s(p,\tau)C_p(\tau)\int_{\tau_0}^{\tau} \frac{d\tau'}{H\tau'}\mathcal{H}^{(2)}_{kk'p}(\tau')\exp\left[i\int_{\tau'}^{\tau} d\tau'' g(\tau'';k,k',p)\right] = -\mathcal{H}^{(4)}_{kk'p}. \tag{45}$$

These equations imply that two of the four interactions (eigenvalues of interaction terms in the Fourier space on the ground state) are not independent. For example, once $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ are given, the others would be completely defined. So, in creating such theories, we are not wholly free to choose interaction terms. One may know if Equations (26)–(29) have solutions such that the entanglement parameter affects on the decoherence process. In other words, the question arises as to whether there is a solution in which decoherence depends on entanglement. Generally, the answer is positive, and this can be seen by choosing $N(\tau) = 0$ and $Q \neq 0$. Here, one should note that the entanglement parameter C is independent of the magnitude of slow-roll parameters in the case of the inflationary universe example. This happens because C just depends on the ratio of interactions. Now, Equation (26) is no longer independent of other equations and should be solved again with the entanglement variable C. Here, we can consider $\mathcal{H}^{(4)}$ as an independent term and use Equation (29) to find F as

$$F_{kpk'} = -\frac{i}{2f_e(k,\tau)C_k(\tau)} 3f_s(p,\tau)C_p(\tau) \int_{\tau_0}^{\tau} \frac{d\tau'}{H\tau'} \mathcal{H}^{(1)}_{kk'p}(\tau') \exp\left[i\int_{\tau'}^{\tau} d\tau'' m(\tau'';k,k',p)\right] - \frac{1}{2f_e(k,\tau)C_k(\tau)} \mathcal{H}^{(3)}_{kk'p}. \quad (46)$$

This solution has two important differences from its non-entangled counterpart in Equation (32). The first happens in the first term where $\mathcal{H}^{(1)}_{kk'p}$ appears in the integral, not $\mathcal{H}^{(2)}_{kk'p}$. Therefore, in the decoherence process in the presence of entanglement, the cubic term ϕ^3 has the main role. The second happens in the last term, where the factor $\frac{1}{C_k}$ appears. We know that the decoherence depends on the imaginary part of F, which means that the imaginary part of C_k has a crucial role in this process. It is exciting because, before this solution, it was known that the physical part of the entanglement parameter is its only real part [11]. If the initial state of the early universe is the entangled one, then the power spectrum should be corrected by terms involving just the real part of C_k. Now, the above relation gives us new insight. If one requests that the decoherence occurs more rapidly, we need the imaginary part of C_k to be much larger than its real part. For example, one can easily show that if the real part is of order 10^{-15}, then to have a large entanglement-decohering effect, the imaginary part should be of order 10^{-5}. The most interesting case occurs when C_k has a vanishing real part. In such a case, this theory with entanglement has the same power spectrum as the standard theory. This is a fair theory because, in addition to the small values of the entanglement parameter, we come back to the expected result for the power spectrum, and we also have a fair decoherence process without specific restrictions on interaction terms. The only important condition for the interaction parts is their eigenvalue, which should be finite and non-zero (this is a reasonable condition in every well-defined interacting theory). Note that in the first term of Equation (46), we have a factor of $\frac{C_p}{C_k}$, which is of order 1, and this factor does not have a significant effect.

With the current precision in the measurement of the temperature fluctuations, we have not currently seen any effects of the entanglement parameter in the power spectrum. This implies that if the early universe had an entangled state, the (real part of the) entanglement parameter was very small. Consequently, this means that the decoherence of this state happened appropriately because of the last term in Equation (46). In the case where the C_k is not very small, the second term is not very important, and we have to focus on the first term and check whether or not $\mathcal{H}^{(1)}_{kk'p}$ has a polynomial form for the scale factor. In other words, we have to restrict ourselves to those interacting theories in which the eigenvalue $\mathcal{H}^{(1)}_{kk'p}$ behaves in a way that makes the first term in the above relation grow very fast at a late time.

5. Conclusions

In this paper, we investigated the decoherence of a field (namely inflaton in cosmology) interacting with the environment or any other independent mode functions (up to the cubic terms in the action). There is no reason one could not consider a more general initial state, which includes the entanglement between the inflaton field and other degrees of freedom. Significantly, one could consider inflation as an effective theory of a fundamental theory, such as quantum gravity, which has multiple fields.

We have two types of entanglements: first, the primary entanglement, which comes from the initial state of the combined system, including the environment and the system under study; second, the secondary entanglement, which comes out during the interaction between the system and environment. To have decoherence during the interaction, the secondary entanglement is necessary.

Let us conclude with some remarks:

- In this paper, we showed that the primary entanglement has a significant role in the decoherence of the quantum state of the system;
- It was also shown that if there is no primary entanglement in a combined system, then the interaction terms responsible for the secondary entanglement are independent in the ground state. In this case, the interaction term, $H_{int} \sim \int_{k',k,p} \mathcal{H}^{(2)}_{kk'p} \chi_k \chi_{k'} \phi_p$, contributes to the decoherence;
- If we have the primary entanglement $C(\tau) \neq 0$, there is a solution of the Schrödinger wave equation, $N(\tau) = Q(\tau) = 0$, in which eigenvalues of the interaction terms in Fourier space cannot be independent. In other words, if one demands an entangled state and the same decoherence (which we have in standard non-entangled theory), then the interaction parts are more restricted and should be chosen consistently with Equations (45) and (44). Such theories have the same decoherence process as the standard theory but have a different result in the power spectrum [11];
- The dependency between the primary entanglement and the interaction terms can have a teleological interpretation. Suppose that the semi-classical picture of inflation theory is an effective low-energy theory of a universal quantum gravity theory (UQGT). Therefore, primary entangled states and interaction terms emerge from the low-energy limit of the UQGT. From this perspective, the entanglement and interaction terms cannot be independent;
- In contrast to the power spectrum of inflation, in which only the real part of the entanglement parameter, C_k, is important (not the imaginary part), in the decoherence process, the imaginary part of the entanglement parameter plays a vital role for speeding up the decoherence. With a significant value of the imaginary part of the entanglement parameter, the decoherence can happen without a specific restriction of the interaction terms;
- At the end, we should emphasize that the difference between theories with different interactions or initial states appear not only through their two-point correlation functions at an early time but also through three-point functions. The contributions of these three-point functions come from a non-Gaussian part of the theories, which is now related to the entanglement parameter C_k in the general solution.

Several directions for future research exist; One can use a multi-fields model [13] (for example, two scalar fields or tensor–scalar field models) with a primary entanglement to look at the decoherence rate of the wave function in the super-horizon and verify whether this entanglement delays the classicalization. Even for single-field inflation, one can look at the action of the third order with different coupling effects between independent modes and see the decoherence of the density matrix in the presence of the primary entanglement. It would also be interesting to study the dynamics of the entangled state in the phase space. With the calculation of the related Wigner function, we can understand the coherence lengths and squeezing at late times and whether the diagonal matrix elements evolve according to the standard Fokker–Planck equation of Starobinsky's stochastic inflation. One can also study the entanglement effects on the redundant records of long-wavelength perturbation during inflation to investigate the squeezing of the quantum stats [21].

Author Contributions: Conceptualization: A.R. Supervision: J.T.F. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: We would like to thank M.M. Sheikh-Jabbari and Hassan Firouzjahi for their valuable comments. Data sharing does not apply to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest: The authors declare no conflict of interest. Authors also identify and declare that there is no any personal circumstances or interest that may be perceived as inappropriately influencing the representation or interpretation of reported research results. Authors also state that "The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results".

Notes

1. For instance, in the case where the long and short wavelength modes have the roles of system and environment respectively, this interaction comes from the cubic terms in the perturbed action [8,19]. Moreover, these interaction terms can come from the extension of the standard model of particle physics or the moduli of compactification in string theory.
2. One may want to know why the normalization factor has not been considered in (24). The reason is that the dynamical equation for normalization is related to the zero order of perturbation and so is irrelevant here. In other words, this consideration has no effect in the derivation of (26)–(29).
3. Here, we have used conformal time instead of cosmological time by substituting $\frac{d}{dt} = -H\frac{d}{d\tau}$.

References

1. Ade, P.A.R. et al. [Planck Collaboration] Planck 2015 results-xiii. cosmological parameters. *Astron. Astrophys.* **2016**, *594*, A13. [CrossRef]
2. Polarski, D.; Starobinsky, A.A. Quantum fluctuations, decoherence of the mean field, and structure formation in the early universe. *Class. Quant. Grav.* **1996**, *13*, 377. [CrossRef]
3. Albrecht, A.; Ferreira, P.; Joyce, M.; Prokopec, T. Inflation and squeezed quantum states. *Phys. Rev. D* **1994**, *50*, 4807. [CrossRef] [PubMed]
4. Kiefer, C.; Polarski, D. Why do cosmological perturbations look classical to us? *Adv. Sci. Lett.* **2009**, *2*, 164. [CrossRef]
5. Martin, J.; Vennin, V. Quantum discord of cosmic inflation: Can we show that CMB anisotropies are of quantum-mechanical origin? *Phys. Rev. D* **2016**, *93*, 023505. [CrossRef]
6. León, G.; Sudarsky, D. Origin of structure: Statistical characterization of the primordial density fluctuations and the collapse of the wave function. *J. Cosmol. Astropart. Phys.* **2015**, *1506*, 020. [CrossRef]
7. León, G.; Bengochea, G.R.; Landau, S.J. Quasi-matter bounce and inflation in the light of the CSL model. *Eur. Phys. J. C* **2016**, *76*, 407. [CrossRef]
8. Nelson, E. Quantum decoherence during inflation from gravitational nonlinearities. *J. Cosmol. Astropart. Phys.* **2016**, *2016*, 022. [CrossRef]
9. Burgess, C.P.; Holman, R.; Tasinato, G.; Williams, M. Decoherence of inflationary primordial fluctuations. *Phys. Rev.* **2008**, *77*, 063534. [CrossRef]
10. Martin-Martinez, E.; Menicucci, N.C. Entanglement in curved spacetimes and cosmology. *Class. Quant. Grav.* **2014**, *31*, 214001. [CrossRef]
11. Albrecht, A.; Bolis, N.; Holman, R. Cosmological consequences of initial state entanglement. *J. High Energy Phys.* **2014**, *2014*, 93. [CrossRef]
12. Rostami, A.; Firouzjaee, J.T. Schrödinger field theory in curved spacetime: In-in formalism and three-point function for the inflationary background. *Int. J. Mod. Phys. D* **2018**, *27*, 1850097. [CrossRef]
13. Starobinsky, A.A.; Tsujikawa, S.; Yokoyama, J. Cosmological perturbations from multi-field inflation in generalized Einstein theories. *Nucl. Phys.* **2001**, *B610*, 383. [CrossRef]
14. Bernardeau, F.; Uzan, J.P. Gravity- and non-gravity-mediated couplings in multiple-field inflation. *Phys. Rev. D* **2002**, *66*, 103506. [CrossRef]
15. Langlois, D.; Renaux-Petel, S. Perturbations in generalized multi-field inflation. *JCAP* **2008**, *0804*, 017. [CrossRef]
16. Bolis, N.; Albrecht, A.; Holman, R. Modifications to cosmological power spectra from scalar-tensor entanglement and their observational consequences. *J. Cosmol. Astropart. Phys.* **2016**, *2016*, 011. [CrossRef]
17. Chen, X.; Wang, Y. Quasi-single field inflation and non-Gaussianities. *JCAP* **2010**, *1004*, 027. [CrossRef]
18. Ghirardi, G.C.; Rimini, A.; Weber, T. Quantum Probability and Applications II. *Phys. Rev. D* **1986**, *34*, 470. [CrossRef]
19. Maldacena, J.M. Non-gaussian features of primordial fluctuations in single field inflationary models. *JHEP* **2003**, *0305*, 013. [CrossRef]
20. Townsend, P.K. Black holes. *arXiv* **1997**, arXiv:gr-qc/9707012.
21. Nelson, E.; Riedel, C.J. Classical entanglement structure in the wavefunction of inflationary fluctuations. *Int. J. Mod. Phys.* **2017**, *26*, 1743006. [CrossRef]