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Noncommutative Corrections to the Minimal Surface Areas of the Pure AdS Spacetime and Schwarzschild-AdS Black Hole

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Abstract: Based on the perturbation expansion, we compute the noncommutative corrections to the minimal surface areas of the pure AdS spacetime and Schwarzschild-AdS black hole, where the noncommutative background is suitably constructed in terms of the Poincaré coordinate system. In particular, we find a reasonable tetrad with subtlety, which not only matches the metrics of the pure AdS spacetime and Schwarzschild-AdS black hole in the commutative case, but also makes the corrections real rather than complex in the noncommutative case. For the pure AdS spacetime, the nocommutative effect is only a logarithmic term, while for the Schwarzschild-AdS black hole, it contains a logarithmic contribution plus both a mass term and a noncommutative parameter related term. Furthermore, we show that the holographic entanglement entropy with noncommutativity obeys a relation which is similar to the first law of thermodynamics in the pure AdS spacetime.

Keywords: noncommutative geometry; holographic entanglement entropy

Citation: Liu, Z.-C.; Miao, Y.-G. Noncommutative Corrections to the Minimal Surface Areas of the Pure AdS Spacetime and Schwarzschild-AdS Black Hole. *Universe* **2022**, *8*, 524. https://doi.org/10.3390/universe8100524

Academic Editor: José Velhinho

Received: 30 August 2022
Accepted: 6 October 2022
Published: 9 October 2022

1. Introduction

One way to understand quantum gravity follows the holographic correspondence. In terms of the open/closed string duality [1–3], the AdS/CFT correspondence connects a $(d+1)$-dimensional AdS space with a d-dimensional CFT which is located at the boundary of the AdS space. Though it is hard to provide a general proof for this kind of duality, some evidence gives rise to ideas on how to connect the classical gravity with quantum gravity. One of the examples is the Ryu–Takayanagi (RT) formula [4,5] in which the entanglement entropy of a CFT in subsystem A[1] is proportional to the minimal surface area γ_A in the AdS spacetime whose boundary is located at $\partial\gamma_A = \partial A$[2]. The proof of the formula elucidates [10–12] an understanding of the AdS/CFT duality in the cosmic brane or the topological black hole background. In addition, the covariant entanglement entropy has also been studied [13], where one can obtain more information about the time evolution of the entanglement entropy. However, there are still many problems that should be solved before the holographic dictionary is completed. Here, we list some of the problems and the literature in which some efforts have been made to try to resolve them.

- How to encode [14] the bulk information from the field theory on the boundary when the information seems to be non-local?
- How to extend [15–19] the RT formula beyond the classical gravity? What are the general conditions that the formula should meet?
- Whether can the AdS/CFT duality be extended [20–23] to the de Sitter spacetime or flat spacetime besides the consideration of symmetries?
- What happens [24,25] in the dual theory if the quantum field is taken into account in the bulk?

Though many of the questions mentioned above have been studied on the side of gravity, the relevant questions on the side of quantum field theory have not been dealt with. Moreover, the entanglement entropy does not always meet the area law. In order to complete the holographic dictionary, the noncommutativity of spacetimes should be considered. Here, we briefly review some interesting models of noncommutative geometry and describe their applications in quantum gravity. Among the attempts to quantize gravity, a natural thought is to consider noncommutative spacetimes, which are early dubbed quantized spacetimes that could be traced back to Snyder's pioneering work [26]. The revival of noncommutative spacetimes about half a century after Snyder's work originated [27] from the low energy effective field theory of string theory. In Ref. [27], the Seiberg–Witten map establishes a connection between one gauge theory on a noncommutative spacetime and another theory on the commutative spacetime. Based on this map, one can investigate [28,29] noncommutative gravity by rewriting it as a gauge theory. According to the coordinate coherent state formalism [30], which records the noncommutativity of coordinates by the spread of coherent states, the black holes in noncummutative spacetimes were constructed [31–35] if the point-like source was replaced with the smearing of objects. One may also refer to the various attempts in the construction of the noncommutative gravity [36,37], noncommutative black holes [38], and noncommutative quantum cosmology [39].

Besides the AdS/CFT correspondence with noncommutativity, we have to study the holographic entanglement entropy with noncommutativity in order to understand the effects of non-locality on quantum entanglement. While the entanglement entropy of a local field theory obeys the area law [40,41], which shows the strong correlations among the states near the boundary, the area law is not necessarily true for a non-local field theory. For instance, the entanglement entropy of the non-local scalar field theory obeys [23,42] the volume law. Furthermore, it is of significance to study the non-leading contributions to holographic entanglement entropy. On the one hand, the non-leading contributions can extend the dictionary beyond the area term. For example, the background entanglement from the one-loop quantum correction contributes [24,25] extra terms to entanglement entropy, and the noncommutative effect contributes [43] an extra divergent term to the holographic entanglement entropy in a noncommutative gauge theory. On the other hand, the information contained in the non-leading contributions is meaningful. The non-leading contributions to entanglement entropy encode [44] the universal terms which discriminate between different phases. Additionally, the higher derivative effect of gravity yields [16] the correct universal term for generalized holographic entanglement entropy in the 4d CFT.

In the present work, we added a non-leading term to the surface area, where this term is induced by the noncommutative structure of spacetimes. We find a specific tetrad and propose a noncommutative relation based on the Poincaré coordinate system. Then, the corresponding Moyal product can be given, which is different from those already constructed [45–47]. Our treatment is a physically straightforward modification to the classical theory of gravity. As it stems from the spacetime structure itself, the noncommutative geometry will show its significance to the holographic entanglement entropy on the side of quantum field theory.

The procedure is as follows. We establish a noncommutative construction by using the Moyal product in the Poincaré coordinate system. Then, according to this construction, we compute the minimal surface areas of the pure AdS spacetime and Schwarzschild-AdS black hole. As the noncommutative parameter is much smaller than one, it is reasonable [48,49] to perturbatively expand a geodesic curve with respect to this parameter. That is, it is important to study the noncommutative contributions to the minimal surface areas by order. We find that the noncommutative geometry contributes a logarithmic divergent term for the pure AdS spacetime, where this term is proportional to the noncommutative parameter. Additionally, we provide a similar discussion for the Schwarzschild-AdS black hole by the additional consideration of the black hole mass, and the result contains both a mass term and a noncommutative parameter-related term besides the logarithmic contribution. Furthermore, for the pure AdS spacetime with noncommutativity, we show that the noncommutative holographic entanglement entropy obeys a relation that is similar to the first law of thermodynamics.

The rest of the paper is organized as follows. In Section 2, we compute the minimal surface area of the pure AdS spacetime with noncommutativity. Then, we turn to the Schwarzschild-AdS black hole in Section 3, where the perturbation of the mass parameter should be performed before the consideration of the noncommutative correction. In Section 4, we consider the noncommutative stress tensor for the pure AdS spacetime and analyze the thermodynamic property of the holographic entanglement entropy with noncommutativity. Finally, we summarize our results in Section 5.

2. The Minimal Surface Area of the Pure AdS Spacetime with Noncommutativity

The metric of the (3+1)-dimensional AdS spacetime in the Poincaré coordinate system takes the following form:

$$ds^2 = L^2 \frac{-dt^2 + dx^2 + dy^2 + dz^2}{z^2}, \tag{1}$$

where L represents the AdS radius. Due to the rotational symmetry in the two models studied in this and the next sections, it is convenient to adopt the polar coordinates, (ρ, ϕ), in the (x, y) plane, and to rewrite the above metric to be

$$ds^2 = L^2 \frac{-dt^2 + d\rho^2 + \rho^2 d\phi^2 + dz^2}{z^2}. \tag{2}$$

In order to rewrite the above metric by using tetrads, there exist many equivalent formulations of tetrads, i.e., any one of them provide the same metric. However, we find that the formulations of tetrads are subtle when the metric is generalized to a noncommutative spacetime. That is, we can write many equivalent formulations of tetrads in the commutative case; however, most of them lead to a complex correction to minimal surface areas in the noncommutative case, which is usually regarded as an unphysical result. Fortunately, we were able to identify a tetrad that meets the requirement in the commutative spacetime and simultaneously provides a real noncommutative correction to minimal surface areas for the pure AdS spacetime and Schwarzschild-AdS black hole. This tetrad can be set to be

$$(k^a)_\mu = (l^a, n^a, m^a, w^a), \tag{3}$$

where the four components are determined[3] in terms of the Kronecker delta as follows:

$$l^a = L\left(\frac{i\delta_0^a}{2z^2} + i\delta_2^a\right), \qquad n^a = L\frac{\delta_1^a}{z}, \qquad m^a = L\left(-i\rho\frac{\delta_0^a}{2z^2} + i\rho\delta_2^a\right), \qquad w^a = L\frac{\delta_3^a}{z}. \tag{4}$$

Therefore, we construct the metric by using the above tetrad as follows,

$$g_{\mu\nu} = \eta_{ab} k_\mu^a k_\nu^b, \tag{5}$$

where η_{ab} is defined by[4]

$$\eta_{ab} = \begin{pmatrix} 0 & 0 & 1+i & 0 \\ 0 & 1 & 0 & 0 \\ 1-i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

We introduce a new variable q which is defined by $q \equiv 1/z$, so that $q \to \infty$ corresponds to the infinity of the AdS spacetime. In particular, such a transformation avoids the ambiguity when the inverse of a coordinate is set to be an operator, i.e., when an ordinary spacetime is generalized to a noncommutative one. Now, we can propose the following noncommutative relation by using the commutator between the operator of q and that of ρ,

$$[\hat{q}, \hat{\rho}] = ih, \tag{7}$$

where h is the noncommutative parameter that is dimensionless.

Under the above noncommutative relation, the multiplication of functions in noncommutative geometry can be realized in terms of the Moyal product, see, for instance, the Moyal product of two functions with respect to two variables u and v,

$$f(u,v) * g(u,v) = f(u,v) e^{i\frac{h}{2}(\overleftarrow{\partial}_u \overrightarrow{\partial}_v - \overleftarrow{\partial}_v \overrightarrow{\partial}_u)} g(u,v), \tag{8}$$

where $u = q$ and $v = \rho$ are specified in the present paper. As a result, we can write the noncommutative metric from the ordinary (commutative) one, Equation (5), as follows:

$$\bar{g}_{\mu\nu} = \eta_{ab} k^a_\mu * k^b_\nu, \tag{9}$$

from which we derive the square of the line element with noncommutativity up to the first order of h,

$$d\bar{s}^2 = L^2 \left[-q^2 dt^2 + hq dt d\phi + q^2 d\rho^2 + (q^2\rho^2 - hq\rho) d\phi^2 + \frac{dq^2}{q^2} \right]. \tag{10}$$

According to the Poincaré coordinate system, the projection of the minimal surface into the plane depicted by the coordinates ρ and q^{-1} is a circle when ϕ is integrated from zero to 2π; therefore, it is convenient to introduce the following polar coordinate system, (r, θ), to compute the minimal surface area

$$\rho = r\cos\theta, \qquad q^{-1} = r\sin\theta, \tag{11}$$

where $\theta \in [\epsilon, \pi/2]$, ϵ is a regularization factor that is close to zero and associated with a lattice length of fields on the boundary, and r represents the spacetime scale.

Now, we can easily write the minimal surface area with the noncommutative deformation using Equations (10) and (11) as follows:

$$A = 2\pi L^2 \int_\epsilon^{\frac{\pi}{2}} \frac{\sqrt{(\dot{r}^2 + r^2)(\cos^2\theta - h\sin\theta\cos\theta)}}{r\sin^2\theta} d\theta, \tag{12}$$

where a dot stands for the differentiation with respect to θ. As the noncommutative parameter h is much smaller than one, we expand the geodesic curve $r(\theta)$ with respect to it,

$$r(\theta) = r_0 + h\bar{r}(\theta) + O(h^2), \tag{13}$$

where r_0 is the initial constant corresponding to the spatial scale and $\bar{r}(\theta)$ represents the first order noncommutative modification to the curve. Because the term related to $\dot{r}^2$ is proportional to h^2, see Equation (13), the minimal surface area, Equation (12), reduces approximately to the following form if only the first order of h is considered in the square root,

$$\begin{aligned} A &= 2\pi L^2 \int_\epsilon^{\frac{\pi}{2}} \frac{\sqrt{\cos^2\theta - h\sin\theta\cos\theta}}{\sin^2\theta} d\theta \\ &\equiv A_0 + \bar{A}, \end{aligned} \tag{14}$$

where A_0 is the ordinary (commutative) minimal surface area and $\bar{A}$ is the first order noncommutative correction to A_0. By applying the Taylor expansion of Equation (14) to h, we can easily obtain A_0 and $\bar{A}$, respectively,[5]

$$A_0 = 2\pi L^2\left(\frac{1}{\sin\epsilon} - 1\right), \tag{15}$$

and

$$\bar{A} = \pi L^2 h \log\frac{\epsilon}{2}, \tag{16}$$

where the latter can also be understood as the nonlocal contribution from the noncommutative spacetime.

In this section, the noncommutative effect is manifested through the Moyal product of our tetrad (Equation (4)), see Equation (9). We calculate the noncommutative minimal surface by using the deformed metric Equation (10). From Equation (16), we see that the noncommutative effect provides a logarithmic divergent term with suppression factor h. Because $\bar{A}$ is minus, the noncommutative effect decreases the minimal surface area of the pure AdS spacetime.

3. The Minimal Surface Area of the Schwarzschild-AdS Black Hole with Noncommutativity

We deal with the Schwarzschild-AdS black hole in two steps. In the first step, we compute its minimal surface area by regarding [48,50] the mass of the Schwarzschild black hole, M, as a perturbative parameter in the pure AdS spacetime. Then, following the method we applied in the above section, we derive in the second step the noncommutative correction to the minimal surface area of the Schwarzschild-AdS black hole. That is, based on the pure AdS spacetime, our result contains the contributions from the pure AdS spacetime together with the mass correction and the noncommutative parameter correction.

The metric of the (3+1)-dimensional Schwarzschild-AdS black hole takes the following form [50]:

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + d\rho^2 + \rho^2 d\phi^2\right), \tag{17}$$

where $f(z)$ is associated with the black hole mass M and reads

$$f(z) = 1 - Mz^3. \tag{18}$$

As mentioned above, M is treated as a perturbative parameter under the condition $Mz^3 \ll 1$, where such a reparametrization has been made that Mz^3 is dimensionless.

In the first step, we derive the minimal surface area of the Schwarzschild-AdS black hole in the polar coordinate (r, θ),

$$\mathcal{A}_0 = 2\pi L^2 \int_{\epsilon}^{\frac{\pi}{2}} \left[r^2 + \dot{r}^2 + Mr^3\sin^3\theta\left(1 + Mr^3\sin^3\theta\right)\left(r^2\cos^2\theta + r\dot{r}\sin 2\theta + \dot{r}^2\sin^2\theta\right)\right]^{1/2} \times \frac{\cos\theta}{r\sin^2\theta}d\theta. \tag{19}$$

By expanding $r(\theta)$ with respect to M, we obtain

$$r(\theta) = l + Ma(\theta) + O(M^2), \tag{20}$$

where l is a constant that is associated with a spacial scale and satisfies the inequality $Ml^3 \ll 1$. Substituting Equation (20) into Equation (19), we then derive the leading contribution up to the first-order of mass M,

$$\begin{aligned} \mathcal{A}_0 &= 2\pi L^2 \int_\epsilon^{\frac{\pi}{2}} \left(1 + \frac{1}{2} M l^3 \sin^3\theta \cos^2\theta\right) \frac{\cos\theta}{\sin^2\theta} d\theta \\ &\equiv A_0 + A_{\rm M}, \end{aligned} \tag{21}$$

where A_0 is the minimal surface area of the pure AdS spacetime, see Equation (15), and $A_{\rm M}$ represents the mass correction to A_0,

$$A_{\rm M} = \frac{1}{4}\pi L^2 M l^3. \tag{22}$$

As a result, we work out the minimal surface area of the Schwarzschild-AdS black hole, $\mathcal{A}_0$, in the approximation of the first-order of M.

Now, we calculate the noncommutative correction to $\mathcal{A}_0$ in the second step by following the method utilized in the above section. Combining Equation (19) with Equation (10), we write the minimal surface area of the Schwarzschild-AdS black hole with noncommutativity as follows:

$$\begin{aligned} \mathcal{A} = 2\pi L^2 \int_\epsilon^{\frac{\pi}{2}} &\left[r^2 + \dot{r}^2 + M r^3 \sin^3\theta (1 + M r^3 \sin^3\theta)(r^2\cos^2\theta + r\dot{r}\sin 2\theta + \dot{r}^2 \sin^2\theta)\right]^{1/2} \\ &\times \left(\cos^2\theta - h\sin\theta\cos\theta\right)^{1/2} \frac{d\theta}{r\sin^2\theta}. \end{aligned} \tag{23}$$

Defining

$$\mathcal{A} \equiv \mathcal{A}_0 + \bar{\mathcal{A}}, \tag{24}$$

and substituting both Equations (13) and (20) into Equation (23), we derive the contribution from the noncommutative modification up to the first order of h,

$$\begin{aligned} \bar{\mathcal{A}} &= -\pi L^2 h \int_\epsilon^{\frac{\pi}{2}} \left(1 + \frac{1}{2} M l^3 \sin^3\theta \cos^2\theta\right) \frac{d\theta}{\sin\theta} \\ &= \pi L^2 h \log\frac{\epsilon}{2} - \frac{\pi}{32}\pi L^2 M l^3 h. \end{aligned} \tag{25}$$

When the mass parameter is set to zero, the above result returns to that of the pure AdS situation, for which see the first term of Equation (25) or Equation (16). On the other hand, besides the noncommutative correction from the noncommutative AdS spacetime, the noncommutativity also modifies the contribution from the mass term of the black hole, for which see the second term of Equation (25).

4. Thermodynamic Property of Holographic Entanglement Entropy with Noncommutativity

To further analyze the noncommutative effects on the thermodynamic property of holographic entanglement entropy, we study the stress tensor under noncommutative deformation. For simplicity, the following analysis is based on the deformed spacetime Equation (10), where the noncommutative effects associated with higher orders of h are ignored. In this way, we regard the noncommutative Minkowski spacetime as the pure AdS spacetime associated with matter. We note that the holographic entanglement entropy is corrected noncommutatively as stated in Sections 2 and 3 when the background metric is deformed by Equation (10), which means that the holographic entanglement entropy remains unchanged whether the noncommutative effect is regarded as matter or not. The reason lies in the fact that holographic entanglement entropy is only sensitive to the background metric.

We start with the gravitational action of the noncommutative $(3+1)$-dimensional pure Ads spacetime [51,52] with matter,

$$S = \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(R+\frac{6}{L^2}\right) + S_{\text{matter}} - \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}} d^3x\sqrt{-\gamma}\,\Theta + \frac{1}{8\pi G_N}S_{\text{ct}}(\gamma_{\mu\nu}), \tag{26}$$

where R is the Ricci scalar of the spacetime, S_{matter} is the matter action equivalent to the noncommutative contribution, γ is the boundary metric, Θ is the trace of the extrinsic curvature $\Theta_{\mu\nu} = \frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu)$ with n_μ the outward pointing normal vector to the boundary $\partial\mathcal{M}$, and S_{ct} is the counterterm action. To obtain a finite stress tensor on the boundary $\partial\mathcal{M}$, the counterterm action S_{ct} is chosen as a covariant function,

$$S_{\text{ct}} = -\frac{2}{L}\int_{\partial M}\sqrt{-\gamma}\left(1-\frac{L^2}{4}R\right)d^3x. \tag{27}$$

Next, we compute the stress tensor,

$$T^{\text{grav}}_{\mu\nu} = -\frac{1}{8\pi G_N}\left(\Theta_{\mu\nu} - \Theta\gamma_{\mu\nu} + \frac{2}{L}\gamma_{\mu\nu} - LG_{\mu\nu}\right), \tag{28}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\gamma_{\mu\nu}$ is the Einstein tensor associated with $\gamma_{\mu\nu}$. Substituting Equation (10) into the above equation, we obtain the energy component

$$-8\pi G_N T^{\text{grav}}_{00} = \frac{hLq}{2\rho} + \frac{Lh^2}{4\rho^2} + O(h^3). \tag{29}$$

We find from this component that the leading contribution without noncommutative corrections is vanishing, which coincides with [51] the usual result in the pure AdS_4 spacetime. The first term in the right-hand side of Equation (29) is divergent when we take $q \to \infty$ at the boundary; however, there are no covariant ways to cancel this because noncommutative correction is coordinate-dependent. In order to cancel the divergent term, we introduce a coordinate-dependent counterterm, which is as follows:

$$S_{\text{hct}} = \frac{h}{2L}\int_{\partial M}\sqrt{-\gamma}\left(\frac{1}{\rho q}\right)d^3x, \tag{30}$$

and add it to the action Equation (26). Thus, we obtain the finite result,

$$-8\pi G_N T^{\text{grav}}_{00} = \frac{Lh^2}{4\rho^2}. \tag{31}$$

From Equation (31), we can see that the noncommutative correction to the stress tensor is of order h^2 and depends only on the radial coordinate. As the stress tensor is divergent around $\rho = 0$, we have to regularize it by replacing the vanishing radial coordinate with a cutoff ϵ'; for further elaboration, see footnote 6 below.

In ref. [53], an analogous relation with the first law of thermodynamics is detailed for a subsystem A when the system is excited,

$$T_{\text{ent}} \cdot \Delta S_A = \Delta E_A, \tag{32}$$

where ΔS_A measures how much S_A (entanglement entropy) is increased in the exited state compared with the ground state of the CFT on the boundary $\partial\mathcal{M}$, and ΔE_A is the increased amount of energy in the subsystem A. Note that T_{ent} means the effective temperature or the so-called entanglement temperature, which is proportional to the inverse of the length scale of subsystem A.

For the pure AdS spacetime with noncommutativity as discussed in Section 2, we obtain the corresponding increased energy of subsystem A by considering Equation (31),

$$\Delta E'_A = \int d^2x \mathcal{T}_{00}^{\text{grav}} = \int_{\epsilon'}^{l} 2\pi\rho \mathcal{T}_{00}^{\text{grav}} d\rho = \frac{Lh^2}{16G_N} \log \frac{\epsilon'}{l}, \tag{33}$$

where we have regularized the stress tensor around the coordinate origin.[6] On the other hand, here, $\Delta S'_A$ is just $\bar{A}$ (see Equation (16)) divided by $4G_N$. Considering Equations (33) and (16), we obtain the relation,

$$T'_{\text{ent}} \cdot \Delta S'_A = \Delta E'_A + \frac{Lh^2}{16G_N} \log \frac{1}{2}, \tag{34}$$

where $T'_{\text{ent}} \equiv \frac{h}{4\pi L}$, which plays a similar role to T_{ent} of Equation (32). We notice that Equation (34) is similar to Equation (32) and the second term on its right-hand side is the subleading term that derives from the noncommutativity.

According to Equation (34), we find that the noncommutative correction of holographic entanglement entropy also obeys an analogous relation to the first law of thermodynamics. Comparing Equation (34) with Equation (32), we believe that the noncommutative correction of holographic entanglement entropy is related to the increased amount of entanglement entropy from a certain excited state [54].

5. Summary

In this paper, we highlight that noncommutative generalization of the minimal surface areas is nontrivial. At the commutative level, there exist many equivalent formulations of tetrads that give rise to the same metric. However, at the noncommutative level, most of them lead to complex noncommutative corrections of minimal surface areas, which is unphysical. Therefore, the construction of a specific tetrad is subtle. Fortunately, we have found such a tetrad and obtained the real noncommutative corrections to the minimal surface areas for the pure AdS spacetime and Schwarzschild-AdS black hole.

The RT formula shows the relation between the entanglement entropy $S_{A'}$ of conformal fields and the minimal surface area in Einstein gravity [4],

$$S_{A'} = \frac{\text{Area}}{4G_N}. \tag{35}$$

While the leading term of the holographic entanglement entropy is not altered by the noncommutativity of spacetimes, we observe that an extra divergent term is induced by the noncommutative geometry. The noncommutative spacetime we consider in the present paper may be interpreted as the perturbation of matter. This leads to the result that the disturbance of the stress tensor at the boundary corresponds to the perturbed state from the vacuum state. Equation (34) shows that the holographic entanglement entropy with noncommutativity satisfies a similar relation to the first law of thermodynamics. While we do not have a direct interpretation of the holographic entanglement entropy with noncommutativity in the boundary theory, we believe that it corresponds to the entanglement entropy of excited states. Here, we only consider the noncommutative holographic duality in the pure AdS spacetime. It is a nontrivial attempt to study the noncommutative holographic correspondence when spacetime is added with extra matter.

Author Contributions: Conceptualization, Z.-C.L.; Formal analysis, Z.-C.L.; Project administration, Y.-G.M.; Supervision, Y.-G.M.; Writing—original draft, Z.-C.L.; Writing—review & editing, Y.-G.M. All authors have read and agreed to the published version of the manuscript.

Funding: The research was funded by the National Natural Science Foundation of China under grant Nos. 11675081 and 12175108.

Acknowledgments: Z.-C.L. would like to thank X. Hao, Y. Sun, and Z.-M. Xu for helpful discussions. Y.-G.M. would like to thank H.P. Nilles for the warm hospitality during his stay at University of Bonn, and acknowledges the financial support from the Alexander von Humboldt Foundation under a renewed research program. In addition, the authors would like to thank the anonymous referees for the helpful comments that improve this work greatly.

Conflicts of Interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Notes

1 The entanglement entropy of subsystem A is defined as the von Neumann entropy S_A when the degrees of freedom in subsystem A's complement, subsystem B, are traced out. It describes how the two subsystems A and B are entangled or correlated each other.

2 A closely related quantity is the black hole entropy which is proportional to the horizon area according to the Bekenstein–Hawking formula. One can refer to Refs. [6–8] for related discussions about the black hole entropy from the viewpoint of quantum field theory. The entanglement entropy computed from the RT formula is equivalent to the black hole entropy in certain cases, such as in the AdS black hole [4] and in the black hole on a brane [9].

3 For instance, we can give another tetrad which will give rise to a complex noncommutative correction to minimal surface areas,

$$l^a = L\left(\frac{\delta_0^a}{z}\right), \qquad n^a = L\frac{\delta_1^a}{z}, \qquad m^a = L\left(\frac{\rho\delta_2^a}{2z^2} + \rho\delta_3^a\right), \qquad w^a = L\left(\frac{\delta_2^a}{2z^2} + \delta_3^a\right).$$

It is equivalent to Equation (4) in the commutative spacetime.

4 For the unphysical tetrad mentioned above in footnote 3, the corresponding η_{ab} takes the form

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

5 According to the RT formula mentioned in Introduction, the undeformed holographic entanglement entropy $(A_0/(4G_N))$ corresponds to the entanglement entropy of the boundary field between disk A and its complementary region B because the spacetime maintains the rotating symmetry.

6 Note $\epsilon' \equiv \epsilon l$, where l is the length scale of subsystem A, as to ϵ, it is dimensionless, see Equation (16).

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