universe

MDPI

Review

Basic Notions of Poisson and Symplectic Geometry in Local Coordinates, with Applications to Hamiltonian Systems

Alexei A. Deriglazov [1,2]

[1] Departamento de Matemática, Universidade Federal de Juiz de Fora, Juiz de Fora 36036-900, Brazil; alexei.deriglazov@ufjf.br

[2] Department of Physics, Tomsk State University, Lenin Prospekt 36, 634050 Tomsk, Russia

Abstract: This work contains a brief and elementary exposition of the foundations of Poisson and symplectic geometries, with an emphasis on applications for Hamiltonian systems with second-class constraints. In particular, we clarify the geometric meaning of the Dirac bracket on a symplectic manifold and provide a proof of the Jacobi identity on a Poisson manifold. A number of applications of the Dirac bracket are described: applications for proof of the compatibility of a system consisting of differential and algebraic equations, as well as applications for the problem of the reduction of a Hamiltonian system with known integrals of motion.

Keywords: Poisson geometry; symplectic geometry; Hamiltonian systems

check for updates

Citation: Deriglazov, A.A. Basic Notions of Poisson and Symplectic Geometry in Local Coordinates, with Applications to Hamiltonian Systems. *Universe* **2022**, *8*, 536. https://doi.org/10.3390/universe8100536

Academic Editor: George Papadopoulos

Received: 30 August 2022
Accepted: 12 October 2022
Published: 17 October 2022

Contents

1. Introduction

In modern classical mechanics, equations of motion for most mechanical and field models can be obtained as extreme conditions for a suitably chosen variational problem. If we restrict ourselves to mechanical models, the resulting system of Euler–Lagrange equations in the general case contains differential second-order and first-order equations, as well as algebraic equations. The structure of this system becomes more transparent after the transition to the Hamiltonian formalism, which studies the equivalent system of equations, with the latter no longer containing second-order equations. For the Euler–Lagrange system consisting only of second-order equations, the transition to the Hamiltonian formalism was already formulated at the dawn of the birth of classical mechanics. For the systems of a general form, the Hamiltonization procedure was developed by Dirac, and is known now as the Dirac formalism for constrained systems [1–4]. In the Dirac formalism, the Hamiltonian systems naturally fall into three classes, depending on the structure of algebraic equations presented in the system. According to the terminology adopted in [3], they are called the non-singular, singular non-degenerate, and singular degenerate theories.

The study of these Hamiltonian systems gave rise to a number of remarkable mathematical constructions. They are precisely the subject of investigation of Poisson and symplectic geometries [5–13]. In particular, the geometry behind a singular non-degenerate theory could be summarized by the diagram (136), that clarifies the geometric meaning of the famous Dirac bracket. This will be explored in Section 7.2 to study the structure of a singular non-degenerate dynamical system. The geometric methods are widely used in current literature, in particular, for the study of massive spinning particles and bodies in external fields as well as in the analysis of propagation of light in dispersive media and in gravitational fields [14–42].

In the rest of this section, we briefly describe the non-singular and singular non-degenerate theories[1].

1.1. Non-Singular Theories

Non-singular theories are mechanical systems that in the Hamiltonian formulation can be described using only first-order differential equations (called Hamiltonian equations)

$$\dot{q}^a = \frac{\partial H}{\partial p_a}, \qquad \dot{p}_a = -\frac{\partial H}{\partial q^a}, \tag{1}$$

where $H(q,p)$ is a given function and $\dot{q}^a = \frac{d}{d\tau}q^a$. The variables $q^a(\tau)$ describe the position of the system, while $p_a(\tau)$ are related to the velocities, and in simple cases are just proportional to them. The equations show that the function $H(q,p)$, called the Hamiltonian, encodes all the information about the dynamics of the mechanical system. The equations can be written in a more compact form if we introduce an operation assigning a new function to every pair of functions $A(q,p)$ and $B(q,p)$, denoted $\{A,B\}_P$, as follows:

$$\{A,B\}_P = \frac{\partial A}{\partial q^a}\frac{\partial B}{\partial p_a} - \frac{\partial B}{\partial q^a}\frac{\partial A}{\partial p_a}. \tag{2}$$

This is called the canonical Poisson bracket of A and B. Then, the Hamiltonian equations acquire the form

$$\dot{z}^i = \{z^i, H\}_P, \tag{3}$$

where $z^i = (q^a, p_b)$, $i = 1, 2, \ldots, 2n$. The equations determine the integral lines $z^i(\tau)$ of the vector field $\{z^i, H\}$ on $\mathbb{R}^{2n}$, created by function H. For smooth vector fields, the Cauchy problem, that is, Equation (1) with the initial conditions $z^i(\tau_0) = z^i_0$, has unique solution in a vicinity of any point $z^i_0 \in \mathbb{R}^{2n}$. The formal solution to these equations in terms of power series is as follows [4]

$$z^i(\tau, z^j_0) = e^{\tau\{z^k_0, H(z^i_0)\}_P \frac{\partial}{\partial z^k_0}} z^i_0. \tag{4}$$

The functions $z^i(\tau, z^j_0)$ depend on $2n$ arbitrary constants z^j_0, and hence represent a general solution to the system (3).

In the Lagrangian formalism, an analogue of this formula is not known. So, Equation (4) can be considered as the first example, showing the usefulness of the transition from the Lagrangian to the Hamiltonian description.

1.2. Singular Non-Degenerate Theories

Consider the system consisting of differential and algebraic equations

$$\dot{z}^i = \{z^i, H\}_P, \qquad \Phi^\alpha(z^i) = 0, \qquad \alpha = 1, 2, \ldots, 2p < 2n, \tag{5}$$

where $H(z^i)$ and $\Phi^\alpha(z^i)$ are given functions. It is supposed that $\Phi^\alpha(z^i)$ are functionally independent functions[2] (constraints), so the equations $\Phi^\alpha(z^i) = 0$ determine $2n - 2p$-dimensional surface $\mathbb{N}$. The system is called the singular non-degenerate theory if the following two conditions are satisfied. The first condition is

$$\det\{\Phi^\alpha, \Phi^\beta\}_P\Big|_{\Phi^\alpha=0} \neq 0, \tag{6}$$

hence the name "non-degenerate system". In the Dirac formalism, functions with the property (6) are called second-class constraints. The second condition is that the functions $\{\Phi^\alpha, H\}_P(z^i)$ vanish on the surface $\mathbb{N}$

$$\{\Phi^\alpha, H\}_P|_{\Phi^\alpha=0} = 0. \tag{7}$$

The two conditions guarantee the existence of solutions to the system (5). To discuss this point, we adopt the following.

Definition 1. *The system (5) is called self-consistent if a solution of the system passes through any point of the surface* $\mathbb{N}$.

For a self-consistent system, its formal solution can be written as in (4), and it is sufficient to take the integration constants z^i_0 on the surface of constraints.

Let us discuss the self-consistency of the system. Given a point of the surface $\mathbb{N}$, there is a unique solution of the first from Equation (5) that passes through this point. It will be a solution of the whole system if it entirely lies on the surface:

$$\dot{z}^i = \{z^i, H\}_P \quad \text{and} \quad \Phi^\alpha(z^i(0)) = 0, \quad \text{implies} \quad \Phi^\alpha(z^i(\tau)) = 0 \quad \text{for all} \quad \tau. \tag{8}$$

This is a strong requirement, and Equations (6) and (7) turn out to be the sufficient conditions for its fulfillment. In a physical context, the proof with use of special coordinates of

$\mathbb{R}^{2n}$ was done in [3]. A more simple proof with use of the Dirac bracket will be presented in Section 7.2.

An example of a self-consistent system as in (5) will be considered in Section 7 (see Affirmation 30).

Here, we discuss the necessity of the condition (7).

Affirmation 1. Consider the system (5) with functionally independent functions Φ^α. Then, $\{\Phi^\alpha, H\}_P|_{z^i(\tau)} = 0$ for any solution $z^i(\tau)$, if any. That is, the algebraic equations $\{\Phi^\alpha, H\}_P = 0$ are consequences of the system.

Proof. Let the system admit the solution $z^i(\tau)$. Then, $\Phi^\alpha(z^i(\tau)) = 0$ for all τ, which implies $\dot{\Phi}^\alpha(z^i(\tau)) = 0$. On other hand, we obtain

$$0 = \dot{\Phi}^\alpha = \left.\frac{\partial \Phi^\alpha}{\partial z^i}\right|_{z^i(\tau)} \dot{z}^i(\tau) = \left.\frac{\partial \Phi^\alpha}{\partial z^i}\right|_{z(\tau)} \{z^i, H\}_P\Big|_{z(\tau)} = \{\Phi^\alpha, H\}_P|_{z(\tau)}. \tag{9}$$

In other words, $\{\Phi^\alpha, H\}_P = 0$ for any solution $z^i(\tau)$. $\square$

Affirmation 2. If the system (5) with functionally independent functions Φ^α is self-consistent, the conditions (7) hold.

Proof. Let z^i_0 be any point of the surface $\Phi^\alpha = 0$. Due to the self-consistency, there is a solution $z^i(\tau)$ that passes through this point, $z^i(0) = z^i_0$. As the equation$\{\Phi^\alpha, H\}_P = 0$ is a consequence of the system, we have $\{\Phi^\alpha, H\}_P(z^i(\tau)) = 0$, in particular $\{\Phi^\alpha, H\}_P(z^i(0)) = \{\Phi^\alpha, H\}_P(z^i_0) = 0$, that is, it vanishes at all points of the surface $\mathbb{N}$. $\square$

Consider the system (5), and now suppose that some of the functions $\{\Phi^\alpha, H\}_P$ do not vanish identically on the surface $\mathbb{N}$. As we saw above, this means that the system is not self-consistent. Then, we can search for a sub-surface of $\mathbb{N}$ where the system could be self-consistent. The procedure is as follows. We separate the functionally independent functions among $\{\Phi^\alpha, H\}_P$, say $\Psi^1, \Psi^2, \ldots, \Psi^k$. As the equations $\Psi^a = 0$ are consequences of the system (5), we add them to the system, obtaining an equivalent system of equations. If the set Φ^α, Ψ^a is composed of functionally independent functions, we repeat the procedure, analyzing the functions $\{\Psi^a, H\}$, and so on. As the number of functionally independent functions cannot be more than $2n$, the procedure will end at some step. If, in addition to this, the resulting set of functions satisfies the condition (6), we arrive at the self-consistent system of equations $\dot{z}^i = \{z^i, H\}_P$, $\Phi^\alpha(z^i) = 0$, $\Psi^a = 0, \ldots$.

It remains to discuss what happens if at some stage, the extended system of algebraic equations consists of functionally dependent functions. Without loss of generality, we assume that the extended system is $\dot{z}^i = \{z^i, H\}_P$, $\Phi^\alpha = 0$, $\Psi \equiv \{\Phi^1, H\} = 0$. By construction, it is equivalent to the original system and the function $\Psi(z^i)$ does not vanish identically on $\mathbb{N}$, and the functions Φ^α, Ψ are functionally dependent. As Φ^α are functionally independent, we present the equations $\Phi^\alpha(z^i) = 0$ in the form $z^\alpha = f^\alpha(z^b)$, and substitute them into the expression for $\Psi(z^i)$, obtaining the system $\dot{z}^i = \{z^i, H\}_P$, $z^\alpha - f^\alpha(z^b) = 0$, $\Psi(z^b, f^\alpha(z^b)) = 0$, which is equivalent to (5). The same is true for the function $\Psi(z^b, f^\alpha(z^b)) \neq 0$. On the other hand, it does not depend on z^b (otherwise we could write it in the form $z^1 = \psi(z^2, z^3, \ldots)$, then the functions $z^\alpha - f^\alpha(z^b)$, $z^1 - \psi(z^2, z^3, \ldots)$ are functionally independent). So, the only possibility is $\Psi = c = const \neq 0$. This means that the system (5) contains the equation $c = 0$, where $c \neq 0$. Hence, the system is contradictory and has no solutions at all.

It should be noted that the outlined procedure for obtaining a self-consistent system lies at the corner of the Dirac method [1,2].

Because all trajectories of the system (5) lie on the surface $\Phi^\alpha(z^i) = 0$ with coordinates, say z^b, a number of questions naturally arise. Can equations for independent variables z^b be written in the form of a Hamiltonian system such as the first equation from (5)? What are the Hamiltonian $H(z^b)$ and the bracket $\{,\}_\mathbb{N}$ in these equations, and how they should be constructed? Is the new bracket a kind of restriction on the original one to $\mathbb{N}$? The answers to these questions will be given in Section 7. In particular, we will show that the new bracket is a restriction of the Dirac bracket to $\mathbb{N}$ and not a restriction of the original bracket.

2. Poisson Manifold

2.1. Smooth Manifolds

In this subsection, we fix our notation and recall some basic notions of the theory of differentiable manifolds that will be useful in what follows.

Notation. Latin indices from the middle of alphabet are used to represent coordinates z^k of a manifold $\mathbb{M}_n$ and run from 0 to n. If coordinates are divided on two groups, we write $y^k = (y^\alpha, y^b)$, that is, Greek indices from the beginning of alphabet are used to represent one group, while Latin indices from the beginning of alphabet represent another group. Notation such as $U_i(z^j)$ means that we work with the functions $U_i(z^1, z^2, \ldots, z^n)$, where $i = 1, 2, \ldots, n$. Notation such as $\partial_i A(z^k)|_{z^k \to f^k(y^j)}$ indicates that in the expression $(\partial A(z^k)/\partial z^i)$, the symbols z^k should be replaced on the functions $f^k(y^j)$. We often denote the inverse matrix ω^{-1} as $\tilde{\omega}$. We use the standard convention of summing over repeated indices. Because we are working in local coordinates, all statements should be understood locally, that is, they are true in some vicinity of the point in question.

Definition 2. *Vector space $\mathbb{V} = \{\vec{V}, \vec{U}, \ldots\}$ is called the Lie algebra if on $\mathbb{V}$ is defined the bilinear mapping $[,] : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ (called the Lie bracket), with the properties*

$$[\vec{V}, \vec{U}] = -[\vec{U}, \vec{V}] \qquad \textit{(antisymmetric)}, \quad (10)$$

$$[\vec{V}, [\vec{U}, \vec{W}]] + [\vec{U}, [\vec{W}, \vec{V}]] + [\vec{W}, [\vec{V}, \vec{U}]] \equiv [\vec{V}, [\vec{U}, \vec{W}]] + cycle = 0 \quad \textit{(Jacobi identity)}. \quad (11)$$

Due to the bilinearity, all properties of the Lie bracket are encoded in the Lie brackets of basic vectors T^i: $[\vec{V}, \vec{U}] = V_i U_j [T^i, T^j]$. Because $[T^i, T^j] = \vec{W} \in \mathbb{V}$, we can expand W on the basis T^i, obtaining

$$[T^i, T^j] = c^{ij}{}_k T^k, \quad (12)$$

where the numbers $c^{ij}{}_k$ are called the structure constants of the algebra in the basis T^i. The conditions (10) and (11) are satisfied if the structure constants obey (Exercise)

$$c^{ij}{}_k = -c^{ji}{}_k, \qquad c^{ij}{}_a c^{ak}{}_b + cycle(i, j, k) = 0. \quad (13)$$

Example 1. *For the three-dimensional vector space with elements $V = v_i T^i$, $i = 1, 2, 3$, let us define $[T^i, T^j] = \epsilon^{ijk} T^k$, where ϵ^{ijk} is the Levi–Chivita symbol with $\epsilon^{123} = 1$. It can be verified that the set $c^{ij}{}_k \equiv \epsilon^{ijk}$ has the properties (13), so the vector space turns into a Lie algebra. It is called the Lie algebra of three-dimensional group of rotations (see Section 1.2 in [4] for details).*

Let $\mathbb{M}_n = \{z, y, \ldots\}$ be an n-dimensional manifold, and $\mathbb{F}_{\mathbb{M}} = \{A, B, \ldots\}$ be a space of scalar functions on $\mathbb{M}_n$, that is, the mappings $A : \mathbb{M}_n \to \mathbb{R}$. Let z^i be local coordinates on $\mathbb{M}_n$, that is, we have an isomorphism $z \in \mathbb{M}_n \to z^i(z) \in \mathbb{R}^n$. If z'^i is another coordinate system, we have the relations

$$z'^i = \varphi^i(z^j), \qquad z^j = \tilde{\varphi}^j(z'^i), \qquad \tilde{\varphi}^i(\varphi^j(z^k)) = z^i. \quad (14)$$

Let, in the coordinates z^i and z'^i, the mapping A be represented by the functions $A(z^i) : \mathbb{R}^n \to \mathbb{R}$ and $A'(z'^i) : \mathbb{R}^n \to \mathbb{R}$. They are related by

$$A'(z'^i) = A(z^j)\Big|_{z^j \to \tilde{\varphi}^j(z'^i)} \equiv A(\tilde{\varphi}^j(z'^i)). \quad (15)$$

We call (15) the transformation law of a scalar function in the passage from z^i to z'^i. In a certain abuse of terminology, we often say "scalar function $A(z^i)$" instead of that "the function $A(z^i)$ is representative of a scalar function $A : \mathbb{M}_n \to \mathbb{R}$ in the coordinates z^i".

Example 2. *Scalar function of a coordinate. Given a coordinate system z^i, define the scalar function $A^1 : z \to z^1$, where z^1 is the first coordinate of the point z in the system z^i. In the coordinates z^i, the mapping is represented by the following function: $A^1(z^1, z^2, \ldots, z^n) = z^1$. In the coordinates $z'^i = \varphi^i(z^j)$, it is represented by $A'^1(z'^1, z'^2, \ldots, z'^n) = z^1|_{z^j \to \tilde{\varphi}^j(z'^i)} = \tilde{\varphi}^1(z'^1, z'^2, \ldots, z'^n)$.*

We often write $z'^i(z^j)$ instead of $\varphi^i(z^j)$, $z^j(z'^i)$ instead of $\tilde{\varphi}^j(z'^i)$, and use the notation $z'^i \equiv z^{i'}$. In the latter case, i' and i, when they appear in the same expression, are considered as two different indexes. For instance, in these notations the scalar function of z^1-coordinate in the system $z^{i'} = z^{i'}(z^j)$ is represented by the function $z^1(z^{i'})$.

Exercise 1. Observe that (14) implies that derivatives of the transition functions φ and $\tilde{\varphi}$ form the inverse matrices

$$\left.\frac{\partial \tilde{\varphi}^i}{\partial z'^k}\right|_{z' \to \varphi(z)} \frac{\partial \varphi^k}{\partial z^j} = \delta^i{}_j \quad \text{or, in short notation} \quad \frac{\partial z^i}{\partial z^{k'}} \frac{\partial z^{k'}}{\partial z^j} = \delta^i{}_j. \tag{16}$$

Given the curve $z^i(\tau) \in \mathbb{M}_n$ with $z^i(0) = z^i_0$, the numbers $V^i = \dot{z}^i(0)$ are called components (coordinates) of tangent vector to the curve at the point z^i_0. If $V^{i'}$ are components of the tangent vector in the coordinates $z^{i'}$, we have the relation $V^{i'} = \left.\frac{\partial z^{i'}}{\partial z^i}\right|_{z_0} V^i$. The set of tangent vectors at z_0 is an n-dimensional vector space denoted $\mathbb{T}_{\mathbb{M}_n}(z_0)$.

We say that we have a vector field $\vec{V}(z)$ on $\mathbb{M}_n$ if in each coordinate system z^j the set of functions $V^i(z^j)$ is defined with the transformation law

$$V^{i'}(z^{j'}) = \left.\frac{\partial z^{i'}}{\partial z^i} V^i(z^k)\right|_{z \to z(z')}. \tag{17}$$

The space of all vector fields on $\mathbb{M}_n$ is denoted $\mathbb{T}_{\mathbb{M}_n}$. In the tensor analysis, $\vec{V}(z)$ is called the contravariant vector field.

We say that we have a covariant vector field $U(z)$ on $\mathbb{M}_n$ if in each coordinate system z^j the set of functions $U_i(z^j)$ is defined with the transformation law

$$U_{i'}(z^{j'}) = \left.\frac{\partial z^i}{\partial z^{i'}} U_i(z^k)\right|_{z \to z(z')}. \tag{18}$$

Gradient of a scalar function A is an example of the covariant vector field. Its components are $U_i = \partial_i A$.

Exercise 2. Let $A(z^i) = \frac{1}{2}[(z^1)^2 + (z^2)^2 + (z^3)^2]$ represent a scalar function in the coordinates z^i. Then, in the coordinates $z^{i'}$, defined by (14), it is represented by $A'(z^{i'}) = \frac{1}{2}[(\tilde{\varphi}^1(z^{i'}))^2 + (\tilde{\varphi}^2(z^{i'}))^2 + (\tilde{\varphi}^3(z^{i'}))^2]$. Gradients of these functions are $U_i = z^i$ and $U_{i'} = \tilde{\varphi}^j(z^{i'}) \frac{\partial \tilde{\varphi}^j(z^{i'})}{\partial z^{i'}}$. Confirm that the two gradients are related by observing Equation (18).

Similarly to this, contravariant tensor of the second rank is a quantity with the transformation law

$$\omega^{i'j'}(z^{k'}) = \left.\frac{\partial z^{i'}}{\partial z^i} \frac{\partial z^{j'}}{\partial z^j} \omega^{ij}(z^m)\right|_{z \to z(z')}, \tag{19}$$

and so on.

Exercise 3. Contraction of ω with covariant vector field *grad A* gives a quantity with the components $V^i = \omega^{ij}\partial_j A$. Confirm that $\vec{V}$ is a contravariant vector field.

The integral line of the vector field $V^i(z^k)$ on $\mathbb{M}_n$ is a solution $z^i(\tau)$ to the system $\frac{dz^i(\tau)}{d\tau} = V^i(z^k(\tau))$. We assume that $V^i(z^k)$ is a smooth field, so a unique integral line $\vec{V}$ passes through each point of the manifold.

Submanifold of $\mathbb{M}_n$. The k-dimensional submanifold $\mathbb{N}_k^{\vec{c}} \in \mathbb{M}_n$ is often defined as a constant-level surface of a set of functionally independent scalar functions $\Phi^\alpha(z)$

$$\mathbb{N}_k^{\vec{c}} = \{z \in \mathbb{M}_n,\ \Phi^\alpha(z^k) = c^\alpha,\ \alpha = 1, 2, \dots n-k\}, \tag{20}$$

where c^α are given numbers.

We recall that the scalar functions $\Phi^\alpha(z)$, $\alpha = 1, 2, \dots, n-k$ are called functionally independent if, for their representatives $\Phi^\alpha(z^i)$ in the coordinates z^i, we have $rank\ (\partial_i \Phi^\alpha) = n-k$. This implies that covariant vectors $V_{(\alpha)}$ with coordinates $V_{(\alpha)i} = \partial_i \Phi^\alpha$ are linearly independent. The equations $\Phi^\alpha(z^i) = c^\alpha$ for the functionally independent functions can be resolved: $z^\alpha = f^\alpha(z^a)$, $a = 1, 2, \dots, k$. So, the coordinates z^i are naturally divided on two groups, (z^α, z^a), and z^a, $a = 1, 2, \dots, k$, which can be taken as local coordinates of the submanifold $\mathbb{N}_k^{\vec{c}}$. Below, we always assume that the coordinates have been grouped in this way, and $\det \frac{\partial \Phi^\alpha}{\partial z^\beta} \neq 0$.

If we have only one function $\Phi(z)$, we assume that it has a non-vanishing gradient, $rank\ (\partial_i \Phi) = 1$.

Taking $c^\alpha = 0$ in (20), we have the surface of level zero

$$\mathbb{N}_k = \{z \in \mathbb{M}_n,\ \Phi^\alpha(z^k) = 0,\ \ \alpha = 1, 2, \dots n-k\}. \tag{21}$$

Let us introduce the notions that will be useful in discussing the Frobenius theorem (see Appendix A.3).

For the curve $z^i(\tau) \subset \mathbb{N}_k \subset \mathbb{M}_n$ with $z^i(0) = z_0^i$, the tangent vector $V^i(z_0) = \frac{dz^i(0)}{d\tau} \in \mathbb{T}_{\mathbb{M}}(z_0)$ is called a tangent vector to $\mathbb{N}_k$ at z_0. The set of all tangent vectors at z_0 is a k-dimensional vector space denoted as $\mathbb{T}_{\mathbb{N}}(z_0)$. For any such vector, the equality $V^i \partial_i \Phi^\alpha(z_0) = 0$ holds[3].

The vector field $V^i(z)$ on $\mathbb{M}_n$ is tangent to $\mathbb{N}_k$ if any integral curve of $V^i(z)$ crossing $\mathbb{N}_k$ lies entirely in $\mathbb{N}_k$: $\Phi^\alpha(z^k(0)) = 0$ implies $\Phi^\alpha(z^k(\tau)) = 0$ for any τ. The vector field $V^i(z)$ on $\mathbb{M}_n$ touches the surface $\mathbb{N}_k$ if $V^i \partial_i \Phi^\alpha|_{z_0} = 0$ for any $z_0 \in \mathbb{N}_k$ the tangent field touches the surface. The converse is not true.

Foliation of $\mathbb{M}_n$. The set $\{\mathbb{N}_k^{\vec{c}},\ \vec{c} \in \mathbb{R}^{n-k}\}$ of the submanifolds (20) is called a foliation of $\mathbb{M}_n$, while $\mathbb{N}_k^{\vec{c}}$ are called leaves of the foliation. Notice that submanifolds with different $\vec{c}$ do not intercept, and any[4] $z \in \mathbb{M}_n$ lies in one of $\mathbb{N}_k^{\vec{c}}$.

There are coordinates, naturally adapted with the foliation: $z^k \to y^k = (y^\alpha, y^a)$, with the transition functions $y^a = z^a$, $y^\alpha = \Phi^\alpha(z^\beta, z^b)$. In these coordinates the submanifolds $\mathbb{N}_k^{\vec{c}}$ appear similar to hyperplanes:

$$\mathbb{N}_k^{\vec{c}} = \{y^i \in \mathbb{M}_n,\ y^\alpha = c^\alpha\}, \tag{22}$$

and $y^a = z^a$ can be taken as local coordinates of $\mathbb{N}_k^{\vec{c}}$. The useful identity is

$$A(z^i(y^j))\Big|_{y^\alpha = 0} = A(f^\alpha(z^a), z^a)|_{z^a \to y^a}. \tag{23}$$

The Lie bracket (commutator) of vector fields is the bilinear operation $[\ ,\ ] : \mathbb{T}_{\mathbb{M}_n} \times \mathbb{T}_{\mathbb{M}_n} \to \mathbb{T}_{\mathbb{M}_n}$, that with each pair of vector fields $\vec{V}$ and $\vec{U}$ of $\mathbb{T}_{\mathbb{M}_n}$ associates the vector field $[\vec{V}, \vec{U}]$ of $\mathbb{T}_{\mathbb{M}_n}$ according to the rule

$$[\vec{V}, \vec{U}]^i = V^j \partial_j U^i - U^j \partial_j V^i. \tag{24}$$

The quantity $[\vec{V},\vec{U}]^i$ is indeed a vector field, which can be verified by direct computation. We have $[\vec{V},\vec{U}]^i = V^{j'}\partial_{j'}(\frac{\partial z^i}{\partial z^{i'}}U^{i'}) - (V \leftrightarrow U) = \frac{\partial z^i}{\partial z^{i'}}(V^{j'}\partial_{j'}U^{i'} - (V \leftrightarrow U)) = \frac{\partial z^i}{\partial z^{i'}}[\vec{V}',\vec{U}']^{i'}$, in agreement with Equation (17). The Lie bracket has the properties (10) and (11) and turns the space of vector fields into infinite-dimensional Lie algebra.

Each vector field determines a linear mapping $\vec{V} : \mathbb{F}_{\mathbb{M}_n} \to \mathbb{F}_{\mathbb{M}_n}$ on the space of scalar functions according to the rule

$$\vec{V} : A \to \vec{V}(A) = V^i\partial_i A. \tag{25}$$

Notice that $\vec{V}(A) = 0$ for all A implies $V^i = 0$. Then, the Lie bracket can be considered as a commutator of two differential operators

$$[\vec{V},\vec{U}](A) = \vec{V}(\vec{U}(A)) - \vec{U}(\vec{V}(A)). \tag{26}$$

Using this formula, it is easy to confirm the Jacobi identity for the Lie bracket (24) by direct computation (11).

2.2. The Mapping of Manifolds and Induced Mappings of Tensor Fields

Given two manifolds $\mathbb{N}_k = \{x^a\}$, $\mathbb{M}_n = \{z^i\}$, consider the functions $z^i = \phi^i(x^a)$. They determine the mapping

$$\phi : \mathbb{N}_k \to \mathbb{M}_n, \qquad x^a \to z^i = \phi^i(x^a) \equiv z^i(x^a). \tag{27}$$

If ϕ is an injective function *rank* $\frac{\partial \phi^i}{\partial x^a} = k$, the image of the mapping is a k-dimensional submanifold of $\mathbb{M}_n$: $\mathbb{N}_k = \{z^i \in \mathbb{M}_n,\ z^\alpha - f^\alpha(z^a) = 0\}$, where the equalities $z^\alpha = f^\alpha(z^a)$ are obtained excluding x^a from the equations $z^i = \phi^i(x^a)$. In some cases [6], the manifold $\mathbb{N}_k$ can be identified with this submanifold of $\mathbb{M}_n$.

Conversely, let $\mathbb{N}_k \subset \mathbb{M}_n$. Then, the parametric equations $z^\alpha = f^\alpha(z^b)$ of the submanifold (21) can be considered as determining the mapping of embedding

$$\eta : \mathbb{N}_k = \{z^b\} \to \mathbb{M}_n = \{z^i\}, \quad z^b \to z^i = (z^\alpha, z^b), \quad \text{where} \quad z^\alpha = f^\alpha(z^b). \tag{28}$$

Using the mapping (27), some geometric objects from one manifold can be transferred to another. We start from the spaces of covariant and contravariant tensors *at the points* x_0 and $z_0 = \phi(x_0)$. Take, for definiteness, the second-rank tensors. Given $U_{ij}(z_0)$, we can construct the induced tensor $U_{ab}(x_0)$

$$\mathbb{T}^{(0,2)}_{\mathbb{M}} \to \mathbb{T}^{(0,2)}_{\mathbb{N}}, \qquad U_{ij}(z_0) \to U_{ab}(x_0) = \frac{\partial z^i(x_0)}{\partial x^a}\frac{\partial z^j(x_0)}{\partial x^b}U_{ij}(z_0). \tag{29}$$

Given $V^{ab}(x_0)$, we can construct the induced tensor $U^{ij}(z_0)$

$$\mathbb{T}^{(2,0)}_{\mathbb{N}} \to \mathbb{T}^{(2,0)}_{\mathbb{M}}, \qquad V^{ab}(x_0) \to V^{ij}(z_0) = \frac{\partial z^i(x_0)}{\partial x^a}\frac{\partial z^j(x_0)}{\partial x^b}V^{ab}(x_0). \tag{30}$$

For the case of the vector, the notion of induced mapping,

$$V^i(z_0) = \frac{\partial z^i(x_0)}{\partial x^a}V^a(x_0), \tag{31}$$

is consistent with the notion of a tangent vector: if $V^a = \frac{dx^a}{d\tau}$ is a tangent vector to the curve $x^a(\tau)$, then V^i, given by (31), is a tangent vector to the image $z^i(x^a(\tau))$

$$V^i(z(\tau)) = \frac{d}{d\tau}z^i(x^a(\tau)). \tag{32}$$

Concerning the fields on the manifolds, ϕ naturally induces the mappings of scalar functions and covariant tensor fields. For the functions, the induced mapping

$$\phi^* : \mathbb{F}_{\mathbb{M}_n} \to \mathbb{F}_{\mathbb{N}_k}, \qquad A(z^i) \to \bar{A}(x^a) = A(z^i(x^a)), \tag{33}$$

is just the composition $\bar{A} = A \circ \phi$. For the covariant tensor fields, we have

$$\phi_* : \mathbb{T}^{(0,2)}_{\mathbb{M}_n} \to \mathbb{T}^{(0,2)}_{\mathbb{N}_k}, \qquad U_{ij}(z^i) \to U_{ab}(x^a) = \frac{\partial z^i}{\partial x^a}\frac{\partial z^j}{\partial x^b}U_{ij}(z^i(x^a)). \tag{34}$$

Notice that the contravariant tensor fields cannot be transferred to another manifold (submanifold) in this manner. As we will see in the next section, the Poisson structure on $\mathbb{M}_n$ is determined mainly by the second-rank contravariant tensor. Hence, it can not be directly transferred to a submanifold. This is found to be possible in a special case of Casimir submanifolds (see Section 4.2) and leads to the Dirac bracket (see Section 6.2).

Note also that if two contravariant fields on the manifolds $\mathbb{M}_n$ and $\mathbb{N}_k$ are given, we can of course compare them using the mapping (27) (see Equation (67) below as an example).

2.3. Poisson Manifold

Let a bilinear mapping on the space of functions $\mathbb{F}_{\mathbb{M}}$ be defined as $\{,\} : \mathbb{F}_{\mathbb{M}} \times \mathbb{F}_{\mathbb{M}} \to \mathbb{F}_{\mathbb{M}}$ (called the Poisson bracket), with the properties

$$\{A, B\} = -\{B, A\} \qquad \text{(antisymmetric)}, \tag{35}$$

$$\{A, \{B, C\}\} + cycle = 0 \qquad \text{(Jacobi identity)}, \tag{36}$$

$$\{A, BC\} = \{A, B\}C + \{A, C\}B \qquad \text{(Leibnitz rule)}. \tag{37}$$

When $\mathbb{F}_{\mathbb{M}}$ is equipped with the Poisson bracket, the manifold $\mathbb{M}_n$ is called the Poisson manifold. Comparing (35) and (36) with (10) and (11), we see that the infinite-dimensional vector space $\mathbb{F}_{\mathbb{M}}$ is equipped with the structure of a Lie algebra.

Exercise 4. Show that constant functions, $A(z) = c$ for any z, have vanishing brackets (commute) with all other functions.

One of the ways to define the Poisson structure on $\mathbb{M}_n$ is as follows.

Affirmation 3. Let $\omega^{ij}(z)$ be the contravariant tensor of second rank on $\mathbb{M}_n$. The mapping

$$\{A, B\} = \partial_i A\, \omega^{ij}\, \partial_j B, \tag{38}$$

determines the Poisson bracket if the tensor ω obeys the properties

$$\omega^{ij} = -\omega^{ji} \qquad \text{(antisymmetric)}, \tag{39}$$

$$\omega^{ip}\partial_p\omega^{jk} + cycle(i, j, k) = 0. \tag{40}$$

In particular, each numeric antisymmetric matrix determines a Poisson bracket. We call ω the Poisson tensor.

Proof. First, we note that (39) implies (35). Second, the mapping (38), being a combination of derivatives, is bilinear and automatically obeys the Leibnitz rule. To complete the proof, we need to show that (40) is equivalent to (36). Using (40), by direct calculation we obtain

$$\begin{aligned}\{A, \{B, C\}\} + cycle(A, B, C) = {} & \partial_i A\partial_j B\partial_k C\omega^{ip}\partial_p\omega^{jk} + cycle(A, B, C) + \\ & \omega^{ip}\omega^{jk}\partial_p\left[\partial_i A\partial_j B\partial_k C\right] + cycle(A, B, C)\end{aligned} \tag{41}$$

By direct calculation, we can show also that in the first term on the right-hand side, the $cycle(A, B, C)$ is equivalent to $cycle(i, j, k)$. So, we write the previous equality as

$$\{A, \{B, C\}\} + cycle(A, B, C) = \partial_i A\partial_j B\partial_k C\left[\omega^{ip}\partial_p\omega^{jk} + cycle(i, j, k)\right] +$$

$$\omega^{ip}\omega^{jk}\partial_p\left[\partial_i A\partial_j B\partial_k C + \partial_j A\partial_k B\partial_i C + \partial_k A\partial_i B\partial_j C\right]. \quad (42)$$

The second line in this equality vanishes identically due to the symmetry properties of this term. Indeed, we write the first term of the line as follows:

$$\omega^{ip}\omega^{jk}\partial_p\left[\partial_i A\partial_j B\partial_k C\right] = \omega^{ip}\omega^{jk}\partial_p\partial_i\left[A\partial_j B\partial_k C\right] -$$
$$\omega^{ip}\omega^{jk}\partial_p\left[A\partial_i\partial_j B\partial_k C + A\partial_j B\partial_i\partial_k C\right] = \omega^{ip}\omega^{jk}\partial_p\left[A\partial_k(\partial_i B\partial_j C - [i\leftrightarrow j])\right]. \quad (43)$$

We write the two remaining terms of the line as

$$\omega^{ip}\omega^{jk}\partial_p\left[\partial_j A\partial_k B\partial_i C + \partial_k A\partial_i B\partial_j C\right] = \omega^{ip}\omega^{jk}\partial_p\left[\partial_k A(\partial_i B\partial_j C - [i\leftrightarrow j])\right] =$$
$$\omega^{ip}\omega^{jk}\partial_p\partial_k\left[A(\partial_i B\partial_j C - [i\leftrightarrow j])\right] - \omega^{ip}\omega^{jk}\partial_p\left[A\partial_k(\partial_i B\partial_j C - [i\leftrightarrow j])\right]. \quad (44)$$

The last terms in (43) and (44) cancel each other out, while the first term in (44) is zero, being the trace of the product of symmetric $D^{ij} \equiv \omega^{ip}\omega^{jk}\partial_p\partial_k$ and antisymmetric $E_{ij} \equiv A(\partial_i B\partial_j C - [i\leftrightarrow j])$ quantities. Thus, we have obtained the identity

$$\omega^{ip}\omega^{jk}\partial_p\left[\partial_i A\partial_j B\partial_k C + \partial_j A\partial_k B\partial_i C + \partial_k A\partial_i B\partial_j C\right] = 0. \quad (45)$$

Taking this into account in (42), we see the equivalence of the conditions (36) and (40). □

Affirmation 4. Let the bracket (38) obey the Jacobi identity in the coordinates z^i. Then, the Jacobi identity is satisfied in all other coordinates.

This is an immediate consequence of tensor character of involved quantities. Indeed, the bracket $\{A,B\} = \partial_i A\omega^{ij}\partial_j B$ is a contraction of three tensors and as such, is a scalar function under diffeomorphisms. Then, the same is true for $\{A,\{B,C\}\}$. Let us denote the left-hand side of the Jacobi identity as $D(z)$. Then, the Jacobi identity is the coordinate-independent statement that the scalar function $D(z)$ identically vanishes for all $z \in \mathbb{M}_n$. This can also be verified by direct computation (see Appendix A). As a consequence, the left-hand side of Equation (40) is a tensor of the third rank[5].

For the scalar functions of coordinates (see Example 2), the Poisson bracket (38) reads

$$\{z^i, z^j\} = \omega^{ij}. \quad (46)$$

In classical mechanics, these equalities are known as fundamental brackets of the coordinates. Observe that the identity (40) can be written as follows: $\{z^i,\{z^j,z^k\}\} + cycle(i,j,k) = 0$.

The bracket (38) is called non-degenerate if $\det\omega \neq 0$, and degenerate when $\det\omega = 0$. Examples will be presented below: (59) is non-degenerate while (63) is degenerate. The structure of the matrix ω depends on its rank, and becomes clear in the so-called canonical coordinates specified by the following theorem:

Generalized Darboux theorem. Let $rank\,\omega = 2k$ at the point $z^i \in \mathbb{M}_n$. Then, there are local coordinates $z^{i'} = (z^{\beta'}, z^{a'})$, $a' = 1,2.\ldots,2k$, $\beta' = 1,2,\ldots,p = n-2k$ such that ω in some vicinity of z^i has the form:

$$\omega^{i'j'} = \begin{pmatrix} 0_{p\times p} & 0 & 0 \\ 0 & 0_{k\times k} & 1_{k\times k} \\ 0 & -1_{k\times k} & 0_{k\times k} \end{pmatrix}, \quad (47)$$

or

$$\omega^{a'b'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \omega^{\beta' j'} = \omega^{j'\beta'} = 0, \quad \text{where} \quad j' = 1,2,\ldots,n. \quad (48)$$

A proof is given in Appendix A.2. We recall that determinant of any odd-dimensional antisymmetric matrix vanishes; this implies that *rank* ω is necessary an even number, as it is written above. Let us further denote $z^{a'} = (q^1, q^2, \ldots, q^k, p_1, p_2, \ldots, p_k)$. Then, in terms of fundamental brackets, the equalities (48) can be written as follows:

$$\{q^{a'}, p_{b'}\} = \delta^{a'}{}_{b'}, \qquad \{q^{a'}, q^{b'}\} = 0, \qquad \{p_{a'}, p_{b'}\} = 0, \qquad \{z^{j'}, z^{\beta'}\} = 0. \tag{49}$$

3. Hamiltonian Dynamical Systems on a Poisson Manifold

3.1. Hamiltonian Vector Fields

Using the Poisson structure (38), with each function $H(z^i) \in \mathbb{F}_{\mathbb{M}}$ we can associate the contravariant vector field $X^i_H \equiv \omega^{ij}\partial_j H = \{z^i, H\} \in \mathbb{T}_{\mathbb{M}}$. That is, we have the mapping

$$\omega : \mathbb{F}_{\mathbb{M}} \to \mathbb{T}_{\mathbb{M}}, \quad \omega : H \to [\omega(H)]^i = \omega^{ij}\partial_j H, \quad \text{we also denote} \quad \omega(H) \equiv \vec{X}_H \in \mathbb{T}_{\mathbb{M}}. \tag{50}$$

$\vec{X}_H$ is called the Hamiltonian vector field of the function H. Then,

$$\dot{z}^i = \{z^i, H\} \equiv \omega^{ij}\partial_j H, \tag{51}$$

are called Hamiltonian equations and the scalar function H is called the Hamiltonian. Solutions $z^i(\tau)$ of the equations are called integral lines of the vector field $\{z^i, H\}$ created by H on $\mathbb{M}_n$. We assume that $\vec{X}_H$ is a smooth vector field, so the Cauchy problem for (51) has a unique solution in a vicinity of any point of $\mathbb{M}_n$. $\vec{X}_H$ at each point is tangent vector to the integral line that passes through this point.

Let $z^k(\tau)$ be integral line of $\vec{X}_A$ and B be scalar function. Then, we can write

$$\frac{d}{d\tau} B(z^k(\tau)) = \{B, A\}|_{z(\tau)}. \tag{52}$$

Using this equality, and the fact that integral lines pass through each point of $\mathbb{M}_n$, it is easy to prove the three affirmations presented below. They will be repeatedly used (and sometimes rephrased, see Section 5) in our subsequent considerations.

Affirmation 5. The integral line of $\vec{X}_H$ entirely lies on one of the surfaces $H(z^k) = c = const$. In classical mechanics, it is simply the law of energy conservation.

Denote $\vec{X}_{(j)}$ as the Hamiltonian vector field associated with scalar function of the coordinate z^j. Its components are $X^i_{(j)} = \omega^{ik}\partial_k z^j = \omega^{ij}$. Hence, the Poisson matrix can be considered[6] to be composed of the columns $\vec{X}_{(j)}$

$$\omega = (\vec{X}_{(1)}, \vec{X}_{(2)}, \ldots, \vec{X}_{(n)}). \tag{53}$$

According to Affirmation 5, the integral lines of the vector $\vec{X}_{(j)}$ lie on the hyperplanes $z^j = const$.

Affirmation 6. Given the scalar functions H and Q^α, $\alpha = 1, 2, \ldots, n-k$, the following two conditions are equivalent:
(A) Integral lines of $\vec{X}_H$ lie in the submanifolds $\mathbb{N}^{\vec{c}}_k = \{z \in \mathbb{M}_n, \quad Q^\alpha = c^\alpha, H = c\}$.
(B) All Q^α commute with H: $\{Q^\alpha, H\} = 0$, for all $z \in \mathbb{M}_n$.

In classical mechanics, the quantities Q^α are called the first integrals (or the conserved charges) of the system.

Affirmation 7. Let A^α, $\alpha = 1, 2, \ldots, n-k$ be functionally independent scalar functions, and denote $\vec{V}_{(\alpha)}$ as the Hamiltonian field of A^α. The following two conditions are then equivalent:
(A) Integral lines of each $\vec{V}_{(\beta)}$ lie in the submanifolds $\mathbb{N}^{\vec{c}}_k = \{z \in \mathbb{M}_n, \quad A^\alpha = c^\alpha\}$.

(B) $\{A^\alpha, A^\beta\} = 0$ on $\mathbb{M}_n$.

3.2. Lie Bracket and Poisson Bracket

Consider the spaces of scalar functions and of vector fields on $\mathbb{M}_n$, which are the infinite-dimensional Lie algebras: $\mathbb{F}_{\mathbb{M}} = \{A, B, \ldots, \{\, ,\, \}\}$ and $\mathbb{T}_{\mathbb{M}} = \{\vec{V}, \vec{U}, \ldots, [\, ,\,]\}$.

Affirmation 8. The mapping (50) respects the Lie products of $\mathbb{F}_{\mathbb{M}}$ and $\mathbb{T}_{\mathbb{M}}$:

$$\omega(\{A, B\}) = -[\omega(A), \omega(B)], \quad \text{or, equivalently} \quad \vec{X}_{\{A,B\}} = -[\vec{X}_A, \vec{X}_B]. \tag{54}$$

According to the last equality, the Hamiltonian vector fields form a subalgebra of the Lie algebra $\mathbb{T}_{\mathbb{M}}$.

Proof. Using the vector notation (25), we can present the Poisson bracket as follows:

$$\{A, B\} = -\vec{X}_A(B). \tag{55}$$

The equality (54) is the Jacobi identity (37), rewritten in the vector notations. Indeed,

$$\{\{A, B\}, C\} = \{A, \{B, C\}\} - \{B, \{A, C\}\}, \quad \text{or} \quad \vec{X}_{\{A,B\}}(C) = \vec{X}_A(\vec{X}_B(C)) - \vec{X}_B(\vec{X}_A(C)), \tag{56}$$

for all C, which is just (54). □

We also note that in the vector notation, the Jacobi identity (40) states that Hamiltonian fields of coordinates form the closed algebra

$$[\vec{X}_{(i)}, \vec{X}_{(j)}] = c_{(i)(j)}{}^{(k)} \vec{X}_{(k)}, \tag{57}$$

with the structure functions $c_{(i)(j)}{}^{(k)} = -\partial_k \omega^{ij}$.

Exercise 5. Show that $\{Q, H\} = const$ implies $[\vec{X}_Q, \vec{X}_H] = 0$.

3.3. Two Basic Examples of Poisson Structures

1. Consider the space $\mathbb{R}^{2n}$, denote its coordinates $z^i = (q^1, q^2, \ldots, q^n, p_1, p_2, \ldots, p_n) \equiv (q^a, p_b)$, $a, b = 1, 2, \ldots, n$, and take the matrix composed from four $n \times n$ blocks as follows:

$$\omega^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{58}$$

In all other coordinate systems $z^{i'}$, we define components of the matrix $\omega^{i'j'}$ according to Equation (19). Then, ω is the contravariant tensor of second rank, which (in the system z) determines the Poisson structure on $\mathbb{R}^{2n}$ according to Equation (38):

$$\{A, B\}_P = \frac{\partial A}{\partial q^a}\frac{\partial B}{\partial p_a} - \frac{\partial B}{\partial q^a}\frac{\partial A}{\partial p_a}, \quad \text{and fundamental brackets are:} \quad \{q^a, p_b\}_P = \delta^a{}_b. \tag{59}$$

As ω is the numeric matrix, the condition (40) is satisfied in the coordinate system (q^a, p_b). According to Affirmation 4, it is then satisfied in all other coordinates. Given the Hamiltonian function H, the Hamiltonian equations acquire the following form:

$$\dot{q}^a = \{q^a, H\}_P = \frac{\partial H}{\partial p_a}, \qquad \dot{p}_a = \{p_a, H\}_P = -\frac{\partial H}{\partial q^a}. \tag{60}$$

It is known (see Section 2.9 in [4]) that they follow from the variational problem for the functional

$$S_H : (q^a(\tau), p_a(\tau)) \to \mathbb{R}; \qquad S_H = \int_{\tau_1}^{\tau_2} d\tau [p_a \dot{q}^a - H(q^a, p_b)]. \tag{61}$$

In classical mechanics, $\mathbb{R}^{2n}$ equipped with the coordinates (q^a, p_b) is called the phase space, the bracket (59) is called the canonical Poisson bracket, and the functional S_H is called the Hamiltonian action.

2. Given the manifold $\mathbb{M}_n$, let $c^{ij}{}_k$ be structure constants of an n-dimensional Lie algebra. We define $\omega^{ij}(z) = c^{ij}{}_k z^k$. Then, the equalities (13) imply (39) and (40), so the tensor ω^{ij} determines a Poisson structure on $\mathbb{M}_n$. The corresponding bracket

$$\{A, B\}_{LP} = \partial_i A c^{ij}{}_k z^k \partial_j B, \quad \text{fundamental brackets:} \quad \{z^i, z^j\}_{LP} = c^{ij}{}_k z^k, \tag{62}$$

is called the Lie–Poisson bracket. In particular, the Lie algebra of rotations determines the Lie–Poisson bracket on $\mathbb{R}^3$

$$\omega^{ij} = \epsilon^{ijk} z^k. \tag{63}$$

Let B^i be coordinates of a constant vector $\mathbf{B} \in \mathbb{R}^3$. Taking $H = z^i B^i$ as the Hamiltonian, we obtain the Hamiltonian equations (called the equations of precession)

$$\dot{z}^i = \epsilon^{ijk} B^j z^k, \quad \text{or} \quad \dot{\mathbf{z}} = \mathbf{B} \times \mathbf{z}, \tag{64}$$

where $\mathbf{B} \times \mathbf{z}$ is the usual vector product in $\mathbb{R}^3$. For any solution $\mathbf{z}(\boldsymbol{\tau})$, the end of this vector lies in a plane perpendicular to $\mathbf{B}$ and describes a circle around $\mathbf{B}$ with an angular velocity equal to the magnitude $|\mathbf{B}|$ of this vector. A compass needle in the earth's magnetic field moves just according to this law.

3.4. Poisson Mapping and Poisson Submanifold

Here, we discuss the mappings which are compatible with Poisson brackets of the involved manifolds. Intuitively, such a mapping turns the bracket of one manifold into the bracket of another. As an instructive example, we first consider the manifolds with the brackets (58) and (62). Introduce the mapping

$$\phi : \mathbb{R}^{2n} \to \mathbb{M}^n, \quad (q^a, p_b) \to z^a = \phi^a(q, p) = -c^{ab}{}_c p_b q^c. \tag{65}$$

Computing the canonical Poisson bracket (59) of the functions $\phi^a(q, p)$, we obtain a remarkable relation between the two brackets (Exercise):

$$\{\phi^a(q,p), \phi^b(q,p)\}_P = c^{ab}{}_c \phi^c(q,p), \quad \text{or} \quad \{\phi^a(q,p), \phi^b(q,p)\}_P = \{z^a, z^b\}_{LP}\Big|_{z \to \phi(q,p)}, \quad \text{or} \tag{66}$$

$$\partial_i \phi^a \omega^{ij} \partial_j \phi^b = \omega^{ab}(z^c)\Big|_{z \to \phi(q,p)}, \quad \text{where} \quad \omega^{ab}(z^c) = c^{ab}{}_c z^c. \tag{67}$$

The relation (67) shows that Poisson structures ω^{ij} and ω^{ab} are related by the tensor-like law (19). The relations (66) show that the Poisson brackets of the special functions ϕ^a on $\mathbb{R}^{2n}$ are the same as fundamental Lie–Poisson brackets (62) of the manifold $\mathbb{R}^n$. We can make these relations hold for an arbitrary scalar functions by using the induced mapping between the functions $A(z^a)$ of $\mathbb{M}^n$ and $\bar{A}(q^a, p_b)$ of $\mathbb{R}^{2n}$

$$\phi^* : A(z^a) \to \bar{A}(q^a, p_b) \equiv \phi^*(A)(q,p) = A(\phi^a(q,p)). \tag{68}$$

This implies the following relation between the Poisson and Lie–Poisson brackets (Exercise):

$$\{\phi^*(A), \phi^*(B)\}_P = \phi^*(\{A, B\}_{LP}). \tag{69}$$

Formalizing this example, we arrive at the notion of a Poisson mapping.

Definition 3. *Consider the Poisson manifolds $\mathbb{N}_k = \{x^a, \{,\}_{\mathbb{N}}\}$ and $\mathbb{M}_n = \{z^i, \{,\}_{\mathbb{M}}\}$. The mapping (27) is called a Poisson mapping if the induced mapping (33) preserves the Poisson brackets*

$$\{\phi^*(A), \phi^*(B)\}_{\mathbb{N}} = \phi^*(\{A, B\}_{\mathbb{M}}). \tag{70}$$

This allows us to compare the Poisson brackets of $\mathbb{M}$ and $\mathbb{N}$. Given two functions A and B of $\mathbb{M}$ and their images $\bar{A}$ and $\bar{B}$, we can compare the bracket $\{\bar{A}, \bar{B}\}_{\mathbb{N}}$ with the image of scalar function $\{A, B\}_{\mathbb{M}}$, that is, with $\phi^*(\{A, B\}_{\mathbb{M}})$. If they coincide, we have the mapping (29) that respects the Poisson structures of the manifolds. The mapping (65) is an example of a Poisson mapping of the canonical Poisson manifold on the Lie–Poisson manifold.

Poisson submanifold of the Poisson manifold. Let the Poisson manifold $\mathbb{N}_k$ be a submanifold of the Poisson manufold $\mathbb{M}_n$, determined by the functionally independent set of scalar functions $\Phi^\beta(z^i)$ of $\mathbb{M}_n$

$$\mathbb{N}_k = \{z^i \in \mathbb{M}; \quad \Phi^\beta(z^i) = 0\}. \tag{71}$$

Solving $\Phi^\beta(z^i) = 0$, we obtain the parametric equations $z^\beta = f^\beta(z^a)$, and take z^a as the local coordinates of $\mathbb{N}_k$. Any scalar function $A(z^i)$ on $\mathbb{M}_n$ is defined, in particular, at the points of $\mathbb{N}_k$, and hence, we can consider the restriction of $A(z^i)$ on $\mathbb{N}_k$

$$\eta^* : \mathbb{F}_{\mathbb{M}} \to \mathbb{F}_{\mathbb{N}}, \qquad A(z^\beta, z^a) \to \bar{A}(z^a) = A(f^\beta(z^a), z^a), \tag{72}$$

The Poisson manifold $\mathbb{N}_k$ is called the Poisson submanifold of $\mathbb{M}_n$ if the mapping η^* turn the bracket of $\mathbb{M}$ into the bracket of $\mathbb{N}$:

$$\eta^*(\{A, B\}_M) = \{\bar{A}, \bar{B}\}_N. \tag{73}$$

or

$$\{A(z^\beta, z^a), B(z^\beta, z^a)\}_{\mathbb{M}}\Big|_{z^\beta \to f^\beta(z^a)} = \{A(f^\beta(z^a), z^a), B(f^\beta(z^a), z^a)\}_{\mathbb{N}}. \tag{74}$$

Various examples of Poisson mappings and Poisson submanifolds will appear in the analysis of dynamical systems in Section 5.2. Notice that η^* determined by (72) is the mapping induced by the embedding mapping (28).

4. Degenerate Poisson Manifold

The affirmations discussed above are equally valid for non-degenerate and degenerate manifolds. Now, we consider some characteristic properties of a Poisson manifold with a degenerate Poisson bracket. Non-degenerate Poisson manifolds will be discussed in Section 6.

4.1. Casimir Functions

A Poisson manifold with a degenerate bracket has the following property: in the space $\mathbb{F}_{\mathbb{M}}$, there is a set of functionally independent functions that have null brackets (commute) with all functions of $\mathbb{F}_{\mathbb{M}}$. They are called the Casimir functions. This allows for the construction of a remarkable foliation of the manifold with the leaves determined by the Casimir functions.

Affirmation 9. Let K_β, $\beta = 1, 2, \ldots, p$, are p functionally independent Casimir functions of a Poisson manifold $\mathbb{M}_n$. Then, ω is degenerated, and *rank* $\omega \leq n - p$.

Proof. K_β commutes with any function, in particular, we can write $\{z^i, K_\beta\} = 0$, or $\omega^{ij}\partial_j K_\beta = 0$. The latter equation means that ω admits at least p independent null-vectors $\vec{V}_{(\beta)}$, so *rank* $\omega \leq n - p$. □

Affirmation 10. Consider a Poisson manifold with *rank* $\omega = n - p$. Then, there are exactly p functionally independent Casimir functions:

$$\{z^i, K_\beta\} = 0, \quad \text{or} \quad \vec{X}_{K_\beta} = 0, \quad i = 1, 2, \ldots, n, \quad \beta = 1, 2, \ldots, p. \tag{75}$$

Proof. First, let us consider the particular case of a $2n+1$-dimensional Poisson manifold with a *rank* $\omega = 2n$. According to the Darboux theorem, there are canonical coordinates $z^{i'}$ such that one of them commutes with all others, e.g., if $z^{1'}$ commutes with all coordinates, $\{z^{i'}, z^{1'}\} \equiv \omega^{i'1'} = 0$. Let us define a scalar function such that at the point $z \in \mathbb{M}_{2n+1}$, its value coincides with the value of the first coordinate of this point in the canonical system: $K(z) = z^{1'}$. In the canonical coordinates, this function is represented by $K'(z^{1'}, z^{2'}, \ldots, z^{(2n+1)'}) = z^{1'}$. Then, according to Equations (15) and (14), it is represented by $K(z^i) = z^{1'}(z^1, z^2, \ldots, z^{2n+1})$ in the original coordinates. Let us confirm that $K(z)$ is a Casimir function. Using the transformation laws (15), (18) and (19), we obtain

$$\{z^i, K(z)\} = \omega^{ij}(z)\partial_j K(z) = \left.\frac{\partial z^i}{\partial z^{i'}}\right|_{z'(z)} \omega^{i'j'} \left.\frac{\partial z^j}{\partial z^{j'}}\right|_{z'(z)} \frac{\partial z^{k'}}{\partial z^j} \left.\frac{\partial K'(z')}{\partial z^{k'}}\right|_{z'(z)} =$$
$$\left.\left[\frac{\partial z^i}{\partial z^{i'}}\omega^{i'j'}\frac{\partial z^{1'}}{\partial z^{j'}}\right]\right|_{z'(z)} = \left.\frac{\partial z^i}{\partial z^{i'}}\right|_{z'(z)} \omega^{i'1'} = 0. \tag{76}$$

Let us return to the general case with *rank* $\omega = n - p$. According to Equation (49), in the Darboux coordinates the functions of $z^{\beta'}$ are Casimir functions. As the complete set of functionally independent Casimir functions, we can take the coordinates $z^{\beta'}$ themselves. More than p functionally independent Casimir functions would exist, in contradiction with Affirmation 9. □

Consider the foliation with the leaves determined by the Casimir functions, $\mathbb{N}^{\vec{c}}_{n-p} = \{z \in \mathbb{M}_n, \quad K_\beta(z^i) = c_\beta\}$. Then, Equation (75) has the following remarkable interpretation: for any function $A \in \mathbb{F}_{\mathbb{M}_n}$, the Hamiltonian vector field $X^i_A = \omega^{ij}\partial_j A$ is tangent to the hypersurfaces $\mathbb{N}^{\vec{c}}_{n-p}$, that is, its integral lines lie in $\mathbb{N}^{\vec{c}}_{n-p}$. Indeed, let $z^i(\tau)$ be an integral line of X^i_A. We obtain: $\frac{d}{d\tau}K_\beta(z^i(\tau)) = X^i_A \partial_i K_\beta(z^i)|_{z(\tau)} = \{K_\beta, A\}|_{z(\tau)} = 0$. Then, $K_\beta(z^i(\tau)) = c_\beta = \text{const}$, that is, $z^i(\tau)$ lies on one of the surfaces, so $\vec{X}_A \in \mathbb{T}_{\mathbb{N}}$.

Exercise 6. Observe that $K = z^i z^i$ is the Casimir function of (63).

4.2. Induced Bracket on the Casimir Submanifold

Consider the degenerate Poisson manifold $\mathbb{M}_n = \{z^i;\ \{A, B\} = \partial_i A \omega^{ij} \partial_j B,\ rank\ \omega = n - p\ \}$, and let $K_\beta(z^i)$ be a subset of Casimir functions (we can take either all functionally independent Casimirs, $\beta = 1, 2, \ldots, p$, or some part of them). Consider the submanifold determined by K_β

$$\mathbb{N} = \{z^i \in \mathbb{M}_n, \quad K_\beta(z^i) = 0\}. \tag{77}$$

For brevity, we call $\mathbb{N}$ the Casimir submanifold. We will show that the Poisson bracket on $\mathbb{M}_n$ can be used to construct a natural Poisson bracket $\{,\}_{\mathbb{N}}$ on $\mathbb{N}$.

Induced bracket in special coordinates. As the functions $K_\beta(z^i)$ are functionally independent, we can take the coordinate system where they turn into a part of coordinates, say $\tilde{z}^i = (\tilde{z}^\beta = K_\beta, \tilde{z}^a)$. On the surface $\mathbb{N}$, we have $\tilde{z}^\beta = 0$, so $\tilde{z}^a$ are the coordinates of $\mathbb{N}$. The Poisson tensor $\tilde{\omega}^{ij} = \{\tilde{z}^i, \tilde{z}^j\}$ of $\mathbb{M}_n$ in these coordinates has the following special form: $\{\tilde{z}^a, \tilde{z}^b\} = \omega^{ab}(\tilde{z}^\beta, \tilde{z}^c)$, $\{\tilde{z}^\beta, \tilde{z}^i\} = \{K_\beta, \tilde{z}^i\} = 0$, for any i. Because $\tilde{\omega}^{ij}$ obeys (39) and (40) for any value of the coordinates $\tilde{z}^\beta$, we obtain $\omega^{ab}(0, z^c) = -\tilde{\omega}^{ba}(0, z^c)$ and

$$\tilde{\omega}^{ap}(\tilde{z}^\beta, \tilde{z}^c)\partial_p\tilde{\omega}^{bc}(\tilde{z}^\beta, \tilde{z}^c) + cycle(a,b,c) = 0, \quad \text{or} \quad \{\tilde{z}^a, \{\tilde{z}^b, \tilde{z}^c\}\} + cycle(a,b,c) = 0, \tag{78}$$

where the index p runs over both β and a subsets. However, observing that

$$\begin{gathered}\{\tilde{z}^a, \{\tilde{z}^b, \tilde{z}^c\}\} = \{\tilde{z}^a, \tilde{\omega}^{bc}(\tilde{z}^\beta, \tilde{z}^c)\} = \\ \{\tilde{z}^a, \tilde{z}^\beta\}\partial_\beta\tilde{\omega}^{bc}(\tilde{z}^\beta, \tilde{z}^c) + \{\tilde{z}^a, \tilde{z}^d\}\partial_d\tilde{\omega}^{bc}(\tilde{z}^\beta, \tilde{z}^c) = \omega^{ad}(\tilde{z}^\beta, \tilde{z}^c)\partial_d\omega^{bc}(\tilde{z}^\beta, \tilde{z}^c),\end{gathered} \tag{79}$$

we can write the first equality in (78) as follows:

$$\omega^{ad}(\tilde{z}^\beta, \tilde{z}^c)\partial_d\omega^{bc}(\tilde{z}^\beta, \tilde{z}^c) + cycle(a,b,c) = 0. \tag{80}$$

As it is true for any value of the coordinates $\tilde{z}^\beta$, we can take $\tilde{z}^\beta = 0$. Then,

$$\tilde{\omega}^{ad}(0, \tilde{z}^c)\partial_d\tilde{\omega}^{bc}(0, \tilde{z}^c) + cycle(a,b,c) = 0. \tag{81}$$

Let us define a matrix with elements

$$\bar{\omega}^{ab}(\tilde{z}^c) \equiv \tilde{\omega}^{ad}(0, \tilde{z}^c), \tag{82}$$

in the coordinates $\tilde{z}^c$. In any other coordinate system on $\mathbb{N}$, say $\tilde{z}^{a'}$, we define the elements $\bar{\omega}^{a'b'}$ according to rule (19). Then, $\bar{\omega}$ is a tensor of $\mathbb{N}$. According to our computations, it obeys Equations (39) and (40), and thus determines a Poisson bracket $\{A, B\}_{\mathbb{N}} = \partial_i A(\tilde{z}^c)\bar{\omega}^{ab}(\tilde{z}^c)\partial_j B(\tilde{z}^c)$ on $\mathbb{N}$.

Induced bracket in the original coordinates. Let us solve the same problem in the original coordinates, divided in two subsets, $z^i = (z^\alpha, z^a)$, such that $\det \frac{\partial K_\beta}{\partial z^\alpha} \neq 0$. Notice that in this case, the Equation (75) reads

$$\omega^{i\alpha}\partial_\alpha K_\beta + \omega^{ib}\partial_b K_\beta = 0. \tag{83}$$

Denoting $\partial_\alpha K_\beta = K_{\alpha\beta}$, this allows us to restore the whole $\omega^{ij}(z^k)$ from the known block $\omega^{ab}(z^k)$ as follows:

$$\omega^{a\alpha} = -\omega^{ab}\partial_b K_\gamma (K^{-1})^{\gamma\alpha}, \qquad \omega^{\alpha\beta} = -\omega^{\alpha b}\partial_b K_\gamma (K^{-1})^{\gamma\beta}. \tag{84}$$

Geometric interpretation of these relations will be discussed in Section 7.3.

It is instructive to obtain the induced bracket in the original coordinates in a manner independent of the calculations made in the previous subsection.

Affirmation 11. Let $K_\beta(z^\alpha, z^b)$ be Casimir functions, and $z^\alpha = f^\alpha(z^b)$ is a solution to the equations $K_\beta(z^\alpha, z^b) = 0$. Then,

(a) $z^\alpha - f^\alpha(z^b)$ are Casimir functions;
(b) The Poisson tensor of $\mathbb{M}_n$ satisfies the identity

$$\omega^{i\alpha} = \omega^{ib}\partial_b f^\alpha. \tag{85}$$

Proof. (a) Contracting the expression $\{z^i, z^\alpha - f^\alpha\} = \omega^{i\alpha} - \omega^{ia}\partial_a f^\alpha$ with $\partial_\alpha K_\beta$ and using (83), we obtain $\omega^{i\alpha}\partial_\alpha K_\beta - \omega^{ia}\partial_a f^\alpha \partial_\alpha K_\beta = -\omega^{ia}(\partial_a K_\beta + \partial_a f^\alpha \partial_\alpha K_\beta) = -\omega^{ia}\partial_a K_\beta(f^\alpha, z^a) = 0$ as $K_\beta(f^\alpha, z^a) \equiv 0$. As $\partial_\alpha K_\beta$ is an invertible matrix, the equality $\{z^i, z^\alpha - f^\alpha\}\partial_\alpha K_\beta = 0$ implies $\{z^i, z^\alpha - f^\alpha\} = 0$.

(b) Let K_α be Casimir functions. According to Item (a), $z^\alpha - f^\alpha(z^b)$ are also Casimir functions. Then, $\{z^i, z^\alpha - f^\alpha\} = 0$ or $\omega^{i\alpha} = \omega^{ib}\partial_b f^\alpha$. □

Affirmation 12. For any function $B(z^i)$ and the Casimir functions $z^\beta - f^\beta(z^b)$, there is the identity

$$\omega^{ip}\partial_p B\Big|_{z^\beta=f^\beta(z^c)} = \omega^{ia}(z^c, f^\beta(z^c))\partial_a B(z^c, f^\beta(z^c)), \quad (86)$$

where $p = (1, 2, \ldots, n)$, while $a = (1, 2, \ldots, n-p)$. Note the geometric interpretation of this equality; if two functions B and B' of $\mathbb{F}_{\mathbb{M}}$ coincide on $\mathbb{N}$, their Hamiltonian vector fields also coincide on $\mathbb{N}$: $B|_{\mathbb{N}} = B'|_{\mathbb{N}}$ implies $\vec{X}_B|_{\mathbb{N}} = \vec{X}_{B'}|_{\mathbb{N}}$.

Proof. Let us write

$$\omega^{ip}\partial_p B\Big|_{z^\beta=f^\beta(z^c)} = \omega^{ia}\partial_a B\Big|_{z^\beta=f^\beta(z^c)} + \omega^{i\beta}\partial_\beta B\Big|_{z^\beta=f^\beta(z^c)}. \quad (87)$$

Using the identity (85), we have $\omega^{i\beta}\partial_\beta B = \omega^{id}\partial_d f^\beta(z^c)\partial_\beta B(z^c, z^\beta) \equiv \omega^{id}\left[\partial_d B(z^c, f^\beta(z^c)) - \partial_d B(z^c, z^\beta)|_{z^\beta=f^\beta(z^c)}\right]$. Using this expression for the term $\omega^{i\beta}\partial_\beta B$ in (87), we arrive at the desired identity (86). □

We are ready to construct the induced Poisson structure. We take z^a as the local coordinates of $\mathbb{N}$, and using the ω^{ab}-block of ω^{ij}, introduce the antisymmetric matrix

$$\tilde{\omega}^{ab}(z^c) = \omega^{ab}(f^\beta(z^c), z^c). \quad (88)$$

Let us confirm that $\tilde{\omega}$ obeys the condition (40). We write the condition (40), satisfied for ω^{ij}. We take the indices i, j, k to equal to a, b, c, and substitute $z^\beta = f^\beta(z^c)$. This gives us the identity

$$\omega^{ap}\partial_p\omega^{bc}\Big|_{z^\beta=f^\beta(z^c)} + cycle(a,b,c) = 0. \quad (89)$$

Using the identity (86), we immediately obtain

$$\omega^{ad}(f^\beta(z^c), z^c)\partial_d\omega^{bc}(f^\beta(z^c), z^c) + cycle(a,b,c) = 0, \quad (90)$$

which is simply the Jacobi identity for the tensor $\tilde{\omega}^{ab}$. Thus the bracket

$$\{A(z^a), B(z^a)\} = \partial_a A\tilde{\omega}^{ab}\partial_b B, \quad (91)$$

defined on $\mathbb{N}$, obeys the Jacobi identity. In any other coordinate system on $\mathbb{N}$, say $z^{a'}$, we define the components $\tilde{\omega}^{a'b'}$ according to rule (19):

$$\tilde{\omega}^{a'b'} = \partial_a z^{a'}\partial_b z^{b'}\tilde{\omega}^{ab}\Big|_{z^a(z^{a'})}. \quad (92)$$

Then $\tilde{\omega}$ is a tensor of $\mathbb{N}$, while the expression (91) is a scalar function, as it should be for the Poisson bracket.

Let us confirm that the obtained bracket does not depend on the coordinates of $\mathbb{M}_n$ chosen for its construction. Let $z^i = \phi^i(z^{j'})$ be transition functions between two coordinate systems. For a point of $\mathbb{N}$, this implies the following relation between its local coordinates z^a and $z^{a'}$:

$$z^a = \phi^a(f^{\alpha'}(z^{a'}), z^{a'}). \quad (93)$$

Using these functions in the expression (92), we obtain the components $\tilde{\omega}^{a'b'}(z^{a'})$ of the tensor $\tilde{\omega}^{ab}(z^a)$ in the coordinates $z^{a'}$. On the other hand, using the Poisson tensor $\omega^{i'j'}$ in coordinates $z^{i'}$, we could construct the matrix $\hat{\omega}^{a'b'}(f^{\alpha'}(z^{a'}), z^{a'})$ according to rule (88). The task is to show that $\hat{\omega}^{a'b'}$ coincides with $\tilde{\omega}^{a'b'}$.

As the functions $\hat{\omega}^{a'b'}(f^{\beta'}(z^{a'}), z^{a'})$ are components of the tensor $\omega^{i'j'}$ of $\mathbb{M}$, we use the transformation law (19), and write

$$\hat{\omega}^{a'b'}(f^{\beta'}(z^{a'}), z^{a'}) = \hat{\omega}^{a'b'}(z^{\beta'}, z^{a'})\Big|_{\mathbb{N}} = \partial_k z^{a'} \omega^{kp} \partial_p z^{b'}\Big|_{z^i(z^{i'})}\Big|_{\mathbb{N}}. \tag{94}$$

In the last expression, we have a quantity $D(z^i)$, and need to replace the coordinates z^i by the transition functions $z^i(z^{i'})$ taken at the point of $\mathbb{N}$. Equivalently, we can first restrict D on $\mathbb{N}$, replacing z^β on $f^\beta(z^a)$, and then replace z^a on its expression (93) through coordinates $z^{a'}$. Creating this and then using the identity (86), we obtain

$$\begin{aligned}\hat{\omega}^{a'b'}(f^{\beta'}(z^{a'}), z^{a'}) = \partial_k z^{a'}(z^i)\omega^{kp}(z^i)\partial_p z^{b'}(z^i)\Big|_{z^\beta \to f^\beta(z^a)}\Big|_{z^a(z^{a'})} = \\ \partial_a z^{a'}(z^\beta(z^a), z^a)\omega^{ab}(z^\beta(z^a), z^a)\partial_b z^{b'}(z^\beta(z^a), z^a)\Big|_{z^a(z^{a'})}.\end{aligned} \tag{95}$$

Comparing this expression with (92), we arrive at the desired result: $\hat{\omega} = \bar{\omega}$.

Exercise 7. Confirm that the Poisson manifold $\mathbb{N}$ is the Poisson submanifold of $\mathbb{M}$ in the sense of definition (73).

Consider the Poisson manifold $\mathbb{M}_n$ with *rank* $\omega^{ij} = n - p$, and let the submanifold (77) be determined by a complete set of p functionally independent Casimir functions. Then, the induced Poisson structure is non-degenerate: $\det \bar{\omega}^{ab} \neq 0$. To demonstrate this, suppose an opposite, $\det \bar{\omega}^{ab} = 0$, and let z^i be the canonical coordinates of $\mathbb{M}$. Then, $\bar{\omega}$ is a numeric degenerate matrix, so it has a numeric null vector, $\bar{\omega}^{ab} c_b = 0$. As a consequence, the function $A(z^i) = z^a c_a$ commutes with all coordinates (49) and hence is a Casimir function of $\mathbb{M}$. It depends only on the variables z^a, so it is functionally independent of the Casimir functions $z^\beta - f^\beta(z^a) = 0$. This is in contradiction with the condition *rank* $\omega^{ij} = n - p$, so $\det \bar{\omega}^{ab} \neq 0$.

Let us resume the obtained results. Let $\mathbb{M}$ be a Poisson manifold with a degenerate Poisson bracket ω. Then, on the submanifold $\mathbb{N} \in \mathbb{M}$ determined by any set of functionally independent Casimir functions, there exists the Poisson bracket $\bar{\omega}$ such that the Poisson manifold $\mathbb{N}$ turns into the Poisson submanifold of $\mathbb{M}$. In the original coordinates, divided on two groups according to the structure of Casimir functions (77), $z^i = (z^\beta, z^a)$, elements of the matrix $\bar{\omega}$ coincide with fundamental brackets of coordinates z^a restricted on $\mathbb{N}$:

$$\bar{\omega}^{ab} = \{z^a, z^b\}_{\mathbb{M}}\Big|_{z^\beta \to f^\beta(z^a)}. \tag{96}$$

4.3. Restriction of Hamiltonian Dynamics to the Casimir Submanifold

The degeneracy of a Poisson structure implies that integral lines of *any* Hamiltonian system on this manifold have special properties: any solution started in a Casimir submanifold lies entirely within it. So, the dynamics can be consistently restricted on the submanifold, and the resulting equations are still Hamiltonian. To discuss these properties, we will need the notion of an invariant submanifold.

Definition 4. *The submanifold $\mathbb{N} \in \mathbb{M}_n$ is called an invariant submanifold of the Hamiltonian H if any trajectory of (51) that starts in $\mathbb{N}$, entirely lies in $\mathbb{N}$*

$$z^i(0) \in \mathbb{N}, \quad \rightarrow \quad z^i(\tau) \in \mathbb{N} \quad \textit{for any} \quad \tau. \tag{97}$$

The observation made in Section 4.1 now can be rephrased as follows.

Affirmation 13. A Casimir submanifold of $\mathbb{M}_n = \{z^i, \{,\}\}$ is invariant submanifold of any Hamiltonian $H \in \mathbb{F}_{\mathbb{M}}$.

Affirmation 14. Solutions to the Hamiltonian equations $\dot{z}^i = \omega^{ij}\partial_j H$ that belong to the Casimir submanifold (77), obey the Hamiltonian equations

$$\dot{z}^a = \bar{\omega}^{ab}\partial_b H(z^c, f^\alpha(z^c)), \tag{98}$$

where z^a are the local coordinates and the Poisson tensor $\bar{\omega}^{ab}$ of $\mathbb{N}$ is the restriction of ω^{ij} on $\mathbb{N}$

$$\bar{\omega}^{ab} = \omega^{ab}(z^c, f^\alpha(z^c)). \tag{99}$$

Proof. According to Affirmation 13, we can add the algebraic equations $z^\beta = f^\beta(z^a)$ to the system (51), thus obtaining consistent equations with solutions living on $\mathbb{N}$. In the equations for $\dot{z}^a$, we substitute $z^\beta = f^\beta(z^a)$, and using the identity (86), we obtain the closed system (98) and (99) for determining z^a. Then, the equations for $\dot{z}^\beta$ can be omitted from the system. The Jacobi identity for $\bar{\omega}$ has been confirmed above. □

5. Integrals of Motion of a Hamiltonian System

5.1. Basic Notions

Let $z(\tau)$ be a solution to the Hamiltonian Equation (51). For any function $Q(z)$ we have $\dot{Q}(z(\tau)) = \{Q(z), H(z)\}|_{z(\tau)}$. In other words, functions $Q(z(\tau))$ follow the Hamiltonian dynamics together with $z(\tau)$. The function $Q(z)$ (with a non-vanishing gradient) is called the integral of motion if it preserves its value along the trajectories of (51): $Q(z(\tau)) = const$, or $\dot{Q}(z(\tau)) = 0$. Note that the value of $Q(z(\tau))$ can vary from one trajectory to another.

Affirmation 15. $Q(z)$ is an integral of motion of the system (51) if and only if its bracket with H vanishes

$$\{Q, H\} = 0. \tag{100}$$

Because $\{H, H\} = 0$, the Hamiltonian itself is an example of the integral of motion. So, any Hamiltonian system admits at least one integral of motion. The Casimir functions obey Equation (100) for any H, so they represent the integrals of motion of any Hamiltonian system on a given manifold. As a consequence, a Hamiltonian system on the manifold $\mathbb{M}_n$ with $rank\ \omega = n - p$ has at least $p + 1$ integrals of motion.

Exercise 8. (a) Confirm Affirmation 15. (hint: take into account that the integral lines of (51) cover all of the manifold).
(b) Observe that if Q_1 and Q_2 are integrals of motion, then $c_1Q_1 + c_2Q_2$, $f(Q_1)$ and $\{Q_1, Q_2\}$ are integrals of motion as well. The integral of motion $\{Q_1, Q_2\}$ may be functionally independent of Q_1 and Q_2.

The integrals of motion Q_α can be used to construct the surfaces of the level in $\mathbb{M}_n$. Considering the Hamiltonian equations on the surfaces, it can be found that it is possible to reduce the number of differential equations that we need to solve. This method, called the reduction procedure, is based on the following affirmations.

Affirmation 16. Let $Q_\alpha(z)$, $\alpha = 1, 2, \ldots, p$ be functionally independent integrals of motion of H. Then, $\mathbb{N}_{\mathbf{c}} = \{z \in \mathbb{M}_n, \quad Q_\alpha(z) = c_\alpha = const\}$ is an $n - p$-dimensional invariant submanifold of H.

Indeed, given a solution with $z(0) \in \mathbb{N}_{\mathbf{c}}$ that is $Q_\alpha(z(0)) = c_\alpha$, we have $Q_\alpha(z(\tau)) = Q_\alpha(z(0)) = c_\alpha$ for any τ; therefore, the trajectory $z(\tau)$ entirely lies in $\mathbb{N}_{\mathbf{c}}$. The manifolds $\mathbb{N}_{\mathbf{c}}$ and $\mathbb{N}_{\mathbf{d}}$ with $\mathbf{c} \neq \mathbf{d}$ do not intercept. As such, the Poisson manifold $\mathbb{M}_n$ is covered by p-parametric foliation of the invariant submanifolds $\mathbb{N}_{\mathbf{c}}$.

As the Casimir function is an integral of motion of any Hamiltonian, Affirmation 16 implies, once again, the geometric interpretation of Equation (75): the integral lines of all Hamiltonian vector fields of $\mathbb{M}_n$ lie on the surfaces of Casimir functions.

Affirmation 17. Let the Hamiltonian system

$$\dot{z}^i = \{z^i, H\}, \tag{101}$$

admit p functionally independent integrals of motion $Q_\alpha(z) = c_\alpha$. We present them in the form $z^\alpha = f^\alpha(z^b, c_\alpha)$. Then, the system of n differential Equation (101) is equivalent to the system

$$\dot{z}^b = \{z^b, H\} \equiv h^b(z^c, z^\alpha), \qquad z^\alpha = f^\alpha(z^b, c_\alpha), \tag{102}$$

composed of $n - p$ differential and p algebraic equations.

Proof. Adding the consequences $z^\alpha = f^\alpha(z^b, c_\alpha)$ to Equation (101), we write the resulting equivalent system as follows

$$\dot{z}^\alpha = \{z^\alpha, H\}, \qquad \dot{z}^b = \{z^b, H\}, \qquad z^\alpha = f^\alpha(z^b, c_\alpha). \tag{103}$$

To prove the equivalence of (102) and (103), we need to show that the equation $\dot{z}^\alpha = \{z^\alpha, H\}$ is a consequence of the system (102). Let $z^\alpha(\tau), z^b(\tau)$ be a solution to (102). Computing the derivative of the identity $z^\alpha(\tau) \equiv f^\alpha(z^b(\tau), c_\alpha)$, we have $\dot{z}^\alpha(\tau) = \partial_b f^\alpha(z^b, c_\alpha)\Big|_{z \to z(\tau)} \dot{z}^b = \partial_b f^\alpha(z^b, c_\alpha)\{z^b, H\}\Big|_{z \to z(\tau)} = \{f^\alpha, H\}|_{z \to z(\tau)} = \{z^\alpha, H\}|_{z \to z(\tau)}$. In the last step, we used (100). Hence, the equation $\dot{z}^\alpha = \{z^\alpha, H\}$ is satisfied by any solution to the system (102). □

Example 3. *Using the reduction procedure, any two-dimensional Hamiltonian system can be completely integrated, that is, solving the differential equations is reduced to the evaluation of an integral. Indeed, consider the system $\dot{x} = \{x, H(x,y)\} \equiv h(x,y)$, $\dot{y} = \{y, H(x,y)\}$. We assume that grad $H \neq 0$ (otherwise $H = const$ and the system is immediately integrated). Let $y = f(x,c)$ be a solution to the equation $H(x,y) = c$. As H is an integral of motion, we use Affirmation 17 to present the original system in the equivalent form: $\dot{x} = h(x,y)$, $y = f(x,c)$. Replacing y on $f(x,c)$ in the differential equation, the latter can be immediately integrated. The general solution $x(\tau,c,d)$, $y(\tau,c,d)$ in an implicit form is as follows:*

$$\int \frac{dx}{h(x, f(x,c))} = \tau + d, \qquad y = f(x). \tag{104}$$

There is a kind of multi-dimensional generalization of this example, see Affirmation A2 in Appendix A.2.

5.2. Hamiltonian Reduction to an Invariant Submanifold

As we saw above, when a dynamical system admits an invariant submanifold, its dynamics can be consistently restricted to the submanifold. Then, it is natural to ask whether the resulting equations form a Hamiltonian system. For instance, according to Affirmation 16, we can add the algebraic equations[7] $Q_\alpha(z) = 0$ to the Hamiltonian system (101), thus obtaining consistent equations with solutions living on the invariant submanifold $\mathbb{N} = \{z \in \mathbb{M}_n, Q_\alpha(z) = 0\}$. Using Affirmation 17, we exclude z^α and obtain differential equations on the manifold $\mathbb{N}$ with the local coordinates z^b

$$\dot{z}^b = h^b(z^c) \equiv \{z^b, H\}\Big|_{z^\alpha \to f^\alpha(z^c)}. \tag{105}$$

They have no pre-existing knowledge the ambient space $\mathbb{M}_n$. Hence, we ask if the resulting equations represent a Hamiltonian system on $\mathbb{N}$. That is, we look for the Hamiltonian equations

$$\dot{z}^b = \omega^{ba}(z^c)\partial_a \hat{H}(z^c), \tag{106}$$

that could be equivalent to (105).

Let us list some known cases of the Hamiltonian reduction.

1. Reduction of non-singular theory (3) to the surface of the constant Hamiltonian gives a Hamiltonian system with a time-dependent Hamiltonian. The method is known as the Maupertuis principle (see [4] for details).

2. Hamiltonian reduction to the surface of Casimir functions (see Affirmation 14). The particular example is a Hamiltonian system with a Dirac bracket (see Equation (154) below).

3. Hamiltonian reduction of non-singular theory to the surface of first integrals Φ^α with the property $\det\{\Phi^\alpha, \Phi^\beta\} \neq 0$ (see Equation (162) below).

4. Singular non-degenerate theories (5)–(7) are equivalent to the theory of Item 2, see Affirmations 28 and 30 below. Hence, it admits the Hamiltonian reduction to the surface of constraints.

5. According to the Gitman–Tyutin theorem, the singular degenerate theory admits Hamiltonian reduction to the surface of all constraints (see [3] for details).

6. Symplectic Manifold and Dirac Bracket

6.1. Basic Notions

As we saw in Section 2.3, a Poisson manifold can be defined by choosing a contravariant tensor with the properties (39) and (40). Here, we discuss another way, which works for the construction of non-degenerate Poisson structures on even-dimensional manifolds. Let $\mathbb{M}_{2n}$ be defined as the covariant tensor $\tilde{\omega}_{ij}(z^k)$ on the even-dimensional manifold (called the symplectic form) with the properties

$$\tilde{\omega}_{ij} = -\tilde{\omega}_{ji} \qquad \text{(antisymmetric)}, \tag{107}$$

$$\det\tilde{\omega} \neq 0 \qquad \text{(non-degenerate)}, \tag{108}$$

$$\partial_i\tilde{\omega}_{jk} + cycle = 0 \qquad \text{(closed)}. \tag{109}$$

$\mathbb{M}_{2n}$ equipped with a symplectic form is called the symplectic manifold.

We recall that the determinant of any odd-dimensional matrix vanishes, so (108) implies that we work on the even-dimensional manifold. Some properties of a symplectic form are in order.

Affirmation 18. The inverse matrix ω^{ij} of the matrix $\tilde{\omega}_{ij}$ obeys the properties (39) and (40). So, it determines the Poisson structure (38) on $\mathbb{M}_{2n}$. In other words, any locally symplectic manifold is a Poisson manifold.

Exercise 9. Prove that (109) implies (40).

Conversely, take a Poisson manifold with the non-degenerated bracket, $\det\omega \neq 0$ and let $\tilde{\omega}$ be its inverse. Contracting the condition (40) with $\tilde{\omega}_{ni}\tilde{\omega}_{mj}\tilde{\omega}_{pk}$, we immediately obtain (109).

Affirmation 19. The Poisson manifold with a non-degenerate bracket is a symplectic manifold.

Darboux Theorem. In the vicinity of any point, there are coordinates y^k where $\omega^{ij}(z^k)$ acquires the form

$$\omega'^{ij}(y^k) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{then} \quad \tilde{\omega}'_{ij}(y^k) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{110}$$

Proof is given in Appendix A.2.

Poincare Lemma. In a vicinity of any point, the symplectic form $\tilde{\omega}$ can be presented through some covariant vector field a_j as follows:

$$\tilde{\omega}_{ij} = \partial_i a_j - \partial_j a_i. \tag{111}$$

In the language of differential forms, this is formulated as follows: the closed form is a locally exact form. Conversely, the tensor $\tilde{\omega}$, constructed given a_j according to Equation (111), obeys the condition (109).

Proof. According to the Darboux theorem, there are coordinates $y^i = (x^1, \ldots, x^n, p^1, \ldots, p^n)$ where $\tilde{\omega}_{ij}$ acquires the canonical form (110). Let us identically rewrite it as follows: $\tilde{\omega}'_{ij} = \partial_i a'_j - \partial_j a'_i$, where $a'_i(y^k) = \frac{1}{2}(-p^1, \ldots, -p^n, x^1, \ldots, x^n)$. Returning to the original coordinates, we write $a_i(z^k) = \frac{\partial y^j}{\partial z^i} a'_j(y(z))$, where $a'_j(y(z)) = \frac{1}{2}(-p^1(z^k), \ldots, -p^n(z^k), x^1(z^k), \ldots, x^n(z^k))$. This contravariant vector field satisfies the desired property: $\partial_i a_j - \partial_j a_i = \frac{\partial y^n}{\partial z^i}\frac{\partial y^m}{\partial z^j}\tilde{\omega}'_{nm} = \tilde{\omega}_{ij}$.

The field a_i can equally be obtained by direct integrations:

$$a_i = -\frac{1}{n-1}\sum_{j=1}^{n}\int \tilde{\omega}_{ij}(z^k) dz^j. \tag{112}$$

□

Due to the Poincare lemma, it is easy to construct examples of closed and non-constant form $\tilde{\omega}$. Then, the tensor ω will automatically obey a rather complicated Equation (40). Note also that in the Darboux coordinates y^k, the Poisson bracket acquires the canonical form (59).

Because any symplectic manifold is simultaneously a Poisson manifold, it has all the properties discussed in Section 3. In particular, we have the mapping

$$\omega : \mathbb{F}_{\mathbb{M}} \to \mathbb{T}_{\mathbb{M}}, \quad \omega : A \to X^i_A = [\omega(A)]^i = \omega^{ij}\partial_j A, \tag{113}$$

and the basic relation between the Lie and Poisson brackets

$$\omega(\{A, B\}) = -[\omega(A), \omega(B)], \quad \text{or, equivalently} \quad X_{\{A,B\}} = -[\vec{X}_A, \vec{X}_B]. \tag{114}$$

The symplectic form can be used to determine the mapping $\tilde{\omega} : \mathbb{T}_{\mathbb{M}} \times \mathbb{T}_{\mathbb{M}} \to \mathbb{F}_{\mathbb{M}}$ as follows

$$\tilde{\omega} : \vec{X}, \vec{Y} \to \tilde{\omega}(\vec{X}, \vec{Y}) \equiv \tilde{\omega}_{ij} X^i Y^j, \qquad \text{then} \qquad \tilde{\omega}(\vec{X}, \vec{X}) = 0. \tag{115}$$

Then, the Poisson bracket can be considered to be a composition[8] of the mappings (115) and (113)

$$\{A, B\} = -\tilde{\omega}(\omega(A), \omega(B)) \equiv -\tilde{\omega}(\vec{X}_A, \vec{X}_B). \tag{116}$$

Exercise 10. (a) Prove that $\{Q, H\} = c = const$ if and only if $[\vec{X}_Q, \vec{X}_H] = 0$. (b) Confirm (116).

By analogy with Riemannian geometry, on the symplectic manifold there is the natural possibility of raising and lowering the indices of tensor quantities. It is achieved with the use of the symplectic tensor and its inverse. For instance, the mapping $U_i = \tilde{\omega}_{ij} V^j$ and its inversion $V^i = \omega^{ij} U_j$ establish an isomrphism between the spaces of covariant and contravariant vector fields.

Affirmation 20. V^i is a Hamiltonian vector field if and only if $U_i = \tilde{\omega}_{ij} V^j$ obeys the condition

$$\partial_i U_j - \partial_j U_i = 0. \tag{117}$$

Proof. The equation $\partial_i A = \tilde{\omega}_{ij} V^j \equiv U_i$ for determining of A implies (117) as a necessary condition. Conversely, when (117) is satisfied, the function

$$A = \frac{1}{n} \sum_{j=1}^{n} \int U_j(z^k) dz^j, \tag{118}$$

generates the field V^i: $V^i = \omega^{ij} \partial_j A$. $\square$

As an application of the developed formalism, we mention the following.

Affirmation 21. Consider the Poisson manifold $\mathbb{M}_{2n}$ with the non-degenerated Poisson structure $\det \omega \neq 0$. Let $\tilde{\omega}$ be the corresponding symplectic form and a_i be the contravariant vector field defined in (111). Then, the Hamiltonian Equation (51) follows from the variational problem

$$S_H = \int d\tau \Big[a_i(z) \dot{z}^i - H(z) \Big]. \tag{119}$$

Exercise 11. Prove affirmation[9].

6.2. Restriction of Symplectic Structure to a Submanifold and Dirac Bracket

We recall that the mapping of manifolds $\mathbb{N} = \{x^a\} \to \mathbb{M}_n = \{z^i\}$ given by $x^a \to z^i(x^a)$ induces the mapping $\mathbb{T}^{(0,m)}_{\mathbb{M}} \to \mathbb{T}^{(0,m)}_{\mathbb{N}}$ of covariant tensor fields (see (34)). Let $\mathbb{M}_n = \{z^k, \tilde{\omega}_{ij}(z^k)\}$ be a symplectic manifold and $\mathbb{N}_k$ be a submanifold determined by the functions $\Phi^\alpha(z^k) = 0$ (see (21)), and n and k are even numbers. Consider the embedding $\mathbb{N}_k \to \mathbb{M}_n$, given by $x^a \to z^i = (f^\alpha(x^a), x^a)$. Then, the induced mapping

$$\tilde{\omega}_{\mathfrak{f}ab}(x^c) = \frac{\partial z^i}{\partial x^a} \frac{\partial z^j}{\partial x^b} \tilde{\omega}_{ij}(f^\alpha(x^c), x^c), \tag{120}$$

is called a restriction of the symplectic form $\tilde{\omega}_{ij}(z^k)$ on $\mathbb{N}_k$. If $\tilde{\omega}_{\mathfrak{f}}$ obeys the properties (108) and (109), $\mathbb{N}_k$ turns into a symplectic manifold. The inverse matrix then determines a Poisson bracket on $\mathbb{N}_k$. Here, we discuss the necessary and sufficient conditions under which this occurs. We will need the following matrix identity.

Affirmation 22. Consider an invertible antisymmetric matrix

$$A = \begin{pmatrix} a & b \\ -b^T & c \end{pmatrix}, \quad \text{and its inverse} \quad A^{-1} = \begin{pmatrix} \alpha & \beta \\ -\beta^T & \gamma \end{pmatrix}. \tag{121}$$

Then the matrix γ is invertible if and only if a is invertible. In addition, we have

$$\gamma^{-1} = c + b^T a^{-1} b, \tag{122}$$

$$a^{-1} = \alpha + \beta \gamma^{-1} \beta^T. \tag{123}$$

Proof. Equations (122) and (123) immediately follow from the identity $AA^{-1} = 1$, written in terms of the blocks. $\square$

Affirmation 23. The matrix (120) obeys the properties (108) and (109) if and only if

$$\det\{\Phi^\alpha, \Phi^\beta\} \equiv \det \triangle^{\alpha\beta} \neq 0, \quad \text{on} \quad \mathbb{N}_k. \tag{124}$$

Proof. Consider the problem in the coordinates of $\mathbb{M}_n$

$$y^k = (y^\alpha, y^a), \quad y^\alpha = \Phi^\alpha(z^\beta, z^b), \quad y^a = z^a, \quad a = 1, 2, \ldots, k, \tag{125}$$

adapted with the functions Φ^α (see Section 2.1). Denote $\omega^{ij}(z^k)$ the Poisson tensor of $\mathbb{M}_n$. Using the transformation law (19), we obtain the following expressions for ω'^{ij} and its inverse $\tilde{\omega}'_{ij}$:

$$\omega'^{ij}(y^k) = \begin{pmatrix} \{\Phi^\alpha, \Phi^\beta\} & \{\Phi^\alpha, z^b\} \\ \{z^a, \Phi^\beta\} & \{z^a, z^b\} \end{pmatrix}\Big|_{z^i(y^j)}, \qquad \tilde{\omega}'_{ij}(y^k) = \begin{pmatrix} \tilde{\omega}'_{\alpha\beta}(y^k) & \tilde{\omega}'_{\alpha b}(y^k) \\ \tilde{\omega}'_{a\beta}(y^k) & \tilde{\omega}'_{ab}(y^k) \end{pmatrix}. \quad (126)$$

For the latter use, we make the following observation. The symplectic matrix $\tilde{\omega}'_{ij}(y^\alpha, y^a)$ obeys the identity (109). In particular, we have $\partial_a \tilde{\omega}'_{bc}(y^\alpha, y^a) + cycle = 0$ for any fixed y^α. Considering $\tilde{\omega}'_{bc}(y^a, y^\alpha)$ as a function of y^a, and applying Affirmation 18, we conclude that its inverse, say ω_D^{ab}, obeys the identity $\omega_D^{ad}\partial_d \omega_D^{bc} + cycle(a,b,c) = 0$. Using Affirmation 22 for the matrices (126), the explicit form of the inverse matrix is

$$\omega_D^{ab}(y^\alpha, y^c) = \left(\{z^a, z^b\} - \{z^a, \Phi^\alpha\}\tilde{\triangle}_{\alpha\beta}\{\Phi^\beta, z^b\}\right)\Big|_{z^i(y^j)}. \quad (127)$$

Let us return to the proof. In the adapted coordinates, the embedding $\mathbb{N}_k \to \mathbb{M}_n$ is given by $x^a \to y^i = (y^\alpha, y^a)$, where $y^\alpha = 0$ and $y^a = x^a$. The Equation (120) reads

$$\tilde{\omega}_{\mathbf{f}ab}(x^a) = \tilde{\omega}'_{ab}(y^\alpha, y^a)|_{y^\alpha = 0, y^a \to x^a}, \quad (128)$$

that is, the restriction of $\tilde{\omega}'_{ij}(y^k)$ on $\mathbb{N}_k$ reduces to the setting $y^\alpha = 0$ in a, b-block of the matrix $\tilde{\omega}'_{ij}(y^k)$. We need to confirm that $\tilde{\omega}_{\mathbf{f}}$ is a non-degenerate and closed form. The symplectic matrix $\tilde{\omega}'_{ij}(y^\alpha, y^a)$ obeys the identity (109). In particular, we have $\partial_a \tilde{\omega}'_{bc}(y^\alpha, y^a) + cycle = 0$ for any fixed y^α. Taking $y^\alpha = 0$, we conclude that $\tilde{\omega}_{\mathbf{f}}$ is closed. Further, using Affirmation 22 for the matrices (126), we conclude that the matrix $\tilde{\omega}_{\mathbf{f}}$ is invertible if and only if $\det\{\Phi^\alpha, \Phi^\beta\} \neq 0$. □

As the restriction (120) determines a symplectic structure on $\mathbb{N}_k$, its inverse gives a Poisson bracket. Its explicit expression in terms of the original bracket can be obtained using the representation (128) for $\tilde{\omega}_{\mathbf{f}ab}$. Using Affirmation 22 for the matrices (126) and Equation (23), we can write for the inverse of $\tilde{\omega}_{\mathbf{f}ab}$ the expression

$$\omega_{\mathbf{f}}^{ab}(x^c) = \left(\{z^a, z^b\} - \{z^a, \Phi^\alpha\}\tilde{\triangle}_{\alpha\beta}\{\Phi^\beta, z^b\}\right)\Big|_{z^i(y^j)}\Big|_{y^\alpha = 0, y^a \to x^a} \quad (129)$$

$$= \left(\{z^a, z^b\} - \{z^a, \Phi^\alpha\}\tilde{\triangle}_{\alpha\beta}\{\Phi^\beta, z^b\}\right)\Big|_{z^\alpha \to f^\alpha(z^a)}\Big|_{z^a \to x^a}. \quad (130)$$

Thus, we obtained the following result.
Affirmation 24. Let $\omega^{ij} = \{z^i, z^j\}$ be a non-degenerate Poisson tensor and Φ^α be functionally independent functions with $\det\{\Phi^\alpha, \Phi^\beta\} \neq 0$. Then, the matrix

$$\omega_{\mathbf{f}}^{ab}(z^c) = \left(\{z^a, z^b\} - \{z^a, \Phi^\alpha\}\tilde{\triangle}_{\alpha\beta}\{\Phi^\beta, z^b\}\right)\Big|_{z^\alpha = f^\alpha(z^c)}, \quad (131)$$

where $z^\alpha = f^\alpha(z^c)$ are parametric equations of the surface $\Phi^\alpha = 0$, obeys the Jacobi identity and determines a non-degenerate Poisson bracket on $\mathbb{N}_k$

$$\{A, B\}_{D(\mathbb{N})} = \partial_a A \omega_{\mathbf{f}}^{ab} \partial_b B. \quad (132)$$

There is a bracket on $\mathbb{M}_n$ that induces[10] the bracket (132) on $\mathbb{N}_k$ according to Equation (88). The equality (131) prompts us to consider

$$\{A, B\}_D = \{A, B\} - \{A, \Phi^\alpha\}\tilde{\triangle}_{\alpha\beta}\{\Phi^\beta, B\} =$$
$$\partial_i A\left[\{z^i, z^j\} - \{z^i, \Phi^\alpha\}\tilde{\triangle}_{\alpha\beta}\{\Phi^\beta, z^j\}\right]\partial_j B \equiv \partial_i A\, \omega_D^{ij}\, \partial_j B. \quad (133)$$

This is the famous Dirac bracket [1,2]. The tensor $\omega_D^{ij}(z^k)$ obeys the Jacobi identity (see below), and hence turns $\mathbb{M}_n$ into the Poisson manifold $(\mathbb{M}_n, \{,\}_D)$. The bracket (132) can be found to be the restriction of (133) to $\mathbb{N}_k$. To see this, we first note that for any function $A(z^i)$, Equation (133) implies

$$\{A, \Phi^\alpha\}_D = 0, \tag{134}$$

so Φ^α are Casimir functions of the Dirac bracket. As we saw in Section 4, this implies that all Hamiltonian fields $V^i_A = \omega^{ij}_D \partial_j A$ are tangent to the surfaces $\Phi^\alpha = c^\alpha$, and we can restrict the Dirac tensor ω^{ij}_D on the submanifold $\mathbb{N}_k$ according to Equation (88). This gives the Poisson bracket (132) on $\mathbb{N}_k$ and turns it into a Poisson submanifold of the Poisson manifold $(\mathbb{M}_n, \{,\}_D)$.

It remains to prove the Jacobi identity for the Dirac bracket.

Affirmation 25. Consider the Poisson manifold $\mathbb{M}_n = \{z^i,\ \omega^{ij}(z^k)\}$ with a non-degenerate tensor ω. Let $\Phi^\alpha(z^k)$ be functionally independent functions which obey the condition (124). Then, the Dirac tensor $\omega^{ij}_D(z^k)$, specified in (133), satisfy the identity (40). Hence, the Dirac bracket (133) satisfies the Jacobi identity: $\{A, \{B, C\}_D\}_D + cycle\ (A, B, C) = 0$.

Proof. Consider the problem in the coordinates (125) adapted with the functions Φ^α. Using Equations (19) and (134), we obtain the Dirac tensor in these coordinates

$$\omega'^{ij}_D(y^k) = \begin{pmatrix} 0 & 0 \\ 0 & \omega^{ab}_D(z^k)\Big|_{z(y)} \end{pmatrix}, \tag{135}$$

where $\omega^{ab}_D(z^k)$ is an a, b-block of the Dirac tensor $\omega^{ij}_D(z^k)$ in original coordinates. Then, $\omega^{ab}_D(z^k)\Big|_{z(y)}$ is just the expression written in (127). The desired Jacobi identity $\omega'^{in}_D \partial_n \omega'^{jk}_D + cycle = 0$ will be fulfilled if the matrix (127) obeys the identity $\omega^{ad}_D \partial_d \omega^{bc}_D + cycle = 0$. However, this was confirmed above (see the discussion below of Equation (126)). □

The results of this subsection can be summarized in the form of diagram (136), which relates geometrical structures on the manifold $\mathbb{M}_n$ (top line), and on its submanifold $\mathbb{N}_k$ (bottom line):

$$\begin{array}{ccccc} \tilde{\omega}_{ij} & \longleftrightarrow & \omega^{ij} & \longrightarrow & \omega^{ij}_D \sim \{,\}_D \\ \downarrow & & & & \downarrow \\ \tilde{\omega}_{\mathbf{f}ab} & \longleftarrow & -- & \longrightarrow & \omega^{ab}_{\mathbf{f}} \sim \{,\}_{D(\mathbb{N})} \end{array} \tag{136}$$

The Dirac bracket appears in the upper right corner of the rectangle, and provides the closure of our diagram.

Discussion of the Dirac bracket in the coordinate-free language can be found in [13,43–47].

6.3. Dirac's Derivation of the Dirac Bracket

Dirac arrived at his bracket in the analysis of a variational problem for singular non-degenerate theories such as (5). Consider the variational problem

$$S = \int d\tau [p_a \dot{q}^a - H_0(q^a, p_b) + \lambda^\alpha \Phi_\alpha(q^a, p_b)], \tag{137}$$

for the set of independent dynamical variables $z^i(\tau) \equiv (q^a, p_b)$, $i = (1, 2, \ldots, 2n)$, and $\lambda^\alpha(\tau)$, $\alpha = (1, 2, \ldots, 2p < 2n)$. H_0, and Φ_α are given functions where Φ_α obeys the condition (124). Variation of the action with respect to z^i and λ^α gives the equations of motion[11]

$$\dot{z}^i = \{z^i, H_0\} + \lambda^\alpha \{z^i, \Phi_\alpha\}, \qquad \Phi_\alpha = 0, \tag{138}$$

where $\{,\}$ is the canonical Poisson bracket on $\mathbb{R}_{2n}$. Let $z^i(\tau), \lambda(\tau)$ be a solution of the system. Computing the derivative of the identity $\Phi_\alpha(z^i(\tau)) = 0$, we obtain the algebraic equations

$$\{\Phi_\alpha, H_0\} + \{\Phi_\alpha, \Phi_\beta\}\lambda^\beta = 0. \tag{139}$$

that must be satisfied for all solutions, that is, they are the consequences of the system. According to this equation, all variables λ^β are determined algebraically: $\lambda^\beta = -\tilde{\triangle}^{\beta\alpha}\{\Phi_\alpha, H_0\}$, where $\tilde{\triangle}$ is the inverse matrix of $\triangle$. Adding the consequences to the system, we obtain the equivalent form

$$\dot{z}^i = \{z^i, H_0\} - \{z^i, \Phi_\alpha\}\tilde{\triangle}^{\alpha\beta}\{\Phi_\beta, H_0\} \equiv \left[\omega^{ij} - \{z^i, \Phi_\alpha\}\tilde{\triangle}^{\alpha\beta}\{\Phi_\beta, z^j\}\right]\partial_j H_0, \tag{140}$$

$$\Phi_\alpha = 0, \qquad \lambda^\beta = -\tilde{\triangle}^{\beta\alpha}\{\Phi_\alpha, H_0\}, \tag{141}$$

where the sectors z^i and λ^β turn out to be separated. The expression that appeared on the right-hand side of (140) suggests the introduction of the new bracket on $\mathbb{M}_{2n}$

$$\{A, B\}_D = \{A, B\} - \{A, \Phi_\alpha\}\tilde{\triangle}^{\alpha\beta}\{\Phi_\beta, B\}, \tag{142}$$

which is simply the Dirac bracket. Then, Equation (140) represents a Hamiltonian system with the Dirac bracket

$$\dot{z}^i = \{z^i, H_0\}_D, \tag{143}$$

with the Hamiltonian being H_0.

7. Poisson Manifold and Dirac Bracket

7.1. Jacobi Identity for the Dirac Bracket

While our discussion of the Dirac bracket in the previous section was based on a symplectic manifold, the prescription (133) can equally be used to generate a bracket $\{A, B\}_D$ starting from a given degenerate Poisson bracket $\{A, B\}$. We show that $\{A, B\}_D$ still satisfies the Jacobi identity. To prove this, we will need the following auxiliary statement.

Affirmation 26. Consider the Poisson manifold
$\mathbb{M}_{m+n} = \{x^K = (x^{\bar{\alpha}}, x^i),\ \omega^{IJ}(x^K),\ rank\ \omega = n\}$. Let $K^{\bar{\alpha}}(x^I)$, $\bar{\alpha} = 1, 2, \ldots, m$ be functionally independent Casimir functions and $\Phi^\alpha(x^I)$, $\alpha = 1, 2, \ldots, p < n$ be functionally independent functions which obey the condition (124). Then,
(A) The $m + p$ functions $K^{\bar{\alpha}}, \Phi^\beta$ are functionally independent.
(B) In the coordinates

$$z^I = (z^{\bar{\alpha}}, z^i), \quad \text{where} \quad z^{\bar{\alpha}} = K^{\bar{\alpha}}(x^I), \quad z^i = x^i, \tag{144}$$

the functions $\Phi^\alpha(z^{\bar{\alpha}}, z^i)$ obey the condition $rank\ \frac{\partial\Phi^\alpha}{\partial z^i} = p$. In other words, Φ^α that are considered as functions of z^i are functionally independent.

Proof. (A) In the coordinates (144), our functions are $z^{\bar{\alpha}}$ and $\Phi^\alpha(z^{\bar{\alpha}}, z^i)$. We will show that functional dependence of the set implies that the matrix $\{\Phi^\alpha, \Phi^\beta\}$ is degenerate. Then, nondegeneracy implies functional independence of the set—the desired result.

Consider $(m + p) \times (m + n)$ matrix

$$J = \frac{\partial(z^{\bar{\alpha}}, \Phi^\alpha(z^{\bar{\alpha}}, z^i))}{\partial(z^{\bar{\alpha}}, z^i)} = \begin{pmatrix} \mathbf{1}_{m\times m} & \mathbf{0} \\ \frac{\partial\Phi^\alpha}{\partial z^{\bar{\beta}}} & \frac{\partial\Phi^\alpha}{\partial z^i} \end{pmatrix}. \tag{145}$$

If $z^{\bar{\alpha}}, \Phi^\alpha(z^{\bar{\alpha}}, z^i)$ are functionally dependent, we have $rank\ J < m + p$, then some linear combination of rows of the matrix J vanishes: $c_{\bar{\alpha}}\delta^{\bar{\alpha}}{}_I + c_\alpha\frac{\partial\Phi^\alpha}{\partial z^I} = 0$ for all I. This equation, together with explicit expression (145) for J, implies

$$c_\alpha \frac{\partial \Phi^\alpha}{\partial z^i} = 0, \quad \vec{c} \neq 0. \tag{146}$$

Consider now the matrix $\{\Phi^\alpha, \Phi^\beta\}$ in the coordinates (144). Using the Poisson tensor

$$\omega'^{IJ}(z^K) \equiv \begin{pmatrix} \omega'^{\bar{\alpha}\bar{\beta}} & \omega'^{\bar{\alpha}j} \\ \omega'^{i\bar{\beta}} & \omega'^{ij} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{m\times m} & \mathbf{0} \\ \mathbf{0} & \omega^{ij}(x^K)|_{x(z)} \end{pmatrix}, \tag{147}$$

we obtain $\{\Phi^\alpha, \Phi^\beta\} = \partial_i \Phi^\alpha \omega'^{ij} \partial_j \Phi^\beta$. Then, (146) implies the degeneracy of the matrix $\{\Phi^\alpha, \Phi^\beta\} c_\beta = 0$.

(B) Item (A) implies that *rank* $J = m + p$. Then, from the explicit form (145) for J it follows that *rank* $\frac{\partial \Phi^\alpha}{\partial z^i} = p$. □

Affirmation 27. The Dirac bracket (133) constructed on the base of a degenerate Poisson bracket $\{A, B\}$ satisfies the Jacobi identity.

Proof. We use the notation specified in Affirmation 26. The original Poisson tensor in the coordinates (144) is written in Equation (147). According to Affirmation 10, its block ω'^{ij} is a non-degenerate matrix. $\omega'^{IJ}(z^K)$ satisfies the Jacobi identity, that due to the special form (147) of this tensor reduces to the expression

$$\omega'^{in} \frac{\partial}{\partial z^n} \omega'^{jk} + cycle = 0. \tag{148}$$

Using the prescription (133), we use $\omega'^{IJ}(z^K)$ to write the Dirac tensor

$$\omega'^{IJ}_D(z^K) = \begin{pmatrix} \mathbf{0}_{m\times m} & \mathbf{0} \\ \mathbf{0} & \omega'^{ij}_D \end{pmatrix}, \quad \text{where} \quad \omega'^{ij}_D = \omega'^{ij} - \omega'^{in} \partial_n \Phi^\alpha \tilde{\triangle}_{\alpha\beta} \partial_k \Phi^\beta. \tag{149}$$

The Jacobi identity for $\omega'^{IJ}_D(z^K)$ will be satisfied if

$$\omega'^{in}_D \frac{\partial}{\partial z^n} \omega'^{jk}_D + cycle = 0. \tag{150}$$

Note that $z^{\bar{\alpha}}$ enters into the expressions (148)–(150) as the parameters. In particular, the derivative $\frac{\partial}{\partial z^{\bar{\alpha}}}$ falls out of all these expressions. According to item (B) of Affirmation 26, the functions $\Phi^\alpha(z^{\bar{\beta}}, z^i)$, considered as functions of z^i, are functionally independent. Taking this into account, we can apply Affirmation 25 to the matrices specified by (148) and (149) and conclude that (150) holds. □

7.2. Some Applications of the Dirac Bracket

With a given scalar function A, we associate the function

$$A_d = A - \{A, \Phi^\alpha\} \tilde{\triangle}_{\alpha\beta} \Phi^\beta. \tag{151}$$

The two functions coincide on the surface $\Phi^\alpha = 0$. There is a remarkable relation between the Dirac bracket of the original functions and the Poisson bracket of the deformed functions,

$$\{A, B\}_D = \{A_d, B_d\} + O(\Phi^\alpha), \tag{152}$$

which means that the two brackets also coincide on the surface. This property can be reformulated in terms of vector fields as follows. Given a scalar function A, integral lines of the Hamiltonian field $V^i = \omega^{ij} \partial_j A_d$ that cross the surface $\Phi^\alpha = 0$ lie entirely on it.

Below, we use the Dirac bracket to analyze some Hamiltonian systems consisting of both dynamical and algebraic equations.

1. Consider the Hamiltonian system $\dot{z}^i = \{z^i, H\}_D$ on the Poisson manifold $(\mathbb{M}, \{,\}_D)$. As $\{\Phi^\alpha, H\}_D = 0$, the functions Φ^α are integrals of motion of the system. According to Affirmation 16, all the submanifolds $\mathbb{N}_k^{\vec{c}} = \{z \in \mathbb{M}_n, \quad \Phi^\alpha(z) = c^\alpha\}$ are invariant submanifolds, that is, any trajectory that starts on $\mathbb{N}_k^{\vec{c}}$ lies entirely on it. In particular, we have:

Affirmation 28. The equations

$$\dot{z}^i = \{z^i, H\}_D, \qquad \Phi^\alpha = 0, \tag{153}$$

form a self-consistent system in the sense of Definition 1.1.

Furthermore, according to Affirmation 17, these equations are equivalent to the system $z^\alpha - f^\alpha(z^a) = 0, \dot{z}^b = \{z^b, H(z^i)\}_D\Big|_{z^\alpha \to f^\alpha(z^b)}$. We replace z^α on $f^\alpha(z^b)$ using Equations (86) and (88). This gives

$$z^\alpha = f^\alpha(z^b), \quad \dot{z}^b = \{z^b, \hat{H}(z^b)\}_{D(\mathbb{N})}, \tag{154}$$

where $\hat{H}(z^b) = H(z^b, z^\alpha(z^b))$, and $\{,\}_{D(\mathbb{N})}$ is the bracket (131) on $\mathbb{N}$ induced by the Dirac bracket. This shows that the variables z^b obey the Hamiltonian equations on the submanifold $\mathbb{N}$.

2. Let us rewrite the system (153) in terms of the original bracket as follows: $\dot{z}^i = \{z^i, H - \Phi^\alpha \tilde{\triangle}_{\alpha\beta}\{\Phi^\beta, H\}\} + \Phi^\alpha\{z^i, \tilde{\triangle}_{\alpha\beta}\{\Phi^\beta, H\}\}$, $\Phi^\alpha = 0$, or, equivalently

$$\dot{z}^i = \{z^i, H - \Phi^\alpha \tilde{\triangle}_{\alpha\beta}\{\Phi^\beta, H\}\}, \qquad \Phi^\alpha = 0. \tag{155}$$

Note that the functions Φ^α are not the Casimir functions of the original bracket. As the systems (155) and (153) are equivalent, we obtained an example of a self-consistent theory of the type of (5).

Affirmation 29. Given a Poisson manifold $(\mathbb{M}_n, \{,\})$, let H be a given function and let Φ^α be a set of functionally independent functions that obey the condition $\det\{\Phi^\alpha, \Phi^\beta\}\big|_{\Phi^\alpha=0} \equiv \det \triangle^{\alpha\beta} \neq 0$. Then, the equations

$$\dot{z}^i = \{z^i, \tilde{H}\}, \qquad \Phi^\alpha = 0. \tag{156}$$

with the Hamiltonian $\tilde{H} = H - \Phi^\alpha \tilde{\triangle}_{\alpha\beta}\{\Phi^\beta, H\}$ form a self-consistent system.

3. Affirmation 30. The singular non-degenerate theory defined by Equation (5) with the properties (6) and (7) is self-consistent and is equivalent to (153).

Proof. Using (6), we rewrite the system (5) in the equivalent form as follows:

$$\dot{z}^i = \{z^i, H\}_D + \{z^i, \Phi^\alpha\}\tilde{\triangle}_{\alpha\beta}\{\Phi^\beta, H\}, \qquad \Phi_\alpha = 0. \tag{157}$$

Take any point of the submanifold $\Phi^\alpha = 0$. According to Affirmation 28, there is a solution $z^i(\tau)$ of (153) that passes through this point. Due to the condition (7), we have $\{\Phi^\beta, H\}\big|_{z^i(\tau)} = 0$. Then, the direct substitution of $z^i(\tau)$ into (157) shows that it is a solution of this system. □

4. Example of Hamiltonian reduction. Let the Hamiltonian system $\dot{z}^i = \omega^{ij}\partial_j H(z^k)$ with $\det\omega \neq 0$ admit the first integrals $\Phi^\alpha(z^k)$, $\alpha = 1, 2, \ldots, n-k$ with the properties

$$\{\Phi^\alpha, H\} = 0, \qquad \det\{\Phi^\alpha, \Phi^\beta\} \equiv \triangle^{\alpha\beta} \neq 0. \tag{158}$$

Then, the dynamics can be consistently restricted on any one of invariant surfaces $\mathbb{N}_k^{\vec{c}} = \{z^k \in \mathbb{M}_n,\ \Phi^\alpha = c^\alpha\}$. Without a loss of generality, we consider a reduction on $\mathbb{N}_k^{\vec{0}}$. Then, $\Phi^\alpha = 0$ implies $z^\alpha = f^\alpha(z^a)$, while the independent variables obey the equations

$$\dot{z}^a = \omega^{aj}\partial_j H\Big|_{z^\alpha = f^\alpha(z^a)}. \tag{159}$$

We do the substitution indicated in this equation and show that the result is a Hamiltonian system. Consider the problem in the adapted coordinates (125). Then, $\Phi^\alpha = 0$ turns into $y^\alpha = 0$, while instead of (159) we have

$$\dot{y}^a = \omega'^{aj}\partial_j H\Big|_{y^\alpha=0} = \omega'^{ab}\partial_b H'\Big|_{y^\alpha=0} + \omega'^{a\beta}\partial_\beta H'\Big|_{y^\alpha=0}, \tag{160}$$

where the explicit form of ω' is given by (126), and $H'(y^i) = H(z^k(y^i))$. The equation $\{\Phi^\alpha, H\} = 0$ in the coordinates y^k gives $0 = \{y^\alpha, H'\} = \omega'^{\alpha a}\partial_a H' + \triangle^{\alpha\beta}\partial_\beta H'$ or $\partial_\beta H' = -\tilde{\triangle}_{\beta\gamma}\omega'^{\gamma a}\partial_a H'$. Using this expression in (160) we obtain

$$\dot{y}^a = \Big[\{z^a, z^b\} - \{z^a, \Phi^\beta\}\tilde{\triangle}_{\beta\gamma}\{\Phi^\beta, z^b\}\Big]\partial_b H(z^i)\Big|_{z^i(y^j)}\Big|_{y^\alpha=0}. \tag{161}$$

Now, note that $A(z^i(y^j))\big|_{y^\alpha=0} = A(f^\alpha(z^a), z^a)|_{z^a\to y^a}$, so the equations of motion read as

$$\dot{z}^a = \omega_D^{ab}(f^\alpha(z^a), z^a)\partial_b H(f^\alpha(z^a), z^a), \tag{162}$$

where $\omega_D^{ab}(f^\alpha(z^a)z^a)$ is the (a,b)-block of the Dirac tensor (see (131)). According to Affirmation 24, it obeys the Jacobi identity, so the Equation (162) represents a Hamiltonian system, which is equivalent to (159).

7.3. Poisson Manifold with Prescribed Casimir Functions

Let $K^\alpha(z^\beta, z^b)$ with $\det \frac{\partial K^\alpha}{\partial z^\beta} \neq 0$ scalar functions in local coordinates $z^i = (z^\beta, z^b)$ of the manifold $\mathbb{M}_n$, where $\beta = 1, 2, \ldots, p$, $b = 1, 2, \ldots, n-p$. Without loss of generality, we assume that $n - p$ is an even number: $n - p = 2k$. The task is to construct a Poisson bracket on $\mathbb{M}_n$ that has K_α as the Casimir functions. One possible solution of this task can be found by using a coordinate system where the functions K_α turn into a part of coordinates.

Introduce the following coordinates on $\mathbb{M}_n$:

$$z^{j'} = \varphi^{j'}(z^i) = (K^\alpha(z^i), z^a). \tag{163}$$

Construct the matrix a with elements $a_i{}^{j'} = \frac{\partial z^{j'}}{\partial z^i} = \partial_i \varphi^{j'}$. Its inverse is denoted as $\tilde{a} \equiv a^{-1}$. In the local coordinates $z^{j'}$, define the bracket

$$\{A, B\} = \partial_{i'} A\, W_0^{i'j'} \partial_{j'} B, \qquad W_0^{i'j'}(z^{i'}) = \begin{pmatrix} 0_{p\times p} & 0 \\ 0 & \omega_0(z^{j'}) \end{pmatrix}, \tag{164}$$

where ω_0 is a $2k \times 2k$ matrix with the elements $\omega_0^{a'b'}(z^{\alpha'}, z^{c'})$ satisfying the identity (40) with respect to $z^{c'}$. From this matrix we can take any known Poisson structure $\omega_0^{a'b'}(z^{c'})$ on the submanifold $K^\alpha(z^\beta, z^b) = 0$. For instance, we could take it in the canonical form

$$\omega_0^{a'b'} = \begin{pmatrix} 0_{k\times k} & 1_{k\times k} \\ -1_{k\times k} & 0_{k\times k} \end{pmatrix}. \tag{165}$$

According to Equation (19), in the original coordinates, z^i, the bracket reads

$$\{A, B\} = \partial_i A \omega^{ij} \partial_j B, \qquad \omega^{ij} = \Big[\tilde{a}^T W_0(K^\alpha(z^i), z^a)\tilde{a}\Big]^{ij}. \tag{166}$$

Then, Affirmation 4 guarantees that it satisfies the Jacobi identity. Hence, it turns $\mathbb{M}_n$ into a Poisson manifold.

Affirmation 31. K^α are Casimir functions of the bracket (166).

Proof. Consider, for instance, $\{A, K^1\} = \partial_i A(\tilde{a}^T W_0 \tilde{a})^{ij} \partial_j K^1$. Compute the term: $(W_0\tilde{a})^{ij}\partial_j K^1 = (W_0\tilde{a})^{ij} a_j{}^1 = W_0^{ij}\delta_j{}^1 = W_0^{i1} = 0$. □

In summary, the set of functionally independent functions $K^\alpha(z^i)$ can be found to be the set of Casimir functions of the Poisson manifold with the bracket (166).

Denoting $\partial_\alpha K^\beta = b_\alpha{}^\beta$, $\partial_a K^\beta = c_a{}^\beta$, the Poisson structure (166) can be written in the following form:

$$\omega = \begin{pmatrix} (cb^{-1})^T \omega_0 cb^{-1} & (\omega_0 cb^{-1})^T \\ -\omega_0 cb^{-1} & \omega_0 \end{pmatrix}. \tag{167}$$

Blocks of this matrix can be compared with Equation (84). We can restrict the bracket (166) on the Casimir submanifold, obtaining the bracket (see Equation (88))

$$\{A(z^a), B(z^a)\} = \partial_a A \bar{\omega}^{ab} \partial_b B, \qquad \bar{\omega}^{ab} = \omega_0^{ab}(f^\alpha(z^a), z^a). \tag{168}$$

In particular, if ω_0 in Equation (164) was originally chosen to be independent of the coordinates z^α, we have $\bar{\omega}^{ab} = \omega_0^{ab}$. The Casimir submanifold with the bracket (168) is the Poisson submanifold of $\mathbb{M}_n$ (166) in the sense of the definition (73).

Example 4. *Consider $\mathbb{M}_3$ and the function $K(z^1, z^2, z^3)$ with $\partial_1 K \neq 0$. Then,*

$$a = \begin{pmatrix} \partial_1 K & 0 & 0 \\ \partial_2 K & 1 & 0 \\ \partial_3 K & 0 & 1 \end{pmatrix}, \qquad \tilde{a} = \frac{1}{\det a}\begin{pmatrix} 1 & 0 & 0 \\ -\partial_2 K & 1 & 0 \\ -\partial_3 K & 0 & 1 \end{pmatrix}. \tag{169}$$

Taking

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{170}$$

we obtain the Poisson structure on $\mathbb{M}_3$ that has $K(z)$ as the Casimir function

$$\omega = \tilde{a}^T W \tilde{a} = \frac{1}{\det a}\begin{pmatrix} 0 & \partial_3 K & -\partial_2 K \\ -\partial_3 K & 0 & 1 \\ \partial_2 K & -1 & 0 \end{pmatrix}, \quad or \quad \omega^{ij} = \frac{1}{\det a}\epsilon^{ijk}\partial_k K. \tag{171}$$

If V_i and U_j are contravariant vectors, the quantity $\frac{1}{\det a}\epsilon^{ijk}\partial_k K\, V_i\, U_j$ is a scalar function under the diffeomorphisms (14). So, ω^{ij} of Equation (171) is a second-rank covariant tensor, as it should be. Restriction of the bracket (171) on the Casimir submanifold $K = 0$ gives the canonical Poisson bracket: $\{z^2, z^3\} = 1$.

Example 5. *$SO(3)$ Lie–Poisson bracket. Choosing $K = \frac{1}{2}[(z^1)^2 + (z^2)^2 + (z^3)^2] - 1$ (see Example 2) in the expressions of previous example, we obtain the diffeomorphism covariant form of the Lie–Poisson bracket:*

$$\{z^i, z^j\} = \frac{1}{\det a}\epsilon^{ijk} z^k. \tag{172}$$

8. Conclusions

In this short survey, we presented an elementary exposition of the methods of Poisson and symplectic geometry with an emphasis on the construction, geometric meaning, and applications of the Dirac bracket. We have traced the role played by the Dirac bracket in the problem of reducing the Poisson structure of a manifold to the submanifold as defined by scalar functions which form the set of second-class constraints. Then, the Dirac bracket was applied to the study of the Hamiltonian system (5) with second-class constraints (6). Let us briefly describe these results.

Let $\mathbb{M}_n = \{z^k, \omega^{ij}(z^k)\}$ be a non-degenerate Poisson manifold and let $\mathbb{N}_m = \{x^a\}$ be a submanifold determined by the equations $\Phi^\alpha(z^k) = 0$, and n and m are even numbers. Let $z^\alpha = f^\alpha(z^a)$ be the solution to these equations. They determine the embedding $\mathbb{N}_m \to \mathbb{M}_n$ given by $x^a \to z^i = (f^\alpha(x^a), x^a)$. The non-degenerate contravariant tensor ω^{ij} cannot be directly used to induce the Poisson structure on the submanifold. However, we can do this with the help of the symplectic form $\tilde{\omega}_{ij}$ corresponding to the Poisson tensor ω^{ij}. In the case of the submanifold determined by the second-class constraints, $\det\{\Phi^\alpha, \Phi^\beta\}_P\big|_{\Phi^\alpha=0} \neq 0$, the induced mapping (120) determines the symplectic form $\tilde{\omega}_{\mathbf{f}ab}(x^c)$ on $\mathbb{N}_m$. The explicit form of inverse of this matrix is given by Equation (131) and determines a non-degenerate Poisson bracket $\{A, B\}_{D(\mathbb{N})} = \partial_a A \omega_{\mathbf{f}}^{ab} \partial_b B$ on $\mathbb{N}_m$. This solves the reduction problem.

Next, we may wonder about constructing a degenerate Poisson bracket on $\mathbb{M}_n$ that directly induces the bracket on $\mathbb{N}_m$ with use the Casimir functions (see Equation (88)). The explicit form (131) of the Poisson tensor $\omega_{\mathbf{f}}^{ab}(x^c)$ immediately prompts the Dirac bracket (133) as a solution of this task. The described construction can be resumed in the form of diagram (136). The Dirac bracket appears in the upper right corner of the rectangle, and provides the closure of the diagram.

Consider now the Hamiltonian system (5)–(7) on $\mathbb{M}_n$, and the following Hamiltonian system on $\mathbb{N}_m$:

$$\dot{x}^a = \{x^a, H(x^b)\}_{D(\mathbb{N})}, \qquad H(x^b) = H(f^\alpha(x^b), x^b). \tag{173}$$

Using the Dirac bracket, we demonstrated in Section 7.2 that the two systems are equivalent. This implies that the system (5) with second-class constraints (6) and (7) is self-consistent, and its restriction on $\mathbb{N}_m$ is a Hamiltonian system.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The work has been supported by the Brazilian foundation CNPq and the Tomsk State University Competitiveness Improvement Program.

Conflicts of Interest: Not applicable.

Appendix A

Appendix A.1. Jacobi Identity

Affirmation A1. Let the bracket (38) obey the Jacobi identity in the coordinates z^i. Then, the Jacobi identity is satisfied in any other coordinates.

Proof. We need to show that the validity of the identity (36) for the bracket (38) with $\omega^{ij}(z)$ implies its validity for the bracket with $\omega^{i'j'}$ defined in (19).

Given the functions $A(z), B(z), C(z)$, let us consider the auxiliary functions $\tilde{A}(z) \equiv A(z'(z))$ and so on. As the Jacobi identity is satisfied in the coordinates z, we can write

$$\partial_i A(z'(z)) \omega^{ip}(z) \partial_p \Big[\partial_j B(z'(z)) \omega^{jk}(z) \partial_k C(z'(z))\Big] + cycle(A, B, C) = 0. \tag{A1}$$

Computing the derivatives, we present this identity as follows:

$$\partial_{i'} A|_{z'(z)} \frac{\partial z^{i'}}{\partial z^i} \omega^{ip}(z) \partial_p \left[\left[\partial_{j'} B \frac{\partial z^{j'}}{\partial z^j}\bigg|_{z(z')} \omega^{jk}(z(z')) \frac{\partial z^{k'}}{\partial z^k}\bigg|_{z(z')} \partial_{k'} C \right] \bigg|_{z'(z)} \right] + cycle(A,B,C) = 0.$$

Using the identity $\partial_p \left[D(z')|_{z'(z)} \right] = \frac{\partial z^{p'}}{\partial z^p} \partial_{p'} D(z') \Big|_{z'(z)}$, we obtain

$$\left[\partial_{i'} A(z') \omega^{i'p'}(z') \partial_{p'} \left[\partial_{j'} B(z') \omega^{j'k'}(z') \partial_{k'} C(z') \right] + cycle(A,B,C) \right] \bigg|_{z'(z)} = 0, \quad \text{(A2)}$$

which is simply the Jacobi identity for the bracket $\{A(z'), B(z')\} = \partial_{i'} A \omega^{i'j'}(z') \partial_{j'} B$. □

Appendix A.2. Darboux Theorem

Lemma A1. *(On the rectification of a vector field). Let $V^i(z^j)$ be vector field, non-vanishing at the point $z_0 \in \mathbb{M}_n$. Then, there are coordinates*[12] *y^i such that $V^i(y^j) = (1, 0, \dots, 0)$ at all points y^j in some vicinity of z_0. The coordinate y^1 has a simple geometric meaning: its integral lines are just the integral lines of $\vec{V}$: $V^i(y^j) = \frac{dy^i}{d\tau}$, where $y^i(\tau) = (y^1 = \tau, y^a = C^a)$, and $C^2, \dots, C^n$ are fixed numbers.*

Proof. Without loss of generality, we take $z_0 = (0, 0, \dots, 0)$, $V^1(z_0) \neq 0$, and $V^1(z) \neq 0$ in some vicinity of z_0. Write the equations for integral lines as

$$\frac{dz^i}{d\tau} = V^i(z^j(\tau)), \quad \text{(A3)}$$

and solve them with the following initial conditions on the hyperplane $z^1 = 0$:

$$z^1(0) = 0, \quad z^2(0) = y^2, \quad \dots, \quad z^n(0) = y^n, \quad \text{where} \quad y^2, \dots, y^n \quad \text{are fixed numbers.} \quad \text{(A4)}$$

Denote by

$$z^i(\tau) = f^i(\tau, y^2, \dots, y^n), \quad \text{(A5)}$$

that the integral line at $\tau = 0$ passes through the point $(0, y^2, \dots, y^n)$. This determines the non-degenerate mapping

$$f: \quad (\tau, y^2, \dots, y^n) \quad \to \quad z^i = f^i(\tau, y^2, \dots, y^n). \quad \text{(A6)}$$

The nondegeneracy follows from (A4) and (A6) as follows:

$$\det \frac{\partial(z^1, \dots, z^n)}{\partial(\tau, y^2, \dots, y^n)}\bigg|_{z_0} = \det \begin{pmatrix} V^1(z_0) & 0 & \dots, & 0 \\ V^a(z_0) & & \delta^a{}_b & \end{pmatrix} = V^1(z_0) \neq 0. \quad \text{(A7)}$$

So, we can take the set

$$y^1 = \tau, \; y^2, \; \dots, \; y^n, \quad \text{(A8)}$$

as new coordinates of $\mathbb{M}_n$, and then the transition functions are given by Equation (A6). For the latter use, we note that

$$\frac{\partial z^2}{\partial y^1}\bigg|_{z_0} = V^2(z_0), \quad \frac{\partial z^2}{\partial y^2}\bigg|_{z_0} = 1, \quad \frac{\partial z^2}{\partial y^\alpha}\bigg|_{z_0} = 0, \quad \alpha = 3, 4, \dots, n. \quad \text{(A9)}$$

According to (A5), integral line of the field $\vec{V}$ in the new system is $y^i(\tau) = (y^1 = \tau, \; y^a = C^a)$, that is, it coincides with the coordinate line of y^1, then $V^i(y^j) = \frac{dy^i}{d\tau} = (1, 0, \dots, 0)$. □

Lemma A2. *Let $\mathbb{M}_n = \{z^k,\ \omega^{ij}(z^k)\}$ be a Poisson manifold with rank $\omega(z_0) \neq 0$. Then, there is a pair of scalar functions, say $q \in \mathbb{F}_{\mathbb{M}}$ and $p \in \mathbb{F}_{\mathbb{M}}$, with the property $\{q, p\} = 1$. Their Hamiltonian fields $\vec{V}_q$ and $\vec{U}_p$ are linearly independent and have a vanishing Lie bracket, $[\vec{V}_q, \vec{U}_p] = 0$.*

Proof. Without loss of generality we take $\omega^{12} \neq 0$. As the function $q(z^i)$, we take the scalar function of the coordinate z^2, its representative in the system z^i is $q(z^i) = z^2$. Then, its Hamiltonian field is

$$V^i_q(z^k) = \omega^{ik}\frac{\partial}{\partial z^k}z^2 = \omega^{i2} = (\omega^{12}, 0, \omega^{32}, \dots, \omega^{n2}). \tag{A10}$$

In particular, $V^1_q = \omega^{12} \neq 0$. We rectify this field according to Lemma A1. Then, its components in the system y^j are[13]

$$V^i_q(y^j) = (1, 0, \dots, 0). \tag{A11}$$

The representative of the function q in the system y^j is $q(y^j) = z^2(y^j)$, so its bracket with any other function reads

$$\{q(y^j), B(y^j)\} = V^i_q(y^j)\frac{\partial}{\partial y^i}B = \frac{\partial}{\partial y^1}B. \tag{A12}$$

Taking as the function p the scalar function of the coordinate y^1: $p(y^j) = y^1$, we obtain the desired pair of functions:

$$\{z^2(y^i), y^1\} = 1, \quad \text{or, in initial coordinates,} \quad \{z^2, y^1(z^j)\} = 1. \tag{A13}$$

In the coordinate system y^j, the Hamiltonian fields of these functions are

$$V^i_q(y^j) = (1, 0, \dots, 0), \qquad U^i_p(y^j) = \omega'^{ik}\frac{\partial}{\partial y^k}y^1 = \omega'^{i1} = (0, \omega'^{21}, \omega'^{31}, \dots, \omega'^{n1}). \tag{A14}$$

From their manifest form, they are linearly independent. Additionally, as the Hamiltonian field of a constant vanishes, we have $[\vec{V}_q, \vec{U}_p] = -\vec{W}_{\{q,p\}} = -\vec{W}_1 = 0$. □

Lemma A3. *(On the existence of a pair of canonical coordinates). Let $\mathbb{M}_n = \{z^k,\ \omega^{ij}(z^k)\}$ be Poisson manifold with rank $\omega(z_0) \neq 0$. Then there are coordinates $q, p, \xi^3, \dots, \xi^n$ with the properties*

$$\{q, p\} = \omega'^{12} = 1, \qquad \{q, \xi^\alpha\} = \omega'^{1\alpha} = 0, \qquad \{p, \xi^\alpha\} = \omega'^{2\alpha} = 0, \tag{A15}$$

$$\{\xi^\alpha, \xi^\beta\} = \omega'^{\alpha\beta}(\xi^\gamma), \quad \textit{that is} \quad \omega'^{\alpha\beta} \quad \textit{do not depend on} \quad q, p. \tag{A16}$$

In addition, Jacobi identity for ω^{ij} and Equations (A15) and (A16) imply the Jacobi identity for $\omega'^{\alpha\beta}$: $\omega'^{\alpha\rho}\partial_\rho\omega'^{\beta\gamma} + cycle = 0$.

Proof. (A) We take $q(z^i) = z^2$, and rectify the vector field V^i_q using the Lemma A1. In the process, we obtain the coordinates y^i, the components of the field $V^i_q(y^j) = (1, 0, \dots, 0)$ in these coordinates, and the scalar function $p(y^j) = y^1$ which obeys

$$\{q, p\} = 1. \tag{A17}$$

(B) Let $U^i_p(y^j)$ be components of Hamiltonian vector field of the function p in the coordinates y^j. According to Lemma B2, $\vec{V}_q$ and $\vec{U}_p$ are commuting fields, then

$$0 = [\vec{V}_q, \vec{U}_p]^i = V_q^k \frac{\partial}{\partial y^k} U_p^i - U_p^k \frac{\partial}{\partial y^k} V_q^i = \frac{\partial U_p^i}{\partial y^1}, \quad \text{implies} \quad U_p^i = U_p^i(y^2, \dots, y^n), \tag{A18}$$

that is, $\vec{U}_p$ does not depend on q and p. Consider the integral lines of the field $\vec{U}_p$. Taking into account that $U_p^1(y^j) = 0$, we have

$$\frac{dy^1}{d\lambda} = 0, \quad \text{then} \quad y^1 = C = const, \tag{A19}$$

$$\frac{dy^a}{d\lambda} = U_p^a(y^2(\lambda), \dots, y^n(\lambda)). \tag{A20}$$

For definiteness, we assume $U_p^2(z_0) \neq 0$. We apply Lemma B2 to the field $U_p^a(y^b)$, with $a, b = 2, 3, \dots, n$, that is, we solve Equation (A20) with initial conditions on the surface $y^2 = 0$:

$$y^2(0) = 0, \quad y^3(0) = \xi^3, \quad \dots, \quad y^n(0) = \xi^n, \tag{A21}$$

Denote the solution of the problem as

$$y^a(\lambda) = g^a(\lambda, \xi^3, \dots, \xi^n), \qquad a = 2, 3, \dots, n. \tag{A22}$$

These equations are invertible, as (A20)–(A22) imply (here $\alpha, \beta = 3, 4, \dots, n$)

$$\det \left. \frac{\partial(y^2, \dots, y^n)}{\partial(\lambda, \xi^3, \dots, \xi^n)} \right|_{z_0} = U_p^2(z_0) \det \frac{\partial y^\alpha(\lambda = 0)}{\partial \xi^\beta} = U_p^2(z_0) \det \mathbf{1} = U_p^2(z_0) \neq 0. \tag{A23}$$

We denote the inverse formulas as follows:

$$\lambda = \tilde{g}(y^2, \dots, y^n), \quad \xi^3 = \tilde{g}^3(y^2, \dots, y^n), \quad \dots, \quad \xi^n = \tilde{g}^n(y^2, \dots, y^n), \tag{A24}$$

and introduce the new coordinates

$$(y^1, y^2, y^3, \dots, y^n) \quad \to \quad (y^1, \lambda(y^a), \xi^\alpha(y^a)), \qquad a = 2, 3, \dots n, \quad \alpha = 3, 4, \dots, n, \tag{A25}$$

with the transition functions (A24). Integral lines of the fields U and V in the new coordinates are $(C, \lambda, \xi^3, \dots \xi^n)$ and $(y^1 = \tau, \tilde{g}(y^2, \dots, y^n), \tilde{g}^\alpha(y^2, \dots, y^n)$. Along the integral lines of U, only the second coordinate λ changes. Along the integral lines of V changes the first coordinate, $y^1 = \tau$, while λ and ξ^α, being functions of $y^2, \dots, y^n$, remain constants. Therefore, in these coordinates, both fields are straightened: $V_q^i = (1, 0, 0, \dots, 0)$, $U_p^i = (0, 1, 0, \dots, 0)$.

(C) The Poisson brackets of q and p with scalar functions of the coordinates ξ^α, $\alpha = 3, 4, \dots, n$ vanish

$$\{q, \xi^\alpha\} = V_q(\xi^\alpha) = \frac{\partial \xi^\alpha}{\partial \tau} = 0, \qquad \{p, \xi^\alpha\} = V_p(\xi^\alpha) = \frac{\partial \xi^\alpha}{\partial \lambda} = 0. \tag{A26}$$

So, the functions q p, and ξ^α obey the Equation (A15).

(D) The last step is to introduce the mapping

$$(y^1, y^2, y^3, \dots, y^n) \quad \to \quad (q = z^2(y^j), \; p = y^1, \; \xi^\alpha = \tilde{g}(y^2, \dots, y^n)). \tag{A27}$$

Its invertibility follows from the direct computation

$$\det \frac{\partial(q,p,\xi^3,\ldots,\xi^n)}{\partial(y^1,y^2,y^3,\ldots,y^n)}\Big|_{z_0} = \det \begin{pmatrix} \frac{\partial z^2}{\partial y^1} & \frac{\partial z^2}{\partial y^2} & \cdots & & \frac{\partial z^2}{\partial y^n} \\ \frac{\partial y^1}{\partial y^1} & \frac{\partial y^1}{\partial y^2} & \cdots & & \frac{\partial y^1}{\partial y^n} \\ \cdots & \cdots & & & \\ \frac{\partial \xi^\alpha}{\partial y^1} & \frac{\partial \xi^\alpha}{\partial y^2} & & \frac{\partial \xi^\alpha}{\partial y^\beta} & \\ \cdots & \cdots & & & \end{pmatrix}\Bigg|_{z_0}$$

$$= \det \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ \cdots & \cdots & & & \\ 0 & \frac{\partial \xi^\alpha}{\partial y^2} & & \mathbf{1} & \\ \cdots & \cdots & & & \end{pmatrix} = -1. \quad \text{(A28)}$$

In the computation, we used the Equations (A9), (A10), (A24), and (A23). In particular: $\frac{\partial z^2}{\partial y^1}\Big|_{z_0} = \frac{\partial f^2(\tau,y^2,\ldots,y^n)}{\partial \tau}\Big|_{z_0} = V_q^2|_{z_0} = \omega^{22} = 0$. Therefore, we can take q, p, ξ^α as a coordinate system on $\mathbb{M}_n$. As we saw above, the coordinates obey the desired property (A15). To confirm (A16), we use $\{q,\xi^\alpha\} = 0$ in the Jacobi identity, obtaining

$$\{q,\{\xi^\alpha,\xi^\beta\}\} = -\{\xi^\alpha,\{\xi^\beta,q\}\} - \{\xi^\beta,\{q,\xi^\alpha\}\} = 0, \quad \text{or} \quad \frac{\partial}{\partial p}\{\xi^\alpha,\xi^\beta\} = 0. \quad \text{(A29)}$$

As such, $\{\xi^\alpha,\xi^\beta\} \equiv \omega'^{\alpha\beta}$ does not depend on p. The similar computation of $\{p,\{\xi^\alpha,\xi^\beta\}\}$ implies, that $\omega'^{\alpha\beta}$ does not depend on q. □

If *rank* $\omega'^{\alpha\beta}(\xi^\gamma) \neq 0$, the manifold $\mathbb{M}_{n-2} = \{\xi^\gamma,\ \omega'^{\alpha\beta}(\xi^\gamma)\}$, in turn, satisfies the conditions of Lemma B3.

Generalized Darboux Theorem. Let $\mathbb{M}_n = \{z^k,\ \omega^{ij}(z^k)\}$ be a Poisson manifold with *rank* $\omega = 2k$ at the point $z_0^i \in \mathbb{M}_n$. Then, there are local coordinates where ω has the form:

$$\omega' = \begin{pmatrix} 0_{p\times p} & 0 & 0 \\ 0 & 0_{k\times k} & 1_{k\times k} \\ 0 & -1_{k\times k} & 0_{k\times k} \end{pmatrix}, \qquad p = n - 2k, \quad \text{(A30)}$$

at all points in some vicinity of z_0^i.

Proof. The proof is carried out by induction on the pairs of canonical coordinates constructed in Lemma B3. After k steps, we obtain the coordinates ξ^α, q^b, p^c, $\alpha = 1,2,\ldots,n-2k$, $b,c = 1,2,\ldots,k$, in which the tensor ω has the block-diagonal form

$$\omega' = \begin{pmatrix} \omega'^{\alpha\beta} & 0 & 0 \\ 0 & 0_{k\times k} & 1_{k\times k} \\ 0 & -1_{k\times k} & 0_{k\times k} \end{pmatrix}, \quad \text{(A31)}$$

and $\omega'^{\alpha\beta} = \{\xi^\alpha,\xi^\beta\}$. From the rank condition and from the manifest form (A31) of the matrix ω', we have $2k = \mathit{rank}\ \omega' = \mathit{rank}\ \omega'^{\alpha\beta} + 2k$, or $\mathit{rank}\ \omega'^{\alpha\beta} = 0$. This implies that $\omega'^{\alpha\beta} = 0$ for all α and β. □

Affirmation A2. Let $Q(z^i)$ be the first integral of the Hamiltonian system $\dot{z}^i = \omega^{ij}\partial_j H$ with a non-degenerate tensor ω^{ij}. Then, solution of this system of n equations reduces to the solution of a Hamiltonian system composed by $n-2$ equations.

Proof. Introduce the coordinates z'^i: $z'^1 = z^1, z'^2 = Q(z^i), z'^3 = z^3, \ldots, z'^n = z^n$, thus turning Q into the second coordinate of the new system. Applying Lemmas B2 and B3, we construct the coordinates q, p, ξ^α with $q = Q$. The Poisson tensor in these coordinates has the form

$$\omega' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \omega'^{\alpha\beta}(\xi^\gamma) \end{pmatrix}. \tag{A32}$$

Consider our Hamiltonian equations in these coordinates. The equation $\dot{q} = \partial_p H'$ together with $q = c_2 = const$ implies that H' does not depend on p: $H' = H'(q, \xi^\gamma)$. Then, on the surface $q = c_1 = const$, the original system is equivalent to

$$\dot{p} = -\partial_q H'(q, \xi^\gamma)\big|_{q=c_2}, \tag{A33}$$

$$\dot{\xi}^\alpha = \omega'^{\alpha\beta}\partial_\beta H'(c_2, \xi^\gamma), \qquad \alpha = 3, 4, \ldots, n. \tag{A34}$$

The $n-2$ Hamiltonian equations (A34) can be solved separately from (A33), let $\xi^\alpha(\tau, c_2, \ldots, c_n)$ be their general solution. Using these functions in Equation (A33), the latter is solved by direct integration: $p = -\int d\tau\, \partial_q H'(q, \xi^\gamma)\big|_{q=c_2, \xi=\xi(\tau, c_2, \ldots, c_n)}$. □

It should be noted that the range of applicability of this affirmation in applications is rather restricted. Indeed, to find manifest form of the Equation (A34), we need to rectify two vector fields. For this, it is necessary to solve the system of equations as in the original system twice.

Appendix A.3. Frobenius Theorem

The equation $\partial_x X(x, y, z) = 0$ has two functionally independent solutions: $X_1 = y$ and $X_2 = z$. The Frobenius theorem can be thought as a generalization of this result to the case of the system of first-order partial differential equations $A^i_a(z^k)\partial_i X(z^k) = 0$. The theorem can also be reformulated in a purely geometric language (see the end of this section).

We will need some properties of vector fields and their integral lines on a smooth manifold $\mathbb{M}_n = \{z^i,\ i = 1, 2, \ldots, n\}$. We recall that the integral line of the vector field $V^i(z^k)$ on $\mathbb{M}_n$ is a solution $z^i(\tau)$ to $\frac{dz^i(\tau)}{d\tau} = V^i(z^k(\tau))$. As before, we assume that through each point of the manifold passes unique integral line of $\vec{V}$. By $\{\mathbb{N}^{\vec{c}}_k,\ \vec{c} \in \mathbb{R}^{n-k}\}$, we denote a foliation of $\mathbb{M}_n$ (see Section 2.1), with the leaves

$$\mathbb{N}^{\vec{c}}_k = \{z^i \in \mathbb{M}_n,\ F^\alpha(z^i) = c^\alpha\}. \tag{A35}$$

Affirmation A3. Let $\vec{V}(z^k)$ be a vector field on $\mathbb{M}_n$ and $F(z^k)$ be a scalar function with a non-vanishing gradient. The following two conditions are equivalent:
(A) $\vec{V}$ touches the surfaces $F(z^k) = c = const$: $V^i\partial_i F = 0$ at each point $z^k \in \mathbb{M}_n$.
(B) $\vec{V}$ is tangent[14] to the surfaces $F(z^k) = c = const$, that is, the integral lines of $\vec{V}$ lie on the surfaces.

Proof. Let $z^k(\tau)$ be an integral line of $\vec{V}$. Then, the Affirmation follows immediately from the equality

$$\frac{d}{d\tau}F(z^k(\tau)) = V^i(z^k)\partial_i F(z^k)\big|_{z(\tau)}. \tag{A36}$$

□

Evidently, the same is true for a set of vector fields:

Affirmation A4. Let $\vec{A}_1(z^k), \ldots, \vec{A}_k(z^k)$ be vector fields on $\mathbb{M}_n$ linearly independent at each point $z \in \mathbb{M}_n$, and let $\{\mathbb{N}^{\vec{c}}_k,\ \vec{c} \in \mathbb{R}^{n-k}\}$ be a foliation of $\mathbb{M}_n$. The following two conditions are equivalent:
(A) The vectors $\vec{A}_a$ touch $\mathbb{N}^{\vec{c}}_k$ at each point $z^k \in \mathbb{M}_n$: $A^i_a\partial_i F^\alpha = 0$ at each point $z \in \mathbb{M}_n$.

(B) The vectors $\vec{A}_a$ are tangent to $\mathbb{N}_k^{\vec{c}}$, that is, each integral line of each $\vec{A}_a$ lies in one of the submanifolds $\mathbb{N}_k^{\vec{c}}$ (hence, $\vec{A}_a(z^k)$ forms a basis of $\mathbb{T}_{\mathbb{N}_k^c}(z^k)$).

Lemma A4. *There is a set of k linearly independent vector fields $\vec{U}_a(z^k)$ on $\mathbb{M}_n$ with the following properties.*
(A) *For any $z^k \in \mathbb{M}_n$, the vectors $\vec{U}_a(z^k)$ touch the submanifold $\mathbb{N}_k^{\vec{c}}$ that passes through this point:*

$$U_a^i \partial_i F^\alpha = 0. \tag{A37}$$

At each point, they form a basis of tangent space to the submanifold.
(B) *Integral lines of $\vec{U}_a$ that pass through $z^k \in \mathbb{M}_n$, lye in $\mathbb{N}_k^{\vec{c}}$ that passes through this point.*
(C) *$\vec{U}_a$ are commuting fields*

$$[\vec{U}_a, \vec{U}_b] = 0. \tag{A38}$$

Proof. Introduce the coordinates, adapted with the foliation: $z^k \to y^k = (y^\alpha, y^a)$, with the transition functions $y^a = z^a$, $y^\alpha = F^\alpha(z^\beta, z^b)$. In these coordinates, the sumanifolds $\mathbb{N}_k^{\vec{c}}$ appear as hyperplanes:

$$\mathbb{N}_k^{\vec{c}} = \{y^i \in \mathbb{M}_n,\ y^\alpha = c^\alpha\}, \tag{A39}$$

and y^a can be taken as local coordinates of $\mathbb{N}_k^{\vec{c}}$. Consider the vector fields $\vec{U}_a$ on $\mathbb{M}_n$, which in the system y^k have the following components: $U_a^i(y^k) = \delta_a{}^i$. Their integral lines are simply lines of the coordinates y^a of the submanifolds $\mathbb{N}_k^{\vec{c}}$. Evidently, the fields obey conditions (A)–(C) of the Lemma. Their explicit form in the original coordinates is as follows:

$$U_a^i(z^k) = \left[\frac{\partial z^i}{\partial y^j} U_a^j(y^k)\right]\Big|_{y(z)} = (U_a^b,\ U_a^\beta) = \left(\delta_a{}^b,\ \frac{\partial f^\beta(z^c, y^\gamma)}{\partial z^a}\Big|_{y^\gamma \to F^\gamma(z^b, z^\beta)}\right), \tag{A40}$$

where $f^\beta(z^c, y^\gamma)$ is a solution to the system $F^\beta(f^\beta, z^c) = y^\gamma$. As (A37) and (A38) are covariant equations, the fields (A40) satisfy them in the original coordinates z^k. □

Lemma A5. *An invertible linear combination of vector fields with closed algebra also form a closed algebra:*

$$\text{if} \quad \vec{V}_a = b_a{}^b \vec{U}_b, \quad \det b \neq 0, \quad \text{and} \quad [\vec{U}_a, \vec{U}_b] = c_{ab}{}^c \vec{U}_c, \quad \text{then} \quad [\vec{V}_a, \vec{V}_b] = \gamma_{ab}{}^c \vec{V}_c. \tag{A41}$$

Proof. This follows from direct calculation, which also implies

$$\gamma_{ab}{}^c = b_a{}^d b_b{}^e c_{de}{}^f \tilde{b}_f{}^c + V_a^j(\partial_j b_b{}^f)\tilde{b}_f{}^c - (a \leftrightarrow b), \tag{A42}$$

where $\tilde{b}$ is inverse for b. □

Lemma A6. *Let $\vec{A}_1, \ldots, \vec{A}_k$ is a set of linearly independent vector fields on $\mathbb{M}_n$, with a closed algebra of commutators*

$$[\vec{A}_a, \vec{A}_b]^i = c_{ab}{}^d(z^k) A_d^i. \tag{A43}$$

Then, there is a set of linearly independent fields $\vec{V}_a$, which are linear combinations of $\vec{A}_a$ and have the vanishing commutators

$$[\vec{V}_a, \vec{V}_b] = 0. \tag{A44}$$

Proof. The components $A_a^i = (a_a{}^b, b_a{}^\beta)$ of linearly independent fields form $k \times n$ matrix with rank equal k. Without a loss of generality, we assume $\det a_a{}^b \neq 0$, and let $\tilde{a}_a{}^b$ be the inverse matrix. We show that $\vec{V}_a \equiv \tilde{a}_a{}^b \vec{A}_b$ are the desired fields.

The expressions (A43) with components $i = c$ can be solved with respect to $c_{ab}{}^d$ as follows: $[\vec{A}_a, \vec{A}_b]^c = c_{ab}{}^d a_d{}^c$ implies $c_{ab}{}^c = [\vec{A}_a, \vec{A}_b]^d \tilde{a}_d{}^c$. Using this equality, we exclude $c_{ab}{}^c$ from the expressions (A43) with $i = \beta$, obtaining $[\vec{A}_a, \vec{A}_b]^\beta = [\vec{A}_a, \vec{A}_b]^d \tilde{a}_d{}^c b_c{}^\beta$. In more detail, this reads

$$\begin{aligned} A_a{}^i \partial_i b_b^\beta - (a \leftrightarrow b) = A_a{}^i (\partial_i a_b{}^d) \tilde{a}_d{}^c b_c{}^\beta - (a \leftrightarrow b) = \\ A_a{}^i \partial_i (a_b{}^d \tilde{a}_d{}^c b_c{}^\beta) - A_a{}^i a_b{}^d \partial_i (\tilde{a}_d{}^c b_c{}^\beta) - (a \leftrightarrow b) = \\ A_a{}^i \partial_i b_b^\beta - A_a{}^i a_b{}^d \partial_i (\tilde{a}_d{}^c b_c{}^\beta) - (a \leftrightarrow b), \end{aligned} \tag{A45}$$

which implies $A_a{}^i a_b{}^d \partial_i (\tilde{a}_d{}^c b_c{}^\beta) - (a \leftrightarrow b) = 0$. Contraction of this equality with $\tilde{a}_e{}^a \tilde{a}_f{}^b$ gives the following relation between components of the fields with closed commutator algebra:

$$\tilde{a}_a{}^c A_c{}^i \partial_i (\tilde{a}_b{}^d b_d{}^\beta) - (a \leftrightarrow b) = 0. \tag{A46}$$

Now, the fields $\vec{V}_a \equiv \tilde{a}_a{}^c \vec{A}_c$ with the components $V_a{}^i = (V_a{}^b,\ V_a{}^\beta) = (\delta_a{}^b,\ \tilde{a}_a{}^c b_c{}^\beta)$ satisfy the conditions of the Lemma. Indeed, $[\vec{V}_a, \vec{V}_b]^c = V_a{}^i \partial_i \delta_b^c - (a \leftrightarrow b) = 0$, and $[\vec{V}_a, \vec{V}_b]^\beta = V_a{}^i \partial_i (\tilde{a}_b{}^d b_d{}^\beta) - (a \leftrightarrow b) = 0$ due to (A46). □

Given the vector field $V^i(z^k)$, let us denote $\varphi^i(\tau, z_0)$ as the unique solution to the problem

$$\frac{dz^i}{d\tau} = V^i(z^k(\tau)), \qquad z^i(0) = z_0^i. \tag{A47}$$

For any fixed value of τ, the integral lines $\varphi^i(\tau, z)$, $z \in \mathbb{M}_n$ determine the transformation

$$\varphi_\tau : \mathbb{M}_n \to \mathbb{M}_n, \qquad z^i \to \varphi^i(\tau, z^k). \tag{A48}$$

Sometimes we will also use the coordinate-free notation $\varphi_\tau(z)$ for the integral line $\varphi^i(\tau, z^k)$. The composition of two transformations has the property

$$\varphi_\tau \circ \varphi_s = \varphi_{\tau+s}. \tag{A49}$$

Indeed, $\varphi^i(\tau, \varphi^j(s, z^k))$ and $\varphi^i(\tau + s, z^k)$ as functions of τ obey the problem (A47) with $z_0^i = \varphi^i(s, z^k)$. Because the problem has a unique solution, they coincide. So, the set of transformations $\{\varphi_\tau,\ \tau \in \mathbb{R}\}$ is a one-parametric Lie group with the group product being the composition law (A49).

Let φ_τ and ψ_λ be the one-parametric groups created by the linearly independent fields $V^i(z^k)$ and $U^i(z^k)$. There is a remarkable relation between the commutativity of the transformations and that of the vector fields.

Lemma A7. *The following two conditions are equivalent:* **(A)** $\varphi_\tau \circ \psi_\lambda(z^k) = \psi_\lambda \circ \varphi_\tau(z^k)$ *for all* τ, λ *and* z^k. **(B)** $[\vec{V}(z), \vec{U}(z)] = 0$ *for all* z.

Proof. (A) $\to$ (B). Expanding the Taylor series, we obtain $[\varphi_\tau \circ \psi_\lambda(z^k) - \psi_\lambda \circ \varphi_\tau(z^k)]^i = \varphi^i(\tau, \psi^j(\lambda, z^k)) - \psi^i(\lambda, \varphi^j(\tau, z^k)) = [\vec{V}(z), \vec{U}(z)]^i \tau\lambda + O^2(\tau) + O^2(\lambda) + O^3(\tau, \lambda)$. Because the left-hand side vanishes for any τ and λ, we conclude $[\vec{V}(z), \vec{U}(z)] = 0$.

(B) $\to$ (A). Consider the fields $\vec{V}$ and $\vec{U}$ in the coordinates y^k of the Lemma A1. Then, $\vec{V}(y^k) = (1, 0, \ldots, 0)$ and its integral line through the point y^k is

$$\varphi^i(\tau, y^k) = (y^1 + \tau, y^2, \ldots, y^n). \tag{A50}$$

Additionally, the condition (B) reads $0 = [\vec{V}, \vec{U}]^i = \frac{\partial U^i}{\partial y^1}$, that is, the field $\vec{U}$ does not depend on y^1. Consider $\psi_\lambda \circ \varphi_\tau(z^k)$ and $\varphi_\tau \circ \psi_\lambda(z^k)$ in the system y^k as functions of λ. Using (A50), we can write

$$\psi^i(\lambda, \varphi^j(\tau, y^k)) = (\psi^1,\ \psi^2, \dots,\ \psi^n), \quad \text{then} \quad \psi^i(0, \varphi^j(\tau, y^k)) = \varphi^i(\tau, y^k) = (y^1 + \tau,\ y^2,\ \dots,\ y^n), \tag{A51}$$

$$\varphi^i(\tau, \psi^j(\lambda, y^k)) = (\psi^1 + \tau,\ \psi^2, \dots,\ \psi^n), \quad \text{then} \quad \varphi^i(\tau, \psi^j(0, y^k)) = (y^1 + \tau,\ y^2,\ \dots,\ y^n). \tag{A52}$$

By construction, $\psi^i(\lambda)$ satisfy the equation

$$\frac{dx^i}{d\lambda} = U^i(x^2, x^3, \dots, x^n). \tag{A53}$$

As the right-hand side of this equation does not depend on x^1, the function $\varphi^i(\lambda)$ also satisfies this equation. In addition, $\psi^i(\lambda)$ and $\varphi^i(\lambda)$ satisfy the same initial conditions (see (A51) and (A52)). Hence, they coincide. □

Any set of coordinate lines, say the lines of the coordinates $z^1, z^2, \dots, z^k$, can be used to construct a set of commuting vector fields. They are the tangent fields to the coordinate lines. The following Lemma is an inversion of this statement. It also generalizes the Lemma A1 to the case of several fields.

Lemma A8. *(On rectification of the commuting vector fields). Let $\vec{V}_1, \vec{V}_2, \dots, \vec{V}_k$ be linearly independent and commuting vector fields in vicinity of $z_0 \in \mathbb{M}_n$: $[\vec{V}_a, \vec{V}_b] = 0$. Then:*
There are coordinates $y^i = (y^a, y^\alpha)$, $\alpha = k+1, \dots, n$, where the fields $\vec{V}_a$ are tangent to the coordinate lines y^a: $V^i_a(y^j) = \delta^i_a$, $a = 1, 2, \dots, k$.

Notice the immediate consequences of the Lemma: through each point $z_1 \in \mathbb{M}_n$ passes a surface $\mathbb{N}_k$ such that $\vec{V}_1(z), \vec{V}_2(z), \dots, \vec{V}_k(z)$ form a basis of the tangent spaces $\mathbb{T}_\mathbb{N}(z)$ at any point $z \in \mathbb{N}_k$. The integral lines of the fields $\vec{V}_a$ that cross $\mathbb{N}_k$ lie entirely in $\mathbb{N}_k$. Evidently, in the coordinates y^k, these surfaces are given by the equations $y^\alpha = c^\alpha = const$.

Proof. Without a loss of generality, we assume that the point z_0 has null coordinates. Selecting the appropriate $n-k$ vectors among the basic vectors of coordinate lines, say $\vec{e}_\alpha$, with coordinates $e^i_\alpha = \delta^i_\alpha$, $\alpha = k+1, \dots, n$, we complete the vectors $\vec{V}_a(z_0)$ up to a basis of $\mathbb{T}_\mathbb{M}(z_0)$. Then, determinant of the matrix composed from components of the basic vectors is not equal to zero at z_0

$$\det(\vec{V}_1, \dots, \vec{V}_k, \vec{e}_{k+1}, \dots, \vec{e}_n)|_{z_0=0} \neq 0. \tag{A54}$$

Denote φ_{τ_a} the one-parametric group (A48) created by the field $\vec{V}_a$. Consider the mapping $h : O(\vec{0}) \in \mathbb{R}^n \to \mathbb{M}_n$ defined according to rule

$$z = h(\tau_1, \dots \tau_k, y^1, \dots, y^{n-k}) = \varphi_{\tau_1} \circ \dots \circ \varphi_{\tau_k}(0, \dots, 0, y^{k+1}, \dots y^n). \tag{A55}$$

Derivatives of this function at the point $\tau_a = y^\alpha = 0$ are $\frac{dh}{d\tau_a}\Big|_0 = \frac{d}{d\tau_a}\varphi_{\tau_a}(0, \dots, 0, 0, \dots, 0)\Big|_{\tau_a=0} = \vec{V}_a(0)$ and $\frac{dh}{dy^\alpha}\Big|_0 = \frac{d}{dz^\alpha}(0, \dots, 0, 0, \dots, y^\alpha, \dots, 0)\Big|_{z^\alpha=0} = (0, \dots, 0, 0, \dots, 1, \dots, 0) = \vec{e}_\alpha$. Then $\det \frac{\partial(z^1, z^2, \dots, z^n)}{\partial(\tau_1, \dots \tau_k, y^{k+1}, \dots, y^n)}\Big|_0 = \det(\vec{V}_1, \dots, \vec{V}_k, \vec{e}_{k+1}, \dots, \vec{e}_n)|_0 \neq 0$, see (A54). So, the mapping (A55) is invertible, and we can take $y^i \equiv (\tau_a, y^\alpha)$ as a coordinate system on $\mathbb{M}_n$. The transition functions are given by Equation (A55).

Consider the integral line $\varphi_{s_a}(z)$ of the field $\vec{V}_a$ through some point z. According to Lemma A7, the commutativity of the fields implies the commutativity of their one-parametric groups, so we have

$$\varphi_{s_a}(z) = \varphi_{s_a} \circ \varphi_{\tau_1} \circ \dots \varphi_{\tau_a} \dots \circ \varphi_{\tau_k}(0, y^\alpha) = \varphi_{\tau_1} \circ \dots \varphi_{\tau_a + s_a} \dots \circ \varphi_{\tau_k}(0, y^\alpha) = h(\tau_1, \dots, \tau_a + s_a, \dots, \tau_k, y^\alpha). \quad \text{(A56)}$$

This shows that integral lines of $\vec{V}_a$ are the coordinate lines of the y^a -coordinate of the new system. Hence, the integral lines lie in the submanifolds $\mathbb{N}_k = \{y^k \in \mathbb{M}_n,\ y^\alpha = c^\alpha = const\}$.

To find the equations of these surfaces in the original coordinates, denote $\tilde{h}$ as the inverse mapping of (A55). Let the point z_1 have the coordinates $\tau_1, \dots, \tau_k, c^{k+1}, \dots c^n$ in the system y^i. Then, the submanifold is $\mathbb{N}_k = \{z \in \mathbb{M}_n,\ \tilde{h}^\alpha(z^i) = c^\alpha\}$. $\square$

Frobenius Theorem. Let $A^i_a(z^k)$, $a = 1, 2, \dots k$ be a set of functions with *rank* $A = k$. The system of first-order partial differential equations

$$A^i_a(z^k)\partial_i X(z^k) = 0, \quad \text{(A57)}$$

has $n - k$ functionally independent solutions if and only if the vectors $\vec{A}_a$ form a set with closed algebra

$$[\vec{A}_a(z^k), \vec{A}_b(z^k)] = c_{ab}{}^c(z^k)\vec{A}_a(z^k). \quad \text{(A58)}$$

Proof. Let the functions $F^\alpha(z^k)$, $\alpha = 1, 2, \dots, n - k$ represent the solutions:

$$A^i_a(z^k)\partial_i F^\alpha(z^k) = 0. \quad \text{(A59)}$$

Consider the foliation $\{\mathbb{N}^{\vec{c}}_k,\ \vec{c} \in \mathbb{R}^{n-k}\}$ determined by F^α according to Equation (A35), and let $\vec{U}_a(z^k)$ be vector fields described in Lemma A4.

Denoting $z^i_a(\tau)$ integral lines of $\vec{A}_a(z^k)$, we have $\frac{d}{d\tau}F^\alpha(z^i_a(\tau)) = A^i_a(z^k)\partial_i F^\alpha(z^k)\Big|_{z^i_a(\tau)} = 0$ according to (A59). Then, $F^\alpha(z^i_a(\tau)) = c^\alpha = const$, that is, the integral lines of $\vec{A}_a(z^k)$ lie in $\mathbb{N}^{\vec{c}}_k$, and $\vec{A}_a(z^k)$ are tangent vectors to this submanifold at each point. Then, we can present them through the basic vectors $\vec{U}_b$: $\vec{A}_a = b_a{}^b\vec{U}_b$ of Lemma A4. According to Lemma A4, $[\vec{U}_a, \vec{U}_b] = 0$. According to Lemma A5, this implies (A58).

Let (A58) be satisfied. Assuming $A^i_a = (a_a{}^b, b_a{}^\beta)$ with $\det a \neq 0$ (see Lemma A6), we write the system (A57) in the equivalent form: $\tilde{a}_a{}^b A^i_b(z^k)\partial_i X(z^k) \equiv V^i_a(z^k)\partial_i X(z^k) = 0$. According to Lemma A6, we have $[\vec{V}_a, \vec{V}_b] = 0$. According to Lemma A8, there are coordinates y^k where $V^i_a(y^k) = \delta_a{}^i$. In these coordinates, our system acquires the form $\frac{\partial}{\partial y^a}X'(y^\beta, y^b) = 0$. The functions $F^\beta(y^\beta, y^b) = y^\beta$ give $n - k$ functionally independent solutions. $\square$

Frobenius theorem, geometric formulation. Let $\vec{A}_1(z^k), \dots, \vec{A}_k(z^k)$ be linearly independent vector fields on $\mathbb{M}_n$. The following two conditions are equivalent:
(A) The fields $\vec{A}_a$ form the closed algebra:

$$[\vec{A}_a(z^i), \vec{A}_b(z^i)] = c_{ab}{}^c(z^i)\vec{A}_c(z^i). \quad \text{(A60)}$$

(B) There is a foliation $\{\mathbb{N}^{\vec{c}}_k,\ \vec{c} \in \mathbb{R}^{n-k}\}$ of $\mathbb{M}_n$ such that the fields $\vec{A}_a(z^k)$ touch the leaf $\mathbb{N}^{\vec{c}}_k$ (see Equation (A35)) at each point $z^k \in \mathbb{M}_n$ (hence, $\vec{A}_a$ form a basis of $\mathbb{T}_{\mathbb{N}^{c}_k}(z^k)$ (see Affirmation A4)).

Proof. $(B) \to (A)$. Consider $z_0 \in \mathbb{M}_n$ and let $z_0 \in \mathbb{N}^{\vec{c}}_k$, where $\mathbb{N}^{\vec{c}}_k$ is one of submanifolds specified in (B). Let $z^i(\tau)$ be the integral line of the field $[\vec{A}_a, \vec{A}_b]^i$, which at $\tau = 0$ passes through z_0. We obtain:

$$\frac{d}{d\tau}F^\alpha(z^i(\tau)) = [\vec{A}_a,\vec{A}_b]^i\partial_i F^\alpha\Big|_{z^i(\tau)} = \Big[\vec{A}_a(\vec{A}_b(F^\alpha)) - (a \leftrightarrow b)\Big)\Big|_{z^i(\tau)} = 0, \quad \text{(A61)}$$

as $\vec{A}_b(F^\alpha) = A^i_b\partial_i F^\alpha = 0$. The equality (A61) implies that the integral line of the field $[\vec{A}_a,\vec{A}_b]^i$ through z_0 lies entirely in $\mathbb{N}^{\vec{c}}_k$, so the vector $[\vec{A}_a,\vec{A}_b]^i(z_0)$ is tangent to $\mathbb{N}^{\vec{c}}_k(z_0)$. Hence, it can be presented through the basic vectors $\vec{A}_a$, which gives the desired result (A60).

$(A) \to (B)$. Let (A60) be satisfied. Using Lemma A6, we construct k linearly independent and commuting fields $\vec{V}_a$. According to Lemma A8, there are coordinates y^k where $V^i_a(y^k) = \delta_a{}^i$. Consider the foliation $\{\mathbb{N}^{\vec{c}}_k,\ \vec{c} \in \mathbb{R}^{n-k}\}$ where $\mathbb{N}^{\vec{c}}_k = \{z^k \in \mathbb{M}_n, y^\alpha = c^\alpha = const\}$. By construction, $\vec{V}_a \in \mathbb{T}_{\mathbb{N}^c_k}$ and forms a basis of $\mathbb{T}_{\mathbb{N}^c_k}$ at each point $z^k \in \mathbb{M}_n$. According to Lemma A6, the linearly independent vectors $\vec{A}$ are linear combinations of $\vec{V}_a$, so they also form a basis of $\mathbb{T}_{\mathbb{N}^c_k}$ at each point $z^k \in \mathbb{M}_n$. □

Notes

1 Singular degenerate theories usually arise if we work within a manifestly covariant formalism, when basic variables of the theory transform linearly under the action of the Poincare group. Their descriptions can be found in [1–4].

2 We recall that the functional independence of functions Φ^α guarantees that the system (5) can be resolved with respect to $2p$ variables z^α among z^i, then $z^\alpha = f^\alpha(z^b)$ are parametric equations of the surface $\Phi^\alpha = 0$.

3 In three-dimensional Euclidean space, this equality has simple geometric meaning: vector $grad\ F(x,y,z)$ in $\mathbb{R}^3$ is orthogonal to the surfaces of level $F(x,y,z) = c$ of the scalar function $F(x,y,z)$.

4 Recall that all our assertions hold locally.

5 This is a non-trivial affirmation, as $\partial_p\omega^{jk}$ is not a covariant object.

6 Notice that it is an example of coordinate-dependent statement.

7 Without loss of generality, we have taken $c_\alpha = 0$.

8 In the coordinate-free formulation of the Poisson geometry, the equality $\tilde{\omega}(\omega(A),\omega(B)) = -\{A,B\}$ is taken as the definition of the symplectic form $\tilde{\omega}$.

9 While formal variation of (119) leads to (51), the following point should be taken into account. Formulating a variational problem, we fix two points in phase space and then look for an extremal trajectory between them. The first-order system (51) has a unique solution for the given initial "position": $z^i(\tau_1) = z^i_1$. This implies that the position at a future instant τ_2 is uniquely determined by the initial position of the system. So, if we look for the extremal trajectory between two arbitrary chosen points $z^i(\tau_1) = z^i_1$ and $z^i(\tau_2) = z^i_2$, the variational problem (119) generally will not have a solution.

10 With this respect, see the comment at the end of Section 2.2.

11 It is instructive to compare the systems (138) and (102). The constraints $\Phi_\alpha = 0$ should not be confused with the first integrals. Indeed, first integrals represent the first-order differential equations which are consequences of a special form of the original equations, $c_{\alpha i}[\dot{z}^i - \{z^i,H\}] = \frac{d}{d\tau}Q_\alpha(z) = 0$, whereas constraints are the algebraic equations. As a consequence, solutions of the systems (138) and (102) have very different properties. Solutions of the system (102) pass through any point of $\mathbb{R}$, while all solutions of (138) live on the submanifold $\Phi_\alpha = 0$.

12 In this section, we use the notation $V^i(z^j)$ and $V^i(y^j)$ instead of V^i and V'^i to denote components of the vector $\vec{V}$ in different coordinate systems.

13 Compare this discussion with that near Equation (53).

14 See the definition of a vector field tangent to a submanifold on page 7.

References

1. Dirac, P.A.M. Generalized Hamiltonian Dynamics. *Can. J. Math.* **1950**, *2*, 129. [CrossRef]
2. Dirac, P.A.M. *Lectures on Quantum Mechanics*; Yeshiva University: New York, NY, USA, 1964.
3. Gitman, D.M.; Tyutin, I.V. *Quantization of Fields with Constraints*; Springer: Berlin, Germany, 1990.
4. Deriglazov, A. *Classical Mechanics: Hamiltonian and Lagrangian Formalism*, 2nd ed.; Springer: Berlin, Germany, 2017.
5. Arnold, V.I. *Mathematical Methods of Classical Mechanics*, 2nd ed.; Springer: New York, NY, USA, 1989.
6. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1986.
7. Fomenko, A.T.; Trofimov, V.V. *Integrable Systems on Lie Algebras and Symmetric Spaces*; Series: Advanced Studies in Contemporary Mathematics; Gordon and Breach Science Publishers: New York, NY, USA, 1988.
8. Perelomov, A.M. *Integrable Systems of Classical Mechanics and Lie Algebras*; Birkhauser Verlag: Basel, Switzerland, 1990; Volume 1.

9. Marsden, J.E.; Abraham, R.H. *Foundations of Mechanics*, 2nd ed.; Benjamin-Cummings Publishing Company, Inc.: Reading, MA, USA, 1978.
10. Crainic, M.; Fernandes, R.L.; Mărcut, I. *Lectures on Poisson Geometry*; Series: Graduate Studies in Mathematics; American Mathematical Society: Providence, RI, USA, 2021; Volume 217.
11. Guillemin, V.; Sternberg, S. *Symplectic Techniques in Physics*; Cambridge University Press: Cambridge, UK, 1984.
12. Vaisman, I. *Lectures on the Geometry of Poisson Manifolds*; Series: Progress in Mathematics; Birkhauser: Basel, Switzerland; Boston, MA, USA, 1994; Volume 118.
13. Karasev, M.V.; Maslov, V.P. *Nonlinear Poisson Brackets Geometry and Quantization*; Series: Translations of Mathematical Monographs; American Mathematical Society: Providence, RI, USA, 1993; Volume 119.
14. Wu, K. Multi-messenger astrophysics of a millisecond pulsar orbiting around a massive black hole. *Universe* **2022**, *8*, 78. [CrossRef]
15. Kassandrov, V.V.; Markova, N.V. New symmetries, conserved quantities and gauge nature of a free Dirac field. *Symmetry* **2021**, *13*, 2288. [CrossRef]
16. Ciaglia, F.M.; Cosmo, F.D.; Ibort, A.; Marmo, G.; Schiavone, L.; Zampini, A. Symmetries and covariant Poisson brackets on presymplectic manifolds. *Symmetry* **2022**, *14*, 70. [CrossRef]
17. Davtyan, M.; Gevorkian, Z.; Nersessian, A. Maxwell fish eye for polarized light. *Phys. Rev. A* **2021**, *104*, 053502. [CrossRef]
18. Kim, J.-W. Quantum corrections to frame-dragging in scattering amplitudes. *arXiv* **2022**, arXiv:2207.04970.
19. Chen, W.M.; Chung, M.Z.; Huang, Y.T.; Kim, J.W. Lense-Thirring effects from on-shell amplitudes. *arXiv* **2022**, arXiv:2205.07305.
20. Ghezelbash, M. Bianchi IX geometry and the Einstein–Maxwell theory. *Class. Quant. Grav.* **2022**, *39*, 075012. [CrossRef]
21. Dahal, P.K.; Terno, D.R. Polarization rotation and near-Earth quantum communications. *Phys. Rev. A* **2021**, *104*, 042610. [CrossRef]
22. Chakraborty, C. Gravitational analog of Faraday rotation in the magnetized Kerr and Reissner-Nordström spacetimes. *Phys. Rev. D* **2022**, *105*, 064072. [CrossRef]
23. Lee, H. Relativistic massive particle with spin-1/2: A vector bundle point of view. *J. Math. Phys.* **2022**, *63*, 012201. [CrossRef]
24. Lee, H. Bundle theoretic descriptions of massive single-particle state spaces; W = with a view toward relativistic quantum information theory. *arXiv* **2022**, arXiv:2201.04324.
25. Läänemets, D.; Hohmann, M.; Pfeifer, C. Observables from spherically symmetric modified dispersion relations. *Int. J. Geom. Meth. Mod. Phys.* **2022**, *19*, 2250155. [CrossRef]
26. Beck, J.L. Relativistic mechanics theory for electrons that exhibits spin, zitterbewegung, superposition and produces Dirac's wave equation. *arXiv* **2021**, arXiv:2108.07267
27. Giri, S.; Sheoran, P.; Nandan, H.; Shaymatov, S. Chaos motion and Periastron precession of spinning test particles moving in the vicinage of a Schwarzschild black hole surrounded by a quintessence matter field. *arXiv* **2022**, arXiv:2207.06007.
28. Abdulxamidov, F.; Benavides-Gallego, C.A.; Han, W.; Rayimbaev, J.; Abdujabbarov, A. Spinning test particle motion around a rotating wormhole. *Phys. Rev. D* **2022**, *106*, 2. [CrossRef]
29. Zhang, Y.P.; Zeng, Y.B.; Wang, Y.Q.; Wei, S.W.; Liu, Y.X. Equatorial orbits of spinning test particles in rotating boson stars. *Eur. Phys. J. C* **2022**, *82*, 809. [CrossRef]
30. Bubenchikov, M.A.; Kaparulin, D.S.; Nosyrev, O.D. Chiral effects in classical spinning gas. *J. Phys. A* **2022**, *55*, 395006. [CrossRef]
31. Vergeles, S.N.; Nikolaev, N.N.; Obukhov, Y.N.; Silenko, A.J.; Teryaev, O.V. General relativity effects in precision spin experimental tests of fundamental symmetries. *arXiv* **2022**, arXiv:2204.00427.
32. Zhang, Z.; Fan, G.; Jia, J. Effect of particle spin on trajectory deflection and gravitational lensing. *J. Cosmol. Astropart. Phys.* **2022**, *2022*, 061. [CrossRef]
33. Mikóczi, B.; Keresztes, Z. Spin dynamics of moving bodies in rotating black hole spacetimes. *Ann. Phys.* **2022**, *534*, 2100444. [CrossRef]
34. Ootsuka, T.; Yahagi, R. Generalized Mathisson-Papapetrou-Tulczyjew-Dixon equations. *arXiv* **2022**, arXiv:2204.11430.
35. Compère, G.; Druart, A. Complete set of quasi-conserved quantities for spinning particles around Kerr. *SciPost Phys.* **2022**, *12*, 012. [CrossRef]
36. Herrera, J.; Rosa, M.D.; Rubio, R.M. Relativistic particles with torsion in three-dimensional non-vacuum spacetimes. *J. Math. Phys.* **2021**, *62*, 062502. [CrossRef]
37. Abyaneh, M.Z.; Farhoudi, M. Current density of Majorana bound states. *Phys. Lett. A* **2022**, *453*, 128475. [CrossRef]
38. Awobode, A.M. Precision measurement of the electron orbital gyromagnetic factor: Relativistic contributions from zitterbewegung. *Can. J. Phys.* **2021**, *99*, 696. [CrossRef]
39. Juneghani, S.N.; Bagheri, M.; Shokri, B. Study Zitterbewegung effect in a quasi one-dimensional relativistic quantum plasma by Dirac-Heisenberg-Wigner formalization. *J. Cosmol. Astropart. Phys.* **2021**, *2021*, 2. [CrossRef]
40. Chuprikov, N.L. A variational principle, wave-particle duality, and the Schrödinger equation. *arXiv* **2022**, arXiv:2206.14601.
41. Otalora, G.; Saridakis, E. N. Effective dark energy through spin-gravity coupling. *arXiv* **2022**, arXiv:2210.06598.
42. Aoki, S.; Onogi, T. Conserved non-Noether charge in general relativity: Physical definition versus Noether's second theorem. *Int. J. Mod. Phys. A* **2022**, *35*, 2250129. [CrossRef]
43. Courant, T.J. Dirac manifolds. *Trans. Am. Math. Soc.* **1990**, *319*, 631. [CrossRef]
44. Śniatycki, J. Dirac brackets in geometric dynamics. *Ann. Inst. H. Poincaré A***1974**, *20*, 365.
45. Zambon, M. Submanifolds in Poisson geometry: A survey. In Proceedings of the Complex and Differential Geometry, Leibniz Universität, Hannover, Germany, 14–18 September 2009; p. 403.

46. Bursztyn, H. A brief introduction to Dirac manifolds. *arXiv* **2011**, arXiv:1112.5037.
47. Meinrenken, E. Poisson geometry from a Dirac perspective. *Lett. Math. Phys.* **2018**, *108*, 447. [CrossRef]