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Ricci Soliton and Certain Related Metrics on a Three-Dimensional Trans-Sasakian Manifold

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Abstract: In this article, a Ricci soliton and $*$-conformal Ricci soliton are examined in the framework of trans-Sasakian three-manifold. In the beginning of the paper, it is shown that a three-dimensional trans-Sasakian manifold of type (α, β) admits a Ricci soliton where the covariant derivative of potential vector field V in the direction of unit vector field ξ is orthogonal to ξ. It is also demonstrated that if the structure functions meet $\alpha^2 = \beta^2$, then the covariant derivative of V in the direction of ξ is a constant multiple of ξ. Furthermore, the nature of scalar curvature is evolved when the manifold of type (α, β) satisfies $*$-conformal Ricci soliton, provided $\alpha \neq 0$. Finally, an example is presented to verify the findings.

Keywords: Ricci flow; Ricci soliton; $*$-conformal Ricci soliton; trans-Sasakian manifold

MSC: 53C25; 53D15; 53E20

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1. Introduction

Richard S. Hamilton introduced the concept of Ricci flow (for details see [1]) which was named after the great Italian mathematician Gregorio Ricci-Curbastro. Later, Grigori Perelman [2–4] found it very useful to solve Poincare conjecture. If we take a smooth closed (compact without boundary) Riemannian manifold M equipped with a smooth Riemannian metric g then the Ricci flow is defined by the geometric evolution equation,

$$\frac{\partial g(t)}{\partial t} = -2S(g(t)), \tag{1}$$

where S is the Ricci curvature tensor of the manifold and $g(t)$ is a one-parameter family of metrics on M.

A Riemannian manifold (M, g) is called a Ricci soliton if there exists a vector field V and a constant λ such that the following equation holds, [5]

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda g = 0, \tag{2}$$

where $\mathcal{L}_V$ denotes Lie derivative along the direction of V. The vector field V is called the potential vector field and λ is called the soliton constant. The Ricci soliton, which is a natural extension of the Einstein manifold, is a self-similar solution of Ricci flow. When establishing the characteristics of the soliton, the potential vector field V and the soliton constant λ are crucial factors. According to whether $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, the soliton is said to be shrinking, steady or expanding. The Ricci soliton reduces to Einstein manifold if V is Killing vector field. Compact Ricci solitons are the fixed points of the Ricci flow (1) projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds.

By simply generalizing the classical Ricci flow equation and changing the unit volume constraint to a scalar curvature constraint, in 2005 A. E. Fischer [6] introduced conformal Ricci flow. The conformal Ricci flow equation was given by

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg,$$
$$r(g) = -1,$$

where g is a dynamically evolving metric, r is the scalar curvature, p is the scalar non-dynamical field and n is the dimension of the manifold. The conformal Ricci soliton, which is an extension of the Ricci soliton, was introduced by N. Basu and A. Bhattacharyya [7] in 2015 in relation to the conformal Ricci flow equation. The conformal Ricci soliton equation was given by,

$$\mathcal{L}_V g + 2S + [2\lambda - (p + \frac{2}{n})]g = 0. \tag{3}$$

The definition of the Ricci soliton was modified in 2014 by G. Kaimakamis and K. Panagiotidou [8] who substituted the $*$-Ricci tensor S^*, introduced by S. Tachibana [9] and T. Hamada [10], respectively, for the Ricci tensor S. The $*$-Ricci tensor S^* is defined by

$$S^*(X, Y) = \frac{1}{2}(trace\{\phi.R(X, \phi Y)\}),$$

for arbitrary vector fields X and Y on M, where R is the Riemannian curvature tensor and ϕ is a $(1, 1)$ tensor field. Within the context of real hypersurfaces of a complex space form, the $*$-Ricci soliton notion has been applied. A pseudo-Riemannian metric g is called a $*$-Ricci soliton if there exists a constant λ and a vector field V such that,

$$\mathcal{L}_V g + 2S^* + 2\lambda g = 0.$$

Note that, $*$-Ricci soliton is trivial if the vector field V is Killing, and in this case, the manifold becomes $*$-Einstein. By a $*$-Einstein manifold we mean that the $*$-Ricci tensor (S^*) is proportional to the metric g. Thus, it is considered a natural generalization of $*$-Einstein metric. A $*$-Ricci soliton is said to be almost $*$-Ricci soliton if λ is a smooth function on M. Moreover, an almost $*$-Ricci soliton is called shrinking, steady, and expanding according to as λ is negative, zero, and positive, respectively. It was demonstrated by G. Kaimakamis et al. [8] that a real hypersurface in a complex projective space does not admit a $*$-Ricci soliton by studying real hypersurfaces of a non-flat complex space that admit a $*$-Ricci soliton whose potential vector field is the structure vector field. They also proved that a real hypersurface of complex hyperbolic space admitting a $*$-Ricci soliton is locally congruent to a geodesic hypersphere.

With the aid of (3), P. Majhi and D. Dey [11] further modified the aforementioned definition of $*$-Ricci soliton in 2020 and defined $*$-conformal Ricci soliton as follows,

$$\mathcal{L}_V g + 2S^* + [2\lambda - (p + \frac{2}{n})]g = 0. \tag{4}$$

Ricci solitons have been studied in many contexts: on Kähler manifolds [12], on contact and Lorentzian manifolds [13,14], on K-contact manifolds [15], etc. by many authors. Later, H. G. Nagaraja and C. R. Premalatha [16] studied the nature of Ricci soliton on a three-dimensional trans-Sasakian manifold; C. Călin and M. Crasmareanu [17] on f-Kenmotsu manifold; C. He and M. Zhu [18] on Sasakian manifold and G. Ingalahalli and C. S. Bagewadi [19] on α-Sasakian manifold. Recently, in 2017, Y. Wang [20] proved that if a three-dimensional cosymplectic manifold M^3 admits a Ricci soliton, then either M^3 is locally flat or the potential vector field is an infinitesimal contact transformation. Furthermore, S. Pahan and A. Bhattacharyya gave some insight into the trans-Sasakian manifold [21]. In 2016, T. Dutta et al. studied conformal Ricci soliton on a three-dimensional trans-Sasakian manifold [22].

As shown in the literature, $*$-Ricci soliton on contact geometry was studied by many authors: on Sasakian and (κ,μ)-contact manifold by A. Ghosh and D. S. Patra [23], on $(\kappa,\mu)'$-almost Kenmotsu manifolds by X. Dai, Y. Zhao and U. C. De [24], on contact 3-manifolds by Y. Wang [25], etc. It is worthy to mention that in [26], D. Dey and P. Majhi considered $*$-Ricci soliton on a three-dimensional trans-Sasakian manifold and proved that if the metric of the manifold represents $*$-Ricci soliton and if it satisfies a certain condition then the manifold reduces to a β-Kenmotsu manifold. Furthermore, very recently generalizations of $*$-Ricci soliton on contact geometry were studied by [5,27–31]. Moreover, some of the latest connected studies can be seen in [32–91].

Motivated from the above-mentioned well-praised works we studied the behaviour of Ricci soliton and $*$-conformal Ricci soliton on a three-dimensional trans-Sasakian manifold. In the later sections, we revisit some definitions and important properties of three-dimensional trans-Sasakian manifold and after that the main result of this paper, containing two theorems are described. We also provide an example to justify our findings.

Physical Motivation

The Ricci soliton has extensive applications, not only in mathematical physics but also in quantum cosmology, quantum gravity, and black holes as well. The Ricci soliton can be considered as a kinematic solution in fluid space–time, whose profile develops a characterization of spaces of constant curvature along with the locally symmetric spaces. It also expresses geometrical and physical applications with relativistic viscous fluid space–time admitting heat flux and stress, dark and dust fluid general relativistic space–time, and radiation era in general relativistic space–time. Ricci soliton has applications in the renormalization group (RG) flow for the bosonic nonlinear sigma model from two-dimensional space–time to a curved Riemannian tangent manifold. A two-dimensional Ricci soliton can be used to discuss the behavior of mass under Ricci flow. Ricci soliton is important as it can help in understanding the concepts of energy or entropy in general relativity. This property is the same as that of the heat equation due to which an isolated system loses the heat for thermal equilibrium.

As an application to cosmology and general relativity by investigating the kinetic and potential nature of relativistic space–time, we can present a physical model of three classes, namely, shrinking, steady, and expanding of perfect and dust fluid solutions of Ricci solitons space–time. The first case shrinking $(\lambda < 0)$ which exists on a minimal time interval $-1 < t < b$ where $b < 1$, steady $(\lambda = 0)$ which exists for all time or expanding $(\lambda > 0)$ which exists on maximal time interval $a < t < 1$, $a > -1$. These three classes give examples of ancient, eternal, and immortal solutions, respectively. From [92,93] (briefly discussed in the above section), we can think more about the physical applications of Ricci soliton.

2. Preliminaries

According to D. E. Blair [94], a differentiable manifold M of dimension $(2n+1)$ is said to have an almost contact structure or (ϕ,ξ,η) structure if M permits a $(1,1)$ tensor field ϕ, a vector field ξ, an 1-form η satisfying

$$\phi^2 = -I + \eta\otimes\xi, \tag{5}$$

$$\eta(\xi) = 1, \tag{6}$$

where I is the identity mapping. A Riemannian metric g is said to be a compatible metric if it satisfies,

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{7}$$

for any vector fields X and Y on M. A manifold having almost contact structure along with compatible Riemannian metric is called almost contact metric manifold.

In an almost contact metric manifold the following conditions are satisfied for arbitrary $X, Y \in \chi(M)$, where $\chi(M)$ denotes the set of all vector fields on M [94]:

$$\phi\xi = 0, \quad (8)$$
$$\eta \circ \phi = 0, \quad (9)$$
$$g(X,\xi) = \eta(X), \quad (10)$$
$$g(\phi X, Y) = -g(X, \phi Y). \quad (11)$$

Let M be a $(2n+1)$-dimensional almost contact manifold. Then we define an almost complex structure J on $M \times \mathbb{R}$ by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, where X is a tangent to M, t is the coordinate on $\mathbb{R}$, and f a C^∞ function on $M \times \mathbb{R}$. Clearly, $J^2 = -I$. If J is integrable then the almost contact structure is said to be normal. The normality of an almost contact metric manifold is equivalent to the vanishing of the tensor field $[\phi,\phi] + 2d\eta \otimes \xi$, where $[\phi,\phi]$ is the Nijenhuis torsion tensor of ϕ (for more details see [94]).

In 1985, J. A. Oubiña [95] introduced a new class of almost contact metric manifolds known as trans-Sasakian manifolds. Trans-Sasakian manifolds arose naturally from the classification of almost contact metric structures and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds. An almost contact metric manifold M is called a trans-Sasakian manifold if $(M \times \mathbb{R}, J, G)$, where G is the product metric on $M \times \mathbb{R}$, belongs to the class W_4 (see [96]). If there are smooth functions α, β on an almost contact metric manifold (M, ϕ, ξ, η, g) satisfying [97]

$$(\nabla_X \phi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \quad (12)$$

where $X, Y \in \chi(M)$ are arbitrary and ∇ is the Levi-Civita connection of g on M, then the manifold is called trans-Sasakian manifold of type (α, β). α, β are called structure functions of the manifold. Trans-Sasakian manifolds of type $(0,0)$, $(\alpha, 0)$, $(0, \beta)$ are called cosymplectic, α-Sasakian, and β-Kenmotsu manifolds, respectively. Then from (12), we can deduce that,

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \quad (13)$$
$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \quad (14)$$

J. C. Marrero [98] showed that a trans-Sasakian manifold of dimension ≥ 5 is either cosymplectic or α-Sasakian or β-Kenmotsu. Therefore, proper trans-Sasakian manifold exists only for dimension 3. In a 3-dimensional trans-Sasakian manifold the following relations hold, [21]

$$\begin{aligned} R(X,Y)Z = {} & \Big(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)\Big)[g(Y,Z)X - g(X,Z)Y] - g(Y,Z)[\Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big) \\ & \eta(X)\xi - \eta(X)(\phi D\alpha - D\beta) + ((X\beta) + (\phi X)\alpha)\xi] + g(X,Z)[\Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big) \\ & \eta(Y)\xi - \eta(Y)(\phi D\alpha - D\beta) + ((Y\beta) + (\phi Y)\alpha)\xi)] - [((Z\beta) + (\phi Z)\alpha)\eta(Y) + \\ & ((Y\beta) + (\phi Y)\alpha)\eta(Z) + \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\eta(Y)\eta(Z)]X + [((Z\beta) + (\phi Z)\alpha)\eta(X) \\ & + ((X\beta) + (\phi X)\alpha)\eta(Z) + \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\eta(X)\eta(Z)]Y, \end{aligned} \quad (15)$$

$$\begin{aligned} S(X,Y) = {} & \Big(\frac{r}{2} + (\xi\beta) - (\alpha^2 - \beta^2)\Big)g(X,Y) - \Big(\frac{r}{2} + (\xi\beta) - 3(\alpha^2 - \beta^2)\Big)\eta(X)\eta(Y) - \\ & ((Y\beta) + (\phi Y)\alpha)\eta(X) - ((X\beta) + (\phi X)\alpha)\eta(Y), \end{aligned} \quad (16)$$

$$S(X,\xi) = (2(\alpha^2 - \beta^2) - (\xi\beta))\eta(X) - (X\beta) - (\phi X)\alpha, \quad (17)$$

where Df denotes the gradient of the smooth function f defined on M and R, S, r are the Riemannian curvature tensor, Ricci tensor of type $(0,2)$, and scalar curvature of the manifold, respectively, and α, β are smooth functions on the manifold.

Here in this paper, we restricted the smooth functions α, β to be constant functions. Then we obtained some special relations compatible with our restrictions,

$$R(X,Y)\xi = (\alpha^2-\beta^2)[\eta(Y)X-\eta(X)Y], \tag{18}$$

$$S(X,Y) = \left(\frac{r}{2}-(\alpha^2-\beta^2)\right)g(X,Y)-\left(\frac{r}{2}-3(\alpha^2-\beta^2)\right)\eta(X)\eta(Y), \tag{19}$$

$$S(X,\xi) = 2(\alpha^2-\beta^2)\eta(X), \tag{20}$$

$$QX = \left(\frac{r}{2}-(\alpha^2-\beta^2)\right)X-\left(\frac{r}{2}-3(\alpha^2-\beta^2)\right)\eta(X)\xi, \tag{21}$$

where Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$. The expression of $*$-Ricci tensor (for details see Lemma 3.1 of [26]) on a three-dimensional trans-Sasakian manifold for arbitrary vector fields X and Y of $\chi(M)$ is given by,

$$S^*(X,Y) = \left(\frac{r}{2}-2(\alpha^2-\beta^2)\right)[g(X,Y)-\eta(X)\eta(Y)]. \tag{22}$$

3. Results

In this section, we consider the metric of a three-dimensional trans-Sasakian manifold as a Ricci soliton and a $*$-conformal Ricci soliton and prove the following two results.

Theorem 1. *Let M be a three-dimensional trans-Sasakian manifold of type (α,β) admitting a Ricci soliton where the structure functions α and β are non-zero constant. Then the following relations are satisfied,*

1. *If $\nabla_\xi V$ is orthogonal to ξ, then the soliton is shrinking for $\alpha^2<\beta^2$, steady for $\alpha^2=\beta^2$, and expanding for $\alpha^2>\beta^2$.*
2. *If $\alpha^2=\beta^2$, then the covariant derivative of the potential vector field V in the direction of ξ is a constant multiple of ξ.*

Proof. In a three-dimensional trans-Sasakian manifold where α and β are non-zero constant, we know from (21) that the Ricci operator can be written as,

$$QX = \left(\frac{r}{2}-(\alpha^2-\beta^2)\right)X-\left(\frac{r}{2}-3(\alpha^2-\beta^2)\right)\eta(X)\xi, \tag{23}$$

where $X\in\chi(M)$ is any vector field. The aforementioned equation implies that it is an η-Einstein manifold. Now taking the covariant derivative of (23) along an arbitrary $Y\in\chi(M)$, we have

$$\begin{aligned}(\nabla_Y Q)X =& \frac{1}{2}(Yr)X-\frac{1}{2}(Yr)\eta(X)\xi-\left(\frac{r}{2}-3(\alpha^2-\beta^2)\right)[-\alpha g(\phi Y,X)\xi+\beta g(X,Y)\xi- \\ &\alpha\eta(X)(\phi Y)+\beta\eta(X)Y-2\beta\eta(X)\eta(Y)\xi].\end{aligned} \tag{24}$$

Contracting X and using the well-known formula $trace\{X\to(\nabla_X Q)Y\} = \frac{1}{2}(Yr)$ in (24), we obtain

$$\xi r = -2r\beta+12(\alpha^2-\beta^2)\beta. \tag{25}$$

Using (19) in the definition of Ricci soliton (2), we acquire

$$(\mathcal{L}_V g)(Y,Z) = (2\lambda-r+2(\alpha^2-\beta^2))g(Y,Z)+(r-6(\alpha^2-\beta^2))\eta(Y)\eta(Z), \tag{26}$$

for any vector fields $Y,Z\in\chi(M)$. Now taking the covariant derivative of (26) along an arbitrary vector field $X\in\chi(M)$,

$$\begin{aligned}(\nabla_X\mathcal{L}_V g)(Y,Z) =& -(Xr)g(Y,Z)+(Xr)\eta(Y)\eta(Z)+(r-6(\alpha^2-\beta^2))[-\alpha g(\phi X,Y)\eta(Z) \\ &+\beta g(\phi X,\phi Y)\eta(Z)-\alpha g(\phi X,Z)\eta(Y)+\beta g(\phi X,\phi Z)\eta(Y)].\end{aligned} \tag{27}$$

Again for any vector fields $X, Y, Z \in \chi(M)$, we know [99]

$$(\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\mathcal{L}_V \nabla)(X,Y),Z) - g((\mathcal{L}_V \nabla)(X,Z),Y).$$

Since ∇ is Riemannian metric connection, $\nabla g = 0$. So the above equation reduces to, $(\nabla_X \mathcal{L}_V g)(Y,Z) = g((\mathcal{L}_V \nabla)(X,Y),Z) + g((\mathcal{L}_V \nabla)(X,Z),Y)$. Again, using symmetry of $(\mathcal{L}_V \nabla)$, i.e., $(\mathcal{L}_V \nabla)(X,Y) = (\mathcal{L}_V \nabla)(Y,X)$, we rewrite the last relation as

$$2g((\mathcal{L}_V \nabla)(X,Y),Z) = (\nabla_X \mathcal{L}_V g)(Y,Z) + (\nabla_Y \mathcal{L}_V g)(Z,X) - (\nabla_Z \mathcal{L}_V g)(X,Y). \tag{28}$$

Using (27) in the above equation, we obtain

$$\begin{aligned}(\mathcal{L}_V \nabla)(X,Y) = &-\frac{1}{2}(Xr)Y - \frac{1}{2}(Yr)X + \frac{1}{2}g(\phi X, \phi Y)Dr + \frac{1}{2}(Xr)\eta(Y)\xi + \frac{1}{2}(Yr)\eta(X)\xi \\ &+ (r - 6(\alpha^2 - \beta^2))[-\alpha\eta(Y)\phi X - \alpha\eta(X)\phi Y + \beta g(\phi X, \phi Y)\xi],\end{aligned} \tag{29}$$

for all vector fields X and Y on M. Covariant derivative of (29) along an arbitrary vector field yields,

$$\begin{aligned}&(\nabla_X \mathcal{L}_V \nabla)(Y,Z) = -\frac{1}{2}g(Z, \nabla_X Dr)Y - \frac{1}{2}g(Y, \nabla_X Dr)Z + \frac{1}{2}g(\phi Y, \phi Z)(\nabla_X Dr) - \\ &\alpha\eta(Z)(Xr)\phi Y - \alpha\eta(Y)(Xr)\phi Z + \frac{1}{2}[(Zr)\eta(Y) + (Yr)\eta(Z)](\nabla_X \xi) + \frac{1}{2}[g(Y, \nabla_X Dr)\eta(Z) - \\ &\alpha(Yr)g(\phi X, Z) + \beta g(\phi X, \phi Z)(Yr) + g(Z, \nabla_X Dr)\eta(Y) - \alpha(Zr)g(\phi X, Y) + \beta g(\phi X, \phi Y)(Zr) \\ &+ 2\beta g(\phi Y, \phi Z)(Xr)]\xi + \frac{1}{2}[\alpha g(\phi X, Y)\eta(Z) - \beta g(\phi X, \phi Y)\eta(Z) + \alpha g(\phi X, Z)\eta(Y) - \\ &\beta g(\phi X, \phi Z)\eta(Y)]Dr + (r - 6(\alpha^2 - \beta^2))[\{\alpha^2 g(\phi X, Z) - \alpha\beta g(\phi X, \phi Z)\}\phi Y + \{\alpha^2 g(\phi X, Y) - \\ &\alpha\beta g(\phi X, \phi Y)\}\phi Z - \alpha\eta(Z)((\nabla_X \phi)Y) - \alpha\eta(Y)((\nabla_X \phi)Z) + \beta g(\phi Y, \phi Z)(\nabla_X \xi) + \\ &\{\alpha\beta g(\phi X, Y)\eta(Z) - \beta^2 g(\phi X, \phi Y)\eta(Z) + \alpha\beta g(\phi X, Z)\eta(Y) - \beta^2 g(\phi X, \phi Z)\eta(Y)\}\xi].\end{aligned}$$

From K. Yano [99], we know $(\mathcal{L}_V R)(X,Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y,Z) - (\nabla_Y \mathcal{L}_V \nabla)(X,Z)$. Using this formula in the above equation we obtain,

$$\begin{aligned}&(\mathcal{L}_V R)(X,Y)Z = \frac{1}{2}g(Z, \nabla_Y Dr)X - \frac{1}{2}g(Z, \nabla_X Dr)Y - \frac{1}{2}[\alpha(Yr)g(\phi X, Z) + \beta(Yr)g(\phi X, \phi Z) \\ &- g(Z, \nabla_X Dr)\eta(Y) + \alpha(Zr)g(\phi X, Y) - \alpha(Xr)g(\phi Y, Z) - \beta g(\phi Y, \phi Z)(Xr) + g(Z, \nabla_Y Dr) \\ &\eta(X) - \alpha(Zr)g(\phi Y, X)]\xi + \alpha\eta(Z)(Yr)\phi X - \alpha\eta(Z)(Xr)\phi Y + \alpha\{\eta(X)(Yr) - \eta(Y)(Xr)\}\phi Z \\ &+ \frac{1}{2}\{\alpha g(\phi X, Y)\eta(Z) + \alpha g(X, \phi Y)\eta(Z) - \alpha g(\phi X, Z)\eta(Y) - \beta g(\phi X, \phi Z)\eta(Y) - \alpha g(\phi Y, Z) \\ &\eta(X) + \beta g(\phi Y, \phi Z)\eta(X)\}Dr + \frac{1}{2}\{(Yr)\eta(Z) + (Zr)\eta(Y)\}(\nabla_X \xi) - \frac{1}{2}\{(Xr)\eta(Z) + \\ &(Zr)\eta(X)\}(\nabla_Y \xi) + \frac{1}{2}g(\phi Y, \phi Z)(\nabla_X Dr) - \frac{1}{2}g(\phi X, \phi Z)(\nabla_Y Dr) + (r - 6(\alpha^2 - \beta^2)) \\ &[\{\alpha\beta g(\phi Y, \phi Z) - \alpha^2 g(\phi Y, Z)\}\phi X - \{\alpha\beta g(\phi X, \phi Z) - \alpha^2 g(\phi X, Z)\}\phi Y + 2\alpha^2 g(\phi X, Y)\phi Z + \\ &\{2\alpha\beta g(\phi X, Y)\eta(Z) + \alpha\beta g(\phi X, Z)\eta(Y) - \beta^2 g(\phi X, \phi Z)\eta(Y) - \alpha\beta g(\phi Y, Z)\eta(X) + \\ &\beta^2 g(\phi Y, \phi Z)\eta(X)\}\xi + \beta g(\phi Y, \phi Z)(\nabla_X \xi) - \beta(\nabla_Y \xi)g(\phi X, \phi Z) - \alpha\eta(Z)((\nabla_X \phi)Y) - \\ &\alpha\eta(Y)((\nabla_X \phi)Z) + \alpha\eta(Z)((\nabla_Y \phi)X) + \alpha\eta(X)((\nabla_Y \phi)Z).\end{aligned} \tag{30}$$

The above equation holds for any $X, Y, Z \in \chi(M)$. Contracting X in (30), we achieve

$$(\mathcal{L}_V S)(Y,Z) = \left(\frac{\Delta r}{2} - 6\alpha^4 + 12\alpha^2\beta^2 - 6\beta^4 + r\alpha^2 - r\beta^2\right) g(\phi Y, \phi Z), \tag{31}$$

for any $Y, Z \in \chi(M)$. Again, from (19), we obtain

$$(\mathcal{L}_V S)(Y,Z) = \frac{1}{2}g(\phi Y,\phi Z)(Vr) + \left(\frac{r}{2} - (\alpha^2-\beta^2)\right)\{g(\nabla_Y V, Z) + g(Y,\nabla_Z V)\} - \left(\frac{r}{2} - 3(\alpha^2-\beta^2)\right)\{\eta(Z)((\nabla_V\eta)Y) + \eta(Y)((\nabla_V\eta)Z) + \eta(Z)\eta(\nabla_Y V) + \eta(Y)\eta(\nabla_Z V)\}. \tag{32}$$

Comparison of (31) with (32) yields,

$$\left(\frac{\Delta r}{2} - 6\alpha^4 + 12\alpha^2\beta^2 - 6\beta^4 + r\alpha^2 - r\beta^2\right)g(\phi Y,\phi Z) = \frac{1}{2}\{g(\phi Y,\phi Z)(Vr) + \left(\frac{r}{2} - (\alpha^2-\beta^2)\right)\{g(\nabla_Y V, Z) + g(Y,\nabla_Z V)\} - \left(\frac{r}{2} - 3(\alpha^2-\beta^2)\right)\{\eta(Z)((\nabla_V\eta)Y) + \eta(Y)((\nabla_V\eta)Z) + \eta(Z)\eta(\nabla_Y V) + \eta(Y)\eta(\nabla_Z V)\}. \tag{33}$$

Now, letting $Y = Z = \xi$ gives rise to $(\alpha^2-\beta^2)\eta(\nabla_\xi V) = 0$. From here, two cases arise, either $\eta(\nabla_\xi V) = 0$ or $(\alpha^2-\beta^2) = 0$. From the definition of Ricci soliton (2), we have

$$\frac{1}{2}(g(\nabla_X V, Y) + g(\nabla_Y V, X)) + S(X,Y) = \lambda g(X,Y), \tag{34}$$

for any vector fields X and Y. In first case, $\eta(\nabla_\xi V) = 0$ which implies $\nabla_\xi V$ is orthogonal to ξ, putting $X = Y = \xi$ in (34) gives $2(\alpha^2-\beta^2) = \lambda$. It directly implies that the soliton is shrinking if $\alpha^2 < \beta^2$, steady if $\alpha^2 = \beta^2$ and expanding if $\alpha^2 > \beta^2$.

For the second case where $\alpha^2 = \beta^2$, then it follows directly from (34) that $\nabla_\xi V = \lambda\xi$, i.e., the covariant derivative of the potential vector field V in the direction of ξ is λ-multiple of ξ. □

Theorem 2. *Let M be a three-dimensional trans-Sasakian manifold of type (α,β) where the structure functions α and β are constant with $\alpha \neq 0$. If the metric g represents a $*$-conformal Ricci soliton then the scalar curvature of the manifold is given by $r = \left(1 - \frac{\beta^2}{\alpha^2}\right)\left(\frac{p}{2} + \frac{1}{3} - \lambda + 4\alpha^2\right)$.*

Proof. Since the metric g represents a $*$-conformal Ricci soliton, using (22) in the definition of $*$-conformal Ricci soliton (4), we obtain

$$(\mathcal{L}_V g)(X,Y) = \left(p + \frac{2}{3} + 4(\alpha^2-\beta^2) - r - 2\lambda\right)g(X,Y) + (r - 4(\alpha^2-\beta^2))\eta(X)\eta(Y), \tag{35}$$

for all vector fields X and Y on M. If we consider covariant derivative with respect to arbitrary vector field Z, then (35) reduces to

$$(\nabla_Z \mathcal{L}_V g)(X,Y) = (Zr)[\eta(X)\eta(Y) - g(X,Y)] - (r - 4(\alpha^2-\beta^2))[\alpha g(\phi Z, X)\eta(Y) - \beta g(\phi X,\phi Z)\eta(Y) + \alpha g(\phi Z, Y)\eta(X) - \beta g(\phi Y,\phi Z)\eta(X)], \tag{36}$$

for all $X, Y \in \chi(M)$. Using (11) and (28) in (36), we obtain

$$(\mathcal{L}_V\nabla)(X,Y) = \frac{1}{2}(Dr)[g(X,Y) - \eta(X)\eta(Y)] - \frac{1}{2}(Xr)[Y - \eta(Y)\xi] - \frac{1}{2}(Yr)[X - \eta(X)\xi] + (r - 4(\alpha^2-\beta^2))[\beta g(\phi X,\phi Y)\xi - \alpha\eta(Y)(\phi X) - \alpha\eta(X)(\phi Y)], \tag{37}$$

for arbitrary vector fields X and Y on M. Setting $Y = \xi$ in (37), we have

$$(\mathcal{L}_V\nabla)(X,\xi) = -\frac{1}{2}(\xi r)[X - \eta(X)\xi] - \alpha(r - 4(\alpha^2-\beta^2))(\phi X). \tag{38}$$

Applying covariant derivative along an arbitrary vector field Y and making use of (12)–(14), we obtain

$$(\nabla_Y \mathcal{L}_V \nabla)(X,\xi) = \alpha(\mathcal{L}_V \nabla)(X,\phi Y) - \beta(\mathcal{L}_V \nabla)(X,Y) - \frac{1}{2}(Y(\xi r))[X - \eta(X)\xi] + \frac{1}{2}(\xi r)[\alpha g(\phi X,Y)\xi + \beta g(\phi X,\phi Y)\xi - \alpha\eta(X)(\phi Y) + \beta\eta(X)Y - \beta\eta(Y)X] - \alpha(Yr)(\phi X) - \alpha(r - 4(\alpha^2 - \beta^2))[\alpha g(X,Y)\xi - \alpha\eta(X)Y + \beta g(\phi Y,X)\xi - \beta\eta(X)(\phi Y) + \beta\eta(Y)(\phi X)]. \tag{39}$$

From K. Yano [99], we know $(\mathcal{L}_V R)(X,Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y,Z) - (\nabla_Y \mathcal{L}_V \nabla)(X,Z)$. Using (39) in this formula, we obtain

$$(\mathcal{L}_V R)(X,Y)\xi = \alpha(\mathcal{L}_V \nabla)(\phi X,Y) - \alpha(\mathcal{L}_V \nabla)(X,\phi Y) - \frac{1}{2}(X(\xi r))[Y - \eta(Y)\xi] + \frac{1}{2}(Y(\xi r)) [X - \eta(X)\xi] + \frac{1}{2}(\xi r)[2\alpha g(X,\phi Y)\xi - \alpha\eta(Y)(\phi X) + \alpha\eta(X)(\phi Y) + 2\beta\eta(Y)X - 2\beta\eta(X)Y] - \alpha(Xr)(\phi Y) + \alpha(Yr)(\phi X) - \alpha(r - 4(\alpha^2 - \beta^2)) [\alpha\eta(X)Y - \alpha\eta(Y)X + 2\beta g(\phi X,Y)\xi + 2\beta\eta(X)(\phi Y) - 2\beta\eta(Y)(\phi X)].$$

Setting $Y = \xi$ in the foregoing equation, we acquire

$$(\mathcal{L}_V R)(X,\xi)\xi = \frac{1}{2}(\xi(\xi r))[X - \eta(X)\xi] + \beta(\xi r)[X - \eta(X)\xi] - 2\alpha(r - 4(\alpha^2 - \beta^2))[-\alpha X + \alpha\eta(X)\xi - \beta(\phi X). \tag{40}$$

Again, Lie differentiation of Equation (18) along soliton vector field V and use of (15) and (18) leads to,

$$(\mathcal{L}_V R)(X,\xi)\xi = (\alpha^2 - \beta^2)[g(X,\mathcal{L}_V \xi)\xi - ((\mathcal{L}_V \eta)X)\xi - 2\eta(\mathcal{L}_V \xi)X], \tag{41}$$

which holds for arbitrary vector field X on M. Setting $Y = \xi$ in (35) implies,

$$(\mathcal{L}_V \eta)X - g(X,\mathcal{L}_V \xi) = \left(p + \frac{2}{3} - 2\lambda\right)\eta(X). \tag{42}$$

Taking (42) into account, Lie derivative of $\eta(\xi) = 1$ along the direction of V leads to

$$2\eta(\mathcal{L}_V \xi) = -\left(p + \frac{2}{3} - 2\lambda\right). \tag{43}$$

After using (42) and (43), the Equation (41) reduces to

$$(\mathcal{L}_V R)(X,\xi)\xi = (\alpha^2 - \beta^2)\left(p + \frac{2}{3} - 2\lambda\right)[X - \eta(X)\xi], \tag{44}$$

for all $X \in \chi(M)$. Comparing (40) with (44) we acquire,

$$(\alpha^2 - \beta^2)\left(p + \frac{2}{3} - 2\lambda\right)[X - \eta(X)\xi] = \frac{1}{2}(\xi(\xi r))[X - \eta(X)\xi] + \beta(\xi r) [X - \eta(X)\xi] - 2\alpha(r - 4(\alpha^2 - \beta^2))[-\alpha X + \alpha\eta(X)\xi - \beta(\phi X), \tag{45}$$

for any $X \in \chi(M)$. Inner product of the foregoing equation with arbitrary vector field Y gives,

$$\left[\frac{1}{2}(\xi(\xi r)) + \beta(\xi r) + 2\alpha^2(r - 4(\alpha^2 - \beta^2)) - (\alpha^2 - \beta^2)\left(p + \frac{2}{3} - 2\lambda\right)\right] [g(X,Y) - \eta(X)\eta(Y)] + 2\alpha\beta(r - 4(\alpha^2 - \beta^2))g(\phi X,Y) = 0. \tag{46}$$

Anti-symmetrizing the foregoing equation yields,

$$\left[\frac{1}{2}(\xi(\xi r)) + \beta(\xi r) + 2\alpha^2(r - 4(\alpha^2 - \beta^2)) - (\alpha^2 - \beta^2)\left(p + \frac{2}{3} - 2\lambda\right)\right] g(\phi X, \phi Y) = 0. \tag{47}$$

Since this equation holds for arbitrary vector fields ϕX and ϕY and as we know from (25) that $\xi r = -2r\beta + 12\beta(\alpha^2 - \beta^2)$ holds in a three-dimensional trans-Sasakian manifold, we conclude that the scalar curvature of the manifold satisfies $r = \left(1 - \frac{\beta^2}{\alpha^2}\right)\left(\frac{p}{2} + \frac{1}{3} - \lambda + 4\alpha^2\right)$. □

4. Example of a Three-Dimensional Trans-Sasakian Manifold Admitting Ricci Soliton

In this section, we provide an example to verify our outcomes.

Example 1. *We consider the manifold as $M = \{(x, y, z) \in \mathbb{R}^3 : y \neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields as defined bellow,*

$$e_1 := e^{2z}\frac{\partial}{\partial x}, \qquad e_2 := e^{2z}\frac{\partial}{\partial y}, \qquad e_3 := \frac{\partial}{\partial z},$$

are linearly independent at each point of M. We define the Riemannian metric g as,

$$g_{ij} = g(e_i, e_j) := \begin{cases} 1, & \text{if } i = j \text{ and } i, j \in \{1, 2, 3\} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\xi = e_3$. Then the 1-form η is defined by $\eta(Z) := g(Z, e_3)$, for arbitrary $Z \in \chi(M)$, then we have the following relations,

$$\eta(e_1) = 0, \qquad \eta(e_2) = 0, \qquad \eta(e_3) = 1.$$

Let us define the (1,1)-tensor field ϕ as

$$\phi e_1 := e_2, \qquad \phi e_2 := -e_1, \qquad \phi e_3 := 0,$$

then the relations (5), (6), and (7) are satisfied. Thus, (ϕ, ξ, η, g) defines an almost contact metric structure on M. We can now easily conclude,

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = -2e_2, \qquad [e_1, e_3] = -2e_1.$$

Let ∇ be the Levi-Civita connection of g. Then from Koszul's formula, $2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y])$, we can have

$$\begin{aligned} &\nabla_{e_1} e_1 = 2e_3, && \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_3 = -2e_1, \\ &\nabla_{e_2} e_1 = 0, && \nabla_{e_2} e_2 = 2e_3, && \nabla_{e_2} e_3 = -2e_2, \\ &\nabla_{e_3} e_1 = 0, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_3 = 0. \end{aligned}$$

From here we can easily verify that the relations (12) and (13) are satisfied. Hence M becomes trans-Sasakian manifold of type $(0, -2)$. The components of Riemannian curvature tensor are given by,

$$\begin{aligned} &R(e_1, e_2)e_1 = -4e_3, && R(e_1, e_2)e_2 = -4e_1, && R(e_1, e_2)e_3 = 0, \\ &R(e_1, e_3)e_1 = 4e_2, && R(e_1, e_3)e_2 = 0, && R(e_1, e_3)e_3 = -4e_1, \\ &R(e_2, e_3)e_1 = 0, && R(e_2, e_3)e_2 = -4e_2, && R(e_2, e_3)e_3 = -4e_2. \end{aligned}$$

And the components of Ricci tensor are given by,

$$S(e_1, e_1) = 0, \qquad S(e_2, e_2) = 0, \qquad S(e_3, e_3) = -8.$$

From here we can easily deduce that the scalar curvature of the manifold $r = \sum_{i=1}^{3} S(e_i, e_i) = -8$*. Let us define a vector field V by,* $V := \xi$*. Then we can obtain,*

$$(\mathcal{L}_V g)(e_1, e_1) = -4, \qquad (\mathcal{L}_V g)(e_2, e_2) = -4, \qquad (\mathcal{L}_V g)(e_3, e_3) = 0.$$

Contracting (2) and using the result $r = -8$*, we deduce* $\lambda = 4$*. So g defines a Ricci soliton on this trans-Sasakian manifold for* $\lambda = 4$*.*

5. Conclusions

In this article, we used the methods of local Riemannian geometry to interpret solutions of (2) and (4) and impregnate Einstein metrics in a large class of metrics of Ricci soliton and $*$-conformal Ricci soliton on a trans-Sasakian manifold of the third dimension. Our results will not only play an indispensable and incitement role in contact geometry but also make a significant and motivational contribution in the area of further research of complex geometry, especially on Kähler and para-Kähler manifolds, etc. Some questions arise from our article to study in further research:

(i) What will be the outcomes if we consider the structure functions α and β to satisfy $\phi D\alpha = D\beta$?

(ii) Do the above results hold without assuming any restrictions on structure functions?

(iii) How do the aforementioned outcomes differ for the $*$-η Ricci soliton and the $*$-conformal η-Ricci soliton?

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