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Scattering in Geometric Approach to Quantum Theory

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Abstract: We define inclusive scattering matrix in the framework of a geometric approach to quantum field theory. We review the definitions of scattering theory in the algebraic approach and relate them to the definitions in the geometric approach.

Keywords: inclusive scattering matrix; geometric approach; convex cone

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1. Introduction

The geometric approach to quantum theory where the starting point is the set of states was suggested in [1,2]. In this approach, one can work with convex set $\mathcal{C}_0$ of normalized states or with convex cone $\mathcal{C}$ of not necessarily normalized states (proportional points of the cone $\mathcal{C}$ specify equivalent states)[1]. In present paper, we discuss scattering theory in the geometric approach. Our starting point is a convex cone $\mathcal{C}$ and a subgroup $\mathcal{V}$ of the group of automorphisms of this cone.

We notice at the end of the paper that one can use also a subsemiring $\mathcal{W}$ of the semiring of endomorphisms $End\mathcal{C}$. (Endomorphisms of cone $\mathcal{C}$ form a semiring $End\mathcal{C}$ because the set of endomorphisms is closed with respect to addition and composition. Notice that the semiring $End\mathcal{C}$ is closed also with respect to multiplication by a non-negative number; we assume that $\mathcal{W}$ also has this property.)

We review geometric and algebraic approaches to quantum theory and the relation between these approaches. We give definitions of scattering matrix and inclusive scattering in algebraic approach. This makes the present paper independent of papers [1,2] and of the papers [3,4] devoted to the scattering in algebraic approach.

Let us recall the relation of the geometric approach with the algebraic approach to quantum theory [2]. In algebraic approach, a starting point is an associative algebra $\mathcal{A}$ with involution * (a $*$-algebra). The cone $\mathcal{C}$ of not necessarily normalized states is defined as a set of linear functionals on $\mathcal{A}$ obeying $f(A^*A) \geq 0$. Every element $B \in \mathcal{A}$ specifies two operators on $\mathcal{A}^\vee$ (on the dual space); one of them, denoted by the same symbol B, transforms a functional $f(A)$ into the functional $f(AB)$, another, denoted by the symbol $\tilde{B}$, transforms $f(A)$ into the functional $f(B^*A)$. The operator $\tilde{B}B$ is an endomorphism of the cone $\mathcal{C}$. We define $\mathcal{V}$ as the group of all involution preserving automorphisms of $\mathcal{A}$ acting in natural way on $\mathcal{C}$. The semiring $\mathcal{W}$ is defined as the minimal set of endomorphisms of $\mathcal{C}$ containing all endomorphisms of the form $\tilde{B}B$ and closed with respect to addition and composition (it is closed also with respect to multiplication by a non-negative number as all semirings we consider).

To define scattering in any approach to quantum field theory, we need notions of time and spatial translations. In the algebraic approach, translations (as any symmetries) are automorphisms of the algebra $\mathcal{A}$; these automorphisms induce automorphisms of the cone $\mathcal{C}$ and other objects related to the algebra $\mathcal{A}$. In the geometric approach, translations should be regarded as elements of the group $\mathcal{V}$ consisting of automorphisms of the cone $\mathcal{C}$; their action on the cone should induce an action on the semiring $\mathcal{W}$.

Particles and quasiparticles are defined as elementary excitations of stationary translation-invariant state ω.

In the algebraic approach, one can define the notion of scattering matrix of elementary excitations. Probably, it is impossible to generalize this notion to the geometric approach; however, in the geometric approach, one can give a very natural definition of *inclusive* scattering matrix of elementary excitations of stationary translation-invariant state ω. It is easy to show that this notion agrees with the analogous notion in the algebraic approach.

Notice that our constructions can be applied also to the scattering of quasiparticles in equilibrium and non-equilibrium statistical physics. (The conventional scattering matrix does not make sense in this situation, but the inclusive scattering matrix does; see [3,5]).

In [6], we apply the notions of present paper to define scattering in the framework of Jordan algebras.

2. Geometric Approach

In the geometric approach to quantum theory, we start with a convex closed cone $\mathcal{C}$ of (non-normalized) states in Banach space $\mathcal{L}$ (or, more generally, in complete topological linear space $\mathcal{L}$). We fix a subgroup $\mathcal{V}$ of the group of automorphisms of the cone $\mathcal{C}$. (By definition, an endomorphism of $\mathcal{C}$ is a continuous linear operator in $\mathcal{L}$ transforming the cone into itself. An automorphism is an invertible endomorphism.)

In some cases, it is useful to add to these data a subsemiring $\mathcal{W}$ of the semiring $\mathrm{End}(\mathcal{C})$ of endomorphisms of the cone; we assume that $\mathcal{W}$ is invariant with respect to the action of the group $\mathcal{V}$.

The dynamics in quantum theory is governed by a one-parameter group of time translations T_τ acting on the cone $\mathcal{C}$. We assume that $T_\tau \in \mathcal{V}$. (Here, τ stands for a real number.) Time translations can be considered also as transformations of $\mathcal{W}$ denoted by the same symbol T_τ. If $A \in \mathcal{V}$ or $A \in \mathcal{W}$, the time translation acts as a conjugation: $T_\tau(A) = T_\tau A T_{-\tau}$; we will use the notation $T_\tau(A) = A(\tau)$.

Quantum field theory in the geometric approach is specified by a cone $\mathcal{C}$ with the action of spatial translations $T_{\mathbf{x}}$ where $\mathbf{x} \in \mathbb{R}^d$ and time translations T_τ (the translations should constitute a commutative subgroup of the group $\mathcal{V}$.) The same data specify statistical physics in the space $\mathbb{R}^d$ where d stands for the dimension of the group of spatial translations. We use the notations

$$T_\tau T_{\mathbf{x}}(A) = T_\tau T_{\mathbf{x}} A T_{-\tau} T_{-\mathbf{x}} = A(\tau, \mathbf{x})$$

for an operator A acting in $\mathcal{L}$.

Let us discuss the relation of the above definitions to the quantum theory in the algebraic approach. In this approach, as in the geometric one, we need time and spatial translations to define elementary excitations and scattering. The time translations T_τ and spatial translations $T_{\mathbf{x}}$ act as automorphisms of $\mathcal{A}$; these automorphisms induce automorphisms of the cone $\mathcal{C}$ and of the semiring $\mathcal{W}$ denoted by the same symbols. If $\omega \in \mathcal{C}$ is a translation-invariant stationary state, we can consider a representation of $\mathcal{A}$ in a pre-Hilbert space $\mathcal{H}$ such that there exists a cyclic vector $\theta \in \mathcal{H}$ obeying $\omega(A) = \langle \theta, A\theta \rangle$. This representation is called GNS (Gelfand–Naimark–Segal) representation. We denote an operator in this representation corresponding to $A \in \mathcal{A}$ by the same symbol A. (Notice that these operators are bounded.) We can consider also the representation of $\mathcal{A}$ in the Hilbert space $\bar{\mathcal{H}}$ (in the completion of $\mathcal{H}$). Time and spatial translations descend to $\mathcal{H}$ and to $\bar{\mathcal{H}}$.

For every vector Ψ in the Hilbert space $\bar{\mathcal{H}}$, we define the corresponding state σ by the formula $\sigma(A) = \langle \Psi, A\Psi \rangle$. If $\Psi = \theta$, we have $\sigma = \omega$; if $\Psi = B\theta$, we have $\sigma = \tilde{B}B\omega$.

3. Elementary Excitations

Let us repeat the definitions and statements from [2] with small modifications.

We consider a translation-invariant stationary state $\omega \in \mathcal{C}$. Let us start with the definition of excitation of ω in geometric approach. *We say that $\sigma \in \mathcal{C}$ is an excitation of ω if $T_{\mathbf{x}}\sigma$ tends to $C\omega$ as $\mathbf{x}$ tends to ∞ for some constant C.* (We have in mind weak convergence in this definition. Recall that u is a weak limit of $u_\alpha \in \mathcal{L}$ if for every $f \in \mathcal{L}^\vee$ (in the dual space) the limit of $f(u_\alpha)$ is equal to $f(u)$.) We say that proportional elements of a cone specify the same state; hence, this condition means that for large $\mathbf{x}$, the state $T_{\mathbf{x}}\sigma$ is close to ω.

To define the notion of elementary excitation, we need a notion of elementary space.

Recall that *elementary space* $\mathfrak{h}$ is defined as a space of smooth real-valued or complex-valued functions on $\mathbb{R}^d \times \mathcal{I}$ with all derivatives decreasing faster than any power (here, $\mathcal{I}$ denotes a finite set consisting of m elements). One can identify this space with $\mathcal{S}^m$ (with the direct sum of m copies of Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$. The space $\mathfrak{h}$ can be regarded as pre-Hilbert space (as a dense subspace of L^2). The spatial translations act naturally on $\mathfrak{h}$ (shifting the argument); we assume that the time translations also act on $\mathfrak{h}$ and commute with spatial translations. In momentum representation, an element ϕ of $\mathfrak{h}$ should be considered as a complex function of $\mathbf{k} \in \mathbb{R}^d$ and discrete variable $i \in \mathcal{I}$. If $\mathfrak{h}$ consists of real-valued functions, then in momentum representation, we should impose the condition $\bar{\phi}(-\mathbf{k}) = \phi(\mathbf{k})$. The spatial translation $T_{\mathbf{x}}$ is represented as multiplication by $e^{i\mathbf{x}\mathbf{k}}$ and the time translation T_τ is represented as a multiplication by a matrix $e^{-i\tau E(\mathbf{k})}$ where $E(\mathbf{k})$ is a non-degenerate Hermitian matrix. We assume that $E(\mathbf{k})$ is a smooth function of at most polynomial growth; then, the multiplication by $E(\mathbf{k})$ is an operator acting in $\mathfrak{h}$. The eigenvalues of $E(\mathbf{k})$ are denoted by $\epsilon_s(\mathbf{k})$.

We need some facts about the time evolution of elements of $\mathfrak{h}$ in coordinate representation. If

$$|(T_\tau\phi)(\mathbf{x}, j)| < C_n(1 + |\mathbf{x}|^2 + \tau^2)^{-n}$$

for all $\mathbf{x} \in \mathbb{R}^d$ obeying $\frac{\mathbf{x}}{\tau} \notin U$ and all $n \in \mathbb{N}$, we say that τU is an *essential support* of $T_\tau\phi$ in coordinate representation. Notice that the set U is not defined uniquely; if U' is a subset of $\mathbb{R}^d$ containing U and τU is an essential support of $T_\tau\phi$ in coordinate representation, then $\tau U'$ is also an essential support of $T_\tau\phi$.

Let us consider functions $f_1, \dots, f_n \in \mathfrak{h}$ and essential supports τU_i of functions $T_\tau(f_i)$ in coordinate representation. *We say that these functions do not overlap if the distances between sets U_i are positive (the distances between essential supports grow linearly with τ).*

> *ASSUMPTION. We assume that collections $(f_1, \dots, f_n)$ of non-overlapping functions are dense in $\mathfrak{h}^n = \mathcal{S}^{mn}$*

It is easy to verify that this assumption is almost always satisfied (in particular, it is satisfied if all functions $\epsilon_s(\mathbf{k})$ are strictly convex). The proof can be based on the following lemma.

Lemma 1. *Let us denote by U_ϕ , where $\phi \in \mathfrak{h}$, an open subset of $\mathbb{R}^d$ containing all points having the form $\nabla\epsilon_s(\mathbf{k})$ where $\mathbf{k}$ belongs to $\operatorname{supp}(\phi) = \cup_j \operatorname{supp}(\phi_j)$ (to the union of supports of the functions $\phi(\mathbf{k}, j)$ in momentum representation).*

Let us assume that $\operatorname{supp}(\phi)$ is a compact subset of $\mathbb{R}^d$. Then, for large $|\tau|$, we have

$$|(T_\tau\phi)(\mathbf{x}, j)| < C_n(1 + |\mathbf{x}|^2 + \tau^2)^{-n}$$

where $\frac{\mathbf{x}}{\tau} \notin U_\phi$, the initial data $\phi = \phi(\mathbf{x}, j)$ is the Fourier transform of $\phi(\mathbf{k}, j)$, and n is an arbitrary integer. (In other words, τU_ϕ is an essential support of $T_\tau\phi$ in coordinate representation.)

The proof of this lemma (Lemma 2 in [2]) can be given by means of the stationary phase method; see Section 4.2 of [4] for more detail.

An elementary excitation of ω is defined as a map $\sigma : \mathfrak{h} \to \mathcal{C}$ of an elementary space $\mathfrak{h}$ into the set of excitations of ω. This map should commute with translations and satisfy the following additional requirement: one can define a map $L : \mathfrak{h} \to End(\mathcal{L})$ obeying $\sigma(\phi) = L(\phi)\omega$.

Notice that the conditions we imposed on $L(\phi)$ do not specify it uniquely. Later, we impose some extra conditions on these operators. Not very precisely, one can say that the operators $L(\phi)$ and $L(\psi)$ should almost commute if supports of ϕ and ψ are far away (see (11) for precise formulation). Still, these extra conditions leave some freedom in the choice of L. We assume that the operators $L(\phi)$ are chosen in some way.

In the algebraic approach, we define an excitation of ω as a vector in the space of GNS representation $\mathcal{H}$; assuming cluster property, one can verify that the state corresponding to such a vector is an excitation in the sense of the geometric approach. An elementary excitation of ω is defined as an isometric map Φ of elementary space $\mathfrak{h}$ into $\mathcal{H}$ commuting with time and spatial translations. This definition agrees with the definition of the geometric approach. To verify this fact, we notice that the assumption that θ is a cyclic vector implies the existence of operators $B(\phi)$ obeying $\Phi(\phi) = B(\phi)\theta$. (Here, $\phi \in \mathfrak{h}$.) We define a map $\sigma : \mathfrak{h} \to \mathcal{C}$ saying that $\sigma(\phi)$ is a linear functional on $\mathcal{A}$ assigning a number $\langle \Phi(\phi), A\Phi(\phi)\rangle$ to $A \in \mathcal{A}$. The map σ is quadratic if we are working over $\mathbb{R}$, and it is Hermitian if we are working over $\mathbb{C}$. It commutes with time and spatial translations. Representing $\sigma(\phi)$ in the form $\sigma(\phi) = L(\phi)\omega$ where $L(\phi) = \tilde{B}(\phi)B(\phi) \in End(\mathcal{C})$, we obtain that this map specifies an elementary excitation in the geometric approach.

We assume that $B(\phi)$ is linear with respect to ϕ; then, $L(\phi)$ is quadratic or Hermitian.

We say that a map σ of real vector spaces is quadratic if the expression $\sigma(u+v) - \sigma(u) - \sigma(v)$ is linear with respect to u and v. A map σ of complex vector spaces is Hermitian if $\sigma(u+v) - \sigma(u) - \sigma(v)$ is linear with respect to u and antilinear with respect to v. If V is a real vector space, then the corresponding cone $C(V)$ is defined as a convex envelope of the set of vectors of the form $v \otimes v$ in the tensor square $V \otimes V$. (If we are dealing with topological vector spaces, there exist different definitions of tensor product and of topology in the tensor product. In this case, we should consider the closure of convex envelope in the appropriate topology of the tensor product.) A quadratic map $V \to V'$ induces a linear map of the cone $C(V) \to V'$; a quadratic map of V into a cone $C' \subset V'$ induces a linear map of cones $C(V) \to C'$. Similar statements are true for complex vector spaces and Hermitian maps. (The cone corresponding to complex vector space is defined as a convex envelope of the set of vectors of the form $f \otimes \bar{f}$ in the tensor product $V \otimes \bar{V}$.) If V is a Hilbert space, the corresponding cone can be identified with the cone of positive definite self-adjoint operators belonging to the trace class.

It is natural to assume that in the geometric approach, the maps σ and L are quadratic or Hermitian, but this assumption is not used in most of our statements.

Elementary excitations should be identified with particles or quasiparticles. Notice that particles and quasiparticles can be unstable; this means that we should consider also objects that only approximately obey the conditions we imposed on elementary excitations. The definition of inclusive scattering matrix given in the next section works also for such objects, but instead of the time τ tending to $\pm\infty$, we should consider large but finite τ. (This is true also for the conventional scattering matrix in algebraic approach; see Appendix to [4] for detail.)

4. Scattering Møller Matrices

Let us consider the scattering of elementary excitations defined by the map $\sigma(f) = L(f)\omega$. We define the operator $L(f, \tau)$ where $f \in \mathfrak{h}$ by the formula

$$L(f, \tau) = T_\tau(L(T_{-\tau} f)) = T_\tau L(T_{-\tau} f) T_{-\tau}.$$

(We are using the same notation for time translations in $\mathcal{C}$ and in $\mathfrak{h}$. The time translation acts on operators as conjugation with T_τ.) *We assume that* $\sup_{\tau \in \mathbb{R}} ||T_\tau|| < \infty$ *and the operators* $L(f)$

are bounded, hence $\sup_{\tau\in\mathbb{R}} ||L(f,\tau)|| < \infty$. (Here and in what follows, we assume that $\mathcal{L}$ is a Banach space. If $\mathcal{L}$ is a a topological vector space specified by a system of seminorms, we should impose the above conditions for every seminorm.)

Notice that $L(f,\tau)\omega$ *does not depend on* τ. (Using the fact that the map σ commutes with translations, we obtain that $L(f,\tau)\omega = T_\tau\sigma(T_{-\tau}f) = \sigma(f)$.). This means that

$$\dot{L}(f,\tau)\omega = 0 \tag{1}$$

where the dot stands for the derivative with respect to τ.

Let us introduce the notation

$$\Lambda(f_1,\cdots,f_n|-\infty) = \lim_{\tau_1\to-\infty,\cdots,\tau_n\to-\infty} \Lambda(f_1,\tau_1,\cdots,f_n,\tau_n) \tag{2}$$

where

$$\Lambda(f_1,\tau_1,\ldots,f_n,\tau_n) = L(f_1,\tau_1)\ldots L(f_n,\tau_n)\omega.$$

We say that (2) is an *in*-state.

For large negative τ, the state

$$T_\tau\Lambda(f_1,\cdots,f_n|-\infty)$$

can be described as a collection of particles with wave functions $T_\tau f_i$. To prove this fact, we use the formulas

$$T_\tau(L(f,\tau')) = T_{\tau+\tau'}L(T_{-\tau'}f)T_{-\tau-\tau'} = L(T_\tau f,\tau+\tau'),$$
$$T_\tau\Lambda(f_1,\cdots,f_n|-\infty) = \Lambda(T_\tau f_1,\cdots,T_\tau f_n|-\infty).$$

For $f_1,\cdots,f_n$ in a dense subset of $\mathfrak{h}\times\cdots\times\mathfrak{h}$, the distance between essential supports of wave functions $T_\tau f_i$ tends to ∞ as $\tau\to-\infty$. This follows from the assumption in the preceding section.

This remark allows us to say that for arbitrary τ, the state $T_\tau\Lambda(f_1,\cdots,f_n|-\infty)$ describes a collision of particles with wave functions $(f_1,\cdots,f_n)$.

It is obvious that *the in-state (2) is symmetric with respect to* $f_1,\ldots,f_n$ *if*

$$\lim_{\tau\to-\infty} ||[L(f_i,\tau),L(f_j,\tau)]|| = 0. \tag{3}$$

One can replace (3) by

$$||[L(\phi),L(\psi)]|| \le \int d\mathbf{x}d\mathbf{x}' D^{ab}(\mathbf{x}-\mathbf{x}')|\phi_a(\mathbf{x})|\cdot|\psi_b(\mathbf{x}')| \tag{4}$$

where $D^{ab}(\mathbf{x})$ tends to zero faster than any power as $\mathbf{x}\to\infty$.

Then, the *in*-state is symmetric if the wave functions $f_1,\ldots,f_n$ do not overlap.

Let us give conditions for the existence of the limit

$$\lim_{\tau_1\to-\infty,\cdots,\tau_n\to-\infty} \Lambda(f_1,\tau_1,\cdots,f_n,\tau_n). \tag{5}$$

For simplicity, we consider the case when $\tau_1=\cdots=\tau_n=\tau$.

Lemma 2. *Let us assume that for* $\tau\to-\infty$*, the commutators* $[\dot{L}(f_i,\tau),L(f_j,\tau)]$ *are small. More precisely, the norms of these commutators should be bounded from above by a summable function of* τ *:*

$$||[\dot{L}(f_i,\tau),L(f_j,\tau)]|| \le c(\tau), \int |c(\tau)|d\tau < \infty. \tag{6}$$

Then, the vector $\Lambda(\tau) = \Lambda(f_1, \tau, \cdots, f_n, \tau)$ *has a limit as* $\tau \to -\infty$.

It is sufficient to check that the norm of the derivative of this vector with respect to τ is a summable function of τ. (Then, $\Lambda(\tau_2) - \Lambda(\tau_1) = \int_{\tau_1}^{\tau_2} \Lambda(\tau) d\tau$ tends to zero as $\tau_1, \tau_2 \to -\infty$.)

Calculating $\dot{\Lambda}(\tau)$ by means of Leibniz rule, we obtain n summands; each summand has one factor with $\dot{L}$. The assumption about the behavior of commutators allows us to move the factor with a derivative to the right if we neglect the terms tending to zero faster than a summable function of τ. It remains to be noticed that the expression with the derivative in the rightmost position vanishes due to (1).

If $\mathcal{L}$ is a complete topological linear space with the topology specified by a system of seminorms, we can generalize the above proof assuming an analog of (6) for every seminorm.

Instead of (6), we can assume that

$$||[L(f_i, \tau') - L(f_i, \tau), L(f_j, \tau)]|| \le c(\tau), \int |c(\tau)| d\tau < \infty. \tag{7}$$

where $|\tau' - \tau|$ is bounded from above.

We can slightly strengthen (6) assuming that

$$||[\dot{L}(f_i, \tau), L(f_j, \tau_1)]|| \le c(\tau), \int |c(\tau)| d\tau < \infty. \tag{8}$$

where $\tau - \tau_1$ is bounded from above. Then, we can derive (7) from (8) integrating over τ.

It is easy to derive from (7) that

$$||[L(f_i, \tau') - L(f_i, \tau), L(f_j, \tau)]|| \to 0 \tag{9}$$

as $\tau, \tau' \to \infty$ or $\tau, \tau' \to -\infty$.

Lemma 3. *The condition (9) implies the existence of the limit (2). Hence, the existence of this limit follows also from (7) or (8).*

We should check that the difference

$$L(f_1, \tau_1') \dots L(f_n, \tau_n')\omega - L(f_1, \tau_1) \dots L(f_n, \tau_n)\omega$$

tends to zero as $\tau_i', \tau_i \to -\infty$.

It is sufficient to consider the expression

$$L(f_1, \tau_1) \dots (L(f_i, \tau_i') - L(f_i, \tau_i)) \dots L(f_n, \tau_n)\omega. \tag{10}$$

(One can go from $L(f_1, \tau_1) \dots L(f_n, \tau_n)\omega$ to $L(f_1, \tau_1', \dots L(f_n, \tau_n')\omega$ in n steps changing one variable at every step.) Using (9), we can move the factor $L(f_i, \tau_i') - L(f_i, \tau_i)$ to the rightmost position in (10). It remains to be noticed that this factor gives zero acting on ω.

Notice that the distance between essential supports of functions $T_\tau f_i$ grows linearly as $\tau \to -\infty$ if the sets U_{f_i} do not overlap. This allows us to derive the existence of the limit for $f_1, \cdots, f_n$ in a dense subset of $\mathfrak{h} \times \cdots \times \mathfrak{h}$ if we assume that the commutator $[T_\alpha(L(T_{-\tau'} f)), L(T_{-\tau} g)]$ is small when the essential supports of $T_{\tau'} f$ and $T_\tau g$ are far away for $\tau, \tau' \to \infty$. One can make this statement precise in various ways.

For example, applying Lemma 3, we can prove the following theorem

Theorem 1. *Let us assume that*

$$||[T_\alpha(L(\phi)), L(\psi)]|| \leq \int d\mathbf{x}d\mathbf{x}' D^{ab}(\mathbf{x}-\mathbf{x}')|\phi_a(\mathbf{x})| \cdot |\psi_b(\mathbf{x}')| \tag{11}$$

where $D^{ab}(\mathbf{x})$ tends to zero faster than any power as $\mathbf{x} \to \infty$ and α runs over a finite interval. Then, the limit (2) exists if the functions f_i do not overlap (hence, it exists for $f_1, \ldots, f_n$ in a dense subset of $\mathfrak{h} \times \ldots \times \mathfrak{h}$).

Applying (11), we obtain estimates for commutators $[T_\alpha(L(T_{-\tau}f)), L(T_{-\tau}g)]$ that are sufficient to prove the inequality (7); hence, the existence of the limit (2). (We are using the relation

$$||[L(f,\tau'), L(g,\tau)]|| = ||[T_{\tau'}(L(T_{-(\tau')}f), T_\tau(L(T_{-\tau}g))]|| \leq \tag{12}$$
$$C||[T_{\tau'-\tau}(L(T_{-\tau'}f), L(T_{-\tau}g)]||$$

and its particular case for $\tau' = \tau$).

Let us review shortly the scattering theory in the algebraic approach modifying slightly the considerations of [3][2]. Recall that in this approach, an elementary excitation of translation-invariant stationary state ω is specified by an isometric map $\Phi : \mathfrak{h} \to \mathcal{H}$ commuting with translations and obeying $\Phi(f) = B(f)\theta$ where $B(f) \in \mathcal{A}$. (Here, θ stands for a vector corresponding to ω in the space $\mathcal{H}$ of GNS representation.)

Let us define the operator $B(f,\tau)$ by the formula

$$B(f,\tau) = T_\tau(B(T_{-\tau}f)) = T_\tau B(T_{-\tau}f))T_{-\tau}.$$

Notice that $B(f,\tau)\theta$ does not depend on τ. This follows from the remark that ω is stationary; hence, $T_{-\tau}\theta = \theta$ and $B(f,\tau)\theta = T_\tau\Phi(T_{-\tau}f) = \Phi(f)$.

Lemma 4. *Let us assume that*

$$||[\dot{B}(f_i,\tau), B(f_j,\tau)]|| \leq c(\tau)$$

where $c(\tau)$ is a summable function. Then, the vector

$$\Psi(\tau) = B(f_1,\tau)\ldots B(f_n,\tau)\theta$$

has a limit in $\bar{\mathcal{H}}$ as τ tends to $-\infty$.

Theorem 2. *Let us assume that*

$$||[\dot{B}(\phi), B(\psi)]|| \leq \int d\mathbf{x}d\mathbf{x}' D^{ab}(\mathbf{x}-\mathbf{x}')|\phi_a(\mathbf{x})| \cdot |\psi_b(\mathbf{x}')| \tag{13}$$

where $D^{ab}(\mathbf{x})$ tends to zero faster than any power as $\mathbf{x} \to \infty$. Then, for $f_1, \ldots, f_n$ in a dense subset of $\mathfrak{h} \times \ldots \times \mathfrak{h}$, the vector

$$\Psi(f_1,\tau_1,\ldots,f_n,\tau_n) = B(f_1,\tau_1)\ldots B(f_n,\tau_n)\theta$$

has a limit in $\bar{\mathcal{H}}$ as τ_j tends to $-\infty$; this limit will be denoted by

$$\Psi(f_1,\ldots,f_n|-\infty)$$

The proof of Lemma 4 is very similar to the proof of Lemma 2. To prove Theorem 2, we use the analog of (12) to verify the analogs of (8), (7) and (9); using the analog of (9), we apply the method used in the proof of Lemma 3.

Let us introduce the asymptotic bosonic Fock space $\mathcal{H}_{as}$ as a Fock representation of canonical commutation relations

$$[b(\rho), b(\rho')] = [b^+(\rho), b^+(\rho')] = 0, [b(\rho), b^+(\rho')] = \langle\rho, \rho'\rangle$$

where $\rho, \rho' \in \mathfrak{h}$.

We define Møller matrix S_- as a linear map of $\mathcal{H}_{as}$ into $\bar{H}$ that transforms $b^+(f_1) \dots b^+(f_n)|0\rangle$ into $\Psi(f_1, \dots, f_n| - \infty)$. (Here, $|0\rangle$ stands for the Fock vacuum.) Imposing some additional conditions, one can prove that the operator S_- can be extended to isometric embedding of $\mathcal{H}_{as}$ into $\bar{H}$ (see [3]).

Replacing $-\infty$ by $+\infty$ in the definition of S_-, we obtain the definition of the Møller matrix S_+. If both Møller matrices are surjective maps, we say that the theory has particle interpretation. We can define the scattering matrix of elementary excitations (particles) as an operator in $\mathcal{H}_{as}$ by the formula $S = S_+^* S_-$; if the theory has particle interpretation, this operator is unitary.

Let us define the *in*-operators a_{in}^+ by the formula

$$a_{in}^+(f) = \lim_{\tau \to -\infty} B(f, \tau). \tag{14}$$

This limit exists as a strong limit on vectors $\Psi(f_1, \dots, f_n| - \infty)$ if there exists the limit $\Psi(f, f_1, \dots, f_n| - \infty)$.

Operators a_{out}^+ (*out*-operators) are defined by the formula

$$a_{out}^+(f) = \lim_{\tau \to +\infty} B(f, \tau). \tag{15}$$

Equivalently, the Møller matrix S_- can be defined as a map $\mathcal{H}_{as} \to \overline{\mathcal{H}}$ obeying

$$a_{in}^+(\rho) S_- = S_- b^+(\rho), S_-|0\rangle = \theta.$$

The operators $a_{in}(\rho), a_{out}(\rho)$ (Hermitian conjugate to $a_{in}^+(\rho)$ and $a_{out}^+(\rho)$) obey

$$a_{in}(\rho) S_- = S_- b(\rho), a_{out}(\rho) S_+ = S_+ b(\rho).$$

Notice that spatial and time translations act naturally in $\mathcal{H}_{as}$. The Møller matrix commutes with translations.

There exists an obvious relation between our considerations in the geometric and algebraic approach. It is clear that the operator $L(f, \tau)$ in the space of states corresponds to the operator $B(f, \tau)$ in $\bar{\mathcal{H}}$ (i.e., $L(f, \tau) = \tilde{B}(f, \tau) B(f, \tau)$.) It follows that the state $\Lambda(f_1, \tau_1, \cdots, f_n, \tau_n)$ corresponds to vector $\Psi(f_1, \tau_1, \cdots, f_n, \tau_n)$, and the state $\Lambda(f_1, \cdots, f_n| - \infty)$ (the *in*-state) corresponds to the vector $\Psi(f_1, \cdots, f_n| - \infty)$.

The relation (11) implies that (5) specifies a map of symmetric power of $\mathfrak{h}$ into the cone $\mathcal{C}$. This map (defined on a dense subset) will be denoted by $\tilde{S}_-$; it can be regarded as an analog of the Møller matrix S_- in the geometric approach. The above statements allow us to relate $\tilde{S}_-$ with S_- for theories that can be formulated algebraically. In this case, S_- maps a symmetric power of $\mathfrak{h}$ considered as a subspace of the Fock space into $\bar{\mathcal{H}}$. Composing this map with the natural map of $\bar{\mathcal{H}}$ into the cone of states $\mathcal{C}$, we obtain $\tilde{S}_-$.

The map $\tilde{S}_-$ is not linear, but in the case when L is quadratic or Hermitian, it induces a multilinear map of the symmetric power of the cone $C(\mathfrak{h})$ corresponding to $\mathfrak{h}$ into the cone $\mathcal{C}$.

Constructing the scattering matrix in the algebraic approach, we imposed some conditions on commutators (for example, the condition (13) in Lemma 5). These conditions can be replaced by similar conditions on anticommutators; the above statements remain correct after slight modifications. (In particular, we should consider the fermionic Fock space instead of the bosonic one.) It is important to notice that operators $L = \tilde{B}B$ (almost) commute not only in the case when operators B (almost) commute but also in the case when operators B (almost) anticommute; hence, our considerations in the geometric approach can be applied not only to bosons but also to fermions.

5. Inclusive Scattering Matrix

Instead of the cone $\mathcal{C}$, one can consider the dual cone $\mathcal{C}^\vee \subset \mathcal{L}^\vee$ (it consists of linear functionals that are non-negative on $\mathcal{C}$). The group $\mathcal{V}$ (in particular the group of translations) and the semiring $\mathcal{W}$ act on $\mathcal{C}^\vee$.

Let us consider a translation invariant stationary element $\alpha \in \mathcal{C}^\vee$ obeying the conditions similar to the conditions we imposed on ω. (In the algebraic approach, we can take $\alpha(\sigma) = \sigma(1)$, the value of σ on the unit of algebra.) Let us assume that $\langle\alpha|L'(g)$ is an elementary excitation of α. (Here, L' maps the elementary space $\mathfrak{h}$ into the space of endomorphisms of $\mathcal{L}$; these endomorphisms can be considered also as endomorphisms of the dual space $\mathcal{L}^\vee$.)

Taking

$$\lim_{\tau_k \to +\infty} \langle\alpha|(L'(g_1,\tau_1)\dots L'(g_m,\tau_m)|\Lambda(f_1,\cdots,f_n|-\infty)\rangle$$

we obtain a number characterizing the result of the collision. We can write this number as

$$\lim_{\tau'_k \to +\infty, \tau_j \to -\infty} \langle\alpha|L'(g_1,\tau'_1)\dots L'(g_m,\tau'_m)L(f_1,\tau_1)\dots L(f_n,\tau_n)|\omega\rangle \tag{16}$$

Let us assume that *operators $L(f)$ obey (11) and operators $L'(g)$ obey similar condition*. Then

Theorem 3. *If both $(f_1,..,f_n)$ and $(g_1,\dots,g_m)$ do not overlap, the limit (16) exists. This limit is symmetric with respect to $(f_1,..,f_n)$ and with respect to $(g_1,\dots,g_m)$.*

The proof of this theorem is similar to the proof of Theorem 1. The second statement follows from the fact that operators $L(f_j,\tau_j)$ and $L(f_{j'},\tau_{j'})$ almost commute in the limit $\tau_j,\tau_{j'} \to -\infty$ and from a similar fact for operators L'.

By the definition of elementary excitation, $\sigma(\phi)$ is a quadratic (or Hermitian) map; hence, it is natural to assume that the map $L(\phi)$ is also quadratic (or Hermitian). Then, it can be extended to a bilinear (or sesquilinear) map $L(\tilde{\phi},\phi)$, and the map $L\phi,\tau)$ can be extended to a map $L(\tilde{\phi},\phi,\tau)$. (If we assume that the bilinear map is symmetric, then these extensions are unique, but in the algebraic approach, it is convenient to consider extensions that are not symmetric. Recall that in the algebraic approach, we define $L(\phi)$ as $\tilde{B}(\phi)B(\phi)$; the extension can be defined by the formula $L(\tilde{\phi},\phi) = \tilde{B}(\tilde{\phi})B(\phi)$.) We assume that L' is also quadratic or Hermitian and extend it to a bilinear or sesquilinear map.

Using these extensions, we can define a functional

$$\sigma(\tilde{g}'_1, g'_1, \dots, \tilde{g}'_{n'}, g'_{n'}, \tilde{g}_1, g_1, \dots, \tilde{g}_n, g_n) =$$

$$\langle\alpha| \lim_{\tau'_i \to +\infty, \tau_j \to -\infty} L'(\tilde{g}'_1, g'_1, \tau'_1)\dots L'(\tilde{g}'_{n'}, g'_{n'}, \tau'_{n'})L(\tilde{g}_1, g_1, \tau_1)\dots L(\tilde{g}_n, g_n, \tau_n)|\omega\rangle \tag{17}$$

that is linear or antilinear with respect to all of its arguments.

Notice that in the case when we take symmetric extensions of L and L', the existence of the limit (17) follows from the existence of the limit (16); in the general case, we should modify slightly the condition (11) to prove a generalization of Theorem 3.

We say that (17) is an inclusive scattering matrix. (If we do not assume that the map $L(\phi)$ is quadratic or Hermitian, the inclusive scattering matrix should be defined by the formula (16))[3]

This terminology comes from the fact that in the algebraic approach, matrix elements of an inclusive scattering matrix are related to inclusive cross-sections. In this approach, one can express an inclusive scattering matrix in terms of on-shell GGreen functions that appear in the formalism of L-functionals (used in [4,5,7]) and in Keldysh formalism [8–10]. Let us sketch the derivation of this expression (see [4,5,7] for more detail).

The functional (17) can be considered as a generalized function

$$\sigma(\tilde{\mathbf{k}}'_1, \tilde{i}'_1, \mathbf{k}'_1, i'_1, \ldots, \tilde{\mathbf{k}}'_{n'}, \tilde{i}'_{n'}, \mathbf{k}'_{n'}, i'_{n'}, \tilde{\mathbf{k}}_1, \tilde{i}_1, \mathbf{k}_1, i_1, \ldots, \tilde{\mathbf{k}}_n, \tilde{i}_n, \mathbf{k}_n, i_n) \tag{18}$$

This generalized function is defined for an open dense subset of its arguments. It is sufficient to require that $\tilde{\mathbf{k}}'_i \neq \tilde{\mathbf{k}}'_j, \mathbf{k}'_i \neq \mathbf{k}'_j, \tilde{\mathbf{k}}_i \neq \tilde{\mathbf{k}}_j, \mathbf{k}_i \neq \mathbf{k}_j$, for $i \neq j$ if we assume that $\mathbf{k} \neq \mathbf{k}'$ implies $\nabla\epsilon_j(\mathbf{k}) \neq \nabla\epsilon'_j(\mathbf{k}')$. (Recall that we use the notation $\epsilon_j(\mathbf{k})$ for eigenvalues of the matrix $E(\mathbf{k})$.) More generally, we can consider the sets $U(\mathbf{k})$ consisting of vectors $\nabla\epsilon_j(\mathbf{k})$ and assume that the sets $U(\mathbf{k})$ and $U(\mathbf{k}')$ do not overlap. Then, the essential support of a function $T_{-\tau}(f)$ is far away from the essential support of a function $T_{-\tau}(f')$ if the support of f lies in the neighborhood of $\mathbf{k}$, the support of f' lies in the neighborhood of $\mathbf{k}' \neq \mathbf{k}$ and $\tau \to \infty$.

One can say that the function (18) gives matrix elements of inclusive scattering matrix.

Let us show that in the algebraic approach, inclusive cross-sections can be expressed in terms of these matrix elements. Notice that in this approach

$$\begin{aligned} \sigma(\tilde{g}'_1, g'_1, \ldots, \tilde{g}'_{n'}, g'_{n'}, \tilde{g}_1, g_1, \ldots, \tilde{g}_n, g_n) = \langle 1| \lim_{\tau'_i \to +\infty, \tau_j \to -\infty} \\ \tilde{B}'(\tilde{g}'_1, \tau'_1) B'(g'_1, \tau'_1) \ldots \tilde{B}'(\tilde{g}'_{n'}, \tau'_n) B'(g'_{n'}, \tau'_{n'}) \tilde{B}(\tilde{g}_1, \tau_1) B(g_1, \tau_1) \ldots \tilde{B}(\tilde{g}_n, \tau_n) B(g_n, \tau_n) |\omega\rangle = \\ \langle a^+_{out}(\tilde{g}'_1) \ldots a^+_{out}(\tilde{g}'_{n'}) \Psi(\tilde{g}_1, \ldots, \tilde{g}_n | -\infty), a^+_{out}(g'_1) \ldots a^+_{out}(g'_{n'}) \Psi(g_1, \ldots, g_n | -\infty)\rangle = \\ \langle a_{out}(g'_{n'}) \ldots, a_{out}(g'_1) a^+_{out}(\tilde{g}'_1) \ldots a^+_{out}(\tilde{g}'_{n'}) \Psi(\tilde{g}_1, \ldots, \tilde{g}_n | -\infty), \Psi(g_1, \ldots, g_n | -\infty)\rangle \end{aligned} \tag{19}$$

We have used Theorem 2, Equation (15) and relations $(\tilde{B}_1 B_2 \omega)(A) = \omega(B_1^* A B_2) = \langle \theta, , B_1^* A B_2 \theta\rangle = \langle B_1\theta, A B_2 \theta\rangle$, $\langle 1|\tilde{B}_1 B_2|\omega\rangle = \langle B_1\theta, B_2\theta\rangle$ in this derivation.

In terms of generalized functions

$$\sigma(\tilde{\mathbf{k}}'_1, \tilde{i}'_1, \mathbf{k}'_1, i'_1, \ldots, \tilde{\mathbf{k}}'_{n'}, \tilde{i}'_{n'}, \mathbf{k}'_{n'}, i'_{n'}, \tilde{\mathbf{k}}_1, \tilde{i}_1, \mathbf{k}_1, i_1, \ldots, \tilde{\mathbf{k}}_n, \tilde{i}_n, \mathbf{k}_n, i_n) = \tag{20}$$

$$\langle a_{out}(\mathbf{k}'_{n'}, i'_{n'}) \ldots a_{out}(\mathbf{k}'_1, i'_1) a^+_{out}(\tilde{\mathbf{k}}'_1, \tilde{i}'_1) \ldots a^+_{out}(\tilde{\mathbf{k}}'_{n'}, \tilde{i}'_{n'}) \Psi(\tilde{\mathbf{k}}_1, \tilde{i}_1, \ldots, \tilde{\mathbf{k}}_n, \tilde{i}_n) | -\infty), \Psi(\mathbf{k}_1, i_1, \ldots, \mathbf{k}_n, i_n | -\infty)\rangle$$

The inclusive scattering matrix can be expressed in terms of generalized Green functions. These functions (GGreen functions) are defined by the formula

$$\langle 1| T(\tilde{B}'(\tilde{g}'_1, \tilde{\tau}'_1) B'(g'_1, \tau'_1) \ldots \tilde{B}'(\tilde{g}'_{n'}, \tilde{\tau}'_n) B'(g'_{n'}, \tau'_{n'}) \tilde{B}(\tilde{g}_1, \tilde{\tau}_1) B(g_1, \tau_1) \ldots \tilde{B}(\tilde{g}_n, \tilde{\tau}_n) B(g_n, \tau_n)) |\omega\rangle \tag{21}$$

where T stands for chronological product (see [3]).

The inclusive cross-section of the process $(M, N) \to (Q_1 \ldots, Q_m)$ is defined as a sum (more precisely, a sum of integrals) of effective cross-sections of the processes $(M, N) \to (Q_1, \ldots, Q_m, R_1, \ldots, R_n)$ over all possible $R_1, \ldots, R_n$. If the theory does not have particle interpretation, this formal definition of an inclusive cross-section does not work, but still, the inclusive cross-section can be defined in terms of probability of the process $(M, N) \to (Q_1, \ldots, Q_n +$ something else) and expressed in terms of the inclusive scattering matrix defined above. To verify this statement, we consider the expectation value

$$\nu(a^+_{out}(\mathbf{p}_1, k_1) a_{out}(\mathbf{p}_1, k_1) \ldots a^+_{out}(\mathbf{p}_m, k_m) a_{out}(\mathbf{p}_m, k_m)) \tag{22}$$

where ν is an arbitrary state.

This quantity is the probability density in momentum space for finding m outgoing particles of the types $k_1, \dots, k_n$ with momenta $\mathbf{p_1}, \dots, \mathbf{p_m}$ plus other unspecified outgoing particles. It gives an inclusive cross-section if ν is an *in*-state.

Comparing this statement with (20), we obtain that the inclusive cross-section can be obtained from the inclusive scattering matrix if $\tilde{\mathbf{k}}_i$ tends to $\mathbf{k}_i$ and $\tilde{\mathbf{k}}'_i$ tends to $\mathbf{k}'_i$. (We assume that the expression

$$\nu(a^+_{out}(\tilde{\mathbf{p}}_1, k_1)a_{out}(\mathbf{p}_1, k_1) \dots a^+_{out}(\tilde{\mathbf{p}}_m, k_m)a_{out}(\mathbf{p}_m, k_m)) \tag{23}$$

tends to (22) as $\tilde{\mathbf{p}}_i$ tends to $\mathbf{p_i}$.)

6. Analogs of Green Functions

Let us consider a functional

$$\langle \alpha | T(L'(\tilde{g}'_1, g'_1, \tau'_1) \dots L'(\tilde{g}'_{n'}, g'_{n'}, \tau'_{n'}) L(\tilde{g}_1, g_1, \tau_1) \dots L(\tilde{g}_n, g_n, \tau_n)) | \omega \rangle \tag{24}$$

where T denotes a chronological product. This expression is linear or antilinear with respect to its arguments g'_i, g_j. We assume that these arguments do not overlap. It follows from this assumption and the second statement of Theorem 3 (or generalization of this theorem) that (24) tends to an inclusive scattering matrix (17) as $\tau'_k \to +\infty, \tau_j \to -\infty$ (the time ordering is irrelevant for the first n' factors and also for the last n factors).

The functional (24) can be considered as a generalized function

$$G_{n',n}(\tilde{\mathbf{k}}'_1, \tilde{i}'_1, \mathbf{k}'_1, i'_1, \tau'_1 \dots, \tilde{\mathbf{k}}'_{n'}, \tilde{i}'_{n'}, \mathbf{k}'_{n'}, i'_{n'}, \tau'_{n'}, \tilde{\mathbf{k}}_1, \tilde{i}_1, \mathbf{k}_1, i_1, \tau_1, \dots, \tilde{\mathbf{k}}_n, \tilde{i}_n, \mathbf{k}_n, i_n, \tau_n) \tag{25}$$

This generalized function is defined for an open dense subset of its arguments.

One can obtain (18) (matrix elements of inclusive scattering matrix) from (25) taking the limit $\tau'_k \to +\infty, \tau_j \to -\infty$.

The function (25) can be considered as an analog of the Green function in $(\mathbf{p}, \mathbf{t})$-representation. Taking Fourier transform with respect to τ'_k, τ_j, we obtain an analog of Green function in $(\mathbf{p}, \omega)$-representation that also can be used to calculate matrix elements of the inclusive scattering matrix. (If a function $f(t)$ has limits as $t \to \pm\infty$, then these limits can be calculated as residues in the poles of the Fourier transform of $f(t)$).

In the algebraic approach, the functional (24) and generalized function (25) are related to the generalized Green function (GGreen function) [3]. Namely, in this approach, one can obtain (24) from (21) taking $\tilde{\tau}'_k = \tau'_k, \tilde{\tau}_j = \tau_j$ and using the relation $L(\tilde{g}, g, \tau) = \tilde{B}(\tilde{g}, \tau)B(g, \tau)$.

7. Discussion

Let us discuss some properties of the above construction of *in*-state and of inclusive scattering matrix.

We start again with elementary excitation $\sigma : \mathfrak{h} \to \mathcal{C}$ of state ω. By definition of elementary excitation, there exists a map $L : \mathfrak{h} \to End(\mathcal{L})$ obeying $\sigma(\phi) = L(\phi)\omega$. The map L is not unique; let us prove that under some conditions, the *in*-state does not change when we are changing L. More precisely, we can prove the following statement:

Let us assume that the maps $L_i : \mathfrak{h} \to End(\mathcal{L})$ can be used to define the in-state and

$$||[L_i(\phi), L_j(\psi)]|| \le \int d\mathbf{x} d\mathbf{x}' D^{ab}(\mathbf{x} - \mathbf{x}')|\phi_a(\mathbf{x})| \cdot |\psi_b(\mathbf{x}')|.$$

where D^{ab} tends to zero faster than any power. Then

$$\Lambda(f_1, \cdots, f_n| -\infty) = \lim_{\tau_1 \to -\infty, \cdots, \tau_n \to -\infty} L_{i_1(}f_1, \tau_1), \dots L_{i_n}(f_n, \tau_n)\omega.$$

(*We assume that the functions f_i do not overlap.)*

To prove this statement, we notice first of all that $L_i(f,\tau)\omega = L_j(f,\tau)\omega$; hence, the choice of the operator L_i in the rightmost position does not matter. Then, we use the fact that one can move every factor to the rightmost position without changing the limit (the commutators are small when $\tau_j \to -\infty$).

A similar statement is true for the inclusive scattering matrix.

Let us consider a Poincaré-invariant theory. Recall that in our definitions, we started with the homomorphism of the translation group $\mathcal{T}$ into group $\mathcal{V}$. We assume that this homomorphism can be extended to a homomorphism of the Poincaré group $\mathcal{P}$. The translation group acts also on the elementary space $\mathfrak{h}$; we assume that this action also can be extended to the action of the Poincaré group and that the elementary excitation of the Poincaré invariant state ω considered as a map $\sigma : \mathfrak{h} \to \mathcal{C}$ commutes with the actions of the Poincaré group on $\mathfrak{h}$ and $\mathcal{C}$: for every $P \in \mathcal{P}$ and $f \in \mathfrak{h}$, we have

$$\sigma(Pf) = P\sigma(f) \tag{26}$$

Then, we say that the theory is Poincaré-invariant.

By the definition of elementary excitation, there exists a map $L : \mathfrak{h} \to \mathcal{W}$ obeying $\sigma(f) = L(f)\omega$. If L commutes with Poincaré transformations, the scattering is obviously Poincaré-invariant. However, one can prove the Poincaré invariance of scattering in a much more general situation. Let us sketch a proof of this fact assuming that

$$\lim_{\tau \to -\infty} ||[L(Pf_i,\tau), L^P(f_j,\tau)|| = 0 \tag{27}$$

(We introduced notation $L^P(f,\tau) = PL(f,\tau)P^{-1}$.)

The generalized Møller matrix $\tilde{S}_-$ is a map of the symmetric power of $\mathfrak{h}$ into $\mathcal{C}$. Let us check that this map commutes with actions of the Poincaré group. (A similar proof can be applied to the inclusive scattering matrix.)

We should identify

$$L(Pf_1,\tau),\dots L(Pf_n,\tau)\omega \tag{28}$$

with

$$PL(f_1,\tau),\dots L(f_n,\tau)\omega = L^P(f_1,\tau),\dots L^P(f_n,\tau)\omega$$

in the limit $\tau \to -\infty$. We will show that we can replace $L(Pf_i,\tau)$ with $L^P(f_i,\tau)$ in any number of factors of (28) without changing the limit. For the rightmost factor, this statement is equivalent to (26). Let us assume that this statement is correct for the last k factors. Then, it is true also for the $(k+1)$-th factor from the right. (To prove this, we interchange the $(k+1)$-th factor with the k-th factor from the right using (27) and use the induction hypothesis.) We proved the statement by induction.

Modifying the considerations of Section 4, we can give various conditions for the Poincaré invariance of scattering theory on a dense subset of $\mathfrak{h} \times \dots \times \mathfrak{h}$.

Until now, we did not use the semiring $\mathcal{W}$ in our considerations. Let us show how it can be used. We need an additional structure on this semiring: we assume that it is represented as a union of subsemirings $\mathcal{W}_V$ corresponding to domains $V \subset \mathbb{R}^d$. If $L_1 \in \mathcal{W}_{V_1}$, $L_2 \in \mathcal{W}_{V_2}$, $||L_1|| = ||L_2|| = 1$ and the domains are far away, we assume that the commutator $[L_1, L_2]$ is small: for every n

$$||[L_1, L_2]|| \leq C_n d(V_1, V_2)^{-n}$$

where $d(V_1, V_2)$ stands for the distance between domains and C_n is a constant factor.

Let us assume that the operators $L(\phi)$ belong to the semiring $\mathcal{W}$. Moreover, we require that in the case when the function $T_\tau\phi$ has essential support in τV, the corresponding operator

$L(T_\tau\phi)$ belongs to $\mathcal{W}_{C\tau V}$ for some constant C. Then, it is easy to check that the inequality (7) is satisfied in the case when functions f_i, f_j do not overlap. This allows us to prove the existence of the limit (2) defining *in*-state in the case when the functions f_i do not overlap.

One can give a formulation of quantum theory in terms of group $\mathcal{V}$ of linear operators acting in topological vector space and semiring $\mathcal{W}$ of linear operators acting in the same space. It seems that such a formulation can be useful in the $BRST$ approach to quantum theory.

One can prove analogs of results of the present paper in the case when the group of spatial translations is discrete. It is natural to assume that this group is isomorphic to $\mathbb{Z}^d$ (free abelian group with d generators). This happens, in particular, for quantum theory on a lattice in d-dimensional space.

The notion of elementary space should be modified: $\mathfrak{h}$ should consist of fast decreasing functions on the lattice $\mathbb{Z}^d$, spatial translations act on this space as shifts of the argument. Equivalently, one can consider elements of $\mathfrak{h}$ as smooth functions on a torus (as smooth periodic functions of d arguments); taking corresponding Fourier series, we come to fast decreasing functions on $\mathbb{Z}^d$.

Working with this version of elementary space, we can modify all definitions and theorems of this paper. One should expect that modified theorems can be applied to gapped lattice systems.

These ideas can be applied also in the case when translation symmetry is spontaneously broken (i.e., the theory is translation invariant, but we consider elementary excitations of a state ω that are invariant only with respect to a discrete subgroup of the translation group.).

Similar modifications can be made when the time is discrete.

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Notes

1 We say that a closed convex set $\mathcal{C}$ is a convex cone if for every point $x \in \mathcal{C}$ all points of the form λx where λ is positive also belong to $\mathcal{C}$. Notice that in our terminology a vector space is a convex cone.

2 Notice that the operators $B(f,\tau)$ of present paper correspond to the operators $B(f\phi^{-1},\tau)$ of [3]. The properties of operators $B(f,\tau)$ that are taken for granted in the present paper are derived in [3] from asymptotic commutativity of the algebra $\mathcal{A}$.

3 Notice that (16) and (17) can be considered either as an inclusive scattering matrix of elementary excitations of state ω or as an inclusive scattering matrix of elementary excitations of state α. A similar statement is true for analogs of green functions introduced in Section 6. It is not clear whether this strange duality has any physical meaning.

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