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Vacuum Energy in Saez-Ballester Theory and Stabilization of Extra Dimensions

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Abstract: In this work, we study a spherically symmetric metric in 5D within the framework of Saez-Ballester Theory, where minimal dark energy-matter interaction occurs. We predict that the expanding isotropic universe will be progressively DE dominated. We estimate few values of the deceleration parameter, very close to the recently predicted values. We obtain the value of the DE EoS parameter as $\omega = -1$. Additionally, we measure the value of the overall density parameter as $\Omega = 0.97(\approx 1)$, in line with the notion of a close to or nearly (not exactly) flat universe. We predict that the model universe starts with the Big-Bang and ends at the Big Freeze singularity. In general, we cannot find conditions for stabilization of extra dimensions in general relativity, and all dimensions want to be dynamical. Here, we present two possible conditions to solve this stabilization problem in general relativity.

Keywords: general relativity; Saez-Ballester Theory; vacuum energy; spherically symmetric; singularity; extra dimensions

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1. Introduction

Since the discovery of dark energy (DE) [1,2] in 1998, it has gained a reputation as one of the topics of paramount importance among the cosmological forums. Despite investing tremendous scientific efforts to explore it, its origin, bizarre nature, and future aspects to modern cosmology are still up for grabs. It is characterized by the distinctive feature of possessing a huge negative pressure opposing gravity resulting in the enigmatic phenomenon of the universe expanding at an expedited rate at late times. This cryptic dark entity is considered to be uniformly distributed and varies slowly or nearly unchanged with time [3–6]. Some worth mentioning studies on this mystic dark component that have not escaped our attention in the last few years are briefly presented below.

Recently, in [7], the authors study a higher dimensional cosmological model to find the origin of DE. They further predict an $f(R,T)$ gravity model as a DE source [8]. A presentation on the evolution of DE considering recent findings can be seen in [9]. In [10], the authors investigate the future of this dark entity beyond the bound of cosmological aspects. In [11], the estimation of DE density is presented. In [12], the authors put forward arguments for the need for DE. Gutierre [13] analyses the status of the experimental data on DE. A fascinating comparison of the speed of DE with that of a photon can be found in [14]. The atom-interferometry constraints on DE are studied in [15]. In [16], DE is obtained from the violation of energy conservation. The prediction of clustering galaxy as a result of stirring effect of DE can be seen in [17]. Lastly, in [18], the author claims that particles with imaginary energy density can lead us to the root of the ambiguous dark component.

Cosmologists have witnessed numerous theoretical attempts to obtain hints as to exactly predict the underlying physics of the miraculous expanding phenomenon of the universe at late times. Two well-appreciated methods have been adapted to explain this mystic phenomenon. Firstly, different possible forms of DE are developed. Secondly, modifying the Einstein theory of gravitation [19,20]. Other than these two, recently, cosmologists and theoretical physicists have been successful in developing other interesting and convincing approaches. In [21], the phenomenon is explained by the infrared corrections. Narain [22] predicts that an Ultraviolet Complete Theory leads to the expansion. A fascinating illustration can be seen in [23] where the expedited expansion occurs in the absence of DE.

To figure out the ambiguous nature of DE in as much detail as possible, the equation of state (EoS) parameter ω is studied with utmost importance. The most recent Planck 2018 results [24], estimates its value to be $\omega = -1.03 \pm 0.03$. The late time expedited expansion of the universe is obtained when $\omega < -\frac{1}{3}$ [25]. $\omega = -1$ corresponds to the natural candidate of DE, the cosmological constant (CC), or in other words, vacuum energy (VE). However, CC or VE comes up short to explain the mystery of the coincidence problem (CP) [26]. After multiple efforts, many other well-appreciated forms of DE are developed [27]. One such candidate that has not escaped our notice is the holographic dark energy (HDE), an outcome of the introduction of the holographic principle (HP) [28] to DE. Accordingly, all the physical quantities inside the universe including the energy density of DE can be illustrated by some quantities on the boundary of the universe [29]. Recent works on some of the different forms of HDE can be seen in [30–33]. Construction of interacting HDE and dark matter (DM) models in spherically symmetric space-time settings can be observed in [34–36]. Interacting models can successfully represent modified gravity in the Einstein frame [37–41]. In [42–45], it also is shown that such interacting models are effective in mollifying the CP.

Due to the fascinating natures of the HDE and VE, a spark of interest has been ignited among cosmologists so that they have started to examine HDE paired with VE. In [46], the authors predict that their HDE model evolved from ΛCDM in early time and approaches to the same ΛCDM in the late time. They further mention that for a fixed value of a coupling parameter involved, their HDE model remains fixed in the ΛCDM model all through. In [47], an accelerating HDE model behaving similarly to the ΛCDM model is presented. An explanation can be seen in [48] in which the HDE model cannot be discriminated from ΛCDM in the high-redshift region. In [49], it is asserted that the vacuum entanglement energy is the probable candidate for HDE, where entanglement energy is the disturbed vacuum energy due to the presence of a boundary [50]. Hu et al. [51] develop a heterotic DE model where the DE has two parts, the cosmological constant and HDE. A study of an HDE model where $\omega = -1$ is obtained can be found in [52]. Lastly, a model can be seen in [53] where HDE ends at ΛCDM in the future.

Saez-Ballester Theory (SBT), introduced by Saez and Ballester [54], can be considered to be the right option to study DE and the accelerating universe. It is a member of the family of Scalar Tensor Theory (STT) of gravitation. In SBT, the metric potentials are coupled with a scalar field φ. Scalar fields are considered to play key roles in gravitation and cosmology as they can illustrate prodigies like DE, DM, etc. [55]. They can be regarded as a possible contributing factor in the late time acceleration of the universe [56]. STT is of direct generalization and extension of general relativity [57]. STT can be considered as a perfect candidate for DE [58]. In [59,60], it is asserted that a scaler field might be responsible for the inflation at the initial epoch. The authors in [61,62] discuss Bianchi Type-V cosmology in SBT obtaining a transit from decelerating universe to accelerating phase. Currently, SBT and general relativity are held to align with observation.

The higher-dimensional model has become one of the good choices among cosmologists and theorological physicists. The idea of such a model was put forward by Kaluza and Klein [63,64]. The authors in [65,66] claim that such a model can explain the late time expanding phenomenon. In [67], it is mentioned that extra-dimensional theories of gravity might explain the early inflation and late-time acceleration of the universe. There is a remarkable improvement in our knowledge and the logical consistency of physics by the introduction of the fifth dimension [68]. A study to validate the existence of the extra dimension is presented by Marciano [69]. There is a chance that the unknown fifth dimension might be related to two the ambiguous and unseen dark components—dark energy and dark matter [70]. According to [71], the employment of an extra dimension makes HDE models more complete and consistent. Some recent worth mentioning studies on higher dimension can be seen in [72–77].

Taking into consideration the above noteworthy related studies, we consider a minimal DE-DM interaction within the framework of SBT using a 5D spherically symmetric space-time. In this work, we present an in-depth discussion on every cosmological parameter obtained. The definition of shear scalar and its physical significance are provided. We discuss the initial and future singularity of the model universe. Additionally, we calculate the present values of the overall density parameter, deceleration parameter, and the dark energy EoS parameter. We also discuss the conditions to solve the stabilization problem of extra dimensions in general relativity. The paper is divided into sections. After the introduction, in Section 2, we present the formulation of the problem with solutions to the parameters. In Section 3, the solutions are discussed with graphical representations. In Section 4, we present the explanation of the solution to the stabilization problem of extra dimensions in GR. Lastly, to sum up the observations, a concluding note is provided in Section 5.

2. Formulation of Problem and Solutions

In our universe, the five-dimensional spherically symmetric metric [78] of following the form is considered

$$ds^2 = dt^2 - e^{\alpha}\left(dr^2 + r^2 d\Theta^2 + r^2 \sin^2\Theta d\phi^2\right) - e^{\beta} dy^2, \tag{1}$$

where α and β are cosmic scale factors which are functions of time only.

We consider the following Saez-Ballester field equations

$$R_{ij} - \frac{1}{2} g_{ij} R - \lambda \varphi^n \left(\varphi_{,i}\varphi_{,j} - \frac{1}{2} g_{ij} \varphi_{,k} \varphi^{,k}\right) = -\left(T_{ij} + S_{ij}\right), \tag{2}$$

where T_{ij} and S_{ij} are the energy momentum tensors for matter and HDE, respectively, R and R_{ij} are, respectively, the Ricci scalar and tensors, whereas the scalar field φ satisfies

$$2\varphi^n \varphi^{,i}_{;i} + n\varphi^{n-1} \varphi_{,k}\varphi^{,k} = 0, \tag{3}$$

where n is an arbitrary constant.

We define T_{ij} and S_{ij} as

$$T_{ij} = \rho_m u_i u_j, \tag{4}$$

$$S_{ij} = (\rho_d + p_d) u_i u_j - g_{ij} p_d, \tag{5}$$

where ρ_m and ρ_d represent the energy densities of matter and HDE, respectively, and p_d represents the pressure of the HDE.

Here, the energy is conserved and obviously, we have

$$T^{ij}_{;j} + S^{ij}_{;j} = 0. \tag{6}$$

By using the co-moving coordinate system, the surviving field equations are obtained as follows

$$\frac{3}{4}\left(\dot{\alpha}^2 + \dot{\alpha}\dot{\beta}\right) + \frac{\lambda}{2}\varphi^n \dot{\varphi}^2 = \rho, \tag{7}$$

$$\ddot{\alpha} + \frac{3}{4}\dot{\alpha}^2 + \frac{\ddot{\beta}}{2} + \frac{\dot{\beta}^2}{4} + \frac{\dot{\alpha}\dot{\beta}}{2} - \frac{\lambda}{2}\varphi^n \dot{\varphi}^2 = -p_d, \tag{8}$$

$$\frac{3}{4}\left(\ddot{\alpha} + \dot{\alpha}^2\right) - \frac{\lambda}{2}\varphi^n \dot{\varphi}^2 = -p_d, \tag{9}$$

and from Equation (6), we have

$$\ddot{\varphi}+\dot{\varphi}\left(\frac{3\dot{\alpha}+\dot{\beta}}{2}\right)+\frac{n}{2}\dot{\varphi}^{2}\varphi^{-1}=0, \tag{10}$$

where an overhead dot represents differentiation w.r.t. t.

Considering ω as the EoS parameter of the dark energy so that we have

$$p_d=\omega\rho_d. \tag{11}$$

Now, the conservation equation is given by

$$\dot{\rho}_d+(1+\omega)\left(\frac{3\dot{\alpha}+\dot{\beta}}{2}\right)\rho_d+\dot{\rho}_m+\rho_m\left(\frac{3\dot{\alpha}+\dot{\beta}}{2}\right)=0. \tag{12}$$

Due to the minimal interaction of HDE and matter, by [79,80], both the components conserve separately thereby obtaining

$$\dot{\rho}_m+\rho_m\left(\frac{3\dot{\alpha}+\dot{\beta}}{2}\right)=0. \tag{13}$$

$$\dot{\rho}_d+(1+\omega)\rho_d\left(\frac{3\dot{\alpha}+\dot{\beta}}{2}\right)=0. \tag{14}$$

Furthermore, we have

$$\dot{\rho}+(\rho+p)\left(\frac{3\dot{\alpha}+\dot{\beta}}{2}\right)=0. \tag{15}$$

From Equations (13) and (14), we have

$$\rho_m=a_0e^{-\left(\frac{3\alpha+\beta}{2}\right)}, \tag{16}$$

$$\rho_d=b_0e^{-(1+\omega)\left(\frac{3\alpha+\beta}{2}\right)}, \tag{17}$$

where a_0 and b_0 are arbitrary constants.

From Equations (8) and (9), we obtain the expression for cosmic scale factors as

$$\alpha=c_1+\log(v\,t-uc_2)^{\frac{u}{v}}, \tag{18}$$

$$\beta=kc_1+\log(v\,t-uc_2)^{\frac{ku}{v}}, \tag{19}$$

where $c_1,\ c_2,\ u,\ v$ and $k\neq 0$ are arbitrary constants.

From Equations (16)–(19), the energy densities of matter and DE are, respectively, obtained as

$$\rho_m=a_0e^{-\frac{(k+3)c_1}{2}}(vt-uc_2)^{-\frac{(k+3)u}{2v}}. \tag{20}$$

$$\rho_d=b_0e^{-\frac{(1+\omega)(k+3)c_1}{2}}(vt-uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}}. \tag{21}$$

Using Equations (18) and (19) in Equation (10), the expression for scalar field is obtain as

$$\varphi=c_2e^{\frac{2\log\left(e^{\frac{u}{2}(k+3)\left(\frac{t}{v^2t-uvc_2}-2c_1\right)-(n+2)\left(uvc_2-v^2t\right)}\right)-\frac{(k+3)u\,t}{v^2-uvc_2}}{n+2}}. \tag{22}$$

From Equations (20) and (21), the expression for energy density of the model universe is obtained as

$$\rho = a_0 e^{-\frac{(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(k+3)u}{2v}} + b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}}. \tag{23}$$

Using Equations (18), (19) and (23) in Equation (15), the expression for pressure of the model universe is obtained as

$$p = -\left(a_0 e^{-\frac{(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(k+3)u}{2v}} + b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}} \right). \tag{24}$$

From Equations (11) and (21), the pressure of dark energy is obtained as

$$p_d = \omega\ b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}}. \tag{25}$$

At any time $t = t_0$, we can assume that $p = p_d$ so that

$$\left(a_0 e^{\frac{\omega(k+3)c_1}{2}} (vt - uc_2)^{\frac{\omega(k+3)u}{2v}} + b_0(1+\omega) \right) e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}} = 0. \tag{26}$$

The expression for ω will be given by Equation (26).

Now, the expressions for the different cosmological parameters are obtained as given below

Spatial volume:

$$v = e^{\frac{3\alpha+\beta}{2}} = e^{\frac{(k+3)c_1}{2}} (vt - uc_2)^{\frac{(k+3)u}{2v}}. \tag{27}$$

Scalar expansion:

$$\theta = u^i_{;j} = \frac{3\dot{\alpha}}{2} + \frac{\dot{\beta}}{2} = \frac{(k+3)u}{2(vt - uc_2)}. \tag{28}$$

Hubble parameter:

$$H = \frac{\theta}{4} = \frac{(k+3)u}{8(vt - uc_2)}. \tag{29}$$

Deceleration parameter:

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = \frac{8v}{(k+3)u} - 1. \tag{30}$$

Shear scalar:

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{72}\left(\frac{16vt^2 - 4(3k + 8c_2 + 9)uvt + 3(3k + 4kc_2 + 12c_2 + 9)u^2 + 16u{c_2}^2}{(vt - uc_2)^2} \right). \tag{31}$$

Anisotropic parameter:

$$A_h = \frac{1}{4}\sum_{i=1}^{4}\left(\frac{\Delta H_i}{H}\right)^2 = 3\left(\frac{k-1}{k+3}\right)^2, \tag{32}$$

where $\Delta H_i = H_i - H, (i = 1, 2, 3, 4)$ are the directional Hubble parameters.

Dark energy density parameter:

$$\Omega_d = \frac{\rho_d}{3H^2} = \frac{64}{3}\left(\frac{b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{2-\frac{(1+\omega)(k+3)u}{2v}}}{3(k+3)^2 u^2} \right). \tag{33}$$

Matter density parameter:

$$\Omega_m = \frac{\rho_m}{3H^2} = \frac{64}{3}\left(\frac{a_0 e^{-\frac{(k+3)c_1}{2}}(vt-uc_2)^{2-\frac{(k+3)u}{2v}}}{3(k+3)^2u^2}\right). \tag{34}$$

Overall density parameter:

$$\Omega = \Omega_d + \Omega_m = \frac{64}{3}\left(\frac{\left(a_0 + b_0 e^{-\frac{\omega(k+3)c_1}{2}}(vt-uc_2)^{-\frac{\omega(k+3)u}{2v}}\right)e^{-\frac{(k+3)c_1}{2}}(vt-uc_2)^{2-\frac{(k+3)u}{2v}}}{3(k+3)^2u^2}\right). \tag{35}$$

From [81], the expression for the state finder diagnostic pair $\{r, s\}$ is given by

$$r = 1 + \frac{3\dot{H}}{H^2} + \frac{\ddot{H}}{H^3}. \tag{36}$$

$$s = \frac{r-1}{3\left(q-\frac{1}{2}\right)}. \tag{37}$$

From Equations (29), (36) and (37), we have

$$\{r, s\} = \{1, 0\}. \tag{38}$$

3. Discussion

In this section, for convenience sake and to achieve realistic outcomes, we opt to choose $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78$ and $v = \frac{1}{2}$. The discussion on the nature of the parameters with respect to cosmic time t are presented in details with graphs as follows.

From Equations (20) and (21), it is obvious that ρ_d and ρ_m are functions of t. Figure 1 shows that ρ_d is almost consistent throughout whereas ρ_m decreases in the entire course of evolution, which are acceptable scenarios as the ambiguous DE varies slowly or is unchanged with time [3–6], on the other hand, DM diminishes continuously as a result of the galaxies scattering away from one another during expansion [5]. Moreover, when $t \to \infty$, $\rho_m \to 0$. From these, it would be appropriate to conclude that the universe will be progressively dominated by this cryptic DE. Similar increasing dominant nature of DE can also be seen in [36,82,83].

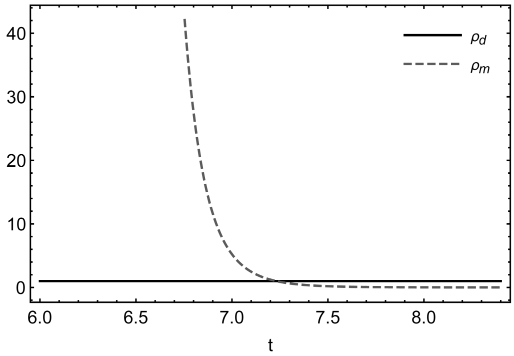

Figure 1. Variation of the energy densities of DE ρ_d and DM ρ_m with t when $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78, v = \frac{1}{2}$.

Figure 2 can be regarded as perfect supporting evidence for the present observation of the spatial expansion of the universe. However, at the initial epoch when $t = 0$, $v = 0$. Furthermore, from Figure 3, we can see that θ initially emerges with a large value, decreases with evolution, and finally, tends to become constant after some finite time which is the indication of the Big-Bang scenario [84]. The prediction of a similar scenario with similar cosmological settings can also be seen in [85]. On considering $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78, v = \frac{1}{2}$ and assuming the present age of the universe to be $t_0 = 13.8$ Gyr which align with the estimated present age by the most recent Planck 2018 results [24], from Equation (26), the value for EoS parameter is measured to be $\omega = -1$. The Planck 2018 results estimates its value to be $\omega = -1.03 \pm 0.03$ [24]. So, the dark energy candidate we are dealing with is the vacuum energy or the cosmological constant. Moreover, from Figure 1, it can be seen that the dark energy density ρ_d remains almost constant throughout evolution, and from Equation (27), $v \to \infty$ when $t \to \infty$. So, it would be a pertinent fact that the universe has no end; expanding forever, ultimately, leading to the Big Freeze singularity in the far future. In a thermodynamic sense, the model universe will enter a point of minimum temperature and maximum entropy. It will be almost as though all astrophysical process is being smothered, as the fuel for growth and reproduction gets so diffuse that it cannot be used [86]. It will be an ending point characterized by increasing isolation, inexorable decay, and an eons-long fade into darkness [87].

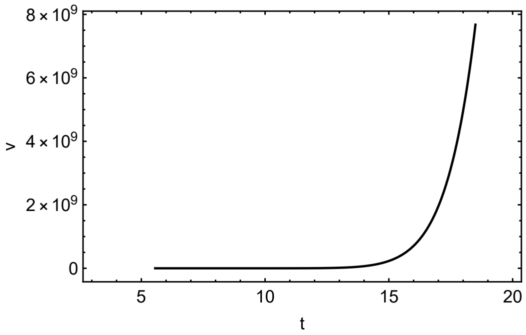

Figure 2. Variation of the spatial volume v with t when $c_1 = c_2 = k = 1, u = 2.78, v = \frac{1}{2}$.

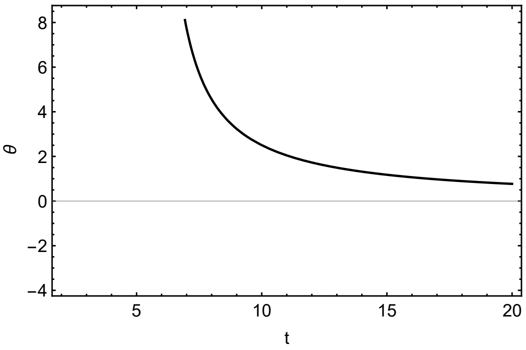

Figure 3. Variation of the expansion scalar θ with t when $c_2 = k = 1, u = 2.78, v = \frac{1}{2}$.

From Figure 4, it is evident that the pressure of DE p_d ranges in the negative plane all through which is in consonance with the mystic property of DE responsible for the accelerated expansion of the universe.

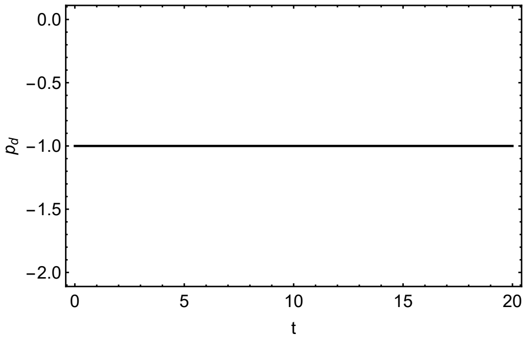

Figure 4. Variation of the DE pressure p_d with t when $b_0 = c_1 = c_2 = k = 1, \omega = -1, u = 2.78, v = \frac{1}{2}$.

From Equation (30), the deceleration parameter q depends on u, v and k. In Table 1, we present different values of q for different values of u, v and k. Recently, Camarena and Marra [88] predict its value as $q = -0.55$, whereas Capozziello et al. [89] estimate the value as $q = -0.644 \pm 0.223$ and $q = -0.6401 \pm 0.187$. With all the values of the constants in Table 1, we obtain the EoS paramter of CC. Since q lies in the range $-1 < q < 0$, the accelerating model universe undergoes exponential expansion [90], in agreement with the present cosmology.

Table 1. Values of deceleration parameter q for different values of u, v and k.

u	v	k	q
2.78	$\frac{1}{2}$	1	-0.64
2.78	$\frac{1}{1.6}$	1	-0.55
2.25	$\frac{1}{2}$	1	-0.55
2.25	$\frac{1}{1.6}$	1.9	-0.54

Figure 5 shows the decreasing nature of Hubble parameter H which is within the limit of the present cosmological scenario [91,92]. Shear scalar σ^2 shows us the rate of deformation of the matter flow within the massive cosmos [93]. The evolution of σ^2 can be seen in Figure 6. It appears to remain constant during the initial epoch, and then it tends to diverge. From these, we can summarize that the model expands with a slow and uniform change of size during the initial evolution, whereas the change becomes faster and faster in late times. This agrees with the present observation of the accelerated expansion of the universe. From Equation (32), the anisotropic parameter $A_h = 0$ for $k = 1$ so that the constructed model is isotropic.

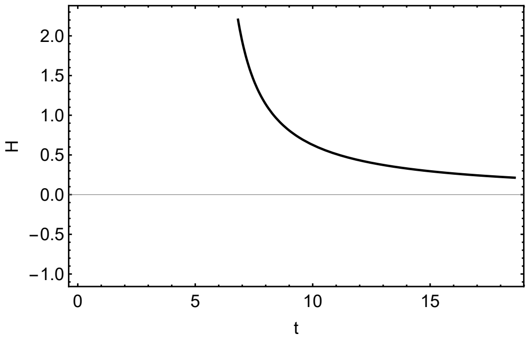

Figure 5. Variation of the Hubble parameter H with t when $c_2 = k = 1, u = 2.78, v = \frac{1}{2}$.

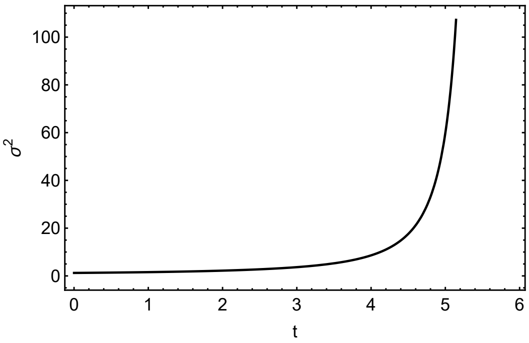

Figure 6. Variation of the shear scalar σ^2 with t when $c_2 = k = 1, u = 2.78$ and $v = \frac{1}{2}$.

Figure 7 shows us the variation of Ω, Ω_d and Ω_m with t. Here, since DE varies slowly or is unchanged with time [3–6], we can see that Ω_d tends to remain constant or increases very slowly. However, Ω_m decreases in the entire course of evolution as a result of the galaxies scattering away from one another leading DM to diminish continuously [5]. Above all, with $k = 1, u = 2.78$ and $v = \frac{1}{2}$, from Equation (35), the overall density parameter is obtained to be $\Omega = 0.97(\approx 1)$. For an exactly flat universe, $\Omega = 1$ [94–96]. Recently, many authors advocate against the belief of an exactly flat universe [94,97–99]. It will be a right conclusion to say that the universe is close to or nearly flat, but not exactly flat [94,99,100]. Above all, the most recent Planck 2018 results [24] obtaining Ω ranging close to unity can be treated as a perfect piece of evidence for a nearly flat universe. Hence, our model obtaining Ω not exactly equal to 1 is justified.

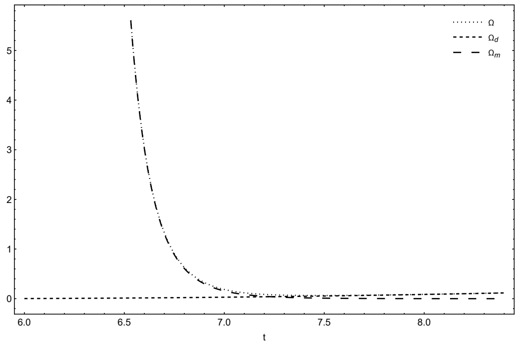

Figure 7. Variation of the overall density parameter Ω, DE density parameter Ω_d and DM density parameter Ω_m with t when $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78$ and $v = \frac{1}{2}$.

Lastly, from Equation (38), we can see that the value of the state finder diagnostic pair $\{r, s\} = \{1, 0\}$ which corresponds to the ΛCDM scenario so that the model universe we are considering is a ΛCDM model. Hence, our interacting HDE model can be considered as an alternate cosmological model to the standard ΛCDM model.

4. Stabilization of Extra Dimensions

The study on the stabilization of extra dimensions can be considered as a phenomenological necessity in higher-dimensional models. The discussion on stabilization is mostly confined to particle physics, supersymmetry, supergravity, string theory, and braneworld models. We require a stabilization mechanism to prevent modification of gravity to an experimentally undesirable manner [101]. The stabilization also makes sure the visible 4D universe with a long lifetime [102]. Another benefit of stabilization is that we can ignore any unwanted outcomes of quantum gravity at Planck length distances [103]. One of the most classic solutions for stabilization is the Goldberger–Wise mechanism [104], where stabilization is achieved in the presence of an additional scalar field. In [105], the authors claim that stabilization can be achieved by introducing a potential of the dilaton field. In [106], we can witness a study of an isotropic 3-brane model where stabilization is achieved with the only value of the EoS $\omega(t) = -\frac{2}{3}$. Another observation of stabilization in an isotropic perfect fluid model in 5D with the value of EoS $\omega > -\frac{1}{3}$ can be seen in [107]. In [108], the authors show that the issue of stabilization can be overcome in a theory of gravity involving high-order curvature invariants. The author in [109] obtains stabilization by quantum corrections from massive matter. In [110], we can find the investigation of a class of dilatonic STT where stabilization is achieved by quantum corrections to the effective 4D Ricci scalar. In [111], we can witness an argument calming that stabilization is attained as soon as inflation ends, on the contrary, the authors in [112] assert that inflation ends if stabilization is attained. According to [113], to achieve a realistic theoretical model, we should assume that the visible three dimensions are expanding isotropically, whereas the extra dimensions are contracting (or contracted for a period during the evolution). Similarly, the authors in [114], predict that the extra dimension contracts with the cosmic time. In [115], the hidden extra dimension is related to scalar fields. The work in [116] also represents the size of the extra dimensions in terms of a scalar field. In [117], the authors investigate 4D gauge theories that dynamically generate a 5D, where stabilization is no longer needed. In their works [118], Tosa studies the Kaluza–Klein cosmology for a torus space with a cosmological constant and matter. He predicts that the number of the extra dimensions should be more than 1, and the extra dimensions should be of small size. However, during recent years, many authors have successfully predicted models with just one extra dimension, where stabilization is obtained [119–124]. Additionally, we can also witness large extra dimensions in [125–127], and infinite-volume extra dimensions in the fourth paragraph of this section.

In our work, we have discussed a 5D spherically symmetric cosmological model in general relativity (GR) with the cosmological constant (CC), or in other words, vacuum energy (VE) as the DE candidate. In GR, generally, we cannot find conditions for stabilization, and all dimensions want to be dynamical [116]. In [128], it is mentioned that in an accelerating model with CC, stabilization cannot be obtained. Therefore, in a trial to solve the stabilization problem in GR, we consider two options. The first one is the Casimir energy and the second is the infinite-volume extra dimension, which are discussed below.

Casimir energy is a DE candidate with the ability to drive the late-time accelerated expansion and stabilize the extra dimensions automatically [124,129]. Casimir energy is VE emerging from imposing boundary conditions on the quantum fluctuations of fields and the EoS's of both Casimir energy and CC are of the same form [124]. Further in [124], we can see the interpretation of Casimir energy as CC. Additionally, the author in [130] equates VE with Casimir energy. In [131], Casimir energy is identified with CC. If the CC is to be created from the Casimir energy, then there will be only one extra dimension [132]. Coincidently, in our spherically symmetric cosmological model with the CC as the DE candidate, there is only one extra dimension.

The study on extra dimensions has been widely considered in braneworld models [133–141], one of which is the DGP model [142], which presents an accelerating 5D scenario with an infinite-volume extra dimension. This infinite-volume extra dimension drives the expedited expansion of the universe at late times [143]. The authors in [144,145] assert that with an infinite-volume extra dimension, one does not need stabilization. They further claim that the infinite-volume scenario can explain to us the late time cosmology and the acceleration of the universe driven by DE, which are one of the core components of GR. According to [146], infinite-volume extra dimensions might result in the emergence of DE. Hence, it would be appropriate to conclude that the extra dimension in our study on 5D spherically symmetric cosmological model is of infinite-volume.

One of the most classic solutions for stabilization is the Goldberger–Wise (GW) mechanism [104]. We can witness the application of the GW mechanism in the field of string theory, M-Theory, and Randall and Sundrum (RS) model in the noteworthy works of [147–152]. In these works, the authors consider a 5D static metric with a 4D Poincare symmetry. To obtain stability, they introduce the proper distance and a massive scalar field and show that the effective radion potential has a minimum. Since the Casimir energy (force) provides a natural alternative to the GW mechanism [153], the stabilization mechanism applied in [147–152] might have some sort of relationship with the Casimir energy stabilization approach which we have predicted above. Above all, one may consider it as an advantage above the GW mechanism that the introduction of an ad hoc classical interaction between the branes is not needed in the Casimir energy approach of stabilization [153]. We may note the work in [154] predicting that the Casimir force will not lead to stabilization to the right value unless a tuning of parameters. Fortunately, the work in [153] shows that this conclusion of [154] is not general, and proves that Casimir energy (force) provides a natural alternative to the GW mechanism in the RS model. There might be more advantages or relationships of our predicted stabilization approaches with the GW mechanism, which we would like to find out in our future works.

We have presented two conditions for stabilization of extra dimensions in GR. Probably, our work might be the first to predict such conditions in GR. Nevertheless, these two conditions are toy models which require further in-depth analysis considering different cosmological aspects. We need more investigation on the reliability of considering, within GR, the identification of Casimir energy with cosmological constant, or in other words, vacuum energy. We also need to verify all the possible outcomes of assuming the extra dimension is of infinite volume in a higher-dimensional vacuum energy model within GR.

5. Conclusions

We have analyzed a cosmological model in spherically symmetric space-time in a 5D setting with minimally interacting matter and HDE in SBT. We predict that the expanding isotropic universe will be progressively DE dominated. The pressure of DE is negative all through. We estimate few values of the deceleration parameter and the values are found very close to the recently predicted values. The Hubble parameter H decreases which agrees with the present cosmological scenario. In the initial epoch, the model universe expands with a very slow and uniform change of shape, but after some finite time, the change becomes faster. Then, it again tends to become very slow and uniform after expanding without any deformation for a finite period. The value of the DE EoS parameter is measured to be $\omega = -1$ indicating that the DE we are dealing with is the vacuum energy or the cosmological constant. The value of the overall density parameter is obtained as $\Omega = 0.97(\approx 1)$, which is not exactly equal to 1, since the universe is close to or nearly flat, but not exactly flat. We observe that the model universe starts with the Big-Bang and ends at the Big Freeze singularity. The value of the state finder diagnostic pair obtained corresponds to the ΛCDM model so that our interacting HDE model can be considered as an alternate cosmological model to the standard ΛCDM model. Lastly, we present two conditions to solve the stabilization problem of extra dimension in GR, the first one is the identification of Casimir energy with cosmological constant, or in other words, vacuum energy and the second is assuming the extra dimension is of infinite volume. Nevertheless, these two conditions are toy models which require further in-depth analysis considering different cosmological aspects.

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