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Hidden Conformal Symmetry in Higher Derivative Dynamics for the Kerr Black Hole

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Abstract: The Kerr/CFT correspondence provides a holographic description of spinning black holes that exist in our universe and the notion of hidden conformal symmetry allows for a formulation of this correspondence that is away from extremality. In this study, we examined how hidden conformal symmetry is manifest when we consider dynamics beyond the Klein–Gordon equation through studying the analytic structure of the higher derivative equations of the motion of a massless probe scalar field on a Kerr background, using the monodromy method. Since such higher derivative dynamics appear in known examples of holographic AdS/logCFT correspondences, we investigated whether or not a Kerr/logCFT correspondence could be possible.

Keywords: women physicists; gravitation; black holes

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1. Introduction

Studying the nature of black holes has led to some of the deepest and most fruitful physical insights of the last century. The realization that black holes possess an entropy that is proportional to their surface area [1,2] was the first step toward identifying the holographic nature of spacetime [3–5], where information stored in any spacetime volume is related to the surface area bounding that region. The most famous concrete realization of this principle in string theory is the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [6,7], which relates a theory of gravity in negatively-curved spacetime to a highly-symmetric quantum field theory in fewer spacetime dimensions. Holographic dualities, such as AdS/CFT, are the workhorse of modern theoretical physics in that they harness black hole physics to study a surprisingly diverse plethora of phenomena, including strongly interacting condensed matter systems [8], the geometrization of entanglement [9–11], Hawking evaporation [12,13] and quantum chaos [14,15].

Remarkably, holographic correspondences now exist for real, physical black holes that appear in our universe, which are maximally (or near-maximally) spinning Kerr black holes [16]. The Kerr/CFT correspondence [17] relies on taking a near-horizon limit in the extremal Kerr metric and showing that the isometries of the resulting metric form a copy of the conformal algebra $SL(2,\mathbb{R})$. A version of the Kerr/CFT correspondence has also been found away from the extremal limit [18]. The authors of [18] found that the price of moving away from extremality is that it is necessary to consider the symmetries of the near-horizon dynamics (the wave equation), rather than only the symmetries of the near-horizon metric, to uncover the underlying conformal algebra. Such symmetries of the dynamics are referred to as hidden symmetries to distinguish them from the explicit isometries of the metric. This notion of hidden conformal symmetry associated with black hole horizons has provided new insights into interesting and potentially observable aspects of black holes, such as black hole shadows [19], tidal Love numbers [20] and near-superradiant geodesics [21]. Furthermore, the presence of hidden symmetries in a gravitational system are known to be responsible for the separability of the equations of motion on that background, as well as the complete integrability of geodesics [22].

A particularly useful method for examining the near-horizon dynamics of a probe scalar field on a black hole background is to study the monodromy properties of solutions to the Klein–Gordon equation [23–26]. The monodromy data encode information about black hole thermodynamics and the hidden conformal symmetry of [18] and provide ample evidence for a two-dimensional CFT description of the thermal properties of black hole microstates. In particular, the fact that hidden conformal symmetry appears to be a feature of a large class of black holes [23,27,28] seems to indicate that it may be sensible to apply a Cardy formula [29] to able to reproduce the Bekenstein–Hawking entropy in scenarios beyond four- and five-dimensional black holes [18,30,31].

The goals of this work were twofold. First, since hidden conformal symmetry is only discernible through studying the dynamics of a probe field, we began this study by asking: how is hidden conformal symmetry manifest when we change the dynamics? That is, we aimed to study an action such that the resulting equations of the motion of our probe field were not the standard Klein–Gordon equation. In addition, we hoped that our chosen dynamics could potentially yield novel physical insight, while at the same time be easily comparable to the known Klein–Gordon case. To this end, we considered the higher derivative action of a massless scalar field Φ :

$$S = -\int d^4x\sqrt{g}\left(\frac{1}{2}\Phi\left(\nabla^\mu\nabla_\mu\right)^n\Phi\right), \tag{1}$$

with the corresponding equation of motion:

$$\left(\nabla^\mu\nabla_\mu\right)^n\Phi = 0. \tag{2}$$

When the integer $n = 1$, we recover the familiar Klein–Gordon equation for a free scalar field. Higher order differential equations such as this arise in other physical settings, e.g., in the study of buoyant thermal convection [32].

Although the equation of motion (2) is a very simple extension of the standard Klein–Gordon case, these higher derivative interactions already possess interesting physical attributes. The holographic duals of logarithmic conformal field theories (logCFTs) [33] are known to involve the higher derivative equations of motion [34]. LogCFTs are interesting in their own right, with applications in percolation [35] and quenched disorder [35–39]. Thus, our second motivating question behind looking for hidden conformal symmetry in the dynamics (2) was: can we construct a holographic logCFT correspondence in the spirit of Kerr/CFT (in which the conformal symmetry exists at the black hole horizon) as opposed to AdS/CFT (in which the CFT is said to exist at the boundary of AdS)? Constructing a logCFT correspondence within this scenario would be particularly interesting due to their non-unitary nature, in which the presence of hidden conformal symmetry has prompted many authors [18,23,25,40] to assume the validity of a Cardy formula and show that it reproduces the Bekenstein–Hawking entropy. However, Cardy's formula is not known to hold in non-unitary settings. From a pragmatic point of view, a longer-term goal of establishing a Kerr/logCFT correspondence is to potentially use the Kerr background (plus a scalar field) to study effects in percolation and quenched disorder, in much the same way that the AdS/CFT correspondence can be harnessed to study aspects of strongly interacting quantum field theories [8].

This article is organized as follows. In Section 2, we review how hidden conformal symmetry is found in the Klein–Gordon equation on a Kerr background. In Section 3, we briefly present a standard form for the Klein–Gordon operator that streamlined our calculations. We note that this standard form holds for a probe field of general spin, in both four and five dimensions. In Section 4, we perform our analysis of hidden conformal symmetry in higher derivative dynamics. We present the calculated monodromy data in Section 4.1 and move to holographic considerations in Section 4.2. We discuss our findings and future work in Section 5. In our Appendix A, we provide more examples of the standard form Klein–Gordon operator presented in Section 3.

2. Hidden Conformal Symmetry from the Klein–Gordon Equation

In this section, we examine the hidden conformal symmetry of the Kerr black hole, as first discovered by [18]. This provides the framework for studying how hidden conformal symmetry is manifest in theories with higher derivative interactions in Section 4. While most of this section is a review, we believe that some of the discussion that we present is new.

The Kerr black hole is described by the following metric:

$$ds^2 = \frac{\rho^2}{\Delta}dr^2 - \frac{\Delta}{\rho^2}(dt^2 - a\,\sin^2\theta d\phi)^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}((r^2+a^2)d\phi - adt)^2, \tag{3}$$

where we have defined:

$$\Delta = r^2 + a^2 - 2Mr\,, \qquad \text{and} \qquad \rho^2 = r^2 + a^2\cos^2\theta\,, \tag{4}$$

and $a \equiv \frac{J}{M}$ is the spin of the black hole of mass M and angular momentum J. This geometry possesses an outer horizon r_+ and inner horizon r_-, the locations of which are determined by the equation $\Delta = 0$. In particular, we have:

$$r_\pm = M \pm \sqrt{M^2 - a^2}\,. \tag{5}$$

There are two Killing vectors associated with (3): ∂_t and ∂_ϕ. These generate explicit isometries of the metric (3).

The additional hidden symmetry generators are not symmetries of the metric, but of the dynamics. Consider a massless scalar field Φ on the background (3). Here, we assume that we can treat the scalar Φ as a probe. The Klein–Gordon equation of motion for this scalar, i.e.,

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\Phi) = 0\,, \qquad \mu = 0,\ldots,3\,, \tag{6}$$

famously separates[1] under the ansatz:

$$\Phi(t,r,\theta,\phi) = e^{i(m\phi - \omega t)}R(r)S(\theta)\,. \tag{7}$$

The radial equation is:

$$\begin{aligned}&\Bigg(\partial_r(\Delta\partial_r) + \frac{(\omega-\Omega_+ m)^2}{4\kappa_+^2}\frac{(r_+-r_-)}{r-r_+} - \frac{(\omega-\Omega_- m)^2}{4\kappa_-^2}\frac{(r_+-r_-)}{r-r_-}\\ &+(r^2+2M(r+2M))\omega^2\Bigg)R(r) = KR(r),\end{aligned} \tag{8}$$

where $\Omega_\pm$ and $\kappa_\pm$ are the angular velocities and surface gravities of the inner and outer horizons,

$$\Omega_\pm = \frac{a}{2Mr_\pm}\,, \qquad \kappa_\pm = \frac{r_+ - r_-}{4Mr_\pm}\,, \tag{9}$$

and K is a separation constant (which also encodes information on the spectrum of the spherical harmonic function $S(\theta)$).

2.1. Near-Region Limit

The authors of [18,40] have argued that a hidden conformal symmetry becomes manifest when we consider only soft hair modes. That is, they consider the following "near-region" limit of the Equation (8):

$$\omega\, M \ll 1\,, \qquad \text{and} \qquad \omega\, r \ll 1\,. \tag{10}$$

As emphasized in [40], this limit can be thought of as a near-horizon limit taken in phase space. The resulting equation is:

$$\left(\partial_r(\Delta\partial_r) + \frac{(\omega-\Omega_+ m)^2}{4\kappa_+^2}\frac{(r_+-r_-)}{r-r_+} - \frac{(\omega-\Omega_- m)^2}{4\kappa_-^2}\frac{(r_+-r_-)}{r-r_-}\right)R(r) = KR(r). \tag{11}$$

The solutions of (11) are hypergeometric functions. As pointed out by [18], the hypergeometric functions transform in representations of $\mathrm{SL}(2,\mathbb{R})$, which is the first hint of the existence of a hidden conformal symmetry; but what are the generators? To find them, we can take our inspiration from Kerr/CFT, noting that the near-horizon extremal Kerr (NHEK) geometry is warped AdS_3 and that, for a particular choice of angle $\theta = \theta_0$, it is exactly the upper-half plane of AdS_3 (up to a conformal factor):

$$ds^2 = F(\theta_0)\left(\frac{dw^+dw^- + dy^2}{y^2}\right). \tag{12}$$

Thus, the existence of the conformal symmetry of black horizons (hidden or otherwise) is tied to the existence of a copy of AdS_3 in the near-horizon limit (either in the metric or in the Klein–Gordon equation).

The isometry group of AdS_3 is $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and we already know the Killing vectors for (12), which are:

$$\begin{aligned} &H_1 = i\partial_+, && H_0 = i\left(w^+\partial_+ + \frac{1}{2}y\partial_y\right), && H_{-1} = i\left(w^{+2}\partial_+ + w^+y\partial_y - y^2\partial_-\right), \\ &\bar{H}_1 = i\partial_-, && \bar{H}_0 = i\left(w^-\partial_- + \frac{1}{2}y\partial_y\right), && \bar{H}_{-1} = i\left(w^{-2}\partial_- + w^-y\partial_y - y^2\partial_+\right). \end{aligned} \tag{13}$$

These generators satisfy the conformal algebra:

$$[H_0, H_{\pm1}] = \mp iH_{\pm1}, \qquad [H_{-1}, H_1] = -2iH_0\,, \tag{14}$$

and have quadratic Casimir:

$$\begin{aligned} \mathcal{H}^2 &= -H_0^2 + \frac{1}{2}(H_1H_{-1} + H_{-1}H_1) \\ &= \frac{1}{4}\left(y^2\partial_y^2 - y\partial_y\right) + y^2\partial_+\partial_-. \end{aligned} \tag{15}$$

Now, the only remaining thing that is needed to identify the hidden symmetry generators of (11) is to find a suitable coordinate transformation between the Boyer–Lindquist coordinates (t, r, ϕ) and the conformal coordinates $(w^\pm, y)$. For Kerr black holes, this turns out to be:

$$\begin{aligned} w^+ &= \left(\frac{r-r_+}{r-r_-}\right)^{1/2} e^{2\pi T_R\phi}\,, \\ w^- &= \left(\frac{r-r_+}{r-r_-}\right)^{1/2} e^{2\pi T_L\phi - \frac{t}{2M}}\,, \\ y &= \left(\frac{r_+-r_-}{r-r_-}\right)^{1/2} e^{\pi(T_L+T_R)\phi - \frac{t}{4M}}\,. \end{aligned} \tag{16}$$

where

$$T_R = \frac{r_+-r_-}{4\pi a}, \qquad T_L = \frac{r_++r_-}{4\pi a}. \tag{17}$$

Much has been written about the conformal coordinates (16) [18,25,40,42] and we observe more directly how to build them in Section 2.2, in terms of monodromy analysis[2]. For now, we just note that they are of the general form:

$$\begin{aligned} w^+ &= f(r)e^{t_R}, \\ w^- &= f(r)e^{-t_L}, \\ y &= g(r)e^{(t_R-t_L)/2}, \end{aligned} \tag{18}$$

and we return to (t_L, t_R) later. It is important to note that plugging the coordinate transformation (16) into the Kerr metric (3) does not exactly reproduce the Poincaré patch metric (12). Rather, near the bifurcation surface $w^\pm = 0$, the Kerr metric becomes:

$$ds^2 = \frac{4\rho_+^2}{y^2}dw^+dw^- + \frac{16J^2\sin^2\theta_0}{y^2\rho_+^2}dy^2 + \rho_+^2 d\theta^2 + \ldots\,, \tag{19}$$

where

$$\rho_+^2 = r_+^2 + a^2\cos^2\theta\,, \tag{20}$$

and the terms in the ellipsis "…" of Equation (19) are at least linear order in the coordinates $w^\pm$. The existence of these higher order terms underscores the fact that the hidden conformal symmetry generators (13) are not isometries of the Kerr metric (3).

For clarity, the expressions of the generators (13) in the Boyer–Lindquist coordinates are [18]:

$$\begin{aligned} H_1 &= ie^{-2\pi T_R\phi}\left(\Delta^{1/2}\partial_r + \frac{1}{2\pi T_R}\frac{r-M}{\Delta^{1/2}}\partial_\phi + \frac{2T_L}{T_R}\frac{Mr-a^2}{\Delta^{1/2}}\partial_t\right), \\ H_0 &= \frac{i}{2\pi T_R}\partial_\phi + 2iM\frac{T_L}{T_R}\partial_t, \\ H_{-1} &= ie^{2\pi T_R\phi}\left(-\Delta^{1/2}\partial_r + \frac{1}{2\pi T_R}\frac{r-M}{\Delta^{1/2}}\partial_\phi + \frac{2T_L}{T_R}\frac{Mr-a^2}{\Delta^{1/2}}\partial_t\right), \end{aligned} \tag{21}$$

with similar expressions for the $\bar{H}$ sector. In these coordinates, the quadratic Casimir (15) becomes exactly the near-region radial Klein–Gordon operator in (11), so that:

$$\mathcal{H}^2\Phi = K\Phi. \tag{22}$$

2.2. Monodromy Method

We now demonstrate that it is possible to find the generators (21) without explicitly taking a "near-region" limit, as in [18]. This subsection will largely follow [25], with some new discussion.

Let us again consider the Klein–Gordon Equation (8). This differential equation has two regular singular points at the horizons $r_\pm$ and one irregular singular point at infinity[3]. Each singular point causes a branch cut, and we were interested in studying the radial solutions $R(r)$ of (8) (now promoted to complex-valued functions) when going around each of the regular singular points. In general, the solutions $R(r)$ develop a monodromy around these singular points. To study this, we posited that $R(r)$ has a series solution of the form:

$$R(r) = (r-r_i)^\beta \sum_{n=0}^{\infty} q_n (r-r_i)^n. \tag{23}$$

Our immediate objective was to determine the monodromy parameter $\beta \equiv i\alpha$ using the Frobenius method. We discuss this in more detail for our more complicated higher derivative case in Section 4. For the case at hand, we just present the answer and refer the reader to [23] for details. The monodromy parameters around the inner and outer horizons are:

$$\alpha_\pm = \frac{\omega - \Omega_\pm m}{2\kappa_\pm}, \tag{24}$$

where $\Omega_\pm$ and $\kappa_\pm$ are as defined in (9).

Next, a crucial step in obtaining the generators (21) was to implement a change of basis:

$$e^{i(m\phi-\omega t)} = e^{-i(\omega_L t_L + \omega_R t_R)}. \tag{25}$$

The choice employed by [25] was:

$$\omega_L = \alpha_+ - \alpha_-, \qquad \omega_R = \alpha_+ + \alpha_-. \tag{26}$$

This particular change of basis (26) is not well motivated in the literature and is often taken as a purely mathematical step to match the results of [18]. To determine whether this was the appropriate basis choice to use in our more complicated higher derivative analysis, presented in Section 4, a deeper physical understanding of this choice was needed, which we outline here[4].

Let us consider how the radial solutions $R(r)$ change when going around the singular point r_+. Near $r = r_+$, our radial solutions are of the form:

$$R(r) = (r - r_+)^{\pm i\alpha_+}\left(1 + \mathcal{O}(r - r_+)\right). \tag{27}$$

When going around the singular point $r - r_+ \to e^{2\pi i}(r - r_+)$, we see that:

$$R(r) \to R(r)e^{\mp 2\pi\alpha_+}. \tag{28}$$

As explained in [23], when going around the singular point r_+ twice, i.e., $r - r_+ \to e^{4\pi i}(r - r_+)$, we expect the wave equation $\Phi = e^{-i\omega t + im\phi}R(r)$ to be invariant [5]. The radial piece of the outgoing solution $R(r) = (r - r_+)^{i\alpha_+}$ picks up the factor $e^{-4\pi\alpha_+}$. Plugging in the value for α_+ in (24), we find that t and ϕ transform in the following way to cancel this factor:

$$(t, \phi) \sim (t, \phi) + \frac{2\pi i}{\kappa_+}(1, \Omega_+). \tag{29}$$

We now see that the basis choice (26) arises from determining the appropriate conjugate variables that lead to more natural thermal and angular transformation properties. For example, around $r = r_+$, we would replace (29) with:

$$(X, Y) \sim (X, Y) + 2\pi i(1, 1). \tag{30}$$

The authors of [23] presciently rename (X, Y) as (t_L, t_R). Similar arguments for the singular point r_- lead us to the transformation properties:

$$(X, Y) \sim (X, Y) + 2\pi i(-1, 1). \tag{31}$$

For the wavefunction $\Phi = e^{-i\omega_L t_L - i\omega_R t_R}R(r)$, we find that by using (30) and (31), the functions (t_L, t_R) that accomplish these identifications are:

$$t_R = 2\pi T_R\phi, \qquad t_L = \frac{1}{2M}t - 2\pi T_L\phi. \tag{32}$$

With (t_L, t_R) in (32), we can now immediately reproduce the zero mode generators of [18]. They are:

$$H_0 = \frac{i}{2\pi T_R}\partial_\phi + 2iM\frac{T_L}{T_R}\partial_t = i\partial_{t_R}, \qquad \bar{H}_0 = -2iM\partial_t = -i\partial_{t_L}. \tag{33}$$

Note that, in this discussion, we avoided making the seemingly arbitrary basis choice (26). Instead, we used (ω_L, ω_R) as fixed by (32) and (25).

It may appear that the monodromy analysis only determines the zero mode generators $(H_0, \bar{H}_0)$ and not $(H_{\pm1}, \bar{H}_{\pm1})$, but we can actually go further. From Equation (18), we see that (t_L, t_R) also fix the conformal coordinates up to a radial factor, which can be recovered from the radial behavior of the hypergeometric solutions of (11) or from the Klein–Gordon operator itself, as we argue in Section 4 and Appendix A. Once we have the proper conformal coordinates, all of the Hs are determined by (13).

3. A Standard Form for the Klein–Gordon Operator

Before we move on to our higher derivative model, it is useful for us to express the Klein–Gordon operator in a standard form. In addition to streamlining our analysis, this form highlights the interesting physical structure of the operator, which persists in both higher dimensional and higher spin settings.

We begin by writing the Klein–Gordon operator on the Kerr background in the following way:

$$\begin{aligned}\nabla^\mu \nabla_\mu = \frac{1}{\rho^2}\Bigg[&\partial_r(\Delta\partial_r) - \frac{(r_+ - r_-)}{(r - r_+)}\left(\frac{\partial_t + \Omega_+\partial_\phi}{2\kappa_+}\right)^2 + \frac{(r_+ - r_-)}{(r - r_-)}\left(\frac{\partial_t + \Omega_-\partial_\phi}{2\kappa_-}\right)^2 \\ &+\frac{1}{\sin^2\theta}\partial_\phi^2 - (a^2\cos^2\theta + 4M^2)\partial_t^2 - (r^2 + 2Mr)\partial_t^2 + \frac{1}{\sin^2\theta}\partial_\theta\left(\sin^2\theta\partial_\theta\right)\Bigg].\end{aligned} \tag{34}$$

This form has several useful features for our analysis. First, if we posit the standard solution $\Phi = e^{i(m\phi - \omega t)}R(r)S(\theta)$, we see that the term

$$\left(\frac{1}{\sin^2\theta}\partial_\phi^2 - (a^2\cos^2\theta + 4M^2)\partial_t^2\right)\Phi = C\Phi \tag{35}$$

produces a version of Carter's constant C [45]. This means that, when we consider a constant θ slice $\theta = \theta_0$, the only dependence on our choice of slice would be in the prefactor ρ^{-2} (recall that $\rho^2 = r^2 + a^2\cos^2\theta$). The presence of this factor means that our higher order equation of motion $\left(\nabla^\mu\nabla_\mu\right)^n\Phi = 0$ appears not to be separable. Nevertheless, we see in Section 4 that at leading-order near $r = r_\pm$, all dependences on ρ^2 (and thus, θ_0) drop out. Thus, we consider a constant θ slice $\theta = \theta_0$, allowing us to study the "radial" operator:

$$\begin{aligned}\nabla^\mu \nabla_\mu = \frac{1}{\rho_0^2}\Bigg[&\partial_r(\Delta\partial_r) - \frac{(r_+ - r_-)}{(r - r_+)}\left(\frac{\partial_t + \Omega_+\partial_\phi}{2\kappa_+}\right)^2 + \frac{(r_+ - r_-)}{(r - r_-)}\left(\frac{\partial_t + \Omega_-\partial_\phi}{2\kappa_-}\right)^2 \\ &C_{t\phi} - (r^2 + 2Mr)\partial_t^2\Bigg],\end{aligned} \tag{36}$$

where $\rho_0^2 = r^2 + a^2\cos^2\theta_0$ and we denote the operator in (35) as $C_{t\phi}$ for convenience.

The form of the operator (36) has a further use. The terms

$$\frac{\partial_t + \Omega_\pm\partial_\phi}{2\kappa_\pm} \tag{37}$$

produce the monodromy parameters (24) introduced in Section 2.2, i.e.,:

$$\left(\frac{\partial_t + \Omega_\pm\partial_\phi}{2\kappa_\pm}\right)\Phi = -i\alpha_\pm\Phi\,. \tag{38}$$

In addition, the Killing vector fields

$$\xi_{\pm} = \kappa_{\pm}\Big(\partial_t + \Omega_{\pm}\partial_{\phi}\Big) \tag{39}$$

are exactly those that vanish on the inner and outer horizons $r_{\pm}$. As such, they are the same vector fields that appear in Wald's formulation [46] of black hole entropy as the integrated Noether charge associated with the Killing vectors vanishing on the horizons. This point was discussed in [23].

Further, the radial factors $\left(\frac{r_+ - r_-}{r - r_\pm}\right)$ can be directly related to the conformal coordinates, as defined in (18), and we can finally write the Klein–Gordon operator (36) acting on Φ as:

$$\nabla^{\mu}\nabla_{\mu}\Phi = \frac{\Phi}{\rho^2 R(r)}\left[\partial_r(\Delta\partial_r) + \alpha_+^2\frac{g^2(r)}{f^2(r)} - \alpha_-^2 g^2(r) + (r^2 + 2Mr)\omega^2 + C_{t\phi}\right]R(r). \tag{40}$$

This form of the Klein–Gordon operator holds for higher spin fields *and* in higher dimensions, even though the forms of $\alpha_{\pm}$, $f(r)$ and $g(r)$ change. This is discussed in Appendix A. The only terms that do change for a higher spin or higher dimension are the non-singular terms:

$$(r^2 + 2Mr)\omega^2 + C_{t\phi}, \tag{41}$$

which are precisely those that are dropped in the near-region limit of Section 2.1.

4. Hidden Conformal Symmetry in Higher Derivative Dynamics

To study how hidden conformal symmetry is manifest in a theory with higher derivative dynamics, we considered the following action for a massless scalar field on a Kerr black hole background:

$$S = -\int d^4x\sqrt{g}\left(\frac{1}{2}\Phi\Big(\nabla^{\mu}\nabla_{\mu}\Big)^n\Phi\right), \tag{42}$$

where n is an integer. The equation of motion resulting from this action was:

$$\Big(\nabla^{\mu}\nabla_{\mu}\Big)^n\Phi = 0, \tag{43}$$

where $n = 1$ is the Klein–Gordon equation. Our motivation for choosing this action was twofold. First, the equations of motion were simple enough to provide a straightforward extension to previous results with $n = 1$ obtained by [18,25]. Though simple, we observed that (43) had already produced interesting complications that provided insights into whether the choice of dynamics affects hidden conformal symmetry. Our second motivation was that the action (42) is of physical interest, since known examples of holographic duals to logarithmic conformal field theories contain higher derivative equations of motion [34]. Thus, (42) provided us with the opportunity both to study the effect of changing the dynamics on hidden conformal symmetry and also to potentially diagnose a new instance of a logCFT correspondence.

The differential Equation (43) can be reformulated in two ways, one of which was of particular use to us. The aim of both approaches is to reduce the system to a series of second-order equations. For example, as was discussed in [47], the equation of motion (43) can be broken up into coupled second-order equations by introducing $n-1$ auxiliary scalar fields:

$$\begin{aligned} &\nabla_{\mu}\nabla^{\mu}\Phi_1 = 0, \\ &\nabla_{\mu}\nabla^{\mu}\Phi_i = \Phi_{i-1}, \qquad \text{for } i = 2, \ldots, n. \end{aligned} \tag{44}$$

The related and more useful alternative to this approach is to repackage the auxiliary scalar fields as higher spin objects. That is, the problem (43) can be expressed as:

$$\begin{aligned}
&\nabla^{\mu}\nabla_{\mu}\Phi = 0, \\
&\nabla^{\mu}\nabla_{\mu}\Phi_{\mu_1\mu_2} = 0, \\
&\nabla^{\mu}\nabla_{\mu}\Phi_{\mu_1\mu_2\mu_3\mu_4} = 0, \\
&\vdots \\
&\nabla^{\mu}\nabla_{\mu}\Phi_{\mu_1\ldots\mu_{2n-2}} = 0.
\end{aligned} \tag{45}$$

In the above expressions, we defined the higher spin fields as:

$$\begin{aligned}
&\Phi_{\mu_1\mu_2} \equiv \nabla_{\mu_1}\nabla_{\mu_2}\Phi, \\
&\Phi_{\mu_1\mu_2\mu_3\mu_4} \equiv \nabla_{\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\nabla_{\mu_4}\Phi, \\
&\vdots \\
&\Phi_{\mu_1\mu_2\ldots\mu_{2n-2}} \equiv \nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{2n-2}}\Phi.
\end{aligned} \tag{46}$$

We return to the significance of this reformulation later, in Section 4.2.1.

We begin this section with a monodromy analysis of higher derivative dynamics on a Kerr background in Section 4.1. As mentioned before, one initial motivation for studying higher derivative dynamics was to potentially identify a new instance of a logCFT correspondence through hidden conformal symmetry, in the spirit of the Kerr/CFT correspondence away from extremality. We present our study of this question in Section 4.2.1, which we begin with a short review of how the higher derivative model (42) is used in AdS/logCFT, followed by a study of whether this is a viable model with which to build a Kerr/logCFT correspondence.

4.1. Monodromy Analysis

In this section, we analyze hidden conformal symmetry in our higher derivative theories in the spirit of Section 2.2. We began our analysis with $n = 2$ in the equation of motion (43). We then treated $n = 3$ and constructed a clear pattern for the monodromy parameters $\alpha_\pm$ for general n.

We immediately encountered the potential issue that, for $n > 1$, the equation of motion (43) appears not to be separable. Perhaps intriguingly, this turned out not to matter at leading-order near $r = r_\pm$. That is, in what follows, we took a constant slice $\theta = \theta_0$ and our results for $\alpha_\pm$ did not depend on the choice of θ_0.

4.1.1. Case $n = 2$

Since we were free to take a constant θ slice, we focused on the behavior of a radial differential equation near its singular points. In particular, we could write the radial equation in standard form near a singular point $r = r_i$:

$$(r-r_i)^4 R^{(4)} + D(r)(r-r_i)^3 R^{(3)} + C(r)(r-r_i)^2 R'' + B(r)(r-r_i)R' + A(r)R = 0. \tag{47}$$

The Frobenius method instructed us to look for series solutions of the form:

$$R(r) = (r-r_i)^{\beta} \sum_{k=0}^{\infty} q_k (r-r_i)^k, \tag{48}$$

and coefficient functions expanded as:

$$\begin{aligned} D(r) &= \sum_{k=0}^{\infty} d_k (r-r_i)^k, & C(r) &= \sum_{k=0}^{\infty} c_k (r-r_i)^k, \\ B(r) &= \sum_{k=0}^{\infty} b_k (r-r_i)^k, & A(r) &= \sum_{k=0}^{\infty} a_k (r-r_i)^k. \end{aligned} \tag{49}$$

In order for (47) to be satisfied, the coefficient of each power of $r - r_i$ had to equal zero. In particular, the coefficient of the $(r - r_i)^\beta$ term produced the fourth-order indicial equation:

$$\beta(\beta-1)(\beta-2)(\beta-3) + \beta(\beta-1)(\beta-2)d_0 + \beta(\beta-1)c_0 + \beta b_0 + a_0 = 0. \tag{50}$$

Without loss of generality, we first studied the analytic structure around $r = r_+$. The zeroth order coefficients of our series expansions (48) were:

$$d_0 = 4, \qquad c_0 = 2(1+\alpha_+^2), \qquad b_0 = 0, \qquad a_0 = \alpha_+^2(1+\alpha_+^2), \tag{51}$$

where α_+ denotes the monodromy parameter for a scalar field on a Kerr blackground near the outer horizon, as defined in (24). Plugging these values back into our indicial equation, we obtained:

$$(\beta^2 + \alpha_+^2)((\beta-1)^2 + \alpha_+^2) = 0. \tag{52}$$

Thus, we obtained:

$$\beta = \{\pm i\alpha_+,\ 1 \pm i\alpha_+\}, \tag{53}$$

where α_+ is given by the expression (24) or, since $\beta \equiv i\alpha$:

$$\alpha_+^{n=2} = \{\pm\alpha_+,\ -i \pm \alpha_+\}. \tag{54}$$

There was a similar result for $\alpha_-^{n=2}$:

$$\alpha_-^{n=2} = \{\pm\alpha_-,\ -i \pm \alpha_-\}, \tag{55}$$

where, again, α_- can be read in Equation (24).

At this point, there are several things to point out regarding (53)–(55). First, we can see why it was useful to reformulate our equation of motion (43) for $n = 2$ as two coupled equations with a higher spin field, as in (45):

$$\begin{aligned} \nabla^\mu \nabla_\mu \Phi &= 0, \\ \nabla^\mu \nabla_\mu \Phi_{\mu_1\mu_2} &= 0. \end{aligned} \tag{56}$$

These equations were coupled in the sense that $\Phi_{\mu_1\mu_2}$ was built from Φ as in (46). The four monodromy parameters $\alpha_+^{n=2}$ associated with our fourth-order equation near the outer horizon r_+ were exactly those found when analyzing the second-order Klein–Gordon equation for a scalar field ($\alpha_\pm$, see Equation (24)) and for a spin-2 field ($-i \pm \alpha_\pm$, see Equation (A6)).

Second, from (53) we could see that two of the exponents β differed from two others by a positive integer. The Frobenius method tells us that, of the four linearly independent solutions around each singular point, two of them *may be* log solutions. For example, near $r = r_+$ we have:

$$\begin{aligned} R(r) &= (r-r_+)^{1 \pm i\alpha_+} \Phi_1^\pm(r), \\ R(r) &= a_\pm (r-r_+)^{1 \pm i\alpha_+} \Phi_1^\pm(r) \log(r-r_+) + (r-r_+)^{\pm i\alpha_+} \Phi_2^\pm(r), \end{aligned} \tag{57}$$

where $a_\pm$ are constants, which can be zero or not. This could signal that, if there is indeed a CFT description of this system, it could be a logCFT. However, it is important to note that there is a subtle difference between the log terms that appear here for Kerr and those that appear in the context of logCFTs dual to an AdS background, as discussed in Section 4.2.1. In the Frobenius method, when two roots are repeated (as in the AdS analysis, see Equation (76)), a logarithmic part of the solution is guaranteed. In contrast, it is a theorem that when two roots differ by a positive integer (as in our case) the coefficients $a_\pm$ could be zero, see e.g., [48]. This depends on the specific and intricate nature of the given differential equation.

Our principal goal was then to study if and how hidden conformal symmetry is manifest in our higher derivative dynamics $\left(\nabla_\mu \nabla^\mu\right)^2 \Phi = 0$. There were several ways to approach this problem. In the usual scenario, that is $\nabla_\mu \nabla^\mu \Phi = 0$, we try to find $SL(2,\mathbb{R})$ generators $(H_0, H_{\pm 1})$, as in (21), that (1) satisfy the commutation relations (14) and (2) form a Casimir that reproduces the near-region Laplacian $\mathcal{H}^2 \Phi = K\Phi$, as in (22). In trying to find equivalent structures in the equation $(\nabla^\mu \nabla_\mu)^2 \Phi = 0$, the role of a quadratic Casimir $\mathcal{H}^2$ would perhaps have to be replaced by a quartic Casimir $\mathcal{H}^4$. However, when we take the equivalent description of our system (56), we can see that hidden conformal symmetry is still visible and presents itself in a natural way.

Since (56) was an equivalent description of our fourth-order equation, we could analyze each equation in (56) separately. The first equation, $\nabla^\mu \nabla_\mu \Phi = 0$, was of course just the standard case that had already been treated in [25]. Then, we turned to the spin-2 equation. For the reader's convenience, we reproduce this equation here from our Appendix A:

$$\left(\partial_r \Delta \partial_r + \alpha^2_{+,s=2} \frac{g^2(r)}{f^2(r)} - \alpha^2_{-,s=2} g^2(r) + \omega^2 r^2 + 2(M\omega + 2i)\omega r + C_{t,\phi}\right) R(r) = 0, \quad (58)$$

where s is the spin of the auxiliary field. Again, we could think of the constant $C_{t,\phi}$ as being absorbed in a separation constant and the terms $\omega^2 r^2 + 2(M\omega + 2i)\omega r$ could be dropped in the near-region limit. Thus, the solutions to

$$\left(\partial_r \Delta \partial_r + \alpha^2_{+,s=2} \frac{g^2(r)}{f^2(r)} - \alpha^2_{-,s=2} g^2(r)\right) R(r) = 0 \quad (59)$$

were also hypergeometric functions, hinting at hidden conformal symmetry.

In our review, Section 2, we introduced several important and interrelated quantities: monodromy exponents $\beta \equiv i\alpha$ (24), the change of basis modes (ω_L, ω_R) (26), their conjugate variables (t_L, t_R) (32), the conformal coordinates $(w^\pm, y)$ (16) and the $SL(2,\mathbb{R})$ generators $(H_0, H_{\pm 1})$ (21). We then asked the question: which of these quantities, if any, needed to be modified from their $n = 1$ values so that we could still obtain the conditions for diagnosing hidden conformal symmetry? Notice that this is equivalent to finding generators that satisfy the commutation relations (14) whose Casimir reproduces the near-region Klein–Gordon operator $\mathcal{H}^2_{n=2} \Phi_{\mu_1 \mu_2} = K\Phi_{\mu_1 \mu_2}$, as in (22). We claim that this was accomplished by modifying only one thing: the change of basis choice (ω_L, ω_R). Let us demonstrates how this worked by discussing each of the above quantities in turn.

First, we claim that (t_L, t_R) did not change from their $n = 1$ values given in (32), since these were obtained by thermal considerations in our review in Section 2. The conformal coordinates $(w^\pm, y)$ should also not change from (16) and (18), since these were purely geometric relations taking the Kerr background to the upper-half plane (to leading-order near the bifurcation surface). Finally, the generators were built directly from the the conformal coordinates via (13), so these should also remain unchanged from their $n = 1$ values. This only left two quantities: the monodromy parameters α, which certainly *did* change, and the basis choice (ω_L, ω_R), which had to change to account for the change in the αs. As mentioned above and in Appendix A, the new αs for the spin-2 equation were:

$$\alpha^{s=2}_\pm = i \pm \alpha^{s=0}_\pm, \quad (60)$$

and the basis choice to accommodate this was modified from (26) to:

$$\omega_L = \alpha_+ - \alpha_- - 2i, \qquad \omega_R = \alpha_+ + \alpha_-. \tag{61}$$

This means that we considered the frequencies $\omega \in \mathbb{C}$.

It is not necessary to write the full equation $\left(\nabla_\mu \nabla^\mu\right)^2 \Phi = 0$ in a standard form (as in Section 3) except to point out one thing. Taking a constant θ slice, the fourth-order equation of motion is of the form:

$$\mathcal{D}\big[R(r)\big] + \left[\alpha_+^2\left(1+\alpha_+^2\right)\frac{g^4(r)}{f^4(r)} + \alpha_-^2\left(1+\alpha_-^2\right)g^4(r) + n.s.\right]R(r) = 0, \tag{62}$$

where $\mathcal{D}\big[R(r)\big]$ stands for all terms involving a derivative of $R(r)$, $n.s.$ represents non-singular terms and the radial functions f and g are as defined in (16) and (18). From Equation (62), we could learn more about the standard form discussed in Section 3: the coefficients of the radial functions were just the Frobenius exponents β (see Equation (53)).

4.1.2. Case $n = 3$ and Higher n

We now present the monodromy calculation for the equation of motion $\left(\nabla^\mu \nabla_\mu\right)^n \Phi = 0$ with $n = 3$ and establish a pattern for general n.

Just as in the $n = 2$ case, our analysis did not depend on our choice of constant θ slice. Therefore, we were free to consider the sixth-order radial equation in standard form:

$$\begin{aligned}&(r-r_i)^6 R^{(6)} + F(r)(r-r_i)^5 R^{(5)} + E(r)(r-r_i)^4 R^{(4)} + \\ &D(r)(r-r_i)^3 R^{(3)} + C(r)(r-r_i)^2 R'' + B(r)(r-r_i)R' + A(r)R = 0.\end{aligned} \tag{63}$$

For concreteness, we again chose to study $r_i = r_+$. Upon positing a series solution for $R(r)$ and the coefficient functions (as in (48) and (49)), we found the indicial equation:

$$(\beta^2 + \alpha_+^2)((\beta-1)^2 + \alpha_+^2)((\beta-2)^2 + \alpha_+^2) = 0. \tag{64}$$

So, all together, our indicial equations close to $r = r_+$ for $\left(\nabla_\mu \nabla^\mu\right)\Phi = 0$, $\left(\nabla_\mu \nabla^\mu\right)^2 \Phi = 0$ and $\left(\nabla_\mu \nabla^\mu\right)^3 \Phi = 0$ were

$$\begin{aligned}&(\beta^2 + \alpha_+^2) = 0, \\ &(\beta^2 + \alpha_+^2)((\beta-1)^2 + \alpha_+^2) = 0, \\ &(\beta^2 + \alpha_+^2)((\beta-1)^2 + \alpha_+^2)((\beta-2)^2 + \alpha_+^2) = 0,\end{aligned} \tag{65}$$

respectively. Equation (65) suggests the monodromy structure of $\left(\nabla^\mu \nabla_\mu\right)^n \Phi = 0$ on the Kerr background for general n:

$$\prod_{j=1}^{n}((\beta - j + 1)^2 + \alpha_+^2) = 0. \tag{66}$$

4.2. Holographic Correspondence with Higher Derivative Dynamics

4.2.1. AdS/logCFT

Here, we begin with a brief review of how higher derivative dynamics in AdS are dual to logCFTs in order to contrast what happened when we considered these dynamics on a Kerr background in the next subsection. Holographic logCFTs have been discussed since the early days of AdS/CFT [49,50]. There has been much progress in this direction, see e.g., [34], and here, we only report the main lesson from it: logCFTs are holographically realized as higher derivative theories in AdS$_{d+1}$ spacetimes. In particular, for scalar fields with mass μ, the action is:

$$S = -\frac{1}{2}\int \mathrm{d}^{d+1}x\sqrt{g}\,\phi\left(\nabla_\mu\nabla^\mu - \mu^2\right)^n\phi, \tag{67}$$

where $n \geq 2$ corresponds to the rank n of the dual logCFT. The equation of motion is then a $2n$-th order differential equation, i.e.,:

$$\left(\nabla_\mu\nabla^\mu - \mu^2\right)^n\phi = 0\,. \tag{68}$$

The above action can also be formulated in terms of auxiliary fields; for example, in the case $n = 2$, we have:

$$S = -\frac{1}{2}\int \mathrm{d}^{d+1}x\sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi_1\partial_\nu\phi_2 + \mu^2\phi_1\phi_2 + \frac{1}{2}\phi_1^2\right). \tag{69}$$

The equation of motion for ϕ_1 and ϕ_2 are

$$\left(\nabla_\mu\nabla^\mu - \mu^2\right)\phi_2 = \phi_1 \qquad \left(\nabla_\mu\nabla^\mu - \mu^2\right)\phi_1 = 0\,, \tag{70}$$

respectively.

It is then clear that the ϕ_2 has to satisfy a "squared" equation, i.e., $\left(\nabla_\mu\nabla^\mu - \mu^2\right)^2\phi_2 = 0$. The action (69) can be generalized to arbitrary rank n [47] and the corresponding equations of motion are given by:

$$\begin{aligned}&\left(\nabla_\mu\nabla^\mu - \mu^2\right)\phi_1 = 0\,,\\&\left(\nabla_\mu\nabla^\mu - \mu^2\right)\phi_i = \phi_{i-1}\,, \qquad i = 2,\ldots,n\,,\end{aligned} \tag{71}$$

from which it follows that the equation of motion for the n-th field is indeed (68). In terms of the auxiliary fields, we can observe a shift symmetry of the equations of motion and the on-shell action

$$\phi_i \to \phi_i + \sum_{p=1}^{i-1}\lambda_p\phi_p\,, \tag{72}$$

for arbitrary constant λ_p.

Here, we briefly illustrate the example of a higher rank wave equation for a scalar field in (Euclidean) AdS$_{d+1}$; see, e.g., [34] for a recent review. We assumed the scalar field to be massless and we used a Poincaré patch, i.e.,:

$$ds^2_{AdS} = \frac{\mathrm{d}\zeta^2 + \mathrm{d}x_i\,\mathrm{d}x^i}{\zeta^2}\,, \tag{73}$$

where $i = 1,\ldots,d$.

Let us use Equation (43) for the case $n = 2$ and in the background (73). Given the high degree of symmetry of the metric (73), the differential equation is particularly simple. We can take $d = 2$ to make a direct comparison to the Kerr black hole metric (3) discussed in this work. Then, the radial differential equation for the radial component $\psi(\zeta)$ of the scalar field is given by:

$$\psi^{(4)}(\zeta) + \frac{2}{\zeta}\psi^{(3)}(\zeta) - 2\left(\mathcal{K}^2 + \frac{1}{\zeta^2}\right)\psi^{(2)}(\zeta) + \left(\frac{1}{\zeta^3} - \frac{2\mathcal{K}^2}{\zeta}\right)\psi^{(1)}(\zeta) + \mathcal{K}^4\psi(\zeta) = 0\,, \tag{74}$$

where $\mathcal{K}$ is a constant that depends on the mode expansions along the directions x^i. At the leading-order in ζ (that is close to the AdS boundary), the radial differential Equation (74) becomes:

$$\psi^{(4)}(\zeta) + \frac{2}{\zeta}\psi^{(3)}(\zeta) - \frac{1}{\zeta^2}\psi^{(2)}(\zeta) + \frac{1}{\zeta^3}\,\psi^{(1)}(\zeta) = 0\,. \tag{75}$$

The main feature to notice here is the absence of a potential term at the leading-order, in contrast to our case (87). Applying the Frobenius method, i.e., assuming $\psi(\zeta) = \zeta^\beta \sum_{n=0}^{\infty} c_n \zeta^n$, at the leading order we obtain the roots[6]:

$$\beta = 0, d\,, \qquad \text{with multiplicity } 2\,. \tag{76}$$

The non-trivial multiplicity means that we obtain the following four linearly independent solutions:

$$\zeta^0,\ \ \zeta^d,\ \ \zeta^0 \log\zeta,\ \ \zeta^d \log\zeta\,. \tag{77}$$

At the next-to-leading order in small ζ, the differential equation becomes:

$$\psi^{(4)}(\zeta) + \frac{2}{\zeta}\psi^{(3)}(\zeta) - 2\left(\mathcal{K}^2 + \frac{1}{\zeta^2}\right)\psi^{(2)}(\zeta) + \left(\frac{1}{\zeta^3} - \frac{2\mathcal{K}^2}{\zeta}\right)\psi^{(1)}(\zeta) = 0\,. \tag{78}$$

The potential term is not present in the equation, but this is specific to the case $d = 2$. This equation can be solved analytically and the four linearly independent solutions are Bessel functions of the first and second kind (I_n, Y_n) whose arguments depend on the dimensions of AdS, the constant $\mathcal{K}$ and the logarithm. Again expanding these solutions around $\zeta = 0$, the Bessel function Y_n gives rise to another explicit logarithmic behavior close to the boundary, in agreement with the leading behavior found in (77).

With this in mind, we took inspiration from the action (67) to start our investigation of higher derivative models in a Kerr black hole background and their hidden symmetries.

4.2.2. Kerr

We start this section by rewriting the radial Klein–Gordon equation in new coordinates and then we examine the squared Klein–Gordon operator in this setting.

Defining

$$z = \frac{r - r_-}{r - r_+}\,, \tag{79}$$

the radial Equation (8) becomes:

$$\begin{aligned} &z(1-z)f''(z) + (1-z)f'(z) \\ &-\left(\alpha_+^2 - \frac{\alpha_-^2}{z} + \frac{K}{1-z}\right. \\ &\left.-\left(\frac{(r_- - r_+)^2}{(1-z)^3} + 2\frac{(r_- - r_+)(M + r_+)}{(1-z)^2} + \frac{4M^2 + 2Mr_+ + r_+^2}{1-z}\right)\omega^2\right) f(z) = 0\,, \end{aligned} \tag{80}$$

where we use the definitions from (24).

The singular points of the original Equation (8), namely $r = r_-$, $r = r_+$ and $r = \infty$ were mapped to $z = 0$, $z = \infty$ and $z = 1$ respectively. In these coordinates (79), the Frobenius analysis became more transparent and it was clear from Equation (80) that close to the regular singular point $z = 0$ $(r = r_-)$, at the leading-order for example, the equation was simply:

$$f''(z) + \frac{1}{z}f'(z) + \frac{\alpha_-^2}{z^2}f(z) = 0\,. \tag{81}$$

The two linear independent solutions were then:

$$z^{i\alpha_-}\,, \qquad z^{-i\alpha_-}\,. \tag{82}$$

Before moving to higher order differential equations, it is useful to examine the second-order Klein–Gordon Equation (A4) for generic spin s in this coordinate system:

$$\begin{aligned}
&z(1-z)f''(z)+(1-z)f'(z)\\
&-\left((\alpha_+^s)^2-\frac{(\alpha_-^s)^2}{z}+\frac{K+s^2}{1-z}+2is\left(\frac{M-r_+}{1-z}-\frac{r_+-r_-}{(1-z)^2}\right)\omega\right.\\
&\left.-\left(\frac{(r_--r_+)^2}{(1-z)^3}+2\frac{(r_--r_+)(M+r_+)}{(1-z)^2}+\frac{4M^2+2Mr_++r_+^2}{1-z}\right)\omega^2\right)f(z)=0\,.
\end{aligned} \tag{83}$$

Notice that the terms proportional to ω^2 are unaffected by the spin s, while there is now a linear term proportional to ω and s. Again, by expanding at leading-order, for example, around $z=0$ ($r=r_-$), we obtain:

$$f''(z)+\frac{1}{z}f'(z)+\frac{(\alpha_-^s)^2}{z^2}f(z)=0\,, \tag{84}$$

where α_-^s is as defined in Equation (A6). The two independent solutions to this equation are:

$$z^{i\alpha_-^s}\,, \qquad z^{-i\alpha_-^s}\,. \tag{85}$$

Similarly, we can consider the next-to-leading order expansion of the full equation and obtain:

$$\begin{aligned}
&f''(z)+\frac{1}{z}f'(z)+\frac{(\alpha_-^s)^2}{z^2}f(z)\\
&-\left(\frac{K+s^2+(\alpha_+^s)^2-(\alpha_-^s)^2+2is(M-r_-)\omega-(4M^2+2Mr_-+r_-^2)\omega^2}{z}\right)f(z)=0\,.
\end{aligned} \tag{86}$$

Notice that:

$$(\alpha_++\alpha_-)(\alpha_+-\alpha_--is)=(\alpha_+^s)^2-(\alpha_-^s)^2\,.$$

The solutions to Equation (86) are modified Bessel functions of the first kind $I_n(z)$. Continuing the expansion of Equation (83) at the next-to-next-to-leading order, when constant terms appear in the potential, we see that the solutions are hypergeometric functions. We should stress that these equations are only valid in the neighborhood of $z=0$ and so, the solutions obtained in this way are not the full solution of the original equation.

Let us now consider Equation (43) for the case of $n=2$. Our starting points were Equations (36) and (40) (after acting on the field Φ). Applying the Klein–Gordon operator (36) to Equation (40) again and changing the coordinate system, as in (79), we obtained a rather lengthy expression. We then chose a constant θ slice, since, as discussed in Section 4, this did not affect the monodromy data. Again, the singular points were $z=0$, $z=\infty$ and $z=1$ and, by performing a series expansion around the regular singular points $z=0$ ($r=r_-$) and $z=\infty$ ($r=r_+$), we obtained the roots (55) and (54), respectively.

We believe it is helpful to examine the fourth-order differential equation obtained in this way, close to a regular singular point at the leading-order. We can focus on $z=0$ ($r=r_-$) for simplicity. Then, the differential equation is given by:

$$f^{(4)}(z)+\frac{4}{z}f^{(3)}(z)+\frac{2(1+\alpha_-^2)}{z^2}f^{(2)}(z)+\frac{\alpha_-^2(1+\alpha_-^2)}{z^4}f(z)=0\,, \tag{87}$$

and the four independent solutions are:

$$z^{1-i\alpha_-}\,,\; z^{1+i\alpha_-}\,,\; z^{-i\alpha_-}\,,\; z^{i\alpha_-}\,. \tag{88}$$

At the next-to-leading order, the structure of the equation is:

$$\begin{aligned} &f^{(4)}(z) + \left(\frac{4}{z} + A_3(\theta_0)\right) f^{(3)}(z) + \left(\frac{2(1+\alpha_-^2)}{z^2} + \frac{A_2(\theta_0)}{z}\right) f^{(2)}(z) + \frac{A_1(\theta_0)}{z^2} f^{(1)}(z) \\ &+ \left(\frac{\alpha_-^2(1+\alpha_-^2)}{z^4} + \frac{A_0(\theta_0)}{z^3}\right) f(z) = 0\,. \end{aligned} \tag{89}$$

The constants A_0, A_1, A_2 and A_3 depend on the black hole parameters and the dynamical inputs K, ω and m, as well as on the choice of the constant slice θ_0, as our notation underlines. We refrain from writing their explicit expression here, since it is not particularly useful. The four linear independent solutions of Equation (89) are hypergeometric functions and Meijer G-functions, which again depend on the monodromy data and the constant As.

We then needed to compare and contrast the AdS case (76) and the Kerr case (88) in order to determine whether an analogous formulation of a Kerr/logCFT was possible. We know from Section 4.2.1 that the fourth-order radial equation in an AdS background admits logarithmic solutions (77). In fact, the logarithmic behavior is *guaranteed* at the leading-order close to the boundary by the degeneracy of the indicial roots, see (75) and (76). However, the Kerr case is more subtle. The crucial difference between Equations (75) and (87) was the presence of a potential term in the Kerr black hole geometry[7]. We can see from Equation (88) that two pairs of indicial roots differed by a positive integer. This signals that a logarithmic solution *may or may not* be present. In the Kerr/CFT construction, the CFT exists at the black hole horizon and so, to determine whether the logarithmic terms were there in the region that interested us, we expanded the Meijer G-functions close to $z = 0$. A general expansion of this function contained terms polynomial in z and also terms such as $z^2 \log z$. The corresponding coefficients were very lengthy expressions, which depended on the A constants. We remind the reader that these were not the solutions of the full Equation (80) and that their validity is within the validity of the expansion of Equation (89) itself and so, the solution showing a logarithmic term beyond its perturbative regime was not meaningful in this context. We investigated the coefficient of the logarithmic term numerically and, interestingly, we were not able to find a non-zero coefficient near the black hole horizon. This seems to indicate that a Kerr/logCFT construction is *not* possible within this framework. We discuss the physical interpretations of this result in the discussion section. It is perhaps worth stressing that a logarithmic solution could still be present outside of our regime of validity, i.e., at order z^2. Again, the presence or absence of a logarithmic term in general depends on specific dynamics, in particular on the slice θ_0 that is chosen. We remind the reader that the full fourth-order Equation (80) is not separable and we chose a specific θ-slice to perform our analysis. This is in contrast to the monodromy data, which did not depend on this choice.

5. Discussion

The goals of this article were to provide a case study of how hidden conformal symmetry is manifest when we change the dynamics on a given background and, in particular, whether we could use variations of the Kerr/CFT correspondence to work toward diagnosing a new instance of a logCFT correspondence. In this section, we review our results, discuss the challenges of a logCFT construction and discuss future directions.

We found the monodromy parameters $\alpha^{(n)}$ for the general number of derivatives $2n$ and showed how they are related to the monodromy exponents of the regular Klein–Gordon equation of higher spin fields. We showed that pairs of the indicial roots $i\alpha^{(n)}$ differed by an integer and thus, a logarithmic contribution to the radial equation could be present. However, we found that when sufficiently close to the black hole horizon, potential logarithmic contributions vanished. This seems to indicate that we cannot construct a Kerr/logCFT correspondence from higher derivative theories, which have been used to construct examples of AdS/logCFT correspondences.

The difficulty in constructing a Kerr/logCFT correspondence is interesting, both from the gravitational perspective and from the field theory perspective. From the field theory perspective, logCFTs have been proven to be relevant in numerous areas of physics. Indeed, they can arise at critical points of various physical systems, such as those describing the quantum Hall plateau transition [51–54], but also in models describing percolation [55], self-avoiding walks [56] and systems with quenched disorder [35–37]. These special conformal field theories are characterized by a logarithmic behavior in correlation functions [33,55,57], which seems to clash with the fact that the theory is scale invariant. However, the presence of these terms is hinged on the reducible but indecomposable representations of the conformal group [33]. The crucial point is that the conformal Hamiltonian is not diagonalizable but rather has a Jordan cell structure (for rank $n \geq 2$), which leads to logarithmic terms in the correlation functions and a lack of unitarity. While this feature would be generally considered a red flag in quantum field theory, it does not pose any threat as a description of statistical mechanical systems, as confirmed by the examples mentioned above. We refer the reader to, e.g., [34,39,58] and references therein for more recent and extensive reviews of logCFTs.

From a gravitational perspective, a natural question is whether hidden conformal symmetries are still visible (or modified) when the dynamics are encoded in higher derivative differential operators, such as those in the action (67). Higher derivative theories break unitarity, hence we could expect that this would be reflected somehow in the hidden symmetry group. Consequently, this could hinder us in our efforts to investigate/study a Cardy-like formula in this setting. Indeed, if we expect a non-unitary CFT, the partition function may not be bounded from below and we may have states with negative norm; thus, it is not clear in which sense we could discuss an entropy. Still, it could be possible to give a description of the density of states, perhaps taking into account anomalies.

There are further challenges with trying to make a logCFT correspondence in the spirit of Kerr/CFT using the model we proposed. First, there is still some progress to be made regarding the creation of a robust holographic correspondence in Kerr/CFT itself. Even though there is ample evidence that the hidden conformal symmetry found in [18] really is described by an underlying CFT (such as the correct computations of scattering cross-sections), many elements are still lacking, such as how to conduct an asymptotic symmetry group analysis when the symmetry generators are not all isometries or when the conformal symmetry acts at the horizon and not the boundary. We leave this interesting problem for future work. As mentioned previously, in the case of a logCFT correspondence in AdS, the scalar field is also guaranteed to have a logarithmic piece near the boundary but in Kerr, the logarithmic piece could vanish near the horizon.

Lastly, we illustrate once again that the monodromy method is really a powerful tool for studying hidden conformal symmetry: it allowed us to study near horizon dynamics without actually taking a near-region limit in the dynamics (although sometimes we did take such a limit, purely for the ease of the calculations). Furthermore, even though the higher order equations $\left(\nabla^{\mu}\nabla_{\mu}\right)^2\Phi = 0$ were no longer separable, it did not matter for the monodromy analysis. Our results were independent of the θ slice we chose. This is not obvious a priori, and we do not have a deep physical understanding of why this would happen. Mathematically, we observed that all terms with a dependence on the angle θ were sub-leading when close enough to the horizons. The leading singular behavior of the solution did not depend on the θ slice. Understanding this would be very interesting and we leave the problem for future works.

There are several interesting opportunities for future work. One important contribution would be to establish a non-extremal analog to the asymptotic symmetry group analysis; for example, that presented in [17]. This would further strengthen the claim of a non-extremal Kerr/CFT correspondence. This paper is a first step toward learning what the monodromy method can tell us about other equations of motion. Another direction that would be interesting, for example, would be to study whether hidden conformal symmetry is somehow encoded in the Dirac or geodesic equations. Further, it was recently shown in [28] that there is a difficulty in constructing conformal coordinates in six spacetime dimensions and higher that do not have branch cuts, unless an explicit near-horizon limit is taken. Thus, it would be interesting to check whether the general form of the Klein–Gordon operator discussed in Section 3 and Appendix A holds in higher dimensions. We leave this for future work.

An intriguing and open question is whether a generalization of the Cardy formula exists for non-unitary theories, particularly those where the underlying conformal field theory is logarithmic [58]. Indeed, exploring this question was one of our initial motivations for this work. The Cardy formula has played a crucial role in the AdS/CFT duality [6,7,59] and, in particular, in the Kerr/CFT correspondence [17]. The Bekenstein–Hawking entropy of an asymptotically AdS_3 black hole exactly reproduces (at high energy) the degeneracy of states governed by the Cardy formula [29] in two-dimensional CFTs, i.e.,:

$$S_{CFT} = 2\pi\sqrt{\frac{c_R L_0}{6}} + 2\pi\sqrt{\frac{c_L \bar{L}_0}{6}}\,, \tag{90}$$

where c_R and c_L are the right and left central charges, respectively, and L_0 and $\bar{L}_0$ are the zero-th generators of the Virasoro algebra. It was later understood that any higher dimensional black holes with an AdS_3 near-horizon geometry obey a Cardy formula, again as a consequence of the symmetry of the given geometry. A step further was made in [60], where the Cardy formula was extended to a warped AdS_3 geometry, i.e., geometries with $SL(2,\mathbb{R}) \times U(1)$ isometries. In particular, these are the isometries of the near-horizon geometry of the extremal Kerr black holes (and, in general, of extremal/near-extremal black holes). Two essential ingredients enter into the derivation of the Cardy formula (90): unitarity and modular invariance. So, the presence of a Cardy formula, even with a reduced symmetry group as in the warped case, tells us that there is still a notion of modular invariance here. We leave the establishment of a Cardy-like formula in non-unitary settings for future work.

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Appendix A. Other Standard Form Examples

In this short appendix, we discuss a general form of the d'Alembertian operator $\nabla_\mu \nabla^\mu$ that persists when acting on fields of higher spin and in higher dimensions. We present this discussion because this form was useful to us in examining monodromies in theories with higher-order equations of motion $(\nabla_\mu \nabla^\mu)^n \Phi = 0$ and because we feel that it highlights important physical structures related to the monodromy parameters and conformal coordinates defined in (23) and (18), respectively.

To set the stage, let us state the results of the four-dimensional case: a scalar field $\Phi = e^{i(m\phi-\omega t)}R(r)S(\theta)$ propagating on a Kerr background (3). The Klein–Gordon equation was:

$$\nabla^\mu\nabla_\mu\Phi = \frac{\Phi}{\rho^2 R(r)}\left[\partial_r(\Delta\partial_r) + \alpha_+^2\frac{g^2(r)}{f^2(r)} - \alpha_-^2 g^2(r) + (r^2+2Mr)\omega^2 + C_{t,\phi}\right]R(r), \quad \text{(A1)}$$

where, as we discussed in Section 2,

$$\alpha_\pm = \frac{\omega - \Omega_\pm m}{2\kappa_\pm} \quad \text{(A2)}$$

are the monodromy parameters, the functions

$$f(r) = \left(\frac{r-r_+}{r-r_-}\right)^{1/2}, \qquad g(r) = \left(\frac{r_+-r_-}{r-r_-}\right)^{1/2} \quad \text{(A3)}$$

define the radial dependence of the conformal coordinates and $C_{t\phi}$ is a constant of motion; see Equations (35) and (36) and the discussion below. In scenarios with higher spin fields and higher-dimensional spacetime backgrounds, the quantities $(\alpha_\pm, f(r), g(r), C_{t,\phi})$ change but the overall form of (A1) does not. Let us see how this works.

The hidden conformal symmetry generators for five-dimensional Myers–Perry black holes [61] were first studied by [31]. The radial equation of motion for a scalar field ansatz $\Phi = e^{i(-\omega t + m_1\phi_1 + m_2\phi_2)}$ can be written as:

$$\left[\frac{\partial}{\partial x}\left(x^2 - \frac{1}{4}\right)\frac{\partial}{\partial x} + \alpha_+^2\frac{g^2(r)}{f^2(r)} - \alpha_-^2 g^2(r) + \frac{x\Delta\omega^2}{4} + \tilde{C}_{t,\phi}\right]\Phi = 0, \quad \text{(A4)}$$

where

$$x \equiv \frac{r^2 - 1/2(r_+^2 + r_-^2)}{(r_+^2 - r_-^2)}$$

is a radial coordinate and the monodromy parameters $\alpha_\pm$ are found in Appendix A of [23]. There are two important points here. First, the functions $f(r)$ and $g(r)$ are precisely those that define the radial behavior of the conformal coordinates in a five-dimensional setting, as presented in [31]. Second, we can see a pattern emerging. The Klein–Gordon equation is expressible as: a derivative piece, pieces involving the monodromy parameters (the form of which are fixed), a constant term and a non-constant r-dependent term that are irrelevant at either horizon.

The monodromy analysis for higher spin s perturbations on a four-dimensional Kerr background was treated in [24]. The equation of motion for such a perturbation can be written as:

$$\left(\partial_r\Delta\partial_r + (\alpha_+^s)^2\frac{g^2(r)}{f^2(r)} - (\alpha_-^s)^2 g^2(r) + \omega^2 r^2 + 2(M\omega + is)\omega r + C_{t,\phi}\right)R(r) = 0. \quad \text{(A5)}$$

Asexpected, the functions $f(r)$ and $g(r)$ that are attached to the monodromy parameters $\alpha_\pm$ are the same as in (A2) and the monodromy parameters themselves are:

$$\alpha_\pm^s = \mp\frac{is}{2} + \frac{2M\omega r_\pm - am}{r_+ - r_-}. \quad \text{(A6)}$$

(There is a typo in the $\alpha_\pm$ reported in [24]). Noticethat when setting $s = 2$ in the expressions (A6), the monodromy parameters reduce to (54) and (55).

Notes

1 (Indeed, the separability of this equation is actually a direct consequence of the existence of the hidden symmetry generators that we were about to build. For a review on this point, see [22,41] and for a more recent discussion, see [28]).

2 (For instances of conformal coordinates in other contexts, see [43,44]).

3 Our discussion only requires the regular singular points. For the treatment of the irregular singular point, see [23,24].

4 The key elements of this discussion were also presented in [23].

5 This basically comes down to the square root in (16) and the analogy between this coordinate transformation and the vacuum between the Minkowski space and the Rindler wedge. In the end, we chose the transformation laws for (t, ϕ), such that this analogy holds.

6 These roots are nothing but the roots of the equation $\Delta(\Delta - d) = m^2$ for a scalar field with mass m in AdS_{d+1}. Here, we set $m = 0$ and used β instead of Δ in order to be consistent with the notation used in this work.

7 We should stress that we are only discussing formal similarities. In the Kerr black hole case, we zoomed in on a region close to the horizon, while in the AdS case, we were interested in the boundary behavior where the CFT lives.

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