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Opinion

Why Is the Mean Anomaly at Epoch Not Used in Tests of Non-Newtonian Gravity?

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Abstract: The mean anomaly at epoch η is one of the standard six Keplerian orbital elements in terms of which the motion of the two-body problem is parameterized. Along with the argument of pericenter ω, η experiences long-term rates of change induced, among other things, by general relativity and several modified models of gravity. Thus, in principle, it may be fruitfully adopted together with ω in several tests of post-Newtonian gravity performed with astronomical and astrophysical binary systems. This would allow us to enhance the gravitational signature we are interested in and to disentangle some competing disturbing effects acting as sources of systematic bias. Nonetheless, for some reasons unknown to the present author, η has never been used so far by astronomers in actual data reductions. This note aims to raise interest in the community about the possible practical use of such an orbital element or, at least, to induce experts in astronomical data processing to explicitly make clear if it is not possible to use η for testing gravitational models and, if this is the case, why.

Keywords: unified astronomy thesaurus concepts; gravitation (661); general relativity (641); relativistic mechanics (1391)

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1. Testing Post-Newtonian Gravity with Orbital Motions

Since the explanation in 1915 [1] of the then anomalous perihelion precession of Mercury [2] by Einstein with his general relativity, orbital motions have been used as effective tools to scrutinize alternative theories of gravitation with respect to the dominant paradigm at the time. Such tests are often presented in terms of secular precessions of the argument of pericentre ω in two-body systems such as Earth's artificial satellites [3,4], the solar system's planets and asteroids [5–8], binary pulsars [9,10] and stars orbiting supermassive black holes [11,12]. In all such cases, a perturbative approach can be adopted in predicting the sought effects since they can be thought as consequences of relatively small additional accelerations with respect to the dominant inverse-square Newtonian monopole.

2. Using the Pericentre

As far as general relativity is concerned, the largest precession, to the first post-Newtonian (pN) level of the order of $\mathcal{O}\left(c^{-2}\right)$, where c is the speed of light in vacuum, is due to the so-called gravitoelectric part of the spacetime metric depending only on the masses $M_{\rm A}$ and $M_{\rm B}$ of the bodies A and B constituting the two-body system at hand [13,14]. It induces a perturbing acceleration $\boldsymbol{A}$ [13,14] lying in the orbital plane, characterized, in general, by both radial A_R and transverse A_T components. When plugged in, the machinery of, e.g., the Gauss equations for the variation of the orbital elements [13–18], it turns out that the semimajor axis a, the eccentricity e, the inclination I and the longitude of the ascending node Ω do not undergo net shifts when the average over one full orbital period $P_{\rm b}$ is taken, contrary to ω which, instead, exhibits a secular rate of change[1] [13,14,16–18]

$$\dot{\omega} = \frac{3\, n_{\rm b}\, \mu}{c^2\, a\left(1-e^2\right)}, \tag{1}$$

where $\mu \doteq GM$ is the product of the Newtonian constant of gravitation G times the sum of the masses of the binary's constituents $M \doteq M_{\mathrm{A}} + M_{\mathrm{B}}$, and $n_{\mathrm{b}} \doteq 2\pi/P_{\mathrm{b}} = \sqrt{\mu/a^3}$ is the Keplerian mean motion.

Long-range modified models of gravity [19–26], devised in recent decades mainly to explain certain astrophysical and cosmological features[2] such as the phenomenology of dark matter [28–30] and the accelerated rate of cosmic expansion [31,32], can be used, in principle, to perform local tests as well. Indeed, for most of them, a correction to the Newtonian gravitational potential in the g_{00} term of the spacetime metric can be explicitly worked out. In turn, this yields a perturbing acceleration which, in most cases, is entirely radial. As a result, only ω experiences a net secular advance.

3. The Impact of the Orbital Perturbations

In fact, the dynamical evolution of the pericentre of any realistic astronomical and astrophysical system of interest is not only determined by the model of gravity one is looking at, but it is also impacted by several other competing Newtonian gravitational[3] effects of different origins (multipole moments of the system's bodies, tides, pulls by bodies external to the system, etc.), which, in the present context, are viewed as sources of systematic bias. Thus, strategies to reduce the systematic errors they induce with respect to the signature of interest must be devised. If the shifts caused by the other dynamical features on the pericentre can be analytically calculated, it is possible, in principle, to disentangle them from the one searched for by using more than one pericentre precession, provided they are accessible to observations. Such an approach was proposed, for the first time, by Shapiro [8], whose goal, at that time, was to separate the Sun-induced pN gravitoelectric perihelion precession from that due to the solar quadrupole mass moment J_2 by using the perihelia of other planets or asteroids on highly eccentric orbits. For a general overview of such a strategy, not limited just to our solar system, see Iorio [33]. Clearly, for each body, it would be better to have at our disposal more than one observable parameter affected by the gravity model under consideration. In such a way, one would be less dependent on the presence or not of other bodies whose pericentres should be used as probes.

4. The Potential Benefits of the Mean Anomaly at Epoch

In fact, this is just the case for general relativity and other alternative gravitational theories. Indeed, there is another orbital parameter which is secularly displaced by the 1 pN gravitoelectric acceleration and by several modified models of gravity: the mean anomaly at epoch η. Let us recall that it is one of the standard six Keplerian orbital elements in terms of which the motion in the two-body problem is parameterized. The Gauss equation for its rate of change is [13–18]

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\frac{2}{n_{\mathrm{b}}\, a} A_R \left(\frac{r}{a}\right) - \frac{\left(1-e^2\right)}{n_{\mathrm{b}}\, a\, e}\left[-A_R \cos f + A_T\left(1+\frac{r}{p}\right)\sin f\right], \tag{2}$$

where r is the mutual distance between A and B, and $p \doteq a\left(1-e^2\right)$ is the semilatus rectum of the Keplerian ellipse. From Equation (2), it can be noted that, in principle, η is shifted with respect to the unperturbed Keplerian case if an in-plane perturbation is present, as is the case for, e.g., the 1 pN gravitoelectric acceleration [13,14] and for several long-range alternative models of gravitation. As an example, it can straightforwardly be worked out that the general relativistic Schwarzschild-like acceleration and Equation (2) yield for η the following net rate of change over one orbital revolution

$$\dot{\eta} = \frac{\mu\, n_{\mathrm{b}}\left[-15 + 6\sqrt{1-e^2} + \nu\left(9 - 7\sqrt{1-e^2}\right)\right]}{c^2\, a\sqrt{1-e^2}}, \tag{3}$$

where

$$\nu \doteq \frac{M_{\mathrm{A}}\, M_{\mathrm{B}}}{M^2}. \tag{4}$$

In general, η is impacted by the same disturbances as ω. This is an important point, since many confuse η with the mean anomaly $\mathcal{M}$ whose rate also includes the mean motion n_b which is plagued by the uncertainty with which μ is known at the time of data analysis and possible non-gravitational effects in a; see Iorio [33] for a discussion of such subtleties.

In principle, using η in conjunction with ω would yield an enhancement of the signal of interest; suffice to say that, in the case of Mercury orbiting the Sun, Equation (1) yields the time-honored 42.98 arcsec cty^{-1}, while Equation (3) provides us with -127.986 arcsec cty^{-1}. Thus, the question arises: why have the mean anomaly at epoch and its secular rate of change never been used in tests of general relativity, and, more generally, pN gravity? The present author does not know the answer, and hopes that experts in data reductions may address such a question, providing the community with a clear and unambiguous answer, even if, for some reasons, it were negative about the possible practical use of η. For example, should η be measured as a delay of the secondary's orbital motion compared to a Keplerian motion, one would need to know the orbital parameters with a sufficiently high precision. Usually, a can be measured accurately, while obtaining e with a comparable accuracy may be not so easy. Such an answer would have a great significance, especially in view of the many systems which are already used or may become adopted in the near future for testing general relativity and other modified theories of gravitation such as extrasolar planets close to their parent stars, binary pulsars and stars orbiting supermassive black holes. Even from the point of view of the history of science, it would be interesting to understand why astronomers missed the much larger general relativistic shift of η of Mercury with respect to that of ω.

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Notes

[1] Here and in the following, the angular brackets $\langle \ldots \rangle$ denoting the average over P_b will be omitted for making the notation less cumbersome.

[2] For a comprehensive overview of such themes, see, e.g., Debono and Smoot [27] and references therein.

[3] In some cases, like Solar System's asteroids and Earth's artificial satellites, non-gravitational perturbations [15] may play a non-negligible role as well.

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