universe MDPI

Review

A Matrix Model of Four-Dimensional Noncommutative Gravity

George Manolakos [1,*,†], Pantelis Manousselis [2,†], Danai Roumelioti [2,†], Stelios Stefas [2,†] and George Zoupanos [2,3,4,5,*,†]

1 Division of Theoretical Physics, Ruđer Bošković Institute, 10000 Zagreb, Croatia
2 Physics Department, National Technical University, 15780 Athens, Greece; pman@central.ntua.gr (P.M.); danai_roumelioti@mail.ntua.gr (D.R.); dstefas@mail.ntua.gr (S.S.)
3 Theory Department, CERN, 1211 Geneva, Switzerland
4 Max-Planck Institut für Physik, 80805 München, Germany
5 Institut für Theoretische Physik, Universität Heidelberg, 69117 Heidelberg, Germany
* Correspondence: georgios.manolakos@irb.hr (G.M.); george.zoupanos@cern.ch (G.Z.)
† These authors contributed equally to this work.

Abstract: In this review, we revisit our latest works regarding the description of the gravitational interaction on noncommutative spaces as matrix models. Specifically, inspired by the gauge-theoretic approach of (ordinary) gravity, we make use of the suggested methodology, modified appropriately for the noncommutative framework, of the well-established formulation of gauge theories on them. Making use of a covariant four-dimensional fuzzy space, we formulate the gauge theory with an extended gauge group due to noncommutativity. In turn, in order to decrease the amount of symmetry we employ a symmetry breaking and result with an action which describes a theory that is a minimal noncommutative extension of the original.

Keywords: gauge theories; gauge theories of gravity; four-dimensional gravity; noncommutative spaces; fuzzy de Sitter; noncommutative gravity; spontaneous symmetry breaking

check for updates

Citation: Manolakos, G.; Manousselis, P.; Roumelioti, D.; Stefas, S.; Zoupanos, G. A Matrix Model of Four-Dimensional Noncommutative Gravity. *Universe* **2022**, *8*, 215. https://doi.org/10.3390/universe8040215

Academic Editors: Athanasios Chatzistavrakidis and Dimitrios Giataganas

Received: 20 January 2022
Accepted: 22 March 2022
Published: 28 March 2022

1. Introduction

Over more than a century, the General theory of Relativity (GR)[1] has been established as the theory describing gravitational interaction, having passed several tests over the years. It is highly emphasized that, contrary to the rest of the fundamental interactions which are established in a gauge-theoretic way, GR is geometrically formulated, rendering gravity as an intrinsic property of the spacetime related to its curvature. This discrepancy on the formulations of the theories of the fundamental interactions has been a major concern of theoretical physicists for several decades; therefore, various attempts on attributing a single description of all interactions has motivated the pursuit of a gauge-theoretic description of gravity[2]. The first step to the above undertaking was taken by Utiyama, whose work [6], although not entirely successful, triggered the production of serious research around the concept of the gauge-theoretic alternative description of GR and, later, extensions of the methodology to other gravitational theories took place, as well. For instance, a gauge-theoretic approach of the four-dimensional Einstein–Hilbert action was successfully formulated as a spontaneously broken gauge theory of the isometry group of the four-dimensional de Sitter (or AdS), induced by the presence of an auxiliary scalar field [7–9]. Later, the four-dimensional Weyl gravity was recovered in a gauge-theoretic way as a gauge theory of the four-dimensional conformal group which becomes explicitly broken by the consideration of specific constraints[3] [12–17]. Moreover, later, it was shown that the three-dimensional Einstein–Hilbert action could be retrieved gauge-theoretically as a three-dimensional Chern–Simons theory, in a straightforward way, i.e., no additional breaking mechanism was necessary [18].

Part of the contemporary theoretical research on gravity is conducted on issues that are related to its behaviour in more extreme conditions, that is in the short-distance regime such that quantum effects may induce non-negligible corrections. Since straightforward quantization of GR is problematic, an alternative way, among others, to approach such a configuration is to assume that in these conditions, for example in Planck scale, it can be assumed that the spacetime is not continuous any more but, rather, discretized due to the presence of the notion of a minimal length. The coordinate systems attributed on such spaces that are governed by the above loss of continuity are such that involve coordinates which obey some kind of uncertainty principle (quantum spaces), leading to the notion of noncommutative geometry. Therefore, noncommutative spaces are highly recommended background spaces to accommodate theories of gravity in such extreme physical conditions. The above argumentation has led to the conduction of serious research within the noncommutative framework towards the direction of formulating theories of noncommutative gravity, realized in several ways, according to the way the classical symmetries (e.g., diffeomorphism invariance) are deformed. Two approaches on introducing the noncommutative spaces are the following:

A first (more mathematical) approach of noncommutative geometry is accomplished through the spectral triples [19,20] (see also [21]). The commutative spectral triples $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is the algebra of the smooth functions on the manifold, $\mathcal{H}$ is the Hilbert space of square integrable spinors on the manifold and D is the Dirac operator of the Levi–Civita connection, provide a correspondence between Riemannian manifolds and abstract algebras. This implies the derivative correspondence of points on the underlying topological space to states on the algebra. If the algebra is taken to be noncommutative (noncommutative spectral triple), then it corresponds to a noncommutative version of a Riemannian manifold.

A second (more physical) approach is along the lines of the quantization of the phase space, in which the canonical position and momentum are replaced by Hermitian operators and, thus, cease to commute (Heisenberg relation). If the coordinates are considered to be parametrized by Hermitian operators, then they will also be noncommutative satisfying the Heisenberg-like commutation relation $[\hat{x}_i, \hat{x}_j] = i\theta_{ij}(x)$, where $\theta_{ij}(x)$ is an antisymmetric tensor parametrizing the noncommutativity of the corresponding space. Having said the above, now there are two paths one can follow in order to work on the noncommutative spaces. The first one is to use the Weyl correspondence, that is an one to one mapping from operators (that do not commute) to regular functions (that commute under the ordinary product) equipped with a deformed product, that is called $\star$-product. This consideration implies that the noncommutative nature of the space will be concentrated exclusively in this new product, that is $[x_i \stackrel{\star}{,} x_j] = i\theta_{ij}(x)$, where x_i, x_j are functions[4]. The second path relies on the matrix representation of the operators, familiar from quantum mechanics. In this realization, the notion of noncommutativity is intrinsic, since the matrix multiplication is noncommutative by definition (matrix geometry [31]). In this case, the defining noncommutative relation becomes $[X_i, X_j] = i\theta_{ij}(X)$, where X_i, X_j are Hermitian matrices. In both pictures, derivation and integration are well-defined operations, therefore noncommutative field theories can be realized (for reviews see [32–34]). Specifically for noncommutative gravitational theories (in both pictures), there are two lines of reasoning one can follow: the first is to focus on the diffeomorphism invariance and try to translate it to the noncommutative framework and the second one is to make use of the gauge-theoretic picture of gravity we mentioned earlier, namely to construct gravitational theories as noncommutative gauge theories, which are well-established in the noncommutative framework [35–37].

Various works have been carried out making use of the above approaches and directions. First, within the spectral triple framework, the main contributions lie in the direction in which the gravitational interaction is coupled to the Standard Model, admitting a purely gravitational noncommutative description [38–45]. Second, within the approach of a quantum spacetime, using the $\star$-product formulation and the Seiberg–Witten map [46], some characterizing works on formulation of noncommutative gravity using the diffeomorphism and/or the gauge-theoretic approach of GR are the following [47–58], while various theories of noncommutative gravity have been constructed using the matrix realization, to name a few: [59–66] where noncommutativity emerges through string theory, [67–70] where, in some cases, quantization of the spacetime can lead to the emergence of noncommutative gravity, [71,72] where the frame formalism gives rise to noncommutative cosmological insights, as well. As far as our interests and contributions is concerned, we focus on the construction of gravitational theories on noncommutative spaces using the gauge-theoretic approach within the matrix formulation of noncommutativity [73–75] (see also [76,77]).

Before we move on to the description of the works we are reviewing, for completion, it is worth noting that there exist some recent works that focus on an alternative construction of noncommutative field theories using the concept of the L_∞ algebras [78,79]. Indicatively, in [80] it is described that noncommutative gauge theories should admit an underlying L_∞ algebra and noncommutative theories of Chern–Simons and Yang–Mills are constructed this way. In [81], the noncommutative deformations are implemented by the quantization technique of Drinfel'd twist leading to braided L_∞ algebras[5]. In this work a braided version of the Einstein–Cartan–Palatini gravitational theory is formulated. Moreover, in [83], braided L_∞ algebras are produced making use of the Drinfel'd twist of the graded Hopf algebra, which, in presence of a compatible codifferential, consists an alternative form of the underlying L_∞ algebra.

Let us now give a brief description of our works which we review. The concept of our approach is the translation of the gauge-theoretic view of gravity to the noncommutative framework, working in its matrix realization. First, we focused on the three-dimensional case [73], in which a noncommutative gravity model was constructed as the gauge theory of $U(2) \times U(2)$ ($GL(2, \mathbb{C})$ in the Lorentzian case) on the covariant noncommutative space $\mathbb{R}^3_\lambda$ [84–87] ($\mathbb{R}^{1,2}_\lambda$ [88]), which is the foliation of the three-dimensional Euclidean space by adjacent fuzzy spheres [89] (foliation of the three-dimensional Minkowski space by adjacent hyperboloids [90]). Then, in the more complicated four-dimensional case, we worked on constructing a covariant four-dimensional noncommutative space which would accommodate the gravitational theory [74,75]. Motivated by works on four-dimensional covariant noncommutative spaces in the literature [71,72,91–98], we moved on with a construction of a fuzzy version of the dS_4 space and then we constructed a gravitational model on it using the toolbox of noncommutative gauge theories, obtaining, among other results, the transformations of the various gauge fields, the expressions of the component tensors of the field strength and the field equations by starting with a topological action and performing a spontaneous symmetry breaking with the inclusion of an auxiliary scalar field. Last, the commutative limit of the theory was the Einstein–Hilbert theory with positive cosmological constant.

The outlook of the current review is the following. First, we recall the various gauge-theoretic approaches of gravity theories (Section 2). Next, we write down the necessary information regarding the construction of gauge theories on noncommutative spaces (Section 3). In turn, we present our gravitational model as a gauge theory on a covariant noncommutative space, i.e., the fuzzy four-sphere (Section 4). Last, we write down our conclusions and future plans (Section 5).

2. Gauge-Theoretic Approach of Gravity Theories

Here, we recall the gauge-theoretic approach of gravity, that is an alternative description to the geometric one, since the employed principles and methodology are important for our purposes. We present this reminder in a retrospective approach attempting to follow the timeline of the developments in the field along the lines of [4] (see also [5]).

The gauge principle has been developed into a fundamental element of the description of the physical world over almost a whole century. In the early days, the concept of the gauge principle was applied on internal symmetries, first by Weyl (1929), with the consideration of gauge invariance under local $U(1)$ transformations of the Dirac Lagrangian augmented by two terms, one involving the related conserved current coupled to the corresponding gauge field encoding the interactions and another one, the kinetic term of the gauge field rendering it as dynamical. Later, in the mid 1950s, Yang and Mills used a similar construction for the local invariance under the $SU(2)$ isospin rotation transformations, which was the first extension of the above to a non-Abelian group [99]. The decisive step towards the translation of the gauge-theoretic technique to spacetime symmetries was made by Utiyama, who extended the Yang–Mills theory to semi-simple groups, thus the localization of the Lorentz group could be realized, a step compatible with the local nature of GR [6].

The tetradic first order formulation of GR had already consisted a suggestive setup pointing to the gauge-theoretic description of GR as a gauge theory of the Lorentz group with the spin connection as the corresponding gauge field which would enter the action through the corresponding field strength (curvature), although in a non-quadratic form. In short, Palatini's first order action principle is an alternative to the Einstein–Hilbert action, but still of the same form $(g^{\mu\nu}R_{\mu\nu}(\Gamma))$, in which the metric and affine connection (not the Levi–Civita one) are treated as independent fields. The translation of the latter in the tangent space language, that is in terms of the frame fields and spin connection, consists the tetradic first order formulation of GR. It is remarkable that the torsionless condition considered in GR emerges now as an equation of motion of the spin connection. Moreover, this translation allows the coupling of gravity to fermions in a natural way, leading to the same results as the Einstein–Cartan (EC) gravity theory (torsionful connection) which is a (viable) generalization of GR in which, due to the reparametrization invariance, energy-momentum tensor sources the curvature, while, due to the local Lorentz invariance on the tangent space, spin-energy potential sources the torsion.

The separation of the two above-mentioned symmetries through the EC theory along with the hints of a gauge theory of the Lorentz group from the tetradic first order formulation of GR and the advances on the gauge principle a la Weyl and Yang–Mills, allowed Utiyama to result with an $SO(1,3)$ gauge theory formulation of GR. Despite the breakthrough, the construction was imperfect as it could not explain the ad hoc introduction of the vierbein, the Riemannian nature of the connection and, last, the involvement of the angular momentum current instead of the energy-momentum one, as it holds for gravity and is confirmed by GR. In the early 1960s, Sciama carried the construction one step further by localizing again the Lorentz group on a Riemannian manifold background and resulted to a spacetime with torsion (Riemann–Cartan) [7]. A serious development on the above setup was made at the same time by Kibble, who chose to gauge the Poincaré group on a Minkowski background, maybe a more intuitive choice as it is the manifold in absence of gravity [8]. In other words, he constructed a gauge theory on a special (and not general—such as Sciama—[6]) relativistic background using as a gauge group the corresponding isometry group with the corresponding algebra expressed by the following commutation relations:

$$[M_{ab}, M_{cd}] = 4\eta_{[a[c}M_{d]b]}\;, \quad [P_a, M_{bc}] = 2\eta_{a[b}P_{c]}\;, \quad [P_a, P_b] = 0\;, \tag{1}$$

where $\eta_{ab} = diag(-1,1,1,1)$ is the Minkowski metric, M_{ab} and P_a the Lorentz and translational generators, respectively. The above consideration renders both the vierbein and the spin connection as gauge fields composing the total gauge connection, but also the corresponding field strength components are identified as the torsion and curvature tensors, respectively. In particular, their expressions are:

$$T_{\mu\nu}{}^{a}(e,\omega) = 2\partial_{[\mu}e_{\nu]}{}^{a} - 2\omega_{[\mu}{}^{ab}e_{\nu]b}\;, \quad R_{\mu\nu}{}^{ab}(\omega) = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^{b}\;. \tag{2}$$

The action proposed was once again of Einstein-Hilbert type, a consideration that is not completely satisfactory since it is considered rather ad hoc. The result was once again a Riemann–Cartan spacetime with the presence of torsion. A nice amendment is that energy-momentum tensor enters now in a natural way through the translational group. The contribution of Sciama and Kibble enhance the Einstein–Cartan theory and that is why the theory is called Einstein–Cartan–Sciama–Kibble.

An improvement regarding the origin of the action was proposed by MacDowell and Mansouri in the late 1970s [14], who exchanged the Poincaré group[7] with the $(A)dS_4$, which has the same amount of generators as the Poincaré group but with the property that it is semi-simple, meaning that generators are treated on equal footing[8]. Effectively, this means that the construction of a Yang–Mills-like quadratic action is now an option. The generators satisfy the commutation relation:

$$[M_{AB}, M_{CD}] = 4\eta_{[A[C}M_{D]B]}\,,$$

where η_{AB} is the mostly positive 5-dim Minkowski metric. The Poincaré group can be viewed as a group contraction of $SO(1,4)$ as the radius, a, goes to infinity (Wigner-Inönü contraction). Considering the change (rescaling) $M_{ab} \to M_{ab}$ and $M_{a5} = -aP_a$ on the dS_4 generators, then the algebra becomes:

$$[M_{ab}, M_{cd}] = 4\eta_{[a[c}M_{d]b]}\,, \quad [M_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b\,, \quad [P_a, P_b] = \frac{1}{a^2}M_{ab}\,.$$

The contraction limit $a \to \infty$, meaning that the translations are much smaller compared to the radius, trivializes the commutator of the generators of the translations and produces the (non-isomorphic) Poincaré algebra of Equation (1). Moreover, the de Sitter space is a maximally symmetric solution of the Einstein field equations in vacuum, thus the radius of the space is related to the cosmological constant as $\Lambda = \frac{3}{a^2}$. Therefore, the contraction limit can be viewed from a more physical perspective, as $\Lambda \to 0$, which means that the spacetime now is the flat Minkowski. The above splitting on the set of the generators induces a splitting on the gauge field to two components identified as the vierbein and the spin connection[9]. Accordingly, the same applies for the field strength tensor and writing it in the Poincaré language of Equation (2), one results:

$$\hat{R}_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} + \frac{2}{a^2}e_\mu{}^{[a}e_\nu{}^{b]}\,, \quad \hat{R}_{\mu\nu}{}^{a} = \frac{1}{a}T_{\mu\nu}{}^{a}\,. \tag{3}$$

As stated above, the main virtue of this construction is that it admits a quadratic action. The invariance under infinitesimal diffeomorphisms suggests the use of a top form including the curvature two-form. Taking into consideration that the only available invariants are the metric and the totally antisymmetric tensor, in order to construct a valid (parity preserving) gauge invariant action, the only candidate is $\hat{R}^{ab} \wedge \hat{R}^{cd}\epsilon_{abcd}$. It is evident that the above term is not invariant under the total $SO(1,4)$ gauge group but only under the Lorentz subgroup. In order to recover a totally invariant combination (only formally but still satisfactory), Mansouri and MacDowell introduced a constant vector, which explicitly breaks the $SO(1,4)$ gauge symmetry to the $SO(1,3)$. The form of the resulting action after the breaking is:

$$S = \alpha \int d^4x \epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\left(\mathcal{L}_{RR} + \frac{1}{a^2}\mathcal{L}_{eeR} + \frac{1}{a^4}\mathcal{L}_{eeee}\right)\,. \tag{4}$$

The first term is topological (Gauss–Bonnet, see [100]), the second term is the Palatini action[10] and the third one is the cosmological constant term.

A very insightful contribution on the breaking of the initial $SO(1,4)$-invariant action was given by Stelle and West who considered the following [9]:

$$S_{\mathrm{SO}(1,4)} = \alpha \int d^4x \left(\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu}{}^{AB}R_{\rho\sigma}{}^{CD}\epsilon_{ABCDE}\frac{\phi^E}{a} + \lambda(\phi^A\phi_A - a^2)\right)\,, \tag{5}$$

where λ is a Lagrange multiplier and ϕ_A transforms trivially under coordinate transformations but as a vector under $SO(1,4)$. The symmetry of the above action breaks spontaneously due to the constraint of the ϕ_A field. Considering the field to be frozen in the fifth direction $\phi^A = a\delta^A_4$, or, in other words, picking the specific gauge $\phi^a(x) = 0$ and $\phi^4(x) = a$, the action functional reduces to the MacDowell–Mansouri one.

Last, it is meaningful, for reasons of completion, to mention the gauge-theoretic approach of another theory of gravity, namely the Teleparallel Equivalent to GR (TEGR), that is a gauge theory of the group of translations constructed on a Weitzenböck spacetime (curvature-less connection) [101]. The construction manifests many important features but presents a problem when it comes to the coupling of the theory to spinors. Additionally, the gauge principle can be applied to retrieve other gravitational theories as well, for instance, one can derive the four-dimensional Weyl gravitational theory considering the four-dimensional conformal group as the initial gauge group on Minkowski spacetime and perform an explicit symmetry breaking on the gauge invariant action (of Yang-Mills type) using specific constraints to result with the action of Weyl gravity [12–17].

3. Gauge Theories and Noncommutativity

Let us now recall the necessary information of the noncommutative version of the gauge-theoretic formulation, as it is contained in [35][11], since it will consist the main ingredient for the construction of our model.

Let us start with a scalar field, $\phi(X)$, where X represents the coordinates of the noncommutative space and suppose that it transforms non-trivially under the gauge transformation produced by a gauge group. For a gauge parameter $\epsilon(X)$, an infinitesimal transformation of the above scalar field will be:

$$\delta\phi(X) = \epsilon(X)\phi(X)\,.$$

Taking into consideration that the coordinates transform trivially under the gauge transformation, the $X_\mu\phi(X)$ quantity transforms as:

$$\delta(X_\mu\phi(X)) = X_\mu\epsilon(X)\phi(X)\,.$$

By inspection, it is straightforward that the above transformation is not covariant, since noncommutativity of the coordinates implies $[X_\mu, \epsilon(X)] \neq 0$, which, in turn, forbids the above transformation to be written covariantly as $\delta(X_\mu\phi) = \epsilon(X)(X_\mu\phi(X))$.

Direct analogy to ordinary gauge theories in which covariant derivative is introduced for similar reasons, in this case too, the covariant coordinate is defined in order that the above quantity obtains a covariant transformation rule:

$$\delta(\mathcal{X}_\mu\phi(X)) \equiv \epsilon(X)\mathcal{X}_\mu\phi(X)\,,$$

which holds if the transformation property of the covariant coordinate is $\delta\mathcal{X}_\mu = [\epsilon(X), \mathcal{X}_\mu]$, that is covariant by definition. The above configuration is achieved by the introduction of a field $A_\mu(X)$ which transforms as:

$$\delta A_\mu(X) = -[X_\mu, \epsilon(X)] + [\epsilon(X), A_\mu(X)]\,.$$

Therefore, the coordinate used in order to build a consistent gauge theory is the covariant one, given as: $\mathcal{X}_\mu = X_\mu + A_\mu(X)$, with the interpretation of the A_μ field as gauge field becoming evident. Furthermore, the introduction of the gauge field implies the definition of the corresponding field strength tensor. In this noncommutative picture, the definition of the field strength tensor is not given only by the commutator of the covariant coordinates, but its definition is upgraded with an extra term which is introduced by necessity to render the transformation as covariant.

A subtle issue in the formulation of non-Abelian gauge theories on noncommutative spaces is the proper treatment of the anticommutators of the various operators. Let us consider the generic situation of the commutator of two elements which belong to an arbitrary gauge algebra with generators T^a, namely $\epsilon(X) = \epsilon^a(X)T_a$ and $\phi(X) = \phi^a(X)T_a$:

$$[\epsilon, \phi] = \frac{1}{2}\{\epsilon^a, \phi^b\}[T_a, T_b] + \frac{1}{2}[\epsilon^a, \phi^b]\{T_a, T_b\}\,.$$

In the ordinary gauge theories, the last term is vanishing as the commutator of $[\epsilon^a, \phi^b]$ contains regular (commutative) functions, therefore the outcomes of the anticommutator of the two generators are completely irrelevant. On the contrary, in the noncommutative case, the commutator is not vanishing since it contains functions which depend on the coordinates which, by definition, do not commute. Therefore, the products of the anticommutator of the generators now matter. In principle, the products of such an anticommutator are operators which do not belong to the considered algebra, i.e., they do not close, but also they are representation-dependent. This is an unwelcome feature and needs to be addressed. One possible scenario is to take into consideration all possible operators coming from the anticommutator and consider them as generators. This leads to the extension of the initial algebra to an infinite-dimensional one (universal enveloping algebra), which, although welcome in other contexts (e.g., in [37,52,54]), it is not optimal for our purposes. Another scenario, which we adopt, is to consider the generators of a specific representation which means that the anticommutators would produce specific and a finite number of operators, therefore, one can incorporate them into the starting algebra and work with the extended, but finite, one.

4. Noncommutative Gravity in Four Dimensions: A Matrix Model

In this section we present our gauge-theoretic construction of the 4-dimensional matrix model of noncommutative gravity. As understood from Section 2, in order to build a (noncommutative) gauge theory, a background (noncommutative) space is required to accommodate it. Therefore, first we focus on the construction of a four-dimensional covariant noncommutative space, which plays the role of the background space and then we present the gravity model, constructed as a noncommutative gauge theory on the above space.

4.1. A Fuzzy Version of the Four-Sphere

The most typical noncommutative space is a fuzzy sphere, S_F^2, which is the discrete matrix approximation of the ordinary, continuous, sphere, S^2, with the property of preserving its isometries [89]. Therefore, its isometry group is the $SO(3)$, which is generated by the three angular momentum operators. The eigenfunctions of the corresponding Laplace operator are the well-known spherical harmonics and by replacing them with another, finite, set of functions, one ends up with a truncated algebra, which does not close under multiplication. In order to recover closure, the common product of those truncated functions may be upgraded to a noncommutative one, namely the matrix product. The above is a way to introduce the fuzzy sphere as a matrix approximation of the ordinary one, with the coordinates defined as dimensionally appropriate rescalings of the $SO(3)$ generators in a high representation.

In the above-mentioned case the construction was straightforward, but that is not the case when a similar approach is applied in higher-dimensional spaces, in the sense that covariance is not automatically satisfied. More specifically, according to the arguments used in the fuzzy sphere case, attempt of the construction of the fuzzy four-sphere, S_F^4, on the same principle, would suggest to consider the $SO(5)$ group, since it consists of the corresponding isometry group, and identify the coordinates with a subset of the generators. Nevertheless, the subalgebra is not closing, and, therefore, covariance is not satisfied [93]. The requirement of the preservation of covariance leads to utilizing a group with larger symmetry, in which it will be possible to incorporate all generators and the noncommutativity in it, with an appropriate identification, and result with a construction in which the coordinates will transform as vectors under the action of the rotational transformations. Extending minimally the symmetry leads to the adoption of the $SO(6)$ group [74,75].

Let us now consider the 15 generators of $SO(6)$, $J_{A,B}$, with $A, B = 1, ..., 6$, which obey the following algebra:

$$[J_{AB}, J_{CD}] = i(\delta_{AC}J_{BD} + \delta_{BD}J_{AC} - \delta_{BC}J_{AD} - \delta_{AD}J_{BC}). \tag{6}$$

Now, let us decompose the above generators in an $SO(4)$ notation and redefine generators as:

$$J_{\mu\nu} = \frac{1}{\hbar}\Theta_{\mu\nu}, \quad J_{\mu 5} = \frac{1}{\lambda}X_\mu, \quad J_{\mu 6} = \frac{\lambda}{2\hbar}P_\mu, \quad J_{56} = \frac{1}{2}\,\mathrm{h}, \tag{7}$$

where $\mu, \nu = 1, ..., 4$. The parameter λ has been introduced for dimensional reasons, h is an operator related with the radius constraint, and $X_\mu, P_\mu, \Theta_{\mu\nu}$ are identified as the coordinates, momenta[12] and noncommutativity tensor, respectively [75]. Coordinates and momenta satisfy the following commutation relations:

$$\begin{gathered}
[X_\mu, X_\nu] = i\frac{\lambda^2}{\hbar}\Theta_{\mu\nu}, \quad [P_\mu, P_\nu] = 4i\frac{\hbar}{\lambda^2}\Theta_{\mu\nu}\,, \\
[X_\mu, P_\nu] = i\hbar\delta_{\mu\nu}\,\mathrm{h}, \quad [X_\mu, \mathrm{h}] = i\frac{\lambda^2}{\hbar}P_\mu\,, \\
[P_\mu, \mathrm{h}] = 4i\frac{\hbar}{\lambda^2}X_\mu,
\end{gathered} \tag{8}$$

where the first two imply that the commutator of coordinates and the commutator of momenta separately close into the $SO(4)$ subalgebra of the total $SO(6)$ symmetry group. The commutation relation of the coordinates and momenta manifests the quantum nature of the noncommutative space. The algebra of spacetime transformations is:

$$\begin{aligned}
[\Theta_{\mu\nu}, \Theta_{\rho\sigma}] &= i\hbar\left(\delta_{\mu\rho}\Theta_{\nu\sigma} + \delta_{\nu\sigma}\Theta_{\mu\rho} - \delta_{\nu\rho}\Theta_{\mu\sigma} - \delta_{\mu\sigma}\Theta_{\nu\rho}\right), \\
[X_\mu, \Theta_{\nu\rho}] &= i\hbar\left(\delta_{\mu\rho}X_\nu - \delta_{\mu\nu}X_\rho\right), \\
[P_\mu, \Theta_{\nu\rho}] &= i\hbar\left(\delta_{\mu\rho}P_\nu - \delta_{\mu\nu}P_\rho\right), \\
[\mathrm{h}, \Theta_{\mu\nu}] &= 0\,.
\end{aligned} \tag{9}$$

The first relation is actually the defining commutation relation of the $SO(4)$ subalgebra, that is the four-dimensional rotational symmetry of the space. The second relation is important since it manifests that the coordinates transform as vectors under rotations in a covariant way (and the momenta as well). The main point to note is that the above algebra admits finite-dimensional representations for X_μ, P_μ and $\Theta_{\mu\nu}$, thus the constructed noncommutative spacetime is actually a finite quantum system[13]. This is the space that is employed for the construction of the four-dimensional gravity model as a noncommutative gauge theory.

4.2. Four-Dimensional Gravity Matrix Model

In the following, we present the formulation of gravity as a gauge theory on the noncommutative space that was constructed above. The procedure that is going to be followed is based on that of Section 2, albeit now we will be using the framework of noncommutative gauge theories, which was presented in Section 3. Drawing lessons from the commutative case (Section 2), in which the isometry group of the underlying space (Poincaré group) was gauged in order to reach the appropriate results, in this case as well, the isometry group of the above, covariant space is gauged, namely the $SO(5)$ group.

4.2.1. Gauge Group and Representation

As it has been noted above, the use of the anticommutators of the generators of the algebra is inevitable in the noncommutative framework. For reasons already presented in Section 3, when dealing with anticommutators of generators of arbitrary representation of an algebra, one does not necessarily end up with generators of the algebra, and this is exactly the case regarding the generators of $SO(5)$. This fact poses a problem, which, in order to be bypassed, one has to pick a specific representation to which the generators of $SO(5)$ belong and then include the operators that the anticommutators produce into the algebra, regarding them as generators of the algebra. That way, one results with an extended gauge algebra; therefore, the corresponding gauge symmetry will be enhanced as compared to the initial one. In our specific case, the above procedure will lead to the extension of the $SO(5)$ to the $SO(6) \times U(1)$ gauge group, whose generators will be represented by 4×4 matrices, namely:

$$M_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b]\,,\ K_a = \frac{1}{2}\Gamma_a\,,\ P_a = -\frac{i}{2}\Gamma_a\Gamma_5\,,\ D = -\frac{1}{2}\Gamma_5\,,\ \mathbf{I}_4\,. \tag{10}$$

In the above expressions of the generators we have used the well-known 4×4 gamma matrices (in the Euclidean signature), which satisfy the relation $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}\mathbf{I}_4$, where $a, b = 1, \ldots, 4$, as well as the matrix $\Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4$.

The algebra and anticommutation relations the above generators satisfy are:

$$\begin{aligned}
&[K_a, K_b] = iM_{ab}, \quad [P_a, P_b] = iM_{ab},\\
&[P_a, D] = iK_a, \quad [K_a, P_b] = i\delta_{ab}D, \quad [K_a, D] = -iP_a,\\
&[K_a, M_{bc}] = i(\delta_{ac}K_b - \delta_{ab}K_c),\\
&[P_a, M_{bc}] = i(\delta_{ac}P_b - \delta_{ab}P_c),\\
&[M_{ab}, M_{cd}] = i(\delta_{ac}M_{bd} + \delta_{bd}M_{ac} - \delta_{bc}M_{ad} - \delta_{ad}M_{bc}),\\
&[D, M_{ab}] = 0,\\
&\{M_{ab}, M_{cd}\} = \frac{1}{8}(\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad})\mathbf{I}_4 - \frac{\sqrt{2}}{4}\epsilon_{abcd}D,\\
&\{M_{ab}, K_c\} = \sqrt{2}\epsilon_{abcd}P_d, \quad \{M_{ab}, P_c\} = -\frac{\sqrt{2}}{4}\epsilon_{abcd}K_d,\\
&\{K_a, K_b\} = \frac{1}{2}\delta_{ab}\mathbf{I}_4, \quad \{P_a, P_b\} = \frac{1}{8}\delta_{ab}\mathbf{I}_4, \quad \{K_a, D\} = \{P_a, D\} = 0,\\
&\{P_a, K_b\} = \{M_{ab}, D\} = -\frac{\sqrt{2}}{2}\epsilon_{abcd}M_{cd}\,.
\end{aligned} \tag{11}$$

4.2.2. Action and Equations of Motion

Following the above, we now try to find the appropriate action for the noncommutative $SO(6) \times U(1)$ gauge theory of gravity. Given the relation that defines the noncommutativity of the background space (8), we first consider the following (topological) action:

$$\mathcal{S} = \mathrm{Tr}\Big([X_\mu, X_\nu] - \kappa^2\Theta_{\mu\nu}\Big)\Big([X_\rho, X_\sigma] - \kappa^2\Theta_{\rho\sigma}\Big)\epsilon^{\mu\nu\rho\sigma}. \tag{12}$$

The above action is considered to have X and Θ as independent fields, while variation with respect to those fields gives the corresponding field equations, respectively:

$$\epsilon^{\mu\nu\rho\sigma}\Big[X_\nu, [X_\rho, X_\sigma] - \kappa^2\Theta_{\rho\sigma}\Big] = 0\,,\ \epsilon^{\mu\nu\rho\sigma}\Big([X_\rho, X_\sigma] - \kappa^2\Theta_{\rho\sigma}\Big) = 0\,. \tag{13}$$

From the above results, the second relation recovers the noncommutative nature and relation of the background space when $\kappa^2 = \frac{i\lambda^2}{\hbar}$ and, in turn, the first one is trivially satisfied. Having in mind the relation between the X and Θ from the construction of the space, (8), we could have started from the very same action considered above, with the only difference of assuming that X and Θ are not independent, i.e., that $\Theta = \Theta(X)$. In this case varying the action with respect to the only independent field, would lead us to the first of the above field equations, which is once more satisfied by the considered fuzzy space.

The next step is to introduce dynamics in the above action and write it in a form which will include the gauge fields of the theory. The way we will go about is by expressing the action in terms of the curvature field strength tensor, since this will allow us to compare the theory with its commutative counterpart and gain intuition in the process. In order to do that, we introduce the gauge fields in the action (12), by considering them as fluctuations of X and Θ, and the action will be written as:

$$\mathcal{S} = \mathrm{Tr}\,\mathrm{tr}\,\epsilon^{\mu\nu\rho\sigma}\Big(\big[X_\mu + A_\mu, X_\nu + A_\nu\big] - \kappa^2\big(\Theta_{\mu\nu} + \mathcal{B}_{\mu\nu}\big)\Big) \cdot \Big(\big[X_\rho + A_\rho, X_\sigma + A_\sigma\big] - \kappa^2\big(\Theta_{\rho\sigma} + \mathcal{B}_{\rho\sigma}\big)\Big), \quad (14)$$

where we have also included a trace over the gauge algebra. Now, if we define:

- $\mathcal{X}_\mu = X_\mu + A_\mu$, the covariant coordinate of the noncommutative gauge theory, where A_μ is the gauge connection and is decomposed on the various generators as:

$$A_\mu(X) = e_\mu{}^a \otimes P_a + \omega_\mu{}^{ab} \otimes M_{ab} + b_\mu{}^a \otimes K_a + \tilde{a}_\mu \otimes D + a_\mu \otimes \mathbf{I}_4\,,$$

  where one gauge field has been attached to each generator;
- $\hat{\Theta}_{\mu\nu} = \Theta_{\mu\nu} + \mathcal{B}_{\mu\nu}$, the covariant noncommutative tensor, with $\mathcal{B}_{\mu\nu}$ the two-form field;
- $\mathcal{R}_{\mu\nu} = \big[\mathcal{X}_\mu, \mathcal{X}_\nu\big] - \kappa^2\hat{\Theta}_{\mu\nu}$, the field strength tensor of the theory,

and replace $\kappa^2 = \frac{i\lambda^2}{\hbar}$ in the above action, we result with:

$$\mathcal{S} = \mathrm{Trtr}\bigg(\big[\mathcal{X}_\mu, \mathcal{X}_\nu\big] - \frac{i\lambda^2}{\hbar}\hat{\Theta}_{\mu\nu}\bigg)\bigg(\big[\mathcal{X}_\rho, \mathcal{X}_\sigma\big] - \frac{i\lambda^2}{\hbar}\hat{\Theta}_{\rho\sigma}\bigg)\epsilon^{\mu\nu\rho\sigma} := \mathrm{Trtr}\,\mathcal{R}_{\mu\nu}\mathcal{R}_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\,. \quad (15)$$

This action is the noncommutative analogue of the four-dimensional Chern–Simons term, and it is the one in which we will introduce the scalar field later on, in order to induce a spontaneous symmetry breaking. Varying the above action with respect to $\mathcal{X}$ and $\mathcal{B}$ will yield the following field equations:

$$\epsilon^{\mu\nu\rho\sigma}\mathcal{R}_{\rho\sigma} = 0\,,\ \epsilon^{\mu\nu\rho\sigma}\big[\mathcal{X}_\nu, \mathcal{R}_{\rho\sigma}\big] = 0\,. \quad (16)$$

The first of these two field equations is just the vanishing of the curvature field strength tensor, while the second can be viewed as the noncommutative counterpart of the Bianchi identity.

For reasons that will become apparent later on (when comparing to the commutative case), we now write the explicit decomposition of the curvature field strength tensor on the generators of the gauge algebra:

$$\mathcal{R}_{\mu\nu}(X) = \tilde{R}_{\mu\nu}{}^a \otimes P_a + R_{\mu\nu}{}^{ab} \otimes M_{ab} + R_{\mu\nu}{}^a \otimes K_a + \tilde{R}_{\mu\nu} \otimes D + R_{\mu\nu} \otimes \mathbf{I}_4\,. \quad (17)$$

Here, we write down the explicit expressions only for the component tensors $R_{\mu\nu}^{ab}$ and $\tilde{R}_{\mu\nu}$ which will be useful later:

$$\begin{aligned} R_{\mu\nu}{}^{ab} = {} & \Big[X_\mu + a_\mu, \omega_\nu{}^{ab}\Big] - \Big[X_\nu + a_\nu, \omega_\mu{}^{ab}\Big] + \frac{i}{2}\Big\{b_\mu{}^a, b_\nu{}^b\Big\} \\ & + \frac{\sqrt{2}}{4}\Big(\Big[b_\mu{}^c, e_\nu{}^d\Big] - \Big[b_\nu{}^c, e_\mu{}^d\Big]\Big)\epsilon_{abcd} - \frac{\sqrt{2}}{4}\Big(\Big[\tilde{a}_\mu, \omega_\nu{}^{cd}\Big] - \Big[\tilde{a}_\nu, \omega_\mu{}^{cd}\Big]\Big)\epsilon_{abcd} \\ & + 2i\Big\{\omega_\mu{}^{ac}, \omega_\nu{}^b\Big\} + \frac{i}{2}\Big\{e_\mu{}^a, e_\nu{}^b\Big\} - \frac{i\lambda^2}{\hbar}B_{\mu\nu}{}^{ab}\,, \end{aligned} \quad (18)$$

$$\begin{aligned} \tilde{R}_{\mu\nu} = {} & \big[X_\mu + a_\mu, \tilde{a}_\nu\big] - \big[X_\nu + a_\nu, \tilde{a}_\mu\big] + \frac{i}{2}\big\{b_{\mu a}, e_\nu{}^a\big\} - \frac{i}{2}\big\{b_{\nu a}, e_\mu{}^a\big\} \\ & - \frac{\sqrt{2}}{8}\epsilon_{abcd}\Big[\omega_\mu{}^{ab}, \omega_\nu{}^{cd}\Big] - \frac{i\lambda^2}{\hbar}\tilde{B}_{\mu\nu}\,. \end{aligned} \quad (19)$$

For the expressions of the rest see [74,75].

4.2.3. Spontaneous Symmetry Breaking of the Noncommutative Action

We now proceed with the induction of a spontaneous symmetry breaking of the action (15). In order to achieve that, we need to modify the aforementioned action, by introducing a scalar field Φ along with a dimensionful parameter λ, which plays the role of the length scale of our theory. Following the above modification, the action is written as[14]:

$$\mathcal{S} = \mathrm{Trtr}_G\, \lambda\Phi(X)\mathcal{R}_{\mu\nu}\mathcal{R}_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} + \eta\Big(\Phi(X)^2 - \lambda^{-2}\mathbf{I}_N \otimes \mathbf{I}_4\Big), \tag{20}$$

where η is a Lagrange multiplier. It is evident, that the above expression will return action (15) when considered on-shell, i.e., when the following constraint equation holds:

$$\Phi^2(X) = \lambda^{-2}\mathbf{I}_N \otimes \mathbf{I}_4.$$

At this point, it is noted that the dimension of η is $[M^{-2}]$, while variation of the above action with respect to it will yield the aforementioned constraint equation, as a field equation. Now, let us consider that the scalar field consists only of the symmetric part of the decomposition on the generators, that is:

$$\Phi(X) = \tilde{\phi}^a(X) \otimes P_a + \phi^a(X) \otimes K_a + \phi(X) \otimes \mathbf{I}_4 + \tilde{\phi}(X) \otimes D\,,$$

where the antisymmetric part, M_{ab}, is absent. The final step towards the spontaneous symmetry breaking of the action is to gauge fix the scalar field. We choose to gauge fix Φ in the direction of the generator D, and more specifically at the value $\tilde{\phi}(X) = -2\lambda^{-1}$. The gauge fixed expression of the scalar field will then take the following explicit form:

$$\Phi(X) = \tilde{\phi}(X) \otimes D|_{\tilde{\phi}=-2\lambda^{-1}} = -2\lambda^{-1}\mathbf{I}_N \otimes D.$$

Using the known anticommutation relations of the algebra generators, we proceed by calculating the traces over the algebra in action (20) and after substituting the gauge fixed scalar field, the surviving terms will comprise the form of the spontaneously broken action:

$$\mathcal{S}_{\mathrm{br}} = \mathrm{Tr}\left(\frac{\sqrt{2}}{4}\epsilon_{abcd}R_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{cd} - 4R_{\mu\nu}\tilde{R}_{\rho\sigma}\right)\epsilon^{\mu\nu\rho\sigma}. \tag{21}$$

Having kept only the symmetric part of the decomposition (symmetric tensor) of the scalar field under $SO(6)$ (uncharged under the initial $U(1)$) means that the remaining symmetry out of the $SO(6) \times U(1)$ initial one is $SO(4) \times U(1)$[15]. This means that, nine out of the total 16 generators break and those are the P_a, which implies the torsionless condition, $\tilde{R}_{\mu\nu}{}^a = 0$, leading to a relation between ω and the independent fields, the K_a ones, that is $R_{\mu\nu}{}^a = 0$, which results to a proportionality relation between the e, b fields and the D one, which is expressed by the gauge fixing of $\tilde{a}_\mu = 0$ [114]. Therefore, according to the above, the gauge group after the symmetry breaking is the $SO(4) \times U(1)$ and the only independent fields of the theory are e and a.

At this point, we write down the explicit expression of the $R_{mn}{}^{ab}$ component tensor, (18), after fixing the conditions of the fields $\tilde{a}_m = 0$, $b_m{}^a = \frac{i}{2}e_m{}^a$:

$$\begin{aligned} R_{mn}{}^{ab} =& \Big[X_m + a_m, \omega_n{}^{ab}\Big] - \Big[X_n + a_n, \omega_m{}^{ab}\Big] + i\Big\{\omega_m{}^{ac}, \omega_n{}^{b}c\Big\} - i\Big\{\omega_m{}^{bc}, \omega_{nc}{}^{a}\Big\} \\ &+ \frac{3i}{8}\Big\{e_m{}^a, e_n{}^b\Big\} - \frac{i\lambda^2}{\hbar}B_{mn}{}^{ab}, \end{aligned}$$

which, as it will be shown in the following, it will survive in the commutative regime, and the first line will coincide with the expression of the curvature two-form $R^{(0)ab}_{mn}$ in the Palatini formulation of GR.

4.2.4. The Commutative Limit

In order to examine if the above noncommutative gravity model is indeed a generalization of the predictions of GR in the energy regime below the Planck scale, we consider the naive commutative limit of the model, that is the vanishing of all noncommutative-related features. In order to achieve that and make direct identification, in this limit, we consider the fuzzy space to have Lorentzian signature (fuzzy dS_4). In order to employ the above, we make the following considerations:

- As long as the noncommutativity of the space ceases to exist, the two-form field $\mathcal{B}_{\mu\nu}$, that was related to the preservation of covariance of the fuzzy space, decouples, as does the a_μ field, which was introduced to extend the gauge group due to the behaviour of the anticommutators in noncommutative gauge theories;
- The commutators of functions vanish, $[f(x), g(x)] \to 0$ while the anticommutators of functions reduce to products, $\{f(x), g(x)\} \to 2f(x)g(x)$;
- The inner derivation reduces to the simple derivative: $[X_\mu, f] \to \partial_\mu f$ and the traces reduce to integrations, $\frac{\sqrt{2}}{4}\,\mathrm{Tr} \to \int d^4x$;
- Additionally, in the specific gauge in which the symmetry breaking occurred, the expression of the D-related component tensor $\tilde{R}_{\mu\nu}$, (19), of the field strength tensor reduces to:
$$\tilde{R}_{\mu\nu} = -\frac{\sqrt{2}}{8}\epsilon_{abcd}\left[\omega_\mu{}^{ab}, \omega_\nu{}^{cd}\right] - \frac{i\lambda^2}{\hbar}\tilde{B}_{\mu\nu}.$$
On that account, when the commutative limit is considered, the second term of the corresponding action, (21), which contains the above tensor, will vanish, since the commutator of the spin connection will be zero and the $\tilde{B}_{\mu\nu}$ will decouple, as mentioned above. Furthermore, since the a_μ field also decouples at the limit, it will not be included in the first term of the aforementioned action.
- In order to exactly match the results of the commutative case, we also need to take into account the following reparametrizations:
$$e_\mu{}^a \to ime_\mu{}^a, \quad P_a \to -\frac{i}{m}P_a, \quad \tilde{R}_{\mu\nu}{}^a \to imT_{\mu\nu}{}^a$$
$$\omega_\mu{}^{ab} \to -\frac{i}{2}\omega_\mu{}^{ab}, \quad M_{ab} \to 2iM_{ab}, \quad R_{\mu\nu}{}^{ab} \to -\frac{i}{2}R_{\mu\nu}{}^{ab},$$
where m is an arbitrary, complex constant of dimensions $[L]^{-1}$, which serves the purpose of keeping the $e_\mu{}^a$ dimensionless in the commutative limit, so that the latter can admit the interpretation of the actual vielbein field.

Regarding the torsion tensor $\tilde{R}_{\mu\nu}{}^a$, after taking into consideration the above limits and reparametrizations, it takes the following form:

$$T_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a - \omega_\mu{}^{ab}e_{\nu b} + \omega_\nu{}^{ab}e_{\mu b} = 0$$

which exactly coincides with the torsionless condition of the first-order formulation of GR. Hence, we understand that the relation between ω and e is exactly the same as in the first-order formulation of GR.

Following the same logic as above, the curvature two-form $R_{\mu\nu}{}^{ab}$, after considering the limits and rescalings above, will be given by the following expression:

$$R_{\mu\nu}{}^{ab} = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} + \omega_\mu{}^{ac}\omega_\nu{}^b{}_c - \omega_\mu{}^{bc}\omega_\nu{}^a{}_c + \frac{3}{2}m^2 e_\mu{}^a e_\nu{}^b = R^{(0)\,ab}_{\mu\nu} + \frac{3}{2}m^2 e_\mu{}^a e_\nu{}^b\,.$$

From the above expression, it is explicitly understood that the curvature two-form in our case is exactly the same as that of the first order formulation of GR, plus an extra term involving only the vielbein fields.

Finally, we comment on the action. As stated above, given the considered limits, the second term of (21) will vanish, and the action will now consist only of the first term. It is also understood that the only invariance that the action is left with, in the commutative limit, is the Lorentz one. After the corresponding calculations are performed, the expression of the action in the commutative limit takes the form given by MacDowell–Mansouri, (4).

5. Conclusions—Future Plans

We reviewed the construction of a gravitational model on a four-dimensional noncommutative space employing the gauge principle and showed that it consists a generalization of GR in a top to bottom way. In the beginning, noncommutativity led to an extended gauge symmetry and the introduction of extra fields in the theory, compared to the ordinary case. However, the employment of a spontaneous symmetry breaking mechanism, reduced the amount of the extra symmetry and eventually we resulted with a model in which the whole noncommutativity modification is manifested through an extra $U(1)$ symmetry.

Based on the above findings, we plan to make contact with phenomenology and estimate the noncommutative scale through compatibility to the experimental gravitational results. To that end, a thorough examination of the commutative limit is necessary in order to find the modifications that noncommutativity induces in the low-energy regime. Moreover, we plan to examine the coupling of the noncommutative gravity to matter and, furthermore, to explore the cosmological implications that can be produced by the above model.

Author Contributions: All authors have contributed equally to the realization of this work. All authors have read and agreed to the published version of the manuscript.

Funding: Affiliated authors of NTUA have been supported by the Basic Research Programme, PEVE2020 of National Technical University of Athens, Greece. One of us (GZ) would like to thank the DFG Exzellenzcluster 2181:STRUCTURES of Heidelberg University, MPP-Munich and CERN-TH for support. The work of GM was supported by the Croatian Science Foundation project IP-2019-04-4168.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We would like to thank Denjoe O'Connor, Paolo Aschieri, Dieter Lüst, Dimitra Karabali, V. Parameswaran Nair, Maja Burić, Ali Chamseddine, Dumitru Ghilencea, Ichiro Oda, Emmanuel Saridakis, Athanasios Chatzistavrakidis, and Harold Steinacker for useful discussion.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

Notes

1. For classical textbooks on the subject see [1–3].
2. For a comprehensive map and a detailed study on this field see [4,5].
3. More recently, further breaking patterns of the Weyl conformal gravity are studied (see [10,11]).
4. See [22–30].
5. For another approach using the Drinfel'd twist see also [82].
6. Sciama's general relativistic background is compensated in Kibble's theory by the presence of the vierbeins. This implies a correlation between the reparametrizations and local translational transformations.
7. The Poincaré group is not semi-simple since it possesses the translations as a normal subgroup. For this reason no non-degenerate invariant form exists and that is why no quadratic action was available [18].
8. Here, we follow the construction for the dS_4 group. The same methodology applies for the AdS_4 case, too.
9. Regarding the vierbein and the spin connection, this is an a posteriori identification as the gauge fields, since, in order to proceed with it, the torsionless condition and the field equation should hold (on-shell state) or else, especially for the vierbein, the transformation is not even $SO(1,4)$ covariant. In other words, reparametrizations and gauge transformations of the fields may be used interchangeably after the symmetry breaking of the considered action and the attainment of the equations of motion.
10. Therefore the torsionless condition is derived as the equation of motion of the Lorentz gauge field.

[11] For previous applications see [102–112].

[12] The momenta operators will become more relevant in our future work, as matter fields is planned to be introduced.

[13] For the initial steps on this construction see Snyder's and Yang's original works [91,92].

[14] See also [113], in which the authors make use of a similar term in their action in the framework of stringy RVM.

[15] Had we taken into consideration the antisymmetric part (antisymmetric tensor) or the adjoint representation, the symmetry breaking would lead to the same gauge symmetry enhanced by a $U(1)$. In case the scalars are charged under the initial $U(1)$, this Abelian gauge group also breaks to a global $U(1)$.

References

1. Carroll, S.M. *Spacetime and Geometry*; Cambridge University Press: Cambridge, UK, 2019.
2. Wald, R.M. *General Relativity*; Chicago University Press: Chicago, IL, USA, 1984. [CrossRef]
3. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*; W.H. Freeman: San Francisco, CA, USA, 1973.
4. Blagojević, M.; Hehl, F.W. (Eds.) *Gauge Theories of Gravitation: A Reader with Commentaries*; World Scientific: Singapore, 2013. [CrossRef]
5. Ortin, T. *Gravity and Strings*; Cambridge Monographs on Mathematical Physics Series; Cambridge University Press: Cambridge, UK, 2004. [CrossRef]
6. Utiyama, R. Invariant theoretical interpretation of interaction. *Phys. Rev.* **1956**, *101*, 1597–1607. [CrossRef]
7. Sciama, D.W. On the analogy between charge and spin in general relativity. In *Recent Developments in General Relativity*; Festschrift for Infeld; Pergamon Press: Oxford, UK; PWN: Warsaw, Poland, 1962; pp. 415–439.
8. Kibble, T.W.B. Lorentz invariance and the gravitational field. *J. Math. Phys.* **1961**, *2*, 212–221. [CrossRef]
9. Stelle, K.S.; West, P.C. Spontaneously Broken De Sitter Symmetry and the Gravitational Holonomy Group. *Phys. Rev. D* **1980**, *21*, 1466. [CrossRef]
10. Oda, I. Planck Scale from Broken Local Conformal Invariance in Weyl Geometry. *Adv. Stud. Theor. Phys.* **2020**, *14*, 9–28. [CrossRef]
11. Ghilencea, D.M. Stueckelberg breaking of Weyl conformal geometry and applications to gravity. *Phys. Rev. D* **2020**, *101*, 045010. [CrossRef]
12. Chamseddine, A.H.; West, P.C. Supergravity as a Gauge Theory of Supersymmetry. *Nucl. Phys. B* **1977**, *129*, 39–44. [CrossRef]
13. Kaku, M.; Townsend, P.K.; van Nieuwenhuizen, P. Gauge Theory of the Conformal and Superconformal Group. *Phys. Lett. B* **1977**, *69*, 304–308. [CrossRef]
14. MacDowell, S.W.; Mansouri, F. Unified Geometric Theory of Gravity and Supergravity. *Phys. Rev. Lett.* **1977**, *38*, 739; Erratum in *Phys. Rev. Lett.* **1977**, *38*, 1376. [CrossRef]
15. Ivanov, E.A.; Niederle, J. Gauge Formulation of Gravitation Theories. 1. The Poincare, De Sitter and Conformal Cases. *Phys. Rev. D* **1982**, *25*, 976. [CrossRef]
16. Kibble, T.W.B.; Stelle, K.S. *Gauge Theories of Gravity and Supergravity*; Elsevier: Amsterdam, The Netherlands, 1985.
17. Fradkin, E.S.; Tseytlin, A.A. Conformal supergravity. *Phys. Rep.* **1985**, *119*, 233–362. [CrossRef]
18. Witten, E. (2+1)-Dimensional Gravity as an Exactly Soluble System. *Nucl. Phys. B* **1988**, *311*, 46. [CrossRef]
19. Connes, A. *Noncommutative Geometry*; InterEditions: Paris, France, 1994; ISBN 9780121858605
20. Chamseddine, A.H.; Connes, A. The Spectral action principle. *Commun. Math. Phys.* **1997**, *186*, 731–750. [CrossRef]
21. Gracia-Bondia, J.M.; Varilly, J.C.; Figueroa, H. *Elements of Noncommutative Geometry*; Birkhauser: Boston, MA, USA, 2001.
22. Bayen, F.; Flato, M.; Fronsdal, C.; Lichnerowicz, A.; Sternheimer, D. Deformation Theory and Quantization. 1. Deformations of Symplectic Structures. *Ann. Phys.* **1978**, *111*, 61. [CrossRef]
23. Kontsevich, M. Deformation quantization of Poisson manifolds *Lett. Math. Phys.* **2003**, *66*, 157–216. [CrossRef]
24. Fritz, C.; Majid, S. Noncommutative spherically symmetric spacetimes at semiclassical order. *Class. Quantum Gravity* **2017**, *34*, 135013. [CrossRef]
25. Buric, M.; Grammatikopoulos, T.; Madore, J.; Zoupanos, G. Gravity and the structure of noncommutative algebras. *JHEP* **2006**, *04*, 054, [CrossRef]
26. Buric, M.; Madore, J.; Zoupanos, G. The Energy-momentum of a Poisson structure. *Eur. Phys. J. C* **2008**, *55*, 489–498. [CrossRef]
27. Buric, M.; Madore, J.; Zoupanos, G. WKB Approximation in Noncommutative Gravity. *SIGMA* **2007**, *3*, 125. [CrossRef]
28. Maceda, M.; Madore, J.; Manousselis, P.; Zoupanos, G. Can noncommutativity resolve the big bang singularity? *Eur. Phys. J. C* **2004**, *36*, 529–534. [CrossRef]
29. Lukierski, J.; Ruegg, H.; Nowicki, A.; Tolstoi, V.N. Q deformation of Poincare algebra. *Phys. Lett. B* **1991**, *264*, 331–338. [CrossRef]
30. Lukierski, J.; Nowicki, A.; Ruegg, H. New quantum Poincare algebra and k deformed field theory. *Phys. Lett. B* **1992**, *293*, 344–352. [CrossRef]
31. Madore, J. *An Introduction to Noncommutative Differential Geometry and Its Physical Applications*; Cambridge University Press: Cambridge, UK, 2000; Volume 257.
32. Landi, G. *An Introduction to Noncommutative Spaces and Their Geometry*; Springer: Berlin/Heidelberg, Germany, 1997; Volume 51. [CrossRef]
33. Szabo, R.J. Quantum field theory on noncommutative spaces. *Phys. Rep.* **2003**, *378*, 207–299. [CrossRef]
34. Douglas, M.R.; Nekrasov, N.A. Noncommutative field theory. *Rev. Mod. Phys.* **2001**, *73*, 977–1029. [CrossRef]

35. Madore, J.; Schraml, S.; Schupp, P.; Wess, J. Gauge theory on noncommutative spaces. *Eur. Phys. J. C* **2000**, *16*, 161–167. [CrossRef]
36. Aschieri, P.; Dimitrijevic, M.; Meyer, F.; Schraml, S.; Wess, J. Twisted gauge theories. *Lett. Math. Phys.* **2006**, *78*, 61–71. [CrossRef]
37. Jurco, B.; Schraml, S.; Schupp, P.; Wess, J. Enveloping algebra valued gauge transformations for nonAbelian gauge groups on noncommutative spaces. *Eur. Phys. J. C* **2000**, *17*, 521–526. [CrossRef]
38. Connes, A. Gravity coupled with matter and foundation of noncommutative geometry. *Commun. Math. Phys.* **1996**, *182*, 155–176. [CrossRef]
39. Chamseddine, A.H.; Connes, A.; Mukhanov, V. Geometry and the Quantum: Basics. *JHEP* **2014**, *12*, 098. [CrossRef]
40. Chamseddine, A.H.; Connes, A.; Mukhanov, V. Quanta of Geometry: Noncommutative Aspects. *Phys. Rev. Lett.* **2015**, *114*, 091302. [CrossRef]
41. Batakis, N.A.; Kehagias, A.A.; Zoupanos, G. Structure and spontaneous symmetry breaking of a gauge theory based on SU(5/1). *Phys. Lett. B* **1993**, *315*, 319–324; Erratum in *Phys. Lett. B* **1993**, *318*, 650–650. [CrossRef]
42. Chamseddine, A.H.; Connes, A.; Marcolli, M. Gravity and the standard model with neutrino mixing. *Adv. Theor. Math. Phys.* **2007**, *11*, 991–1089. [CrossRef]
43. Lambiase, G.; Sakellariadou, M.; Stabile, A. Constraints on NonCommutative Spectral Action from Gravity Probe B and Torsion Balance Experiments. *JCAP* **2013**, *12*, 020. [CrossRef]
44. Glaser, L.; Stern, A.B. Reconstructing manifolds from truncations of spectral triples. *J. Geom. Phys.* **2021**, *159*, 103921. [CrossRef]
45. Bochniak, A.; Sitarz, A. Stability of Friedmann-Lemaître-Robertson-Walker solutions in doubled geometries. *Phys. Rev. D* **2021**, *103*, 044041. [CrossRef]
46. Seiberg, N.; Witten, E. String theory and noncommutative geometry. *JHEP* **1999**, *09*, 032. [CrossRef]
47. Chamseddine, A.H. Deforming Einstein's gravity. *Phys. Lett. B* **2001**, *504*, 33–37. [CrossRef]
48. Chamseddine, A.H. SL(2,C) gravity with complex vierbein and its noncommutative extension. *Phys. Rev. D* **2004**, *69*, 024015. [CrossRef]
49. Nair, V.P. Gravitational fields on a noncommutative space. *Nucl. Phys. B* **2003**, *651*, 313–327. [CrossRef]
50. Abe, Y.; Nair, V.P. Noncommutative gravity: Fuzzy sphere and others. *Phys. Rev. D* **2003**, *68*, 025002. [CrossRef]
51. Cardella, M.A.; Zanon, D. Noncommutative deformation of four-dimensional Einstein gravity. *Class. Quantum Gravity* **2003**, *20*, L95–L104. [CrossRef]
52. Aschieri, P.; Castellani, L. Noncommutative supergravity in D = 3 and D = 4. *JHEP* **2009**, *06*, 087. [CrossRef]
53. Aschieri, P.; Castellani, L. Noncommutative D = 4 gravity coupled to fermions. *JHEP* **2009**, *06*, 086. [CrossRef]
54. Dimitrijević Ćirić, M.; Nikolić, B.; Radovanović, V. Noncommutative $SO(2,3)_\star$ gravity: Noncommutativity as a source of curvature and torsion. *Phys. Rev. D* **2017**, *96*, 064029. [CrossRef]
55. Cacciatori, S.; Klemm, D.; Martucci, L.; Zanon, D. Noncommutative Einstein-AdS gravity in three-dimensions. *Phys. Lett. B* **2002**, *536*, 101–106. [CrossRef]
56. Aschieri, P.; Castellani, L. Noncommutative Chern-Simons gauge and gravity theories and their geometric Seiberg-Witten map. *JHEP* **2014**, *11*, 103. [CrossRef]
57. Banados, M.; Chandia, O.; Grandi, N.E.; Schaposnik, F.A.; Silva, G.A. Three-dimensional noncommutative gravity. *Phys. Rev. D* **2001**, *64*, 084012. [CrossRef]
58. Aschieri, P.; Blohmann, C.; Dimitrijevic, M.; Meyer, F.; Schupp, P.; Wess, J. A Gravity theory on noncommutative spaces. *Class. Quantum Gravity* **2005**, *22*, 3511–3532. [CrossRef]
59. Ishibashi, N.; Kawai, H.; Kitazawa, Y.; Tsuchiya, A. A Large N reduced model as superstring. *Nucl. Phys. B* **1997**, *498*, 467–491. [CrossRef]
60. Banks, T.; Fischler, W.; Shenker, S.H.; Susskind, L. M theory as a matrix model: A Conjecture. *Phys. Rev. D* **1997**, *55*, 5112–5128. [CrossRef]
61. Connes, A.; Douglas, M.R.; Schwarz, A.S. Noncommutative geometry and matrix theory: Compactification on tori. *JHEP* **1998**, *02*, 003. [CrossRef]
62. Aoki, H.; Iso, S.; Kawai, H.; Kitazawa, Y.; Tada, T. Space-time structures from IIB matrix model. *Prog. Theor. Phys.* **1998**, *99*, 713–746. [CrossRef]
63. Hanada, M.; Kawai, H.; Kimura, Y. Describing curved spaces by matrices. *Prog. Theor. Phys.* **2006**, *114*, 1295–1316. [CrossRef]
64. Furuta, K.; Hanada, M.; Kawai, H.; Kimura, Y. Field equations of massless fields in the new interpretation of the matrix model. *Nucl. Phys. B* **2007**, *767*, 82–99. [CrossRef]
65. Nishimura, J. The Origin of space-time as seen from matrix model simulations. *PTEP* **2012**, *2012*, 01A101. [CrossRef]
66. Kim, S.W.; Nishimura, J.; Tsuchiya, A. Expanding (3+1)-dimensional universe from a Lorentzian matrix model for superstring theory in (9+1)-dimensions. *Phys. Rev. Lett.* **2012**, *108*, 011601. [CrossRef]
67. Valtancoli, P. Gravity on a fuzzy sphere. *Int. J. Mod. Phys. A* **2004**, *19*, 361–370. [CrossRef]
68. Nair, V.P. The Chern-Simons one-form and gravity on a fuzzy space. *Nucl. Phys. B* **2006**, *750*, 321–333. [CrossRef]
69. Steinacker, H. Emergent Geometry and Gravity from Matrix Models: An Introduction. *Class. Quantum Gravity* **2010**, *27*, 133001. [CrossRef]
70. Steinacker, H.C. Emergent gravity on covariant quantum spaces in the IKKT model. *JHEP* **2016**, *12*, 156. [CrossRef]
71. Buric, M.; Madore, J. On noncommutative spherically symmetric spaces. *Eur. Phys. J. C* **2014**, *74*, 2820. [CrossRef]
72. Buric, M.; Madore, J. Noncommutative de Sitter and FRW spaces. *Eur. Phys. J. C* **2015**, *75*, 502. [CrossRef]

73. Chatzistavrakidis, A.; Jonke, L.; Jurman, D.; Manolakos, G.; Manousselis, P.; Zoupanos, G. Noncommutative Gauge Theory and Gravity in Three Dimensions. *Fortschritte Phys.* **2018**, *66*, 1800047. [CrossRef]
74. Manolakos, G.; Manousselis, P.; Zoupanos, G. Four-dimensional Gravity on a Covariant Noncommutative Space. *JHEP* **2020**, *08*, 001. [CrossRef]
75. Manolakos, G.; Manousselis, P.; Zoupanos, G. Four-Dimensional Gravity on a Covariant Noncommutative Space (II). *Fortschritte Phys.* **2021**, *69*, 2100085. [CrossRef]
76. Jurman, D.; Manolakos, G.; Manousselis, P.; Zoupanos, G. Gravity as a Gauge Theory on Three-Dimensional Noncommutative spaces. *PoS* **2018**, *CORFU2017*, 162. [CrossRef]
77. Manolakos, G.; Manousselis, P.; Zoupanos, G. Gauge Theories: From Kaluza–Klein to noncommutative gravity theories. *Symmetry* **2019**, *11*, 856. [CrossRef]
78. Lada, T.; Stasheff, J. Introduction to SH Lie algebras for physicists. *Int. J. Theor. Phys.* **1993**, *32*, 1087–1104. [CrossRef]
79. Zwiebach, B. Closed string field theory: Quantum action and the B-V master equation. *Nucl. Phys. B* **1993**, *390*, 33–152. [CrossRef]
80. Blumenhagen, R.; Brunner, I.; Kupriyanov, V.; Lüst, D. Bootstrapping non-commutative gauge theories from L_∞ algebras. *JHEP* **2018**, *05*, 097. [CrossRef]
81. Ćirić, M.D.; Giotopoulos, G.; Radovanović, V.; Szabo, R.J. Braided L_∞-Algebras, Braided Field Theory and Noncommutative Gravity. *arXiv* **2021**, arXiv:2103.08939.
82. Aschieri, P.; Schenkel, A. Noncommutative connections on bimodules and Drinfeld twist deformation. *Adv. Theor. Math. Phys.* **2014**, *18*, 513–612. [CrossRef]
83. Grewcoe, C.J.; Jonke, L.; Kodžoman, T.; Manolakos, G. From Hopf algebra to braided L∞-algebra. 2022, *in press*.
84. Grosse, H.; Presnajder, P. The Construction on noncommutative manifolds using coherent states. *Lett. Math. Phys.* **1993**, *28*, 239–250. [CrossRef]
85. Hammou, A.B.; Lagraa, M.; Sheikh-Jabbari, M.M. Coherent state induced star product on R**3(lambda) and the fuzzy sphere. *Phys. Rev. D* **2002**, *66*, 025025. [CrossRef]
86. Vitale, P.; Wallet, J.C. Noncommutative field theories on R^3_λ: Toward UV/IR mixing freedom. *JHEP* **2013**, *04*, 115, Addendum in *JHEP* **2015**, *03*, 115. [CrossRef]
87. Vitale, P. Noncommutative field theory on $\mathbb{R}^3_\lambda$. *Fortschritte Phys.* **2014**, *62*, 825–834. [CrossRef]
88. KováčIk, S.; Prešnajder, P. The velocity operator in quantum mechanics in noncommutative space. *J. Math. Phys.* **2013**, *54*, 102103. [CrossRef]
89. Madore, J. The Fuzzy sphere. *Class. Quantum Gravity* **1992**, *9*, 69–88. [CrossRef]
90. Jurman, D.; Steinacker, H. 2D fuzzy Anti-de Sitter space from matrix models. *JHEP* **2014**, *01*, 100. [CrossRef]
91. Snyder, H.S. Quantized space-time. *Phys. Rev.* **1947**, *71*, 38–41. [CrossRef]
92. Yang, C.N. On quantized space-time. *Phys. Rev.* **1947**, *72*, 874. [CrossRef]
93. Heckman, J.; Verlinde, H. Covariant non-commutative space–time. *Nucl. Phys. B* **2015**, *894*, 58–74. [CrossRef]
94. Sperling, M.; Steinacker, H.C. Covariant 4-dimensional fuzzy spheres, matrix models and higher spin. *J. Phys. A* **2017**, *50*, 375202. [CrossRef]
95. Sperling, M.; Steinacker, H.C. The fuzzy 4-hyperboloid H^n_4 and higher-spin in Yang–Mills matrix models. *Nucl. Phys. B* **2019**, *941*, 680–743. [CrossRef]
96. Buric, M.; Latas, D.; Nenadovic, L. Fuzzy de Sitter Space. *Eur. Phys. J. C* **2018**, *78*, 953. [CrossRef]
97. Kimura, Y. Noncommutative gauge theory on fuzzy four sphere and matrix model. *Nucl. Phys. B* **2002**, *637*, 177–198. [CrossRef]
98. Abe, Y. Construction of fuzzy S**4. *Phys. Rev. D* **2004**, *70*, 126004. [CrossRef]
99. Yang, C.N.; Mills, R.L. Conservation of Isotopic Spin and Isotopic Gauge Invariance. *Phys. Rev.* **1954**, *96*, 191–195. [CrossRef]
100. Alvarez-Gaume, L.; Kehagias, A.; Kounnas, C.; Lüst, D.; Riotto, A. Aspects of Quadratic Gravity. *Fortschritte Phys.* **2016**, *64*, 176–189. [CrossRef]
101. Møller, C. Conservation laws and absolute parallelism in general relativity. *Mat. Fys. Dan. Vid. Selsk.* **1961**, *1*, 1–50.
102. Aschieri, P.; Madore, J.; Manousselis, P.; Zoupanos, G. Dimensional reduction over fuzzy coset spaces. *JHEP* **2004**, *04*, 034. [CrossRef]
103. Aschieri, P.; Madore, J.; Manousselis, P.; Zoupanos, G. Unified theories from fuzzy extra dimensions. *Fortschritte Phys.* **2004**, *52*, 718–723. [CrossRef]
104. Aschieri, P.; Madore, J.; Manousselis, P.; Zoupanos, G. Renormalizable theories from fuzzy higher dimensions. In Proceedings of the 3rd Summer School in Modern Mathematical Physics, Zlatibor, Serbia, 20–31 August 2004; pp. 135–145.
105. Aschieri, P.; Grammatikopoulos, T.; Steinacker, H.; Zoupanos, G. Dynamical generation of fuzzy extra dimensions, dimensional reduction and symmetry breaking. *JHEP* **2006**, *09*, 026. [CrossRef]
106. Aschieri, P.; Steinacker, H.; Madore, J.; Manousselis, P.; Zoupanos, G. Fuzzy extra dimensions: Dimensional reduction, dynamical generation and renormalizability. *SFIN A* **2007**, *1*, 25–42.
107. Steinacker, H.; Zoupanos, G. Fermions on spontaneously generated spherical extra dimensions. *JHEP* **2007**, *09*, 017. [CrossRef]
108. Chatzistavrakidis, A.; Steinacker, H.; Zoupanos, G. On the fermion spectrum of spontaneously generated fuzzy extra dimensions with fluxes. *Fortschritte Phys.* **2010**, *58*, 537–552. [CrossRef]
109. Chatzistavrakidis, A.; Steinacker, H.; Zoupanos, G. Orbifolds, fuzzy spheres and chiral fermions. *JHEP* **2010**, *05*, 100. [CrossRef]

110. Chatzistavrakidis, A.; Zoupanos, G. Higher-Dimensional Unified Theories with Fuzzy Extra Dimensions. *SIGMA* **2010**, *6*, 063. [CrossRef]
111. Gavriil, D.; Manolakos, G.; Orfanidis, G.; Zoupanos, G. Higher-Dimensional Unification with continuous and fuzzy coset spaces as extra dimensions. *Fortschritte Phys.* **2015**, *63*, 442–467. [CrossRef]
112. Manolakos, G.; Zoupanos, G. The trinification model $SU(3)^3$ from orbifolds for fuzzy spheres. *Phys. Part. Nucl. Lett.* **2017**, *14*, 322–327. [CrossRef]
113. Mavromatos, N.E.; Solà Peracaula, J. Inflationary physics and trans-Planckian conjecture in the stringy running vacuum model: From the phantom vacuum to the true vacuum. *Eur. Phys. J. Plus* **2021**, *136*, 1152. [CrossRef]
114. Chamseddine, A.H. An invariant action for noncommutative gravity in four dimensions. *J. Math. Phys.* **2003**, *44*, 2534. [CrossRef]