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Maxwell's Equations in Homogeneous Spaces for Admissible Electromagnetic Fields

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Abstract: Maxwell's vacuum equations are integrated for admissible electromagnetic fields in homogeneous spaces. Admissible electromagnetic fields are those for which the space group generates an algebra of symmetry operators (integrals of motion) that is isomorphic to the algebra of group operators. Two frames associated with the group of motions are used to obtain systems of ordinary differential equations to which Maxwell's equations reduce. The solutions are obtained in quadratures. The potentials of the admissible electromagnetic fields and the metrics of the spaces contained in the obtained solutions depend on six arbitrary time functions, so it is possible to use them to integrate field equations in the theory of gravity.

Keywords: Maxwell's vacuum equations; Hamilton–Jacobi equation; Klein–Gordon–Fock equation; algebra of symmetry operators; separation of variables; linear partial differential equations

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1. Introduction

A special place in mathematical physics is occupied by the problem of exact integration of the equations of motion of a classical or quantum test particle in external electromagnetic and gravitational fields. This problem is closely related to the study of the symmetry of gravitational and electromagnetic fields in which a given particle moves. A necessary condition for the existence of such symmetry is the admissibility of the algebra of symmetry operators, given by vector and tensor Killing fields, for spacetime and the external electromagnetic field. The algebras of these operators are isomorphic to the algebras of the symmetry operators of the equations of motion of a test particle—Hamilton–Jacobi, Klein–Gordon–Fock, or Dirac–Fock. At present, two methods are known for the exact integration of the equations of motion of a test particle. These are the methods of commutative and noncommutative integration. The first method is based on the use of commutative algebra of symmetry operators (integrals of motion) that form a complete set. The complete set includes linear operators of first and second degree in momentum formed by vector and tensor Killing fields of complete sets of geometric objects of V_4. The method is known as the method of complete separation of variables (in the Hamilton–Jacobi, Klein–Gordon–Fock, or Dirac–Fock equations). The spaces in which the method of complete separation of variables is applicable are called Stackel spaces. The theory of Stackel spaces was developed in [1–12]. A description of the theory and a detailed bibliography can be found in [13–16]. The most frequently used exact solutions of the gravitational field equations in the theory of gravity were constructed on the basis of Stackel spaces (see, e.g., [17–19]). These solutions are still widely used in the study of various effects in gravitational fields (see, e.g., [20–27]).

The second method (noncommutative integration) was developed in [28]. This method is based on the use of algebra of symmetry operators, which are linear in momenta and constructed using Killing vector fields forming noncommutative groups of motion of spacetime G_3 and G_4. The algebras of the symmetry operators of the Klein–Gordon–Fock equation constructed using the algebras of the operators of the noncommutative motion group of space V_4 are complemented to a commutative algebra by the operators of differentiation of the first order in 4 essential parameters. Among these spacetime manifolds, the homogeneous spaces are of greatest interest for the theory of gravity (see, e.g., [29–36]).

Thus, these two methods complement each other to a considerable extent and have similar classification problems (by solving the classification problem, we mean enumerating all metrics and electromagnetic potentials that are not equivalent in terms of admissible transformations). Among these classification problems, the most important are the following.

Classification of all metrics of homogeneous and Stackel spaces in privileged coordinate systems. For Stackel spaces, this problem was solved in building the theory of complete separation of variables in the papers cited above. For homogeneous spaces, this problem was solved in the work of Petrov (see [37]).

Classification of all (admissible) electromagnetic fields applicable to these methods. For the Hamilton–Jacobi and Klein–Gordon–Fock equations, this problem is completely solved in homogeneous spaces (see [38–43]). In Stackel spaces, it is completely solved for the Hamilton–Jacobi equation and partially solved for the Klein–Gordon–Fock equation (see [14–16]).

Classification of all vacuum and electrovacuum solutions of the Einstein equations with metrics of Stackel and homogeneous spaces in admissible electromagnetic fields. This problem has been completely solved for the Stackel metric (see [17–20]). However, this classification problem has not yet been studied for homogeneous spaces.

The solutions to these problems can be viewed as stages of the solution of a single classification problem. In the first two stages, we find all relevant gravitational and electromagnetic fields that are not connected by field equations. In the third stage, using the results of the first two stages, we find metrics and electromagnetic potentials that satisfy the Einstein–Maxwell vacuum equations and have physical meaning.

Thus, for the complete solution to the problem of uniform classification, the Einstein–Maxwell vacuum equations must be integrated using the previously found potentials of admissible electromagnetic fields and the known metrics of homogeneous spaces in privileged (canonical) coordinate systems. This problem can also be divided into two stages. In the first stage, all solutions of Maxwell's vacuum equations for the potentials of admissible electromagnetic fields should be found. The present work is devoted to this stage. In the next stage, the plan is to use the obtained results for the integration of the Einstein–Maxwell equations. This will be the subject of further research. The present work is organized as follows.

Section 2 contains information from the theory of homogeneous spaces, which will be used later, and definitions and conditions for the potentials of admissible electromagnetic fields, written in canonical frames associated with motion groups of a homogeneous space.

In the Section 3 Maxwell's vacuum equations are written in canonical frames.

The Section 4 contains all solutions of Maxwell's vacuum equations for homogeneous spaces admitting groups of motions $G_3(I) - G_3(VI)$.

2. Homogeneous Spaces

By the accepted definition, a spacetime manifold V_4 is a homogeneous space—if a three-parameter group of motions acts on it—whose transitivity hypersurface V_3 is endowed with the Euclidean space signature. Let us introduce a semi-geodesic coordinate system $[u^i]$, in which the metric V_4 has the form:

$$ds^2 = g_{ij}du^i du^j = -du^{0^2} + g_{\alpha\beta}du^\alpha du^\beta, \quad det|g_{\alpha\beta}| > 0. \tag{1}$$

The coordinate indices of the variables of the semi-geodesic coordinate system are denoted by the lower-case Latin letters: $i, j, k, l = 0, 1, \ldots, 3$. The coordinate indices of the variables of the local coordinate system on the hypersurface V_3 are denoted by the lower-case Greek letters: $\alpha, \beta, \gamma, \sigma = 1, \ldots, 3$. A 0 index denotes the temporary variable. Group indices and indices of nonholonomic frames are denoted by $a, d, c = 1, \ldots, 3$. Summation is performed over repeated upper and lower indices within the index range.

There is another (equivalent) definition of a homogeneous space, according to which the spacetime V_4 is homogeneous if its subspace V_3, endowed with the Euclidean space signature, admits a set of coordinate transformations (the group G_3 of motions spaces V_4) that allow the connection of any two points in V_3. (see, e.g., [44]). This definition directly implies that the metric tensor of the V_3 space can be represented as follows:

$$g_{\alpha\beta} = e^a_\alpha e^b_\beta \eta_{ab}, \quad ||\eta_{ab}|| = ||a_{ab}(u^0)||, \quad e^a_{\alpha,0} = 0, \quad det||a_{ab}|| = {l_0}^2, \tag{2}$$

while the form:

$$\omega^a = e^a_\alpha du^\alpha$$

is invariant under the transformation group G_3. The vectors of the frame e^a_α (we call them canonical) define a nonholonomic coordinate system in V_3, and their dual triplet of vectors:

$$e^\alpha_a, \quad e^\alpha_a e^b_\alpha = \delta^b_a, \quad e^\alpha_a e^a_\beta = \delta^\alpha_\beta$$

define the operators of the G_3 algebra group:

$$\hat{Y}_a = e^\alpha_a \partial_a, \quad [\hat{Y}_a, \hat{Y}_b] = C^c_{ab} \hat{Y}_c.$$

The Killing vector fields ξ^α_a and their dual vector fields ξ^a_α form another frame in the space V_3 (we will call it the Killing frame) and another representation of the algebra of group G_3. In the dual frame, the metric of the space V_3 has the form:

$$g_{\alpha\beta} = \xi^a_\alpha \xi^b_\beta G_{ab}, \quad \xi^\alpha_a \xi^b_\alpha = \delta^b_a, \quad \xi^\alpha_a \xi^a_\beta = \delta^\alpha_\beta, \tag{3}$$

where G_{ab} are the nonholonomic components of the $g_{\alpha\beta}$ tensor in this framework. The vector fields ξ^α_a satisfy the Killing equations:

$$g^{\alpha\beta}_{,\gamma} \xi^\gamma_a = g^{\alpha\gamma} \xi^\beta_{a,\gamma} + g^{\beta\gamma} \xi^\alpha_{a,\gamma} \tag{4}$$

and form the infinitesimal group operators of the algebra G_3:

$$\hat{X}_a = \xi^\alpha_a \partial_\alpha, \quad [\hat{X}_a, \hat{X}_b] = C^c_{ab} \hat{X}_c. \tag{5}$$

The Killing equation in the ξ^α_a frame has the following form:

$$G^{ab}_{|c} = G^{ad} C^b_{dc} + G^{bd} C^a_{dc} \quad (|a = \xi^\alpha_a \partial_\alpha). \tag{6}$$

Indeed, substituting the expression:

$$g^{\alpha\beta} = \xi^\alpha_a \xi^\beta_b G^{ab}$$

into Equation (4), we get

$$G^{ab}((\xi^\alpha_{a|c} \xi^\beta_b - \xi^\alpha_a \xi^\beta_{c|b}) + (\xi^\alpha_a \xi^\beta_{b|c} - \xi^\beta_a \xi^\alpha_{c|b})) + \xi^\alpha_a \xi^\beta_b G^{ab}_{|c} = 0.$$

Substituting here the commutation relations (5), we get:

$$(G^{ab}_{|c} - G^{ad} C^b_{dc} - G^{bd} C^a_{dc}) \xi^\alpha_a \xi^\beta_b = 0.$$

The Hamilton–Jacobi equation for a charged test-particle in an external electromagnetic field with potential A_i is:

$$H = g^{ij} P_i P_j = m, \quad P_i = p_i + A_i, \quad p_i = \partial_i \varphi. \tag{7}$$

The integrals of motion of the free Hamilton–Jacobi equation are given using Killing vector fields as follows:

$$X_a = \xi^i_a p_i, \tag{8}$$

Thus, the symmetry of the space given by the Killing vector fields is directly related to the symmetry of the equations of the geodesics given by the integrals of motion. The Hamilton–Jacobi method makes it possible to find these integrals and use them to integrate the geodesic equations. Therefore, the study of the behavior of geodesics is necessary for the study of the geometry of space.

The linear momentum integral of Equation (7) has the following form:

$$X_a = \xi^i_a P_i + \gamma_a, \tag{9}$$

where γ_α are some functions of u^i. Equation (7) admits a motion integral of the form (8) if H and $\hat{X}_a$ commute under Poisson brackets:

$$[H, \hat{X}_a]_P = \frac{\partial H}{\partial p_i}\frac{\partial \hat{X}_a}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial \hat{X}_a}{\partial p_i} = 0 \to g^{i\sigma}(\xi^j_a F_{ji} + \gamma_{a,i})P_\sigma = 0. \tag{10}$$

Hence:

$$\gamma_{a,i} = \xi^j_a F_{ij}, \quad F_{ji} = A_{i,j} - A_{j,i}. \tag{11}$$

Thus, the admissible electromagnetic field is determined from Equation (11) (see [41]). In [39,40] it was proved that in the case of a homogeneous space, the conditions of (11) can be represented as follows:

$$\mathbf{A}_{a|b} = C^c_{ba}\mathbf{A}_c, \tag{12}$$

at the same time:

$$\gamma_a = -\mathbf{A}_a \to \hat{X}_a = \xi^\alpha_a \partial_\alpha.$$

Here, it is denoted that:

$$\mathbf{A}_a = \xi^i_a A_i,$$

It can be shown that Equation (12) forms a completely integrable system. This system specifies the necessary and sufficient conditions for the existence of algebra of integrals of motion that are linear in momenta for Equation (7). Note that in admissible electromagnetic fields given by the conditions (12), the Klein–Gordon–Fock equation:

$$\hat{H}\varphi = (g^{ij}\hat{P}_i\hat{P}_j)\varphi = m^2\varphi, \quad \hat{P}_k = \hat{p}_k + A_k, \quad \hat{p}_k = -\imath\hat{\nabla}_k$$

also admits an algebra of symmetry operators of the form (see [39,41]):

$$\hat{X}_a = \xi^i_a \hat{\nabla}_i$$

$\hat{\nabla}_i$ is the covariant derivative operator corresponding to the partial derivative operator—$\hat{\partial}_i = \imath\hat{p}_i$ in the coordinate field u^i. Function φ is a scalar field, $m = const$. All admissible electromagnetic fields for the homogeneous spacetime are found in [39]. We will use the results of A.Z. Petrov [37]. We follow the notation used in this book with minor exceptions. For example, the nonignorable variable x^4 will be denoted u^0, etc.

3. Maxwell's Equations for an Admissible Electromagnetic Field in Homogeneous Spacetime

Consider Maxwell's equations with zero electromagnetic field sources in homogeneous spacetime in the presence of an admissible electromagnetic field:

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}F^{ij})_{,j} = 0, \quad g = det|g_{\alpha\beta}|. \tag{13}$$

when $i = 0$ from the system (13), the equation follows:

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\alpha\beta}A_{\beta,0})_{,\alpha} = 0. \tag{14}$$

Using the Killing Equations (4) and (5), we can obtain:

$$\frac{g_{|a}}{g} = 2\xi^{\alpha}_{a,\alpha}.$$

Indeed,

$$-\frac{g_{|a}}{g} = g^{\alpha\beta}_{|a}g_{\alpha\beta} = G^{bc}_{|a}G_{bc} + 2\xi^{\alpha}_{a,\alpha} + 2C_a = 2\xi^{\alpha}_{a,\alpha} \quad (C_a = C^b_{ab}).$$

Substituting this expression and the relation (12) into Equation (14), we get:

$$G^{ab}C_b\mathbf{A}_{a,0} = 0. \tag{15}$$

In the case of spaces with groups $G_3(I), G_3(II), G_3(VIII), G_3(IX) C_a = 0$. That is why Equation (15) is satisfied. In the case of the groups $G_3(III), -G_3(VII)$ $C_a = const\delta_{a3}$, and from (15) it follows:

$$\eta^{3a}\tilde{\mathbf{A}}_{a,0} = 0, \quad \tilde{\mathbf{A}}_a = A_\alpha e^\alpha_a. \tag{16}$$

For $i = \alpha$ we have:

$$\frac{1}{\sqrt{g}}(\sqrt{g}g^{\alpha\beta}F_{\beta 0})_{,0} + \frac{1}{\sqrt{g}}(\sqrt{g}g^{\alpha\beta}g^{\gamma\sigma}F_{\beta\sigma})_{,\gamma} = 0. \tag{17}$$

We transform Equation (17) using the (2) frame. The first term then has the form:

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\alpha\beta}F_{\beta 0})_{,0} = -\frac{1}{l_0}(l_0\eta^{ab}\tilde{\mathbf{A}}_{a,0})_{,0}e^\alpha_b, \quad (l_0)^2 = det|\eta_{ab}|.$$

The second term using the (3) frame, the relations (12), and the commutation relations between the operators of the group can be reduced to the following form:

$$\frac{1}{\sqrt{g}}(\sqrt{g}g^{\alpha\beta}g^{\gamma\sigma}F_{\beta\sigma})_{,\gamma} = \frac{1}{2}G^{a_2b_1}C^a_{a_2b_2}(2C_{b_1}G^{bb_2} + C^b_{a_1b_1}G^{a_1b_2})\xi^\alpha_b\xi^\beta_a e^c_\beta\tilde{\mathbf{A}}_c.$$

So Equation (17) can be written as follows:

$$\frac{1}{l_0}(l_0\eta^{ab}\tilde{\mathbf{A}}_{b,0})_{,0} = \tilde{W}^{ba}\tilde{\mathbf{A}}_b, \tag{18}$$

where:

$$\tilde{W}^{ab} = (e^a_\beta\xi^\beta_{a_1})(e^a_\alpha\xi^\alpha_{b_1})W^{a_1b_1}, \quad W^{ab} = \frac{1}{2}G^{a_2b_1}C^a_{a_2b_2}(2C_{b_1}G^{bb_2} + C^b_{a_1b_1}G^{a_1b_2}). \tag{19}$$

Then, Maxwell's equations can be represented as follows:

$$\beta^a_{,0} = l_0\tilde{W}^{ba}\tilde{\mathbf{A}}_b, \tag{20}$$

$$\tilde{\mathbf{A}}_{a,0} = \frac{1}{l_0}\beta^b\eta_{ab}. \tag{21}$$

4. Maxwell's Equations for Spaces Type I–VI According to Bianchi Classification

The group operators in the canonical coordinate set of homogeneous spaces type I–VI according to the Bianchi classification can be represented as follows (see [37]):

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = (ru^1 + \varepsilon u^2)p_1 + nu^2 p_2 - p_3. \tag{22}$$

The values k ε, n for each group take the following values.)

$G(I): \; k = 0, \quad \varepsilon = 0, \quad n = 0.$

$G(II): k = 0, \quad \varepsilon = 1, \quad n = 0.$

$G(III)\; k = 1, \quad \varepsilon = 0, \quad n = 0.$

$G(IV)\; k = 1, \quad \varepsilon = 1, \quad n = 1.$

$G(V): k = 1, \quad \varepsilon = 0, \quad n = 1.$

$G(VI)\; k = 1, \quad \varepsilon = 0, \quad n = 2.$

Structural constants can be represented as follows:

$$C^c_{ab} = k\delta^c_1(\delta^1_a\delta^3_b - \delta^3_a\delta^1_b) + (\varepsilon\delta^c_1 + n\delta^c_2)(\delta^2_a\delta^3_b - \delta^3_a\delta^2_b) \to C_a = -(k+n)\delta^3_a \tag{23}$$

Find the frame vectors $[\xi^\alpha_a], [e^\alpha_a]$ and their dual vectors $[\xi^a_\alpha], [e^a_\alpha]$.

$$\xi^\alpha_a \xi^b_\alpha = e^\alpha_a e^b_\alpha = \delta^b_a, \quad \xi^\alpha_a \xi^a_\beta = e^\alpha_a e^a_\beta = \delta^\alpha_\beta.$$

For this, we use the metrics of homogeneous spaces and the group operators given in [37].

$$\xi^\alpha_a = \delta^1_a\delta^\alpha_1 + \delta^2_a\delta^\alpha_2 + \delta^3_a(\delta^\alpha_1(ku^1 + \varepsilon u^2) + \delta^\alpha_2 nu^2 - \delta^\alpha_3), \tag{24}$$

$$\xi^a_\alpha = \delta^a_1\delta^1_\alpha + \delta^a_2\delta^2_\alpha + \delta^a_3(\delta^1_\alpha(ku^1 + \varepsilon u^2) + \delta^2_\alpha nu^2 - \delta^3_\alpha),$$

$$e^\alpha_a = \delta^1_a\delta^\alpha_1 \exp(-ku^3) + \delta^2_a(-\delta^\alpha_1\varepsilon u^3\exp(-ku^3) + \delta^\alpha_2\exp(-nu^2)) + \delta^\alpha_3\delta^3_a, \tag{25}$$

$$e^\alpha_a = \delta^a_1\delta^1_\alpha \exp(ku^3) + \delta^2_a(\delta^\alpha_1\varepsilon u^3\exp nu^3 + \delta^\alpha_2\exp nu^2)) + \delta^3_\alpha\delta^3_a.$$

With these expressions, we find the matrix $\tilde{W}^{ab}$ (19).

$$\tilde{W}^{ab} = \frac{1}{l_0{}^2}[\delta^a_1\delta^b_1(\varepsilon g_{11} + \varepsilon(n-k)g_{12} - kng_{22})\exp(-2nu^3) \tag{26}$$

$$-(\delta^a_1\varepsilon u^3 + \delta^a_2)(\delta^b_1\varepsilon u^3 + \delta^b_2)kng_{11}\exp(-2ku^3) +$$

$$[\delta^b_1(\delta^a_1\varepsilon u^3 + \delta^a_2)n(g_{12} + \varepsilon g_{11})) + \delta^a_1(\delta^b_1\varepsilon u^3 + \delta^b_2)k(g_{12} - \varepsilon g_{11})].$$

Here (see [37]):

$$g_{11} = a_{11}\exp 2ku^3, \quad g_{12} = (\varepsilon u^3 a_{11} + a_{12})\exp(n+k)u^3, \quad g_{22} = (\varepsilon u^{3^2} a_{11} + 2\varepsilon a_{12} + a_{22})\exp 2nu^3,$$

Maxwell's Equations (20) and (21) become:

$$\dot{\beta}^b = \frac{1}{l_0}[\delta^a_1\delta^b_1(\varepsilon g_{11} + \varepsilon(n-k)g_{12} - kng_{22})\exp(-2nu^3) \tag{27}$$

$$\{-(\delta^a_1\varepsilon u^3 + \delta^a_2)(\delta^b_1\varepsilon u^3 + \delta^b_2)kng_{11}\exp(-2ku^3) +$$

$$[\delta^b_1(\delta^a_1\varepsilon u^3 + \delta^a_2)n(g_{12} + \varepsilon g_{11})) + \delta^a_1(\delta^b_1\varepsilon u^3 + \delta^b_2)k(g_{12} - \varepsilon g_{11})]\tilde{\mathbf{A}}_a,$$

$$\beta^a = l_0\eta^{ab}\tilde{\mathbf{A}}_{b,0}. \tag{28}$$

The dots denote the time derivatives. The components $\tilde{\mathbf{A}}_a$ are defined by the solutions of the (12) $\mathbf{A}_b$ system of equations using the formulas:

$$\tilde{\mathbf{A}}_a = e^\alpha_a \xi^b_\alpha \mathbf{A}_b \tag{29}$$

Further solutions of the system of Equation (27) for homogeneous spaces with groups of motions $G_3(I - VI)$ are given. Spatial metrics are given in the book [37]. Solutions for the system (12) can be found in [38],

$$\alpha_a = \alpha_a(u^0).$$

4.1. Group $G_3(I)$

As the parameters k, n, ε and C^a_{bc} equal zero, $G_3(I)$ is an Abelian group. The components of the vector electromagnetic potential have the form:

$$\mathbf{A}_a = \tilde{\mathbf{A}}_a = A_a = \alpha_a,$$

Substituting these expressions into the system of Equations (27) and (28), we obtain the following system of ordinary differential equations:

$$\dot{\beta}^a = 0 \to \beta^a = c^a = const;$$

$$l_0\dot{\alpha}_a = a_{ba}c^b \quad \to \alpha_q = \int \frac{a_{ab}c^b}{l_0} du^0, \quad {l_0}^2 = det|a_{ab}|.$$

All components of a_{ab} are arbitrary functions of u^0.

4.2. Group $G_3(II)$

For the group $G_3(II)$ the parameters k, n, ε have the following values: $k = n = 0, \varepsilon = 1$.

The components of the vector electromagnetic potential in the frames $[\xi^\alpha_a]$ and $[e^\alpha_a]$ have the form:

$$\mathbf{A}_1 = \alpha_1, \quad \mathbf{A}_2 = \alpha_2 + \alpha_1 u^3, \quad \mathbf{A}_3 = \alpha_1 u^3 - \alpha_3; \quad \tilde{\mathbf{A}}_a = \alpha_a.$$

Substituting these expressions into the system of Equations (27) and (28), we obtain the following system of ordinary differential equations:

$$l_0\dot{\beta}_a = \alpha_1 a_{11}\delta_{1a} \to l_0\dot{\beta}_1 = \alpha_1 a_{11}, \quad \beta_2 = c_2, \quad \beta_3 = c_3 \quad (\beta_a = \delta_{ab}\beta^b); \tag{30}$$

$$l_0\dot{\alpha}_a = a_{1a}\beta_1 + a_{2a}c_2 + a_{3a}c_3, \quad {l_0}^2 = det|a_{ab}| \quad (c_a = const,). \tag{31}$$

Set of equations(30) and (31) contains five equations for 11 functions:

$$l_0, \quad a_{ab}, \quad \alpha_a, \quad \beta_1.$$

We should consider separately the variants $\alpha_1 = 0$ and $\alpha_1 \neq 0$.

1. $\alpha_1 = 0 \to \beta_1 = c_1 = const$. Then the set of Equations (30) and (31) has a quadrature solution:

$$\alpha_q = \int \frac{a_{qb}c_{b_1}\delta^{bb_1}}{l_0} du^0 \quad (q = 2, 3).$$

For $a = 0$, Equation (31) implies a linear dependence of the components a_{1q} :

$$c_1 a_{11} + c_2 a_{12} + c_3 a_{13} = 0.$$

All independent components of a_{ab} are arbitrary functions of u^0.

2. $\alpha_1 \neq 0$. Consider the following Equations (30) and (31) from the system:

$$l_0\dot{\alpha}_1 = (a_{11}\beta_1 + c_2 a_{12} + c_3 a_{13}), \quad l_0\dot{\beta}_1 = a_{11}\alpha_1. \tag{32}$$

Let us take the function a_{11} out of (32). As a result, we obtain:

$$({\alpha_1}^2 - {\beta_1}^2)_{,0} = \frac{2\alpha_1}{l_0}(c_2 a_{12} + c_3 a_{13}).$$

Hence:

$$\beta_1 = \xi\sqrt{{\alpha_1}^2 - 2\int \frac{\alpha_1}{l_0}(c_2 a_{12} + c_3 a_{13})du^0} \quad (\xi^2 = 1).$$

>From the remaining equations of the system, we get:

$$\alpha_q = \int \frac{(a_{1q}\beta_1 + a_{2q}c_2 + a_{3q}c_3)}{l_0}du^0 \quad (q = 2, 3); \quad a_{11} = \frac{l_0\dot{\beta}_1}{\alpha_1}.$$

The functions l_0, α_1, and all components of a_{ab}, except a_{11}, a_{33}, are arbitrary functions of u_0. The component a_{33} results from the equation ${l_0}^2 = det|a_{ab}|$:

$$a_{33} = \frac{{l_0}^2 + a_{11}{a_{23}}^2 + a_{22}{a_{13}}^2 - 2a_{12}a_{13}a_{23}}{a_{11}a_{22} - {a_{12}}^2} \tag{33}$$

4.3. Group $G_3(III)$

For the group $G_3(III)$ the parameters k, n, ε have the following values: $k = 1$, $n = \varepsilon = 0$.

The components of the vector electromagnetic potential in the frames $[\xi^\alpha_a]$ and $[e^\alpha_a]$ have the form:

$$\mathbf{A}_1 = \alpha_1 \exp u^3, \quad \mathbf{A}_2 = \alpha_2, \quad \mathbf{A}_3 = \alpha_1 \exp u^3 - \alpha_3.$$

Substituting these expressions into the system of Equations (27) and (28), we obtain the following system of ordinary differential equations:

$$l_0\dot{\beta}_a = \alpha_1 a_{12}\delta_{2a} \to l_0\dot{\beta}_2 = \alpha_1 a_{12}, \quad \beta_1 = c_1, \quad \beta_3 = 0; \tag{34}$$

$$l_0\dot{\alpha}_a = a_{2a}\beta_2 + a_{1a}c. \tag{35}$$

Here and further, Equation (16) is used, according to which $\beta_3 = 0$. The system of Equations (30) and (31) contains five equations for 11 functions:

$$l_0, \quad a_{ab}, \quad \alpha_a, \quad \beta_2.$$

We should separately consider the variants $\alpha_1 = 0$ and $\alpha_1 \neq 0$.

1. $\alpha_1 = 0 \to \beta_2 = c_2 = const$. In this case the
Then set of equations (30) and (31) has a solution in quadratures:

$$\alpha_q = \int \frac{a_{qb}c_{b_1}\delta^{bb_1}}{l_0}du^0 \quad (q = 2, 3).$$

>From (31) it follows a linear dependence of the components a_{1q} :

$$c_1 a_{13} + c_2 a_{23} = 0 \to a_{12} = ba_{11}, \quad \beta_1 = b, \quad \beta_2 = 1.$$

l_0 and all independent components of a_{ab} are arbitrary functions of u^0. The component a_{33} is found from Equation (33).

2. Let $\alpha_1 \neq 0$. Consider the following equations from system (30) and (31):

$$l_0\dot{\alpha}_1 = a_{12}\beta_2 + c_1 a_{11}, \quad l_0\dot{\beta}_2 = a_{12}\alpha_1. \tag{36}$$

from system (36), it follows:

$$({\alpha_1}^2 - {\beta_2}^2)_{,0} = \frac{2\alpha_1}{l_0} c_1 a_{11}.$$

Hence:

$$\beta_2 = \xi\sqrt{{\alpha_1}^2 - 2\int \frac{\alpha_1}{l_0}(c_1 a_{11} + c_3 a_{13})du^0}(\xi^2 = 1).$$

>From the remaining equations of the system, we get:

$$\alpha_q = \int \frac{(a_{2q}\beta_2 + a_{1q}c_1 + a_{3q}c_3)}{l_0} du^0 \quad (q = 2,3); \quad a_{11} = \frac{l_0\dot{\beta}_2}{\alpha_1}.$$

The functions l_0, α_1 and all components of a_{ab}, except $a_{11}, a_{33},$ are arbitrary functions of u_0. The component a_{33} results from Equation (33).

4.4. Group $G_3(IV)$

For the group $G_3(IV)$ the parameters k, n, ε have the values: $k = n = \varepsilon = 1$.

The components of the vector electromagnetic potential in the frames $[\xi^\alpha_a]$ and $[e^\alpha_a]$ have the form:

$$\mathbf{A}_1 = \alpha_1 \exp u^3, \quad \mathbf{A}_2 = (\alpha_2 + \alpha_1 u^3)\exp u^3,$$
$$\mathbf{A}_3 = (\alpha_1(u^1 + u^2 + u^2u^3) + \alpha_2 u^2)\exp u^3 - \alpha_3;$$
$$\tilde{\mathbf{A}}_a = \alpha_a.$$

Maxwell's Equations (20) and (21) reduce to the following system:

$$l_0\dot{\beta}_a = \delta_{1a}(a_{11}(\alpha_1 + \alpha_2) - \alpha_1 a_{22} + \alpha_2 a_{12}) + \delta_{2a}(\alpha_1 a_{12} - a_{11}(\alpha_1 + \alpha_2)). \tag{37}$$

$$l_0\dot{\alpha}_a = \beta_2 a_{a2} + \beta_1 a_{a1}, \quad \beta_3 = 0. \tag{38}$$

from the system (38) it follows:

$$\dot{\alpha}_3 = \int \frac{\beta_2 a_{32} + \beta_1 a_{31}}{l_0} du^0. \tag{39}$$

Let us now consider the remaining equations.

(A) $\beta_1 \neq 0$.

>From the system (37) it follows:

$$a_{12} = \frac{1}{\beta_1}(l_0\dot{\alpha}_2 - \beta_2 a_{22}) \quad a_{11} = \frac{1}{{\beta_1}^2}(l_0(\dot{\alpha}_1\beta_1 - \dot{\alpha}_2\beta_2) + {\beta_2}^2 a_{22}), \tag{40}$$

Using these relations, we obtain a consequence from the remaining equations of the system (37) and (38):

$$\beta_1\dot{\beta}_2 - \beta_2(\dot{\beta}_1 + \dot{\beta}_2) = \alpha_1\dot{\alpha}_2 - (\alpha_1 + \alpha_2)\dot{\alpha}_1. \tag{41}$$

With Equation (41), the dependent functions α_a, β_a can be expressed in terms of the independent functions. Let us write down the solutions.

1. $(\alpha_1\beta_1 + \beta_2(\alpha_1 + \alpha_2))\beta_2 \neq 0$.

$$\beta_1 = \beta_2(b - \ln\beta_2 - \int \frac{\alpha_1\dot{\alpha}_2 - (\alpha_1 + \alpha_2)\dot{\alpha}_1}{{\beta_2}^2} du^0);$$

$$a_{22}=\frac{l_0(\dot{\alpha}_2(\alpha_1+\alpha_2)-\beta_1(\dot{\beta}_1+\dot{\beta}_2))}{\alpha_1\beta_1+\beta_2(\alpha_1+\alpha_2)}.$$

$l_0, a_{13}, a_{23}, \varphi$ are arbitrary functions of time. The function a_{33} is expressed in terms of these functions using the relation (33)

2. $\alpha_1\beta_1+\beta_2(\alpha_1+\alpha_2)=0, a_{22}$, is an arbitrary function, depending on u^0.

$$\alpha_1=a\exp\varphi+b\exp\varphi,\quad \alpha_2=(1+e)\alpha_1\quad \beta_2=a\exp\varphi-b\exp\varphi,$$

$$\beta_1=e\beta_2\quad (e=const).$$

$l_0, a_{13}, a_{23}, \varphi$ are arbitrary functions of time. The function a_{33} is expressed in terms of these functions using the relation (33).

3. $\beta_2=0$.

$$\alpha_2=\alpha_1(a+\ln\alpha_1),\quad a_{12}=\frac{l_0\dot{\alpha}_2}{\beta_1},\quad a_{11}=\frac{l_0\dot{\alpha}_1}{\beta_1},\quad a_{22}=\frac{l_0(\dot{\alpha}_2(\alpha_1+\alpha_2)-\dot{\beta}_1\beta_1)}{\alpha_1\beta_1}$$

$l_0, a_{13}, a_{23}, \alpha_1, \beta_1$ are arbitrary functions of time. The function a_{33} is expressed in terms of these functions using the relation (33).

(B) $\beta_1=0$. Maxwell's equations take the form:

$$l_0\dot{\beta}_2=\alpha_1a_{12}-(\alpha_1+\alpha_2)a_{11},\quad l_0\dot{\beta}_2=-\alpha_1a_{22}+(\alpha_1+\alpha_2)a_{12};$$

$$l_0\dot{\alpha}_1=\beta_2a_{12},\quad l_0\dot{\alpha}_2=\beta_2a_{22}.$$

The set of equations has the following

(a) $(\alpha_1+\alpha_2)\neq 0$.

$$\beta_2=\xi\sqrt{b+2\int\frac{1}{l_0}(\dot{\alpha}_1(\alpha_1+\alpha_2)-\alpha_1\dot{\alpha}_2)du_0}.\quad a_{12}=\frac{l_0\dot{\alpha}_1}{\beta_2},\quad a_{22}=\frac{l_0\dot{\alpha}_2}{\beta_2}.$$

$$a_{11}=\frac{l_0(\alpha_1\dot{\alpha}_1-\beta_2\dot{\beta}_2)}{\beta_2(\alpha_1+\alpha_2)}$$

$l_0, a_{13}, a_{23}, \alpha_1, \alpha_2$ are arbitrary functions of time. The function a_{33} is expressed in terms of these functions using relation (33).

(b) $\alpha_2=-\alpha_1\to\alpha_1=a\exp\varphi-b\exp\varphi\quad \beta_2=a\exp\varphi+b\exp\varphi,\quad a_{12}=\frac{l_0\dot{\alpha}_1}{\beta_2}$ $a_{22}=\frac{l_0\dot{\alpha}_2}{\beta_2}$.

$l_0, a_{11}, a_{13}, a_{23}, \varphi, \beta_1$ are arbitrary functions of time. The function a_{33} is expressed in terms of these functions using the relation (33).

4.5. Group $G_3(V)$

For the group $G_3(V)$ the parameters k, n, ε have the values: $k=n=1, \varepsilon=0$. The components of the vector electromagnetic potential in the frames $[\xi^\alpha_a]$ and $[e^\alpha_a]$ have the form:

$$\mathbf{A}_1=\alpha_1\exp u^3,\quad \mathbf{A}_2=\alpha_2\exp u^3,\quad \mathbf{A}_3=(\alpha_1u^1+\alpha_2u^2)\exp u^3-\alpha_3;$$

$$\tilde{\mathbf{A}}_a=\alpha_a.$$

Maxwell's Equation (18) reduces to the following system of equations:

$$l_0\dot{\alpha}_a=\beta_2a_{a2}+\beta_1a_{a1},\quad \beta_3=0. \tag{42}$$

$$l_0\dot{\beta}_a = \delta_{1a}(a_{12}\alpha_2 - \alpha_1 a_{22}) + \delta_{2a}(a_{12}\alpha_1 - a_{11}\alpha_2), \tag{43}$$

Hence:

$$\dot{\alpha}_3 = \int \frac{\beta_2 a_{32} + \beta_1 a_{31}}{l_0} du^0,$$

$$l_0\dot{\alpha}_1 = (a_{11}\beta_1 + a_{12}\beta_2), \quad l_0\dot{\alpha}_2 = (a_{12}\beta_1 + a_{22}\beta_2). \tag{44}$$

1. $\alpha_1 \neq 0$. From the set of equations (43) it follows:

$$a_{12} = \frac{1}{\alpha_1}(l_0\dot{\beta}_2 + \alpha_2 a_{11}), \quad a_{22} = \frac{1}{\alpha_1{}^2}(l_0(\dot{\beta}_2\alpha_2 - \dot{\beta}_1\alpha_1) + a_{11}\alpha_2{}^2). \tag{45}$$

Substituting (45) into (44) , we get the corollary:

$$\beta_1\dot{\beta}_2 - \beta_2\dot{\beta}_1 = \alpha_1\dot{\alpha}_2 - \alpha_2\dot{\alpha}_1. \tag{46}$$

$$a_{11}(\alpha_1\beta_1 + \alpha_2\beta_2) = l_0(\dot{\alpha}_1\alpha_1 - \dot{\beta}_2\beta_2). \tag{47}$$

>From (46), it follows:

$$\alpha_2 = \alpha_1(b + \int \frac{\beta_1\dot{\beta}_2 - \beta_2\dot{\beta}_1}{\alpha_1{}^2} du^0),$$

Let us consider (48).

(a) $\alpha_1\beta_1 + \alpha_2\beta_2 \neq 0$. Then, we have:

$$a_{11} = \frac{l_0(\alpha_1\dot{\alpha}_2 - \alpha_2\dot{\alpha}_1)}{\alpha_1\beta_1 + \alpha_2\beta_2};$$

$l_0, a_{13}, a_{23}, \alpha_1, \beta_a$ are arbitrary functions of time. The function a_{33} is expressed in terms of these functions using the relation (33).

(b) $\alpha_1\beta_1 + \alpha_2\beta_2 = 0 \quad \to \alpha_1\dot{\alpha}_1 - \beta_1\dot{\beta}_1 = 0, \quad \alpha_1\dot{\alpha}_2 + \beta_2\dot{\beta}_1 = 0.$
>From this, it follows:

$$\alpha_1 = a\exp\varphi + b\exp\varphi, \quad \beta_2 = a\exp\varphi - b\exp\varphi, \quad \alpha_2 = -l\alpha_1, \quad \beta_1 = l\beta_2,$$

where $a, b, l = const, \varphi = \varphi(u^0)$.
$l_0, a_{11}, a_{13}, a_{23}$ are arbitrary functions of time. The function a_{33} is expressed in terms of these functions using the relation (33).

2. $\alpha_1 = 0$. From the system (43), it follows:

$$a_{12} = \frac{l_0\dot{\beta}_1}{\alpha_2}, \quad a_{11} = -\frac{l_0\dot{\beta}_2}{\alpha_2}, \quad a_{22} = \frac{l_0(\dot{\alpha}_2\alpha_2 - \dot{\beta}_1\beta_1)}{\alpha_2\beta_2}, \quad \beta_1 = a\beta_2,$$

here $a = const, l_0, a_{13}, a_{23}, \alpha_2, \beta_2$ are arbitrary functions of time. The function a_{33} is expressed in terms of these functions using the relation (33).

4.6. Group $G_3(VI)$

For the group $G_3(VI)$, the parameters k, n, ε have the following values: $k = 1$ $n = 2, \varepsilon = 0$. The components of the vector electromagnetic potential in the frames $[\xi^\alpha_a]$ and $[e^\alpha_a]$ have the form:

$$\mathbf{A}_1 = \alpha_1 \exp u^3, \quad \mathbf{A}_2 = \alpha_2 \exp 2u^3, \quad \mathbf{A}_3 = \alpha_1 u^1 \exp u^3 + 2\alpha_2 u^2 \exp 2u^3 - \alpha_3;$$

$$\tilde{\mathbf{A}}_a = \alpha_a.$$

Maxwell's Equation (18) has the form:

$$l_0\dot{\alpha}_a = \beta_2 a_{a2} + \beta_1 a_{a1}. \quad (48)$$

$$l_0\dot{\beta}_a = \delta_{1a}(a_{12}\alpha_2 - 2\alpha_1 a_{22}) + \delta_{2a}(a_{12}\alpha_1 - 2a_{11}\alpha_2), \quad \beta_3 = 0, \quad (49)$$

and from the system (48), it follows:

$$\dot{\alpha}_3 = \int \frac{\beta_2 a_{32} + \beta_1 a_{31}}{l_0} du^0.$$

$$l_0\dot{\alpha}_1 = (a_{11}\beta_1 + a_{12}\beta_2), \quad l_0\dot{\alpha}_2 = (a_{12}\beta_1 + a_{22}\beta_2). \quad (50)$$

I $\beta_1 \neq 0$, from system (48), it follows:

$$a_{12} = \frac{1}{\beta_1}(l_0\dot{\alpha}_2 - \beta_2 a_{22}), \quad a_{11} = \frac{1}{{\beta_1}^2}(l_0(\dot{\alpha}_1\beta_1 - \dot{\alpha}_2\beta_2) + a_{22}{\beta_2}^2). \quad (51)$$

Substituting (51) into (48), we get:

$$a_{22}(\alpha_1\beta_1 + 2\alpha_2\beta_2) = l_0(\alpha_2\dot{\alpha}_2 - \dot{\beta}_1\beta_1), \quad (52)$$

$$(2\alpha_1\beta_1 + \alpha_2\beta_2)(2\alpha_2\dot{\alpha}_1 + \dot{\beta}_2\beta_1) = (\dot{\beta}_1\beta_2 + 2\dot{\alpha}_2\alpha_1)(\alpha_1\beta_1 + 2\alpha_2\beta_2) = 0 \quad (53)$$

Using this relation, we get the following solutions:

(1) $\alpha_1\beta_1 + 2\alpha_2\beta_2 \neq 0$. From (52) it follows:

$$a_{22} = \frac{l_0(\dot{\alpha}_2\alpha_2 - \dot{\beta}_1\beta_1)}{(\alpha_1\beta_1 + 2\alpha_2\beta_2)}.$$

Denote:

$$\alpha_q = a_q \exp\varphi. \quad \beta_q = b_q \exp\varphi \quad (q = 1, 2),$$

where a_q, b_q, φ are functions of u^0. From Equation (53), we get:

$$\dot{\varphi} = \frac{(\dot{b}_1 b_2 + 2\dot{a}_2 a_1)(a_1 b_1 + 2a_2 b_2) - (2a_1 b_1 + a_2 b_2)(2a_2\dot{a}_1 + \dot{b}_2 b_1)}{(2a_1 a_2 + b_1 b_2)(a_1 b_1 - a_2 b_2)};$$

$$a_{12} = \frac{l_0(\dot{\varphi}a_2 + a_2) - b_2 a_{22}}{b_1}; \quad a_{11} = \frac{l_0((a_1 b_1 - a_2 b_2)\dot{\varphi} + \dot{a}_1 b_1 - \dot{a}_2 b_2) + {b_2}^2 a_{22}}{{b_1}^2};$$

$$a_{22} = \frac{l_0((a_2^2 - b_1^2)\dot{\varphi} + \dot{a}_2 a_2 - \dot{b}_1 b_1)}{2a_1 b_1 + a_2 b_2}.$$

$l_0, a_{13}, a_{23}, a_q, b_q$ are arbitrary functions dependent on time. The function a_{33} is expressed by these functions using the relation (33)

(2) $\dot{\alpha}_2\alpha_2 - \dot{\beta}_1\beta_1 = 0 \quad \rightarrow \quad \alpha_1\beta_1 + 2\alpha_2\beta_2 = 0.$ a_{22}—is an arbitrary function from u^0;

$$\alpha_2 = a\exp\varphi - b\exp(-\varphi), \quad \beta_1 = a\exp\varphi + b\exp(-\varphi).$$

>From this, it follows:

(a)

$$\alpha_1 = -\frac{\beta_2}{2}\left(\frac{a\exp\varphi - b\exp(-\varphi)}{a\exp\varphi + b\exp(-\varphi)}\right);$$

$$a_{12} = l_0\dot{\varphi} - \frac{\beta_2 a_{22}}{\beta_1}, \quad a_{11} = \frac{l_0(\dot{\alpha}_1\beta_1 - \dot{\alpha}_2\beta_2) + \beta_2^2 a_{22}}{\beta_1^2}$$

(b) $\dot{\varphi} = 0$

$$\beta_1 = 1, \quad \alpha_2 = -2b, \quad \alpha_1 = -b\beta_2, \quad a_{12} = -\beta_2 a_{22}, \quad a_{11} = -bl_0\dot{\beta}_2 + {\beta_2}^2 a_{22}.$$

where $l_0, a.b = const a_{22}, a_{13}, a_{23}, \beta_2, \varphi$ are arbitrary functions dependent on time.

II $\beta_1 = 0$.
>From (48) and (49) it follows:

$$a_{12} = \frac{2l_0\dot{\alpha}_2\alpha_2}{\beta_2}, \quad a_{22} = \frac{l_0\dot{\alpha}_2}{\beta_2}, \quad a_{11} = \frac{l_0(2b^2\dot{\alpha}_2{\alpha_2}^3 - \beta_2\dot{\beta}_2)}{2\alpha_2\beta_2}, \qquad \alpha_1 = b{\alpha_2}^2. \quad (54)$$

$l_0, a_{22}, a_{13}, a_{23}, \alpha_2\beta_2$ depends arbitrarily on time functions. The function a_{33} is expressed in terms of these functions using the relation (33).

5. Conclusions

The performed classification of admissible electromagnetic fields will be used in the search for electrovacuum solutions of the Einstein–Maxwell equations. As is already known, the components of the Ricci tensor of a homogeneous space in the frame (2) depend only on time. In order for Einstein's equations with matter to be proven as an integrable system of ordinary differential equations, the equations of motion of matter must be subordinated to the conditions of space symmetry. These conditions were fulfilled first by the potentials of the electromagnetic fields determined in this work.

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