

General Deflections in Deflected AMSB

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Abstract: The (extra)ordinary gauge mediation extension of deflected AMSB scenarios can be interesting because it can accommodate together the deflection in the Kahler potential and the superpotential. We derive the analytical expressions for soft SUSY breaking parameters in such EOGM extension of deflected AMSB scenarios with the presence of both types of deflections. The Landau pole and proton decay constraints are also discussed.

Keywords: supersymmetry; AMSB

1. Introduction

Weak-scale supersymmetry (SUSY), which is a leading candidate for physics beyond the standard model (SM), can solve elegantly the hierarchy problem of the Higgs boson by introducing various superpartners at the TeV scale. Moreover, the unification of gauge couplings, which cannot be exact in SM, can be successful in its SUSY extensions. The dark matter (DM) puzzle as well as the baryon asymmetry of universe, can also be explained with proper DM candidates and baryogenesis mechanisms in SUSY. It is worth to note that the Higgs scalar, which was discovered by the ATLAS [1] and CMS [2] collaborations of LHC in 2012, lie miraculously in the small ‘115–135’ GeV window predicted by the low-energy SUSY. (See [3–8] for excellent reviews on SUSY.)

It is well known that the soft SUSY spectrum, including the gaugino and sfermion masses, are determined by the SUSY breaking mechanism. Depending on the way the visible sector ‘feels’ the SUSY breaking effects in the hidden sector, the SUSY breaking mechanisms can be classified into gravity mediation [9–11], gauge mediation [12], anomaly mediation [13,14] scenarios, etc. Gauge-mediated SUSY breaking (GMSB) scenarios, which will not cause flavor and CP problems that bother gravity mediation models, are calculable, predictive, and phenomenologically distinctive with a minimal messenger sector. However, unless additional messenger–matter interactions are introduced, minimal GMSB can hardly explain the 125 GeV Higgs with TeV scale soft SUSY breaking parameters because of the vanishing trilinear terms at the messenger scale. An interesting extension is the (extra)ordinary gauge mediation (EOGM) scenarios [15,16], in which the messenger sector can include all renormalizable, gauge-invariant couplings between the messengers and any number of singlet fields. In fact, many examples in the literature of OGM deformed by mass terms can fall into this category, and their generic properties can be obtained therein.

Gravity can generate the soft SUSY breaking masses by the auxiliary field of the compensator multiplet. Such a ‘pure’ gravity mediation scenario with negligible contributions from direct non-renormalizable contact terms is called the anomaly-mediated SUSY breaking (AMSB). Pure anomaly mediation is bothered by the tachyonic slepton problem [17]. Its non-trivial extensions with messenger sectors, namely the deflected AMSB [18–20], can elegantly solve such tachyonic slepton problems through the deflection of the renormalization group equation (RGE) trajectory [21]. There are two types of deflections in the literature, the deflection in the superpotential [18,20,22–24] and the deflection in the Kahler potential [25–29], respectively. However, it is difficult to determine consistently the deflection parameter ‘ d ’ and soft SUSY parameters if both deflections are present. So, it is



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very interesting theoretically to solve such a difficulty systematically. A special example of EOGM extension of deflected AMSB had been given in [30]. However, no systematical treatments and analytical results of the previously mentioned general case are given yet.

We find that the most general deflection scenario in deflected AMSB can be seen as a special case of the EOGM extension of deflected AMSB. We propose to introduce proper auxiliary spurion superfields and derive successfully the analytical expressions of the soft SUSY breaking parameters in the framework of EOGM extension of the deflected AMSB scenarios with both Kahler and superpotential deflections. Besides, special attentions are paid to the case with multi-scales messengers, especially the case with a vanishing beta function at an intermediate scale and the case with $\det(\lambda_{ij}) = \det(m_{ij}) = 0$.

This paper is organized as follows. In Section 2, we discuss the deflected AMSB scenario with EOGM extension for scenarios with both Kahler and superpotential deflections. In Section 3, we discuss the analytical expressions of soft SUSY parameters in EOGM extended deflected AMSB. In Section 4, constraints from the Landau pole, etc., with multi-scales messengers are given. Section 5 contains our conclusions.

2. Effective EOGM Form in Deflected AMSB

To fully understand the deflected AMSB scenarios with the presence of both the superpotential and Kahler potential deflections, we need to obtain the effective deflection parameter ‘ d ’ to derive the full expressions of the soft SUSY breaking parameters. We find that such an effective deflection parameter ‘ d ’ can be obtained in the framework of EOGM extension of deflected AMSB.

In the deflected AMSB, the Kahler potential can have the following types of deflection with holomorphic terms for messengers

$$K \supseteq \kappa'_{ij} \frac{\phi^\dagger}{\phi} \tilde{P}_i P_j + h.c. , \tag{1}$$

or the deflection from the couplings in the superpotential

$$W \supseteq \lambda'_{ij} \tilde{X} \tilde{Q}_i Q_j + W(\tilde{X}) , \tag{2}$$

with ϕ the conformal compensator field that carries the SUSY breaking information in the SUSY breaking sector

$$\phi = 1 + \theta^2 F_\phi . \tag{3}$$

A proper form of superpotential $W(\tilde{X})$ is assumed for the pseudo-moduli field \tilde{X} to determine the deflection parameter \tilde{d}

$$\frac{F_{\tilde{X}}}{\tilde{X}} = (\tilde{d} + 1) F_\phi , \tag{4}$$

and the messenger threshold scale \tilde{X} . We should note that the messengers \tilde{P}_i, P_i can possibly be identified to be the \tilde{Q}_i, Q_i superfields.

To accommodate both types of deflections, we need to extract the total GMSB contributions in addition to the AMSB-type contributions, especially in the case that several SUSY breaking sources can contribute via the messenger sector. Combining both the Kahler and superpotential deflections, we have

$$\mathcal{L} \supseteq \int d^4\theta \left[\kappa'_{ij} \frac{\phi^\dagger}{\phi} \tilde{P}_i P_j + h.c. \right] + \int d^2\theta \left[\lambda'_{ij} \tilde{X} \tilde{Q}_i Q_j + W(\tilde{X}) \right] + h.c. \tag{5}$$

We propose to rewrite this expression in the form

$$\mathcal{L} \supseteq \int d^2\theta (\lambda'_{ij} \tilde{X} + \kappa'_{ij} \tilde{T}) \tilde{\phi}_i \phi_j + W(\tilde{X}) + h.c. , \quad i, j = 1, \dots, N \tag{6}$$

with \tilde{T} a new spurion superfield (\tilde{T} can have an R-symmetry charge $R(T) = 2$ as the combination $\tilde{P}_i P_j$ has vanishing R-charge) introduced to incorporate the Kahler deflection effects with its VEV

$$\langle \tilde{T} \rangle = F_\phi - \theta^2 F_\phi^2 \tag{7}$$

and ϕ_i being P_i or Q_i . Knowing the spurion VEV, the GMSB contributions could be determined in the framework of EOGM to include several SUSY breaking sources.

We can rotate the spurion superfields \tilde{X} and \tilde{T} so that only one combination X will acquire F-term VEVs while T will acquire only the lowest component VEVs

$$\begin{aligned} X &= \frac{1}{\sqrt{F_{\tilde{X}}^2 + F_\phi^4}} [F_{\tilde{X}} \tilde{X} - F_\phi^2 \tilde{T}] , \\ T &= \frac{1}{\sqrt{F_{\tilde{X}}^2 + F_\phi^4}} [F_\phi^2 \tilde{X} + F_{\tilde{X}} \tilde{T}] . \end{aligned} \tag{8}$$

So, the superpotential can be rewritten as

$$W \supseteq (\lambda_{ij} X + m_{ij}) \tilde{\phi}_i \phi_j , \tag{9}$$

with

$$\begin{aligned} \lambda_{ij} &= \frac{1}{\sqrt{F_{\tilde{X}}^2 + F_\phi^4}} (\lambda'_{ij} F_{\tilde{X}} - \kappa'_{ij} F_\phi^2) , \\ m_{ij} &= \frac{T}{\sqrt{F_{\tilde{X}}^2 + F_\phi^4}} (\lambda'_{ij} F_\phi^2 + \kappa'_{ij} F_{\tilde{X}}) . \end{aligned} \tag{10}$$

So, the coupling of the spurions to the messenger sector takes the form of EOGM.

It is reasonable to assume that the superpotential (9) has a non-trivial R-symmetry

$$\begin{cases} \lambda_{ij} \neq 0, & \text{only if } R(\tilde{\phi}_i) + R(\phi_j) = 2 - R(X) , \\ m_{ij} \neq 0, & \text{only if } R(\tilde{\phi}_i) + R(\phi_j) = 2 , \end{cases} \tag{11}$$

which can prevent destructive D-term contributions to sfermion masses. As noted in [15], we can choose $R(X) = 2$ without loss of generality by the mixing of the R-symmetry with the trivial $U(1)_R$. After integrating out the messenger fields, the messenger determinant is proven by [15] to be a monomial in X

$$\det(\lambda'_{ij} \tilde{X} + \kappa'_{ij} \tilde{T}) \equiv \det(\lambda_{ij} X + m_{ij}) = X^{n_0} G(\lambda_{ij}, m_{ij}) , \tag{12}$$

with

$$n_0 = \frac{1}{R(X)} \sum_{i=1}^n [2 - R(\tilde{\phi}_i) + R(\phi_i)] . \tag{13}$$

Note that the GMSB contribution of the most general deflection scenario in AMSB is quite similar to that of EOGM in GMSB, except that the VEV of X is given by

$$\sqrt{F_{\tilde{X}}^2 + F_\phi^4} \langle X \rangle = F_\phi [(\tilde{d} + 1) \tilde{X}^2 - F_\phi^2] + (F_{\tilde{X}}^2 + F_\phi^4) \theta^2 , \tag{14}$$

from (8).

The previous discussions can be generalized to the case with multiple pseudo-moduli \tilde{X}_a superfields in the superpotential, which are all very flat in the SUSY limit and can couple differently to the messengers. Each \tilde{X}_i will acquire its corresponding F-term VEV to give

additional GMSB-type contributions to the soft SUSY breaking parameters in addition to AMSB contributions. We have

$$\mathcal{L} \supseteq \int d^4\theta \left[\bar{\kappa}_{ij} \frac{\phi^\dagger}{\phi} \tilde{P}_i P_j \right] + \int d^2\theta \sum_a \left[\bar{\lambda}_{ij}^a \tilde{X}_a \tilde{Q}_i Q_j \right] + W(\tilde{X}_1, \dots, \tilde{X}_a), \tag{15}$$

which can be rewritten as

$$\mathcal{L} \supseteq \int d^2\theta \left(\sum_a \bar{\lambda}_{ij}^a \tilde{X}_a + \bar{\kappa}_{ij} \tilde{T} \right) \tilde{\phi}_i \phi_j + W(\tilde{X}_1, \dots, \tilde{X}_a). \quad i, j = 1, \dots, N \tag{16}$$

After associating each dimensional parameter in the superpotential $W(\tilde{X}_1, \dots, \tilde{X}_a)$ with proper power of compensator field ϕ and substituting the expression of the compensator VEV $\phi = 1 + \theta^2 F_\phi$, one can obtain the VEV of each \tilde{X}_a by minimizing the whole scalar potential. The VEV for each \tilde{X}_a can be parameterized as

$$\langle \tilde{X}_a \rangle = M_a + \theta^2 F_a. \tag{17}$$

Similar to (8), one can perform a unitary transformation for \tilde{X}_a, \tilde{T} superfields so that only one of the combinations can have an F-term VEV, while the other combinations only have the lowest component VEVs. So, the resulting superpotential involving the messengers can again be written in the form

$$W \supseteq (\lambda_{ij} X + m_{ij}) \tilde{\phi}_i \phi_j, \tag{18}$$

with similar discussions from (11) to (13).

We will discuss the soft SUSY breaking parameters in our scenarios. The VEV of X , which is determined by the dynamics of X and the unitary transformation of X_k , can take the following form

$$\langle X \rangle = X_0 + \theta^2 F_X. \tag{19}$$

For multiple messenger species with the general form of the coupling

$$W \supseteq (\lambda_{ij} X + m_{ij}) \bar{\Psi}_i \Psi_j, \tag{20}$$

we can in principle diagonalize the non-singular mass matrix $\mathcal{M}_{ij} \equiv \lambda_{ij} X + m_{ij}$ by $U_{ik}^\dagger \mathcal{M}'_{diag;kl} V_{lj}$, so the couplings can be recast into the form

$$W \supseteq \sum_{i=1}^n M_i(X) \bar{\Psi}'_i \Psi'_i, \tag{21}$$

with $M_i(X)$ the corresponding eigenvalues and

$$\bar{\Psi}'_i = U_{ij}^\dagger \bar{\Psi}_j, \quad \Psi'_i = V_{ij} \Psi_j. \tag{22}$$

Simply, we choose $\bar{\Psi}_i, \Psi_i$ to lie in the $\bar{\mathbf{5}}$ and $\mathbf{5}$ representations of SU(5), respectively. Because of the R-symmetry, the eigenvalue $M_i(X)$ has to be a monomial in X , which can only take the following form (see the discussions in Section 3.3):

$$M_i(X) = \begin{cases} a_i X, \\ b_i, \\ c_i X^{-m}, \end{cases}$$

with $0 \leq m \leq [(r_\lambda - n_0)/2]$ and $r_\lambda \equiv rank(\lambda_{ij})$. We can associate each dimensional parameter with a proper power of the compensator field ϕ to obtain the deflection parameter related to the messenger $\bar{\Psi}'_i$ and Ψ'_i

$$d_i = \begin{cases} \frac{F_X}{X F_\phi} - 1 \equiv d_U, & \text{for } M_i(X) = a_i X, \\ 0, & \text{for } M_i(X) = b_i, \\ m - m \frac{F_X}{X F_\phi} = -m \cdot d_U, & \text{for } M_i(X) = c_i X^{-m}, \end{cases} \tag{23}$$

after substituting the VEV $\langle X \rangle$ and ϕ . Without loss of generality, one can assume that the messenger thresholds satisfy

$$M_1(X_0) \equiv M_1 \gtrsim M_2(X_0) \equiv M_2 \gtrsim \dots \gtrsim M_n(X_0) \equiv M_n. \tag{24}$$

These M_k can be hierarchical or lie at the same order.

So, the soft SUSY breaking parameters can be obtained by the wavefunction renormalization [31] approach.

- The gaugino masses are given as

$$\begin{aligned} M_i &= g_i^2 \left(\frac{F_\phi}{2} \frac{\partial}{\partial \ln \mu} - \sum_{k=1}^n \frac{d_k F_\phi}{2} \frac{\partial}{\partial \ln M_k} \right) \frac{1}{g_i^2} (\mu, M_1, \dots, M_n), \\ &\equiv g_i^2 \left(\frac{F_\phi}{2} \frac{\partial}{\partial \ln \mu} - \sum_{k=1}^n \frac{d_U F_\phi}{2} \frac{\partial \ln M_k}{\partial \ln X} \frac{\partial}{\partial \ln M_k} \right) \frac{1}{g_i^2} (\mu, M_1, \dots, M_n). \end{aligned} \tag{25}$$

- The trilinear couplings $A_0^{ijk} \equiv A_{ijk}/y_{ijk}$ are given as

$$\begin{aligned} A_0^{ijk} &= \sum_i \left(-\frac{F_\phi}{2} \frac{\partial}{\partial \ln \mu} + \sum_{k=1}^n \frac{d_k F_\phi}{2} \frac{\partial}{\partial \ln M_k} \right) \ln[Z_i(\mu, M_1, \dots, M_n)] \Big|_{\mu=M_n}. \\ &\equiv \sum_i \left(-\frac{F_\phi}{2} \frac{\partial}{\partial \ln \mu} + \sum_{k=1}^n \frac{d_U F_\phi}{2} \frac{\partial \ln M_k}{\partial \ln X} \frac{\partial}{\partial \ln M_k} \right) \ln[Z_i(\mu, M_1, \dots, M_n)] \Big|_{\mu=M_n}. \end{aligned}$$

- The soft SUSY breaking scalar masses are given as

$$\begin{aligned} &m_{soft}^2 \\ &= - \left| -\frac{F_\phi}{2} \frac{\partial}{\partial \ln \mu} + \sum_{k=1}^n \frac{d_k F_\phi}{2} \frac{\partial}{\partial \ln M_k} \right|^2 \ln[Z_i(\mu, M_1, \dots, M_k)] \Big|_{\mu=M_n}, \tag{26} \\ &= - \frac{F_\phi^2}{4} \left(\frac{\partial^2}{(\partial \ln \mu)^2} + \sum_{k,j=1}^n d_k d_j \frac{\partial^2}{\partial (\ln M_j) \partial (\ln M_k)} - \sum_{k=1}^n 2d_k \frac{\partial^2}{\partial \ln M_k \partial \ln \mu} \right) \ln[Z_i(\mu, M_1, \dots, M_k)]. \end{aligned}$$

Details of the expression involving the derivatives with respect to $\ln M_i$, etc., can be found in [24,32,33].

It is well known that in the $d \rightarrow \infty$ limit, the anomaly mediation contributions in the deflected AMSB are sub-leading and the gauge mediation contributions are dominant. So, we will derive the EOGM contributions first and return to deflected AMSB cases subsequently.

3. Analytical Expressions within EOGM

For multiple messengers, we assume that the mass thresholds of the N messengers can be degenerated and separated into ' p ' groups as

$$\underbrace{(M_1, \dots, M_1)}_{n_1}, \underbrace{(M_2, \dots, M_2)}_{n_2}, \dots, \underbrace{(M_p, \dots, M_p)}_{n_p}, \tag{27}$$

with $\sum_{i=1}^p n_p = N$.

The gauge couplings at a scale μ below all the messenger thresholds are given as

$$\begin{aligned} \frac{1}{g_i^2(\mu, M_a)} &= \frac{1}{g_i^2(\Lambda)} + \frac{b'_i}{8\pi^2} \ln \frac{\Lambda}{M_1} + \frac{b'_i - n_1}{8\pi^2} \ln \frac{M_1}{M_2} + \frac{b'_i - \sum_{k=1}^2 n_k}{8\pi^2} \ln \frac{M_2}{M_3} + \dots + \frac{b'_i - N}{8\pi^2} \ln \frac{M_p}{\mu}, \\ &= \frac{1}{g_i^2(\Lambda)} + \frac{b'_i}{8\pi^2} \ln \Lambda - \frac{1}{8\pi^2} \ln \det \mathcal{M} - \frac{b'_i - N}{8\pi^2} \ln \mu. \end{aligned} \tag{28}$$

Here we assume that the eigenvalues of the messenger mass matrix are given by $M_1 \geq M_2 \geq \dots \geq M_p$. So, the gaugino mass can be given as

$$\begin{aligned} M_i &= g_i^2 \left(\frac{F_\phi}{2} \frac{\partial}{\partial \ln \mu} - \sum_{a=1}^p \frac{d_a F_\phi}{2} \frac{\partial}{\partial \ln M_a} \right) \frac{1}{g_i^2}(\mu, M_a), \\ &= -F_\phi \frac{g_i^2}{16\pi^2} b_i + d_U F_\phi \frac{g_i^2}{16\pi^2} \frac{\partial \ln \det \mathcal{M}}{\partial \ln X}, \\ &= -F_\phi \frac{g_i^2}{16\pi^2} b_i + d_U F_\phi \frac{g_i^2}{16\pi^2} n_0. \end{aligned} \tag{29}$$

Here $b_i = b'_i - N$, which are given by

$$(b_1, b_2, b_3) = (33/5, 1, -3), \tag{30}$$

with $N = N_5 + 3N_{10}$ and n_0 is given in Equation (12). Besides, the previous expressions also agree with the result

$$\sum_{a=1}^p d_a n_a = n_0, \tag{31}$$

with d_a the deflection parameter with respect to messenger threshold M_a , whose expression is given by (23). The previous results can be rewritten in an alternative form by using the identity

$$\ln \det M = \text{Tr} \ln M$$

for non-singular matrix M . If we assume that the lowest component VEV of $\langle X \rangle$ vanishes and the SUSY breaking parameter F_X is small in comparison to the messenger scale, we have

$$\begin{aligned} n_0 &\equiv \frac{\partial \ln \det \mathcal{M}}{\partial \ln X} = \frac{\partial \text{Tr} \ln \mathcal{M}}{\partial \ln X} = \frac{\partial \text{Tr} \ln(\lambda_{ij} X + m_{ij})}{\partial \ln X} \\ &= \frac{\partial \text{Tr} \ln(\lambda_{ij} X + m_{ij})}{\partial \ln X} \approx \frac{\partial \text{Tr} \ln \left[m_{ij} \left(1 + \frac{\lambda_{ij} X}{m_{ij}} \right) \right]}{\partial \ln X} \approx \sum_i \frac{\lambda_{ii}}{m_{ii}}, \end{aligned} \tag{32}$$

in the basis where m_{ij} is real and diagonal.

For the soft sfermion masses and trilinear couplings, we need the dependence of wavefunction Z_i on each messenger threshold M_a . The derivative of $\ln Z_i$ with respect to $\ln M_a$ can be obtained via

$$\begin{aligned} &\frac{d \ln Z_l[\mu, g_i(\mu'), y_i(\mu'), M_a]}{d \ln M_a} \\ &= \sum_{M_j} \sum_{g_i(\mu')} \frac{\partial \ln g_i(\mu')}{\partial \ln M_a} \frac{\partial \ln Z_l[\mu, g_i(\mu'), y_i(\mu'), M_a]}{\partial \ln g_i(\mu')} + [g_i(\mu') \rightarrow y_a(\mu')] \\ &+ \sum_{M_a} \frac{\partial \ln Z_l[\mu, g_i(\mu'), y_i(\mu'), M_a]}{\partial \ln M_a}. \end{aligned} \tag{33}$$

The sum over $g_i(\mu')$, which depends on the messenger thresholds M_i , take the values $g_i(M_1), g_i(M_2), \dots, g_i(M_p)$. The third term always vanishes because the anomalous dimension is continuous across the messenger thresholds (here we assume no messenger–matter interactions are present).

To obtain the expressions for wavefunction Z_i , we need the following classifications:

- In the case $b'_i - \sum_{r=1}^k n_r \neq 0$ for all $0 \leq k \leq p$, the expression will fall into class A.
- In the case $b'_i - \sum_{r=1}^k n_r = 0$ for some $0 \leq k \leq p$, the expression will fall into class B.

3.1. Class A: Partition without Vanishing Gauge Beta Functions

To obtain Z_i , we can construct an invariant by surveying the anomalous dimension of Z_i

$$\frac{d \ln Z_i}{dt} = -\frac{1}{8\pi^2} \left[d_{ij}^k y_{ijk}^2 - 2C(r)g_i^2 \right], \tag{34}$$

and solve the differential equation in the basis of $(y_t^2, y_b^2, y_\tau^2, g_3^2, g_2^2, g_1^2)$ with

$$\frac{d}{dt} \ln Z_i = \sum_{l=g_3, g_2, g_1} 2\tilde{A}_l \frac{d \ln g_l}{dt} + \sum_{l=y_t, y_b, y_\tau} 2B_l \frac{d \ln y_l}{dt}, \tag{35}$$

at the scale Λ . The expressions of the wavefunction can be solved (for example, see Appendix A in Refs. [34,35] for details) as

$$Z_l(\mu, M_a) = Z_l(\Lambda) \left(\frac{y_t^2(\mu)}{y_t^2(\Lambda)} \right)^{B_t} \left(\frac{y_b^2(\mu)}{y_b^2(\Lambda)} \right)^{B_b} \left(\frac{y_\tau^2(\mu)}{y_\tau^2(\Lambda)} \right)^{B_\tau} \prod_{i=1}^3 \left[\left(\frac{g_i^2(M_1)}{g_i^2(\Lambda)} \right)^{\frac{A_i}{b'_i}} \left(\frac{g_i^2(M_2)}{g_i^2(M_1)} \right)^{\frac{A_i}{b'_i - n_1}} \left(\frac{g_i^2(M_3)}{g_i^2(M_2)} \right)^{\frac{A_i}{b'_i - \sum_{i=1}^2 b_i}} \dots \left(\frac{g_i^2(\mu)}{g_i^2(M_p)} \right)^{\frac{A_i}{b'_i - N}} \right], \tag{36}$$

with the corresponding coefficients \tilde{A}_i given as $\tilde{A}_i \equiv A_i/b'_i, A_i/(b'_i - n_1), \dots$ at the energy interval $[M_i, M_{i+1}]$ ($M_0 \equiv \Lambda$), respectively.

If all the Yukawa terms within the wavefunction Z_i are neglected, the A_i will take the value $4C(r)_i$ with $C(r)_i$, the quadratic Casimir invariant for the superfield Φ_i .

So, we have

$$\ln Z_l(\mu, M_a) = \ln Z_l(\Lambda) + \sum_i \left\{ -\frac{A_i}{b'_i} \ln g_i^2(\Lambda) + \left(\frac{A_i}{b'_i} - \frac{A_i}{b'_i - n_1} \right) \ln g_i^2(M_1) + \left(\frac{A_i}{b'_i - n_1} - \frac{A_i}{b'_i - \sum_{k=1}^2 n_k} \right) \ln g_i^2(M_2) + \dots + \frac{A_i}{b'_i - N} \ln g_i^2(\mu) \right\} + \dots \tag{37}$$

From Equation (37), we obtain

$$\left(\frac{\partial \ln Z_l(\mu, M_a)}{\partial \ln g_i(\mu')} \right) = 2 \left(\frac{A_i}{b'_i} - \frac{A_i}{b'_i - n_1}, \frac{A_i}{b'_i - n_1} - \frac{A_i}{b'_i - \sum_{k=1}^2 n_k}, \dots, \frac{A_i}{b'_i - \sum_{k=1}^{p-1} n_k} - \frac{A_i}{b'_i - N}, \frac{A_i}{b'_i - N} \right), \tag{38}$$

with the column indices $g_i(M_1), g_i(M_2), \dots, g_i(M_p), g_i(\mu)$.

From the expressions of gauge coupling at scale μ within each threshold interval, we can obtain an $(p + 1) \times p$ matrix

$$\left(\frac{\partial \ln g_i(\mu')}{\partial \ln M_j}\right)_{\mu',j} = \frac{1}{16\pi^2} \begin{pmatrix} \ln M_1 & \ln M_2 & \dots & \ln M_p \\ b'_i g_i^2(M_1) & 0 & \dots & 0 \\ n_1 g_i^2(M_2) & (b'_i - n_1) g_i^2(M_2) & \dots & 0 \\ n_1 g_i^2(M_3) & n_2 g_i^2(M_3) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ n_1 g_i^2(M_p) & n_2 g_i^2(M_p) & \dots & (b'_i - \sum_{i=1}^{p-1} n_i) g_i^2(M_p) \\ n_1 g_i^2(\mu) & n_2 g_i^2(\mu) & \dots & n_p g_i^2(\mu) \end{pmatrix} \begin{pmatrix} g_i(M_1) \\ g_i(M_2) \\ g_i(M_3) \\ \dots \\ g_i(M_p) \\ g_i(\mu) \end{pmatrix} \tag{39}$$

So, we have

$$U_b \equiv \left(\frac{\partial \ln Z_l(\mu, M_a)}{\partial \ln M_b}\right)^T \equiv - \sum_{i=1,2,3} \frac{A_i}{8\pi^2} U_{b;i}, \tag{40}$$

$$U_{b;i} = \begin{pmatrix} n_1(P_i[1] + Q_i[1]) \\ n_2(P_i[2] + Q_i[2]) \\ n_3(P_i[3] + Q_i[3]) \\ \dots \\ n_p(P_i[p] + Q_i[p]) \end{pmatrix} \equiv \begin{pmatrix} \left[\frac{n_1 g_i^2(M_1)}{b'_i - n_1} - \frac{n_1 g_i^2(\mu)}{b'_i - N} \right] + \sum_{a=2}^p \frac{n_1 n_a g_i^2(M_a)}{(b'_i - \sum_{k=1}^{a-1} n_k)(b'_i - \sum_{k=1}^a n_k)} \\ \left[\frac{n_2 g_i^2(M_2)}{b'_i - \sum_{k=1}^2 n_k} - \frac{n_2 g_i^2(\mu)}{b'_i - N} \right] + \sum_{a=3}^p \frac{n_2 n_a g_i^2(M_a)}{(b'_i - \sum_{k=1}^{a-1} n_k)(b'_i - \sum_{k=1}^a n_k)} \\ \left[\frac{n_3 g_i^2(M_3)}{b'_i - \sum_{k=1}^3 n_k} - \frac{n_3 g_i^2(\mu)}{b'_i - N} \right] + \sum_{a=4}^p \frac{n_3 n_a g_i^2(M_a)}{(b'_i - \sum_{k=1}^{a-1} n_k)(b'_i - \sum_{k=1}^a n_k)} \\ \dots \dots \dots \\ \left[\frac{n_p g_i^2(M_p)}{b'_i - N} - \frac{n_p g_i^2(\mu)}{b'_i - N} \right] \end{pmatrix},$$

with the column indices corresponding to M_1, M_2, \dots, M_p , etc. Here we rewrite our expressions neatly by defining

$$P_i[a] = \frac{g_i^2(M_a)}{b'_i - \sum_{k=1}^a n_k} - \frac{g_i^2(\mu)}{b'_i - N},$$

$$Q_i[a] = \sum_{c=a+1}^p \frac{n_c g_i^2(M_c)}{(b'_i - \sum_{k=1}^{c-1} n_k)(b'_i - \sum_{k=1}^c n_k)}, \tag{41}$$

within which $b_i \equiv b'_i - N$ is just the beta function coefficient below all the messenger thresholds. From the previous expressions, we can check that each row will vanish if we neglect the scale dependence of g_i^2 . This observation agrees with the ordinary conclusion that the trilinear couplings of GMSB vanish if no Yukawa deflections are present.

From the expressions in Equation (40), we can obtain the symmetric matrix (for indices j and k)

$$\left(\frac{\partial^2 \ln Z_l(\mu, M_a)}{\partial \ln M_b \partial \ln M_a}\right)^T \equiv - \sum_{i=1,2,3} \frac{4A_i}{(16\pi^2)^2} K_{ab;i}, \tag{42}$$

with the contributions $K_{ab;i}, i = 1, 2, 3$ for each gauge field

$$\begin{pmatrix} n_1^2 X_i[1], & n_1 n_2 Y_i[2], & n_1 n_3 Y_i[3], & n_1 n_4 Y_i[4], & n_1 n_5 Y_i[5], & \dots, & n_1 n_p Y_i[p] \\ n_1 n_2 Y_i[2], & n_2^2 X_i[2], & n_2 n_3 Y_i[3], & n_2 n_4 Y_i[4], & n_2 n_5 Y_i[5], & \dots, & n_2 n_p Y_i[p] \\ n_1 n_3 Y_i[3], & n_2 n_3 Y_i[3], & n_3^2 X_i[3], & n_3 n_4 Y_i[4], & n_3 n_5 Y_i[5], & \dots, & n_3 n_p Y_i[p] \\ n_1 n_4 Y_i[4], & n_2 n_4 Y_i[4], & n_3 n_4 Y_i[4], & n_4^2 X_i[4], & n_4 n_5 Y_i[5], & \dots, & n_4 n_p Y_i[p] \\ n_1 n_5 Y_i[5], & n_2 n_5 Y_i[5], & n_3 n_5 Y_i[5], & n_4 n_5 X_i[5], & n_5^2 X_i[5], & \dots, & n_5 n_p Y_i[p] \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ n_1 n_p Y_i[p], & n_2 n_p Y_i[p], & n_3 n_p Y_i[p], & n_4 n_p Y_i[p], & n_5 n_p Y_i[p], & \dots, & n_p^2 X_i[p], \end{pmatrix}. \tag{43}$$

The functions within $K_{ab;i}$ are defined as

$$X_i[a] = G_i^E[a] + K_i^E[a], \quad Y_i[a] = F_i^E[a] + K_i^E[a], \tag{44}$$

within which

$$\begin{aligned} F_i^E[a] &= \frac{g_i^A(M_a)}{b'_i - \sum_{i=1}^a n_i} - \frac{g_i^A(\mu)}{b'_i - N}, \\ G_i^E[a] &= \frac{(b'_i - \sum_{i=1}^{a-1} n_i) g_i^A(M_a)}{n_a (b'_i - \sum_{i=1}^a n_i)} - \frac{g_i^A(\mu)}{b'_i - N}, \\ K_i^E[a] &= \sum_{c=a+1}^p \frac{n_c g_i^A(M_c)}{(b'_i - \sum_{i=1}^{c-1} n_i)(b'_i - \sum_{i=1}^c n_i)}. \end{aligned} \tag{45}$$

Here we define the summation to vanish if its index lies out of its definition range. For example, $\sum_{i=1}^0 (\dots) \equiv \sum_{i=p+1}^p (\dots) = 0$. From the previous expressions, we can check that each non-diagonal element of K_{ab} will vanish if we neglect the scale dependence of g_i^2 . Only the diagonal elements of K_{ab} can give non-vanishing values of order

$$K_{aa} \sim n_a^2 \frac{g_i^A}{n_a} = n_a g_i^A. \tag{46}$$

The inclusion of top-Yukawa coupling is straightforward in the analytical expressions. The scale dependence of top-Yukawa in the simplest case, in which only the leading top Yukawa $\alpha_t \equiv y_t^2/4\pi$ and $\alpha_s \equiv g_3^2/4\pi$ are kept in the anomalous dimension, takes the form

$$\frac{d}{dt} \ln \alpha_t = \frac{1}{\pi} \left(3\alpha_t - \frac{8}{3}\alpha_s \right), \quad \frac{d}{dt} \ln \alpha_s = \frac{1}{2\pi} b_3 \alpha_s. \tag{47}$$

Define $A = \ln \left(\alpha_t \alpha_s^{\frac{16}{3b_3}} \right)$, the equation can be written as

$$d[e^{-A}] = -\frac{3}{\pi} \alpha_s^{-\frac{16}{3b_3}} dt = -\frac{6}{b_3} \alpha_s^{-\frac{16}{3b_3}-2} d\alpha_s. \tag{48}$$

So, we can exactly solve the differential equation to obtain

$$\left[\frac{\alpha_t(\mu)}{\alpha_t(\Lambda)} \left(\frac{\alpha_s(\mu)}{\alpha_s(\Lambda)} \right)^{\frac{16}{3b_3}} \right]^{-1} = 1 - \frac{6\alpha_t(\Lambda)}{\frac{16}{3} + b_3} \left[\alpha_s(\Lambda)^{-1} - \left(\frac{\alpha_s(\mu)}{\alpha_s(\Lambda)} \right)^{-\frac{16}{3b_3}} \alpha_s^{-1}(\mu) \right].$$

Expanding the expressions and neglecting high-order terms, we finally have

$$[\ln \alpha_t(\mu) - \ln \alpha_t(\Lambda)] \approx \left[-\frac{8}{3\pi} \alpha_s(\mu) + \frac{3}{\pi} \alpha_t(\mu) \right] \ln \left(\frac{\Lambda}{\mu} \right). \tag{49}$$

It can be observed that, in the leading order approximation, the expression within the square bracket is just the beta function of top Yukawa coupling. As there are no Yukawa deflection contributions related to the introduction of messengers, the Yukawa coupling contributions will not enter the expression within the GMSB part of the generalized dAMSB.

3.2. Class B: Partition with Vanishing Gauge Beta Functions

In previous discussions, apparent poles $A_i/(b'_i - \sum_{a=1}^j n_a)$ may arise if we have vanishing gauge beta function coefficient $b'_i - \sum_{a=1}^j n_a = 0$ between certain messenger scales. (Although such poles will not appear if higher loop order beta functions contributions are included, we still want to check the status of the contribution with tiny beta functions.) For example, with $N > 3$ non-degenerate messengers in $\mathbf{5} \oplus \bar{\mathbf{5}}$ representation, the beta function for α_3 , which is given by $b'_3 = -3 + N$, will vanish after decoupling $N - 3$ family of vector-like messengers at one-loop level. (The beta function for $i = 1, 2$ gauge fields will not encounter such difficulties.) Such an artificial pole can be resolved by revisiting the deduction procedure of wavefunctions.

Assume that all the i -th gauge coupling beta function coefficients $b'_i - \sum_{a=1}^{k-1} n_a$ are non-vanishing for $k < j$. After integrating out the n_j family of vector-like messengers at the M_j scale, the beta function coefficient is assumed to vanish (so that $b'_i - \sum_{a=1}^j n_a = 0$).

The wavefunction at the M_j scale takes value

$$Z_l(M_j, M_a) = Z_l(\Lambda) \prod_{i=1}^3 \left[\left(\frac{g_i^2(M_1)}{g_i^2(\Lambda)} \right)^{A_i/b'_i} \left(\frac{g_i^2(M_2)}{g_i^2(M_1)} \right)^{A_i/(b'_i-n_1)} \dots \left(\frac{g_i^2(M_j)}{g_i^2(M_{j-1})} \right)^{A_i/(b'_i-\sum_{a=1}^{j-1} n_a)} \right],$$

$$\left(\frac{y_i^2(M_j)}{y_i^2(\Lambda)} \right)^{B_i} \left(\frac{y_b^2(M_j)}{y_b^2(\Lambda)} \right)^{B_b} \left(\frac{y_\tau^2(M_j)}{y_\tau^2(\Lambda)} \right)^{B_\tau}. \tag{50}$$

As the i -th beta function vanishes at one-loop level

$$\frac{dg_i}{dt} = 0, \quad M_{j+1} < \mu < M_j \tag{51}$$

it can be seen as a constant between $M_{j+1} < \mu < M_j$. Within this range, the RGE invariant became

$$\frac{d}{dt} \ln \left[Z(\mu) \prod_{l=y_t, y_b, y_\tau} [y_l(\mu)]^{-2B_l} \prod_{k \neq i} [g_k(\mu)]^{-2\tilde{A}_k} \right] = D_i g_i^2(\mu) = D_i g_i^2(M_j), \tag{52}$$

and we can deduce that

$$Z(\mu) \prod_{l=y_t, y_b, y_\tau} [y_l(\mu)]^{-2B_l} \prod_{k \neq i} [g_k(\mu)]^{-2\tilde{A}_k}$$

$$= \left(\frac{\mu}{M_j} \right)^{D_i g_i^2(M_j)} \left[Z(M_j) \prod_{l=y_t, y_b, y_\tau} [y_l(M_j)]^{-2B_l} \prod_{k \neq i} [g_k(M_j)]^{-2\tilde{A}_k} \right]. \tag{53}$$

The value $D_i \equiv A_i/8\pi^2$ with the value A_i given in Appendix A. We will keep using ' D_i ' in this paper to indicate clearly the consequence of vanishing one-loop beta functions.

So, for $M_{j+1} < \mu < M_j$,

$$\begin{aligned} Z(\mu) &= Z(M_j) \left(\frac{\mu}{M_j}\right)^{D_i g_i^2(M_j)} \prod_{l=y_l, y_b, y_\tau} \left[\frac{y_l(M_j)}{y_l(\mu)}\right]^{-2B_l} \prod_{k \neq i} \left[\frac{g_k(M_j)}{g_k(\mu)}\right]^{-2\tilde{A}_k} \\ &= Z(\Lambda) \prod_{l=y_l, y_b, y_\tau} \left[\frac{y_l(\Lambda)}{y_l(\mu)}\right]^{-2B_l} \prod_{k \neq i} \left[\left(\frac{g_k^2(M_1)}{g_k^2(\Lambda)}\right)^{\frac{A_k}{b'_k}} \left(\frac{g_k^2(M_2)}{g_k^2(M_1)}\right)^{\frac{A_k}{b'_k - n_1}} \dots \left(\frac{g_k^2(\mu)}{g_k^2(M_j)}\right)^{\frac{A_k}{b'_k - \sum_{a=1}^j n_a}}\right], \\ &\quad \left[\left(\frac{g_i^2(M_1)}{g_i^2(\Lambda)}\right)^{\frac{A_i}{b'_i}} \left(\frac{g_i^2(M_2)}{g_i^2(M_1)}\right)^{\frac{A_i}{b'_i - n_1}} \dots \left(\frac{g_i^2(M_j)}{g_i^2(M_{j-1})}\right)^{\frac{A_i}{b'_i - \sum_{a=1}^{j-1} n_a}}\right] \left(\frac{\mu}{M_j}\right)^{D_i g_i^2(M_j)}, \end{aligned}$$

and for $\mu < M_p$,

$$\begin{aligned} Z(\mu) &= Z(\Lambda) \prod_{l=y_l, y_b, y_\tau} \left[\frac{y_l(\Lambda)}{y_l(\mu)}\right]^{-2B_l} \\ &\quad \prod_{k \neq i} \left[\left(\frac{g_k^2(M_1)}{g_k^2(\Lambda)}\right)^{\frac{A_k}{b'_k}} \left(\frac{g_k^2(M_2)}{g_k^2(M_1)}\right)^{\frac{A_k}{b'_k - n_1}} \dots \left(\frac{g_k^2(M_j)}{g_k^2(M_{j-1})}\right)^{\frac{A_k}{b'_k - \sum_{a=1}^j n_a}} \dots \left(\frac{g_k^2(\mu)}{g_k^2(M_p)}\right)^{\frac{A_k}{b'_k}}\right], \\ &\quad \left[\left(\frac{g_i^2(M_1)}{g_i^2(\Lambda)}\right)^{\frac{A_i}{b'_i}} \left(\frac{g_i^2(M_2)}{g_i^2(M_1)}\right)^{\frac{A_i}{b'_i - n_1}} \dots \left(\frac{g_i^2(M_j)}{g_i^2(M_{j-1})}\right)^{\frac{A_i}{b'_i - \sum_{a=1}^{j-1} n_a}}\right] \left(\frac{M_{j+1}}{M_j}\right)^{D_i g_i^2(M_j)} \\ &\quad \left(\frac{g_i^2(M_{j+2})}{g_i^2(M_{j+1})}\right)^{\frac{A_i}{b'_i - \sum_{a=1}^{j+1} n_a}} \dots \left(\frac{g_i^2(\mu)}{g_i^2(M_p)}\right)^{\frac{A_i}{b'_i}}. \end{aligned} \tag{54}$$

Therefore, we have for $\mu < M_p$

$$\frac{d \ln Z[\mu, g_i(\mu'), M_j, M_{j+1}]}{d \ln M_a} = \left[\sum_{g_i(\mu')} \frac{\partial g_i(\mu')}{\partial \ln M_a} \frac{\partial}{\partial g_i(\mu')} + \frac{\partial}{\partial \ln M_a} \right] \ln Z[\mu, g_i(\mu'), M_j, M_{j+1}],$$

with the last term giving non-vanishing contributions only for $a = j, j + 1$

$$\frac{\partial}{\partial \ln M_a} \ln Z[\mu, g_i(\mu'), M_j, M_{j+1}] = [\delta_{a,j+1} - \delta_{a,j}] D_i g_i^2(M_j). \tag{55}$$

From the general expressions, we can see that the j -th and $(j + 1)$ -th components will be changed into

$$\left(\frac{\partial \ln Z[\mu; g_i(\mu'), M_a]}{\partial \ln g_i(\mu')}\right)_{j,j+1} = 2 \left(\frac{A_i}{b'_i - \sum_{a=1}^{j-1} n_a} + D_i g_i^2(M_j) \ln \frac{M_{j+1}}{M_j}, -\frac{A_i}{b'_i - \sum_{a=1}^{j+1} n_a} \right),$$

while other columns are unchanged as Equation (38) if $b_i(\mu) = 0$ for $M_{j+1} < \mu < M_j$. The matrix $\partial \ln g_i(\mu')/\partial M_j$, which is given by Equation (39), is unchanged. Then the contribu-

tions from the i -th gauge field, which has vanishing beta functions between $[M_{j+1}, M_j]$, are given as

$$\begin{aligned}
 U_{b;i} &\equiv \left. \frac{d \ln Z[g_i(\mu'), M_n]}{d \ln M_b} \right|_i \\
 &= \left[\frac{\partial g_i(\mu')}{\partial \ln M_b} \frac{\partial}{\partial g_i(\mu')} + \frac{\partial}{\partial \ln M_b} \right] \ln Z[g_i(\mu'), M_j, M_{j+1}] \\
 &= -\frac{A_i}{8\pi^2} \begin{pmatrix} n_1 \left(P_i^S[1] + Q_i^S[1] - \frac{D_i}{A_i} g_i^A(M_j) \ln \frac{M_{j+1}}{M_j} \right), \\ n_2 \left(P_i^S[2] + Q_i^S[2] - \frac{D_i}{A_i} g_i^A(M_j) \ln \frac{M_{j+1}}{M_j} \right), \\ \dots \\ n_j \left(P_i^S[j] + Q_i^S[j] - \frac{D_i}{A_i} g_i^A(M_j) \ln \frac{M_{j+1}}{M_j} \right) + \frac{8\pi^2}{A_i} D_i g_i^2(M_j), \\ n_{j+1} \left(P_i^S[j+1] + Q_i^S[j+1] \right) - \frac{8\pi^2}{A_i} D_i g_i^2(M_j), \\ n_{j+2} \left(P_i^S[j+2] + Q_i^S[j+2] \right), \\ \dots \\ n_p \left(P_i^S[p] + Q_i^S[p] \right). \end{pmatrix}.
 \end{aligned}
 \tag{56}$$

Within the deduction, we use the fact that $g_i(M_j) = g_i(M_{j+1})$ and $b'_i - \sum_{k=1}^j n_k = 0$. Furthermore, we also define

$$\begin{aligned}
 Q_i^S[c] &= \begin{cases} \sum_{a=c+1, a \neq j, j+1}^p \frac{n_a g_i^2(M_a)}{(b'_i - \sum_{k=1}^{a-1} n_k)(b'_i - \sum_{k=1}^a n_k)} + \frac{(n_j + n_{j+1}) g_i^2(M_j)}{(b'_i - \sum_{k=1}^{j-1} n_k)(b'_i - \sum_{k=1}^{j+1} n_k)}, & 1 \leq c \leq j \\ \sum_{a=c+1}^p \frac{n_a g_i^2(M_a)}{(b'_i - \sum_{k=1}^{a-1} n_k)(b'_i - \sum_{k=1}^a n_k)}, & j+1 \leq c \leq p \end{cases} \\
 P_i^S[c] &= \begin{cases} \frac{g_i^2(M_c)}{b'_i - \sum_{k=1}^c n_k} - \frac{g_i^2(\mu)}{b'_i - N}, & c \neq j, j+1 \\ -\frac{g_i^2(\mu)}{b'_i - N}. & c = j, j+1 \end{cases}
 \end{aligned}
 \tag{57}$$

within the expression

$$\frac{(n_j + n_{j+1}) g_i^2(M_j)}{(b'_i - \sum_{k=1}^{j-1} n_k)(b'_i - \sum_{k=1}^{j+1} n_k)} = -\frac{n_j + n_{j+1}}{n_j n_{j+1}} g_i^2(M_j).
 \tag{58}$$

We note that when c takes value $j - 1$ or j in the summation of $Q_i^S[c]$, the sum skip $j, j + 1$ and begins at $a = j + 2$.

From the previous expressions, we can see that each row will vanish if we neglect the scale dependence of g_i^2 and higher-order g_i^6 terms. In fact, with such an approximation, the j -th and $j + 1$ -th row is given by

$$\begin{aligned}
 U_{b;j} &\sim -\frac{g^2}{n_j} n_j + g^2 \sim 0, \\
 U_{b;j+1} &\sim \frac{g^2}{n_{j+1}} n_{j+1} - g^2 \sim 0.
 \end{aligned}
 \tag{59}$$

In the summation

$$\frac{\partial \ln Z_l[\mu, g_i(\mu'), M_n]}{\partial \ln M_b} = \sum_{i=1}^3 \left. \frac{\partial \ln Z_l[\mu, g_i(\mu'), M_n]}{\partial \ln M_b} \right|_i,
 \tag{60}$$

the expressions for $i = 1, 2$ gauge fields (which have no vanishing beta functions) are still given by

$$\left(\frac{\partial \ln Z_l[\mu, g_i(\mu')]}{\partial \ln M_b} \right) \Big|_{i=1,2} \equiv -\frac{A_i}{8\pi^2} \begin{pmatrix} n_1(P_i[1] + Q_i[1]) \\ n_2(P_i[2] + Q_i[2]) \\ n_3(P_i[3] + Q_i[3]) \\ \dots \\ n_p(P_i[p] + Q_i[p]) \end{pmatrix}, \tag{61}$$

from Equation (40).

With previous results, we can derive the expression of $\frac{\partial^2}{\partial \ln M_a \partial \ln M_b} \ln Z[\mu; g_i(\mu'), M_n]$ from i -th (here $i = 3$) gauge fields

$$\begin{aligned} & \frac{\partial}{\partial \ln M_a} \left(\frac{\partial \ln Z[g_i(\mu'), M_n]}{\partial \ln M_b} \Big|_i \right) \equiv -\frac{A_i}{8\pi^2} \left[\frac{\partial \ln g_j(\mu')}{\partial \ln M_a} \frac{\partial}{\partial \ln g_j(\mu')} + \delta_{a;j,j+1} \frac{\partial}{\partial M_{j,j+1}} \right] V_{b;i} \\ & = -\frac{4A_i}{(16\pi^2)^2} K_{ab;i}, \end{aligned} \tag{62}$$

with $K_{ab;i}$ a symmetric matrix given as

$$\begin{pmatrix} n_1^2 J[1] & n_1 n_2 H[2] & \dots & n_1 n_j H[j] & n_1 n_{j+1} H[j+1] & n_1 n_{j+2} H[j+2] & \dots & n_1 n_p H[p] \\ n_1 n_2 H[2] & n_2^2 J[2] & \dots & n_2 n_j H[j] & n_2 n_{j+1} H[j+1] & n_2 n_{j+2} H[j+2] & \dots & n_2 n_p H[p] \\ n_1 n_3 H[3] & n_3 n_2 H[3] & \dots & n_3 n_j H[j] & n_3 n_{j+1} H[j+1] & n_3 n_{j+2} H[j+2] & \dots & n_3 n_p H[p] \\ \dots & \dots \\ n_1 n_j H[j] & n_2 n_j H[j] & \dots & n_j^2 J[j] & n_j n_{j+1} H[j+1] & n_j n_{j+2} H[j+2] & \dots & n_j n_p H[p] \\ n_1 n_{j+1} H[j+1] & n_2 n_{j+1} H[j+1] & \dots & n_j n_{j+1} H[j+1] & n_{j+1}^2 J[j+1] & n_{j+1} n_{j+2} H[j+2] & \dots & n_{j+1} n_p H[p] \\ n_1 n_{j+2} H[j+2] & n_2 n_{j+2} H[j+2] & \dots & n_j n_{j+2} H[j+2] & n_{j+1} n_{j+2} H[j+2] & n_{j+2}^2 J[j+2] & \dots & n_{j+2} n_p H[p] \\ \dots & \dots \\ n_1 n_p H[p] & n_2 n_p H[p] & \dots & n_j n_p H[p] & n_{j+1} n_p H[p] & n_{j+2} n_p H[p] & \dots & n_p^2 J[p] \end{pmatrix}. \tag{63}$$

The functions within $K_{ab;i}$ are defined as

$$\begin{aligned} J[m] &= \begin{cases} G_i^S[m] + K_i^S[m] - 2\frac{D_i}{A_i} g_i^6(M_j) \ln \frac{M_{j+1}}{M_j}, & 1 \leq m \leq j-1 \\ G_i^S[j] + K_i^S[j] + 16\pi^2 \frac{D_i}{n_j A_i} g_i^4(M_j) - 2\frac{D_i}{A_i} g_i^6(M_j) \ln \frac{M_{j+1}}{M_j}, & m = j \\ G_i^S[m] + K_i^S[m], & j+1 \leq m \leq p \end{cases} \\ H[m] &= \begin{cases} F_i^S[m] + K_i^S[m] - 2\frac{D_i}{A_i} g_i^6(M_j) \ln \frac{M_{j+1}}{M_j}, & 1 \leq m \leq j-1 \\ F_i^S[j] + K_i^S[j+1] + 8\pi^2 \frac{D_i}{n_j A_i} g_i^4(M_j) - 2\frac{D_i}{A_i} g_i^6(M_j) \ln \frac{M_{j+1}}{M_j}, & m = j \\ F_i^S[j+1] + K_i^S[j+1] - 8\pi^2 \frac{D_i}{n_{j+1} A_i} g_i^4(M_j), & m = j+1 \\ F_i^S[m] + K_i^S[m], & j+2 \leq m \leq p \end{cases} \end{aligned}$$

with

$$K_i^S[c] = \begin{cases} \sum_{a=c+1; a \neq j, j+1}^p \frac{n_a g_i^4(M_a)}{(b'_i - \sum_{k=1}^{a-1} n_k)(b'_i - \sum_{k=1}^a n_k)} + \frac{(n_j + n_{j+1})g_i^4(M_j)}{(b'_i - \sum_{k=1}^{j-1} n_k)(b'_i - \sum_{k=1}^{j+1} n_k)}, & 1 \leq c \leq j \\ \sum_{a=c+1}^p \frac{n_a g_i^4(M_a)}{(b'_i - \sum_{k=1}^{a-1} n_k)(b'_i - \sum_{k=1}^a n_k)}, & j+1 \leq c \leq p \end{cases}$$

$$F_i^S[c] = \begin{cases} \frac{g_i^4(M_c)}{b'_i - \sum_{k=1}^c n_k} - \frac{g_i^4(\mu)}{b'_i - N}, & c \neq j, j+1 \\ \frac{n_j + n_{j+1}}{b'_i - \sum_{k=1}^{j-1} n_k} \frac{g_i^4(M_c)}{b'_i - \sum_{k=1}^{j+1} n_k} - \frac{g_i^4(\mu)}{b'_i - N}, & c = j \\ -\frac{g_i^4(\mu)}{b'_i - N}, & c = j+1 \end{cases} \tag{64}$$

$$G_i^S[c] = \begin{cases} \frac{(b'_i - \sum_{k=1}^{c-1} n_k)g_i^4(M_a)}{n_c(b'_i - \sum_{k=1}^c n_k)} - \frac{g_i^4(\mu)}{b'_i - N}, & c \neq j, j+1 \\ -\frac{g_i^4(\mu)}{b'_i - N}. & c = j, j+1 \end{cases} \tag{65}$$

From the previous expressions, we can check that each non-diagonal element of $K_{ab}(a, b \neq j, j+1)$ will vanish if we neglect the scale dependence of g_i^2 and higher-order g_i^6 terms. The diagonal elements of $K_{ab}(a, b \neq j, j+1)$ can give non-vanishing values of order

$$K_{aa} \equiv n_a^2 J[a] \sim n_a^2 \frac{g_i^4}{n_a} = n_a g_i^4. \tag{66}$$

The $J[j]$ and $H[j]$ term will be given as

$$\begin{aligned} J[j] &\sim -\frac{g_i^4}{n_j} + \frac{2}{n_j} g_i^4 = \frac{g_i^4}{n_j}, & J[j+1] &\sim \frac{g_i^4}{n_{j+1}}, \\ H[j] &\sim -\frac{g_i^4}{n_j} + \frac{g_i^4}{n_j} \sim 0, & H[j+1] &\sim \frac{g_i^4}{n_{j+1}} - \frac{g_i^4}{n_{j+1}} \sim 0, \end{aligned} \tag{67}$$

if we neglect the scale dependence of g_i^2 and higher-order g_i^6 terms.

The contributions from $i = 1, 2$ gauge fields are still given by Equation (43). The total contributions are given by the sum of $i = 1, 2, 3$ gauge fields.

3.3. Dependence of $\ln M_a$ on $\ln X$

As noted before, with non-trivial $U(1)_R$ symmetry, the messenger determinant is proven by [15] to be a monomial in X

$$\det(\lambda_{ij} X + m_{ij}) = X^{n_0} G(\lambda, m). \tag{68}$$

Knowing the value of the determinant, it is still non-trivial to express the eigenvalues of \mathcal{M} in terms of $\langle X \rangle$. (Here and after, $\langle X \rangle$ is used to denote the scalar component VEV of the superfield X .) Fortunately, the asymptotic behavior will display a simple form. In large $\langle X \rangle$ region, $r_\lambda \equiv \text{rank}(\lambda_{ij})$ messengers acquire masses $\mathcal{O}(\langle X \rangle)$ while the remaining $N - r_\lambda$ messengers acquire masses of order

$$M_i \sim \frac{m^{n_i+1}}{\langle X \rangle^{n_i}}, \quad \sum_{i=1}^{N-r_\lambda} n_i = r_\lambda - n_0, \tag{69}$$

with $n_i \geq 0$. At small $\langle X \rangle$ region, $r_m \equiv \text{rank}(m_{ij})$ messengers acquire masses $\mathcal{O}(m)$ while the remaining $N - r_m$ messengers acquire masses of order

$$M_i \sim \frac{\langle X \rangle^{\tilde{n}_i+1}}{m^{\tilde{n}_i}}, \quad \sum_{i=1}^{N-r_m} (\tilde{n}_i + 1) = n_0, \quad (70)$$

with $\tilde{n}_i \geq 0$.

Depending on the singularity properties of the messenger mass matrix, we have the following discussions.

- Type I: $\det m \neq 0$.

In the basis in which m is diagonal, it can prove [15] that Equation (68) takes the form

$$n_0 = 0, \quad \det(\lambda \langle X \rangle + m) = \det m, \quad (71)$$

which necessarily implies $\det \lambda = 0$. As the matrix is upper triangular, the eigenvalues are m_{ii} that do not depend on $\langle X \rangle$.

So, in this case, we have $d_k \equiv 0$ or

$$\frac{d \ln M_i}{d \ln X} \equiv 0. \quad (72)$$

So, we can see that the gauginos, the trilinear couplings, and the sfermions will not receive any gauge mediation contributions.

- Type II: $\det \lambda \neq 0$.

Similarly, we can obtain an upper triangular matrix with eigenvalues equal to the diagonal elements of diagonalized matrix λ'_{ii} . The determinant is

$$n_0 = N, \quad \det(\lambda \langle X \rangle + m) = \langle X \rangle^N \det \lambda. \quad (73)$$

So, the eigenvalues will be $\lambda'_{ii} \langle X \rangle$ and depend linearly on $\langle X \rangle$. We will arrange λ'_{ii} to obtain the eigenvalues $\tilde{M}_1(\langle X \rangle), \tilde{M}_2(\langle X \rangle), \dots, \tilde{M}_N(\langle X \rangle)$.

Suppose the $T_i \equiv \lambda'_{ii}$ are ordered so as that $T_1 \geq T_2 \geq T_3 \dots \geq T_N$, we define

$$V_i \equiv \frac{d \ln M_i}{d \ln X} \equiv (1, 1, 1, \dots, 1). \quad (74)$$

For degenerate eigenvalues

$$\begin{aligned} \tilde{M}_1 &= \tilde{M}_2 = \dots = \tilde{M}_{n_1} \equiv M_1 = \lambda'_{n_1 n_1} \langle X \rangle, \\ \tilde{M}_{n_1+1} &= \dots = \tilde{M}_{n_2} \equiv M_2 = \lambda'_{n_2 n_2} \langle X \rangle, \\ \tilde{M}_{n_2+1} &= \tilde{M}_{n_2+2} = \dots = \tilde{M}_{n_3} \equiv M_3 = \lambda'_{n_3 n_3} \langle X \rangle, \dots \\ \tilde{M}_{n_{p-1}+1} &= \tilde{M}_{n_{p-1}+2} = \dots = \tilde{M}_{n_p} \equiv M_p = \lambda'_{n_p n_p} \langle X \rangle, \end{aligned} \quad (75)$$

with $M_1 \geq M_2 \geq \dots \geq M_p$, the matrix V_i reduces to a $1 \times p$ matrix

$$V_i \equiv \frac{d \ln M_i}{d \ln X} \equiv (1, 1, \dots, 1). \quad (76)$$

So, the soft SUSY breaking parameters for GMSB can be given as

- The gaugino mass:

$$M_i = \frac{F_X}{X_0} \frac{g_i^2}{16\pi^2} N. \quad (77)$$

- GMSB contributions to trilinear terms:

$$A_i = \frac{A_{ijk}}{y_{ijk}} = \sum_{i,j,k} \frac{F_X}{2X_0} \frac{\partial \ln Z_i(\mu, M_a(X))}{\partial \ln |X|} = \frac{F_X}{2X_0} \sum_{i,j,k} \sum_{v=1}^N U_b V_b. \tag{78}$$

As anticipated, the trilinear term will vanish if we neglect the scale dependence of g_i^2 and higher-order g_i^6 terms.

- Pure GMSB contributions to soft sfermion masses:

$$\begin{aligned} m_i^2 &= -\frac{F_X^2}{4X_0^2} \frac{\partial^2}{\partial \ln |X|^2} \ln Z_i(\mu, M_a(X)) \\ &= \frac{F_X^2}{4X_0^2} \sum_{i=1,2,3} \frac{4A_i}{(16\pi^2)^2} \sum_{a,b} V_a K_{abj} V_b. \end{aligned} \tag{79}$$

This expression can be simplified into

$$m_i^2 = \frac{F_X^2}{4X_0^2} \sum_{i=1,2,3} \frac{4A_i}{(16\pi^2)^2} g_i^4 N, \tag{80}$$

if we neglect the scale dependence of g_i^2 and higher-order g_i^6 terms.

Such GMSB expressions can easily be extended to gauge mediation contributions in deflected AMSB.

- Type III: $\det m = \det \lambda = 0$.

As the matrix $\lambda X + m$ is non-singular, its eigenvalues can be written as x_1, \dots, x_n , which should satisfy

$$\prod_i x_i = \det(\lambda X + m) = X^{n_0} G(\lambda, m), \tag{81}$$

and

$$\sum_i x_i = -\text{Tr}(\lambda X + m) = cX + d. \tag{82}$$

In the large X region in which m_{ij} can be neglected, we can use linear transformation to put λ_{ij} into

$$\begin{pmatrix} a_1 & & & & & \\ & a_2 & & & & \\ & & \dots & & & \\ & & & a_{r_\lambda} & & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix}. \tag{83}$$

There are r_λ messengers with mass of order X . As the trace depends linearly on X , such r_λ messengers had to have a linear dependence on X . The remaining messengers can only be proportional to an inverse power of X or be a constant. From the trace, which contains only the constant and the linear X term, the term with negative power of X should appear in pairs or vanish. It is also obvious that terms with negative power of X have to appear if $r_\lambda \neq n_0$. As the eigenvalues, which contain non-vanishing negative n_i powers, are suppressed by an additional $(m/X_0)^{n_i}$ factor, they need to be the lighter eigenvalues.

As r_λ messengers depend linearly on X , we can approximately use

$$\frac{\partial \ln M_a}{\partial \ln X} \equiv V_a \approx \underbrace{(1, 1, \dots, 1)}_{r_\lambda}, \underbrace{(0, 0, \dots, -n_{i_1}, -n_{i_1}, \dots, -n_{i_k}, -n_{i_k})}_{N-r_\lambda}, \tag{84}$$

with

$$\sum_k 2n_{i_k} = r_\lambda - n_0. \tag{85}$$

For degenerate eigenvalues,

$$\begin{aligned} \tilde{M}_1 &= \tilde{M}_2 = \dots = \tilde{M}_{n_1} \equiv M_1 = a_{n_1} \langle X \rangle, \dots \\ \tilde{M}_{n_{k-1}+1} &= \tilde{M}_{n_{k-1}+2} = \dots = \tilde{M}_{n_k} \equiv M_k = a_{n_k} \langle X \rangle, \\ \tilde{M}_{n_k+1} &= \tilde{M}_{n_k+2} = \dots = \tilde{M}_{n_{k+1}} \equiv M_{k+1} = c_k, \dots \\ \tilde{M}_{n_x+1} &= \tilde{M}_{n_x+2} = \dots = \tilde{M}_{n_{x+1}} \equiv M_{x+1} = c_x, \\ \tilde{M}_{n_{x+1}+1} &= \tilde{M}_{n_{x+1}+2} = \dots = \tilde{M}_{n_{x+2}} \equiv M_{x+2} = b_{n_{x+2}} \langle X \rangle^{-\lambda_{x+2}}, \dots \\ \tilde{M}_{n_{p-1}+1} &= \tilde{M}_{n_{p-1}+2} = \dots = \tilde{M}_{n_p} \equiv M_p = b_{n_p} \langle X \rangle^{-\lambda_p}, \end{aligned} \tag{86}$$

with c_k, \dots, c_x , some constants eigenvalues of $\lambda \langle X \rangle + m$ are independent of X . Assuming $M_1 \geq M_2 \geq \dots \geq M_p$, the matrix V_i reduces to a $1 \times p$ matrix

$$\frac{\partial \ln M_a}{\partial \ln X} \equiv V_a \approx \underbrace{(1, \dots, 1)}_k, \underbrace{(0, \dots, 0)}_{x-k+1}, \underbrace{(-\lambda_{x+2}, \dots, -\lambda_p)}, \tag{87}$$

with

$$\sum_{i=1}^k n_i = r_\lambda, \quad \sum_{k=2}^{p-x} (n_{x+k} - n_{x+k-1}) \lambda_{x+k} = r_\lambda - n_0. \tag{88}$$

The partition of N can be obtained numerically by diagonalizing $\lambda \langle X \rangle + m$ to obtain its eigenvalues as functions of $\langle X \rangle$.

So, we can obtain the GMSB contributions

$$\begin{aligned} A_i &= \frac{A_{ijk}}{y_{ijk}} = \sum_{i,j,k} \frac{F_X}{2M} \frac{\partial \ln Z_l(\mu, M_a(X))}{\partial \ln X} = \frac{F_X}{2M} \sum_{i,j,k} \sum_{v=1}^p U_b V_b, \\ m_i^2 &= -\frac{F_X^2}{4M^2} \frac{\partial^2}{\partial \ln |X|^2} \ln Z_i(\mu, M_a(X)) \\ &= \frac{F_X^2}{4M^2} \sum_{i=1,2,3} \frac{4A_i}{(16\pi^2)^2} \sum_{a,b} V_a K_{ab;i} V_b, \end{aligned} \tag{89}$$

with the V_a taking the value in Equation (87). Again, the trilinear term will vanish if we neglect the scale dependence of g_i^2 and higher-order g_i^6 terms. m_i^2 will reduce to

$$m_i^2 \sim \frac{F_X^2}{4M^2} \sum_{i=1,2,3} \frac{4A_i}{(16\pi^2)^2} \delta_i^4 n_0 \tag{90}$$

if the scale dependence of g_i^2 and higher-order g_i^6 terms are neglected. If the first j -th eigenvalues are proportional to X while the remaining $N - j$ eigenvalues are proportional to m , the summation of the elements in the scalar matrix K_{ij} will be truncated so that only the upper $j \times j$ terms will be taken into account. So, the sfermion soft masses will in general be lighter than that of Type II. It is phenomenologically

attractive to have light slepton masses. The well-known muon $g_\mu - 2$ discrepancy can be solved with large contributions from slepton-electroweakino loops [36].

The inclusion of EOGM in deflected AMSB is straightforward. The AMSB type contributions can be given as

$$\begin{aligned} \frac{\partial}{\partial \ln \mu} \ln[Z_i(\mu, M_a)] &= -\frac{1}{8\pi^2} G_i^- [g_i(\mu), y_i(\mu)], \\ \frac{\partial^2}{\partial (\ln \mu)^2} \ln[Z_i(\mu, M_a)] &= -\frac{1}{8\pi^2} \left[\frac{\partial g_i(\mu)}{\partial \ln \mu} \frac{\partial}{\partial g_i(\mu)} + \frac{\partial y_i(\mu)}{\partial \ln \mu} \frac{\partial}{\partial y_i(\mu)} \right] G_i^- [g_i(\mu), y_i(\mu)] \\ &= -\frac{2}{(16\pi^2)^2} \left[\beta_{g_i} \frac{\partial}{\partial g_i(\mu)} + \beta_{y_i} \frac{\partial}{\partial y_i(\mu)} \right] G_i^- [g_i(\mu), y_i(\mu)], \quad (91) \\ \frac{\partial^2}{\partial \ln X \partial \ln \mu} \ln[Z_i(\mu, M_a)] &= -\frac{1}{8\pi^2} \left(\frac{\partial \ln M_a}{\partial \ln X} \right) \left(\frac{\partial^2}{\partial \ln M_a \partial \ln \mu} \ln[Z_i(\mu, M_a)] \right), \\ \frac{\partial^2}{\partial \ln M_a \partial \ln \mu} \ln[Z_i(\mu, M_a)] &= -\frac{1}{8\pi^2} \left[\frac{\partial g_i(\mu)}{\partial \ln M_a} \frac{\partial}{\partial g_i(\mu)} + \frac{\partial y_i(\mu)}{\partial \ln M_a} \frac{\partial}{\partial y_i(\mu)} \right] G_i^- [g_i(\mu), y_i(\mu)] \end{aligned}$$

with

$$\frac{\partial \lambda_l(\mu, M_a)}{\partial \ln M_a} = \Delta\beta[\lambda_l(\mu, M_a)], \quad \text{with } \lambda_l = g_l, y_l, \quad (92)$$

the discontinuity of beta functions across the threshold M_a .

In [15], the ‘effective messenger number’ is defined as

$$N_{eff} \equiv \frac{\Lambda_G^2}{\Lambda_S^2}, \quad (93)$$

with

$$M_i = \frac{g_i^2}{16\pi^2} \Lambda_G, \quad m_{\tilde{f}}^2 = 2 \frac{g_i^4}{(16\pi^2)^2} \sum_i C_{\tilde{f}}(r) \Lambda_S^2. \quad (94)$$

So, the approximate value of N_{eff} can be given as

$$N_{eff} = \frac{n_0^2 g_i^4}{\sum_{a,b} V_a K_{ab;i} V_b}, \quad (95)$$

by neglecting the scale dependence of g_i and higher-order terms in the expressions of soft SUSY parameters. With previous approximation, the value of N_{eff} in Type II can be calculated to be $N_{eff} = N$ after simplification, while in Type III EOGM, it can be calculated to be $N_{eff} = n_0$. Such a result holds for both Class A and Class B. Taking into account the scale dependence of g_i , N_{eff} can be different to n_0 and N .

The framework of EOGM extension of AMSB can accommodate deflections both in the Kahler potential and in the superpotential. So, it can easily solve the notorious negative square mass problem of sleptons. As the Kahler deflection [29] and superpotential deflection [22,23] in AMSB can easily accommodate the 125 GeV Higgs and be regarded as special cases of our EOGM extensions framework, this general case can also possibly accommodate the 125 GeV Higgs. With our previous setting in [29], it can be seen in Figure 1 that our EOGM extension of AMSB can easily lead to a realistic spectrum in various constrained limits. A detailed numerical scan will be studied in our subsequent works.

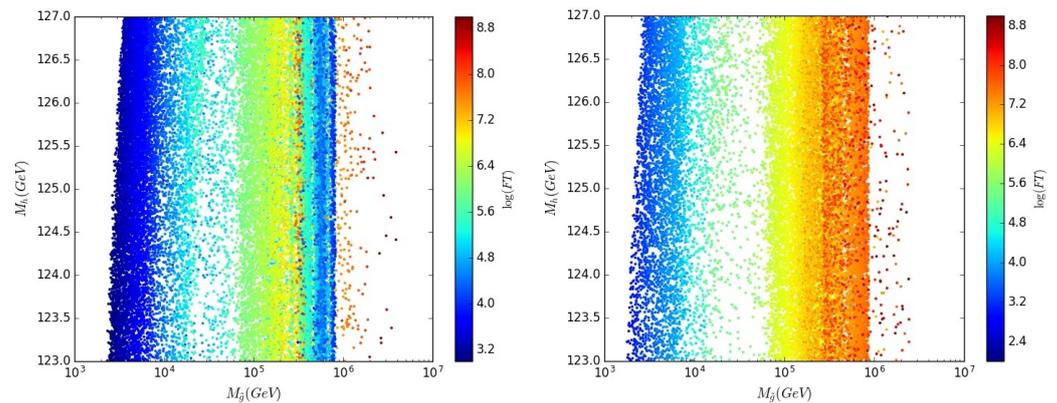


Figure 1. The Kahler deflection and superpotential deflections in AMSB, which can be seen as constrained limits of our EOGM extension of AMSB with multi-scale messengers, can both easily accommodate the 125 GeV Higgs. The upper pannels, with the same setting as our previous work [29], show the predicted Higgs mass range.

EOGM extension of deflected AMSB scenarios can easily adopt light sleptons. In addition, the EOGM extension of deflected AMSB scenarios can alter the gaugino relations into $M_1 : M_2 : M_3 \approx (33/5 - d_{Un}) : 2(1 - d_{Un}) : -6(-3 - d_{Un})$ at the EW scale, with n an integer satisfying $0 \leq n \leq N$ and d_U the deflection parameter. So, with proper choices for d_U and n , light electroweakinos can easily be obtained, given the stringent LHC constraints on gluino mass.

Light sleptons and electroweakinos are welcome to explain the recent muon $g - 2$ anomaly [37] and new CDF II W-boson mass data [38] in the SUSY framework [39]. Such light smuons and electroweakinos can be tested at the LHC. However, some compressed regions can still survive the updated constraints. Detailed simulations on the signals and collider exclusion bounds will be given in our subsequent studies.

Large gaugino ratios, especially between M_3 and M_1, M_2 , can be possible in this EOGM extension of deflected AMSB scenarios. With a large hierarchy, the gluino mass should be heavy due to the stringent lower mass bounds on chargino masses. Consequently, the squarks will always be pushed to be heavy by gluino loops. The heavy first two generation squarks are welcome to evade the LHC constrains. However, heavy stops, which can be welcome to accommodate the 125 GeV Higgs, may increase the involved electroweak fine-tuning (EWFT). Fortunately, EOGM extension of deflected AMSB scenarios can always predict large A_t term, which can lead to small EWFT even for TeV scale stops [40,41]. So, EOGM extension of deflected AMSB scenarios are also interesting phenomenologically.

Wino DM is always predicted in such scenarios. For light wino smaller than approximately 3 TeV, additional DM species are necessarily present to give correct relic abundances. Relevant phenomenology for such wino DM will be given in our subsequent studies.

4. Messengers on GUT and Landau Pole

We must ensure that no Landau pole will be reached below the GUT scale. It is obvious that the gauge coupling unification will be preserved because the messengers are fitted into complete SU(5) representations. The presence of (complete GUT representation) messenger fields at an intermediate scale does not modify the value of M_{GUT} . However, proton decay could possibly set constraints on the gauge couplings at the GUT scale.

We can define the quantity

$$\delta = - \sum_{r=1}^p \frac{n_r}{2\pi} \ln \frac{M_{GUT}}{M_r} , \tag{96}$$

which contributes to the inverse gauge coupling strength. The gauge couplings at the GUT scale are given at one-loop level by

$$\frac{4\pi}{g_i^2(GUT)} = \frac{4\pi}{g_i^2(M_Z)} - \frac{b_i}{2\pi} \ln\left(\frac{M_{GUT}}{M_Z}\right) - \sum_{n_r} \frac{n_r}{2\pi} \ln\left(\frac{M_{GUT}}{M_{n_r}}\right). \tag{97}$$

The first two terms give the gauge couplings at the GUT scale with $\alpha^{-1}(M_{GUT}) = 24.3$ with $M_{GUT} = 2.0 \times 10^{16}$ GeV and the SUSY scale $M_{SUSY} = 2$ TeV. So, the perturbativity of gauge couplings at the GUT scale set a bound on the quantity δ to be

$$|\delta| \lesssim 24.3. \tag{98}$$

Proton decay experiments will also constrain the value of δ . As the proton decay induced from the triplet Higgs depends on the scale of the triplets, we just take constraints from proton decay induced by heavy gauge bosons. The decay channel $p \rightarrow \pi^0 e^+$ has the lifetime

$$\tau(p \rightarrow \pi^0 e^+) = \frac{4f_\pi^2 M_X^4}{\pi m_p \alpha_{GUT}^2 (1 + D + F)^2 \alpha_N^2 [A_{R,1}^2 + (1 + |V_{ud}|^2)^2 A_{R,2}^2]} \left[1 + 2 \frac{m_\pi^2}{m_p^2} \right]. \tag{99}$$

With updated experimental bounds from Super-Kamiokande [42,43] $\tau > 1.67 \times 10^{34}$ years, we can constrain the inputs

$$\alpha_{GUT} \lesssim (5.27)^{-1}, \tag{100}$$

by taking $f_\pi = 131$ MeV, chiral Lagrangian factor $1 + D + F = 2.27$ with $D = 0.80$, $F = 0.47$ [44], the hadronic matrix element $\alpha_N = 0.0112$ GeV³ (at renormalization scale $\mu = 2$ GeV) and $A_{R,1} = A_{R,2} \approx 5$, respectively. This value constrained δ to be

$$|\delta| \lesssim 19. \tag{101}$$

It is known that a measure of gauge unification by experiments

$$B \equiv \frac{\alpha_2^{-1}(M_Z) - \alpha_3^{-1}(M_Z)}{\alpha_1^{-1}(M_Z) - \alpha_2^{-1}(M_Z)} \tag{102}$$

agrees with the one-loop MSSM prediction $B = 5/7$ up to 5% accuracy. In our scenario with multi-scale messengers, the parameter is given by

$$\begin{aligned} B &= \frac{(b_2 - b_3) \ln\left(\frac{M_{GUT}}{M_Z}\right) - \sum_{n_r} n_r \ln\left(\frac{M_{GUT}}{M_{n_r}}\right)}{(b_1 - b_2) \ln\left(\frac{M_{GUT}}{M_Z}\right) - \sum_{n_r} n_r \ln\left(\frac{M_{GUT}}{M_{n_r}}\right)}, \\ &= \frac{(b_2 - b_3) \ln\left(\frac{M_{GUT}}{M_Z}\right) - \ln\left(\frac{M_{GUT}^N}{\det \mathcal{M}}\right)}{(b_1 - b_2) \ln\left(\frac{M_{GUT}}{M_Z}\right) - \ln\left(\frac{M_{GUT}^N}{\det \mathcal{M}}\right)}, \end{aligned} \tag{103}$$

with $N = \sum_{n_r} n_r$ the total species of messenger. So, in order for the deviation of B to not exceed 5%, the parameters should satisfy

$$N \ln\left(\frac{M_{GUT}}{\sqrt[N]{\det \mathcal{M}}}\right) \lesssim 7/8. \tag{104}$$

We should note that $A_{R;1}, A_{R;2}$, which represent the renormalization effects resulting from the anomalous dimensions of the operators, will also be amended by presence of vector-like messengers [45]. They are defined as

$$A_{R;1} = A_L A_{S;1}, \quad A_{R;2} = A_L A_{S;2}, \tag{105}$$

with $A_L, A_{S;i}$ the long and short distance factors, respectively. Here the long-distance contribution A_L is taken to be 1.25. The short distance factors will be changed into

$$\begin{aligned} A_{S;i} &= \left(\prod_j \frac{\alpha_j(M_{SUSY})}{\alpha_j(M_Z)} \right)^{\frac{\gamma_{ji}^{SM}}{b_j^{SM}}} \left(\prod_j \frac{\alpha_j(M_{mess})}{\alpha_j(M_{SUSY})} \right)^{\frac{\gamma_{ji}^{MSSM}}{b_j^{MSSM}}} \left(\prod_j \frac{\alpha_j(M_{GUT})}{\alpha_j(M_{mess})} \right)^{\frac{\gamma_{ji}^{MSSM}}{b_j'}}, \\ &= \left(\prod_j \frac{\alpha_j(M_{SUSY})}{\alpha_j(M_Z)} \right)^{\frac{\gamma_{ji}^{SM}}{b_j^{SM}}} \left(\prod_j \frac{\alpha_j(M_{GUT})}{\alpha_j(M_{SUSY})} \right)^{\frac{\gamma_{ji}^{MSSM}}{b_j^{MSSM}}} \left(\prod_j \frac{\alpha_j(M_{GUT})}{\alpha_j(M_{mess})} \right)^{\frac{\gamma_{ji}^{MSSM}}{b_j'} - \frac{\gamma_{ji}^{MSSM}}{b_j^{MSSM}}}, \\ &\equiv A_{S;i}^0 \left(\prod_j \frac{\alpha_j(M_{GUT})}{\alpha_j(M_{mess})} \right)^{\frac{\gamma_{ji}^{MSSM}}{b_j'} - \frac{\gamma_{ji}^{MSSM}}{b_j^{MSSM}}}. \end{aligned} \tag{106}$$

in the case that one vector-like family of messengers at scale M_{mess} is present. Results with multiple messenger thresholds can be trivially extended. The relevant coefficients within the expressions are given [46] as

$$\begin{aligned} \gamma_{j;1}^{SM} &= (2, 9/4, 11/20), \quad \gamma_{j;2}^{SM} = (2, 9/4, 23/20), \\ \gamma_{j;1}^{MSSM} &= (4/3, 3/2, 11/30), \quad \gamma_{j;2}^{SM} = (4/3, 3/2, 23/30), \end{aligned} \tag{107}$$

with b_j the relevant gauge beta functions upon each threshold. The multiple factor for $A_{S;i}^0$ in the presence of messengers is given approximately by

$$\begin{aligned} F_1 &= \left(\prod_j \frac{\alpha_j(M_{GUT})}{\alpha_j(M_{mess})} \right)^{\frac{\gamma_{ji}^{MSSM}}{b_j'} - \frac{\gamma_{ji}^{MSSM}}{b_j^{MSSM}}} \\ &\approx \left[1 + \frac{b_j'}{2\pi} \alpha_j(M_{GUT}) \ln \frac{M_{GUT}}{M_{mess}} \right]^{\frac{\gamma_{ji}^{MSSM}}{b_j'} - \frac{\gamma_{ji}^{MSSM}}{b_j^{MSSM}}}, \\ &\approx 1 - \frac{\gamma_{ji}^{MSSM}}{2\pi} \frac{\Delta b_j^m}{b_j^{MSSM}} \alpha_j(M_{GUT}) \ln \frac{M_{GUT}}{M_{mess}}, \end{aligned} \tag{108}$$

in which we define $\Delta b_j^m = b_j' - b_j^{MSSM} = n_1$. This multiple factor can be easily extended to include multiple messengers. For example, with additional messenger thresholds at M_2 , the new multiple factor is given by

$$F_2 \approx 1 - \frac{\gamma_{ji}^{MSSM}}{2\pi} \frac{n_2}{b_j^{MSSM} + n_1} \alpha_j(M_{GUT}) \ln \frac{M_{GUT}}{M_2}, \tag{109}$$

with the total multiple factor

$$F = \prod_k F_k \approx 1 - \frac{\gamma_{ji}^{MSSM}}{2\pi} \alpha_j(M_{GUT}) \sum_{k=1}^p \frac{n_k}{b_j^{MSSM} + \sum_{l=1}^{k-1} n_l} \ln \frac{M_{GUT}}{M_k}. \tag{110}$$

As the coefficients $A_{R,1}, A_{R,2}$ depend on the messenger scales, the proton decay constraints will feed back into the constraints on δ . Detailed discussions on constraints for δ will be given in our subsequent studies.

5. Conclusions

The EOGM extension of deflected AMSB scenarios can accommodate the deflected AMSB scenarios with the presence of both Kahler and superpotential deflections. We revisit the EOGM scenario and derive the analytical expressions for soft SUSY breaking parameters with wavefunction renormalization approach in EOGM extension of deflected AMSB scenarios. As EOGM extension scenarios always introduce additional messenger species, we therefore also consider the Landau pole and proton decay constraints on the messenger sector.

Minimal AMSB is always bothered by the tachyonic slepton problem. As deflections by the messengers can introduce additional positive contributions to the slepton squared masses, the slepton squared masses can be tuned to be positive small numbers. Consequently, EOGM extension of deflected AMSB scenarios can easily predict light sleptons. In addition, the EOGM extension of deflected AMSB scenarios can alter the gaugino relations at the EW scale. The gluino masses are constrained by LHC to be heavier than 2.2 TeV. So, the gaugino ratios $M_1:M_2:M_3 \approx 1:2:6$ in ordinary GMSB and mSUGRA always constrain the electroweakino masses to be light. However, recent muon $g - 2$ anomaly [37] and new CDF II W-boson mass data [38] always prefer light electroweakinos [39]. So, spoiled gaugino relations are always welcome. In EOGM extension of deflected AMSB scenarios, the gaugino ratios change approximately into $M_1 : M_2 : M_3 \approx (33/5 - d_U n) : 2(1 - d_U n) : -6(-3 - d_U n)$ at the EW scale, with n an integer satisfying $0 \leq n \leq N$ and d_U the deflection parameter. So, with proper choices for d_U and n , light electroweakinos can be obtained, given the LHC constraints on gluino mass. With light sleptons and electroweakinos (especially light wino), the muon $g - 2$ anomaly and new CDF II W-boson mass data can be explained in the SUSY framework. Such light smuons and electroweakinos will be tested at the LHC and set new constraints on the EOGM extension AMSB type models, although some compressed regions can still survive. Wino DM is always predicted in such scenarios. For light wino, additional DM species are necessary to give correct relic abundances. Relevant phenomenology for such EOGM extension of deflected AMSB scenarios will be discussed in our subsequent studies.

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Appendix A. Coefficients in the Wavefunction of MSSM Superfields

From the anomalous dimension

$$\frac{d}{dt} \ln Z_f = \sum_{l=g_3, g_2, g_1} 2\tilde{A}_l \frac{d \ln g_l}{dt} + \sum_{l=y_t, y_b, y_\tau} 2B_l \frac{d \ln y_l}{dt},$$

in the basis of $(y_t^2, y_b^2, y_\tau^2, g_3^2, g_2^2, g_1^2)$, the coefficients \tilde{A}_l, B_l can be solved. Expressions of the coefficients had already been obtained in our previous paper [34]. The coefficients A_i are listed in Table A1.

Table A1. The gauge field coefficients $A_i (i = 1, 2, 3)$ within the wavefunction for MSSM superfields.

	$A_3(g_3)$	$A_2(g_2)$	$A_1(g_1)$
Q_3	$\frac{128}{61}$	$\frac{87}{61}$	$-\frac{11}{61}$
U_3	$\frac{144}{61}$	$-\frac{108}{61}$	$\frac{144}{305}$
D_3	$\frac{112}{61}$	$-\frac{84}{61}$	$\frac{112}{305}$
L_3	$\frac{80}{61}$	$\frac{123}{61}$	$-\frac{103}{305}$
E_3	$\frac{160}{61}$	$-\frac{120}{61}$	$\frac{61}{32}$
H_u	$-\frac{272}{61}$	$\frac{21}{61}$	$-\frac{89}{305}$
H_d	$-\frac{240}{61}$	$-\frac{3}{61}$	$-\frac{57}{305}$
Q_2	$\frac{16}{3}$	3	$\frac{1}{15}$
U_2	$\frac{16}{3}$	0	$\frac{16}{15}$
D_2	$\frac{16}{3}$	0	$\frac{4}{15}$
L_2	0	3	$\frac{3}{5}$
E_2	0	0	$\frac{12}{5}$

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