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Quantum Capacity and Vacuum Compressibility of Spacetime: Thermal Fields

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Abstract: An important yet perplexing result from work in the 1990s and 2000s is the near-unity value of the ratio of fluctuations in the vacuum energy density of quantum fields to the mean in a collection of generic spacetimes. This was carried out by way of calculating the noise kernels which are the correlators of the stress-energy tensor of quantum fields. In this paper, we revisit this issue via a quantum thermodynamics approach, by calculating two quintessential thermodynamic quantities: the heat capacity and the quantum compressibility of some model geometries filled with a quantum field at high and low temperatures. This is because heat capacity at constant volume gives a measure of the fluctuations of the energy density to the mean. When this ratio approaches or exceeds unity, the validity of the canonical distribution is called into question. Likewise, a system's compressibility at constant pressure is a criterion for the validity of grand canonical ensemble. We derive the free energy density and, from it, obtain the expressions for these two thermodynamic quantities for thermal and quantum fields in 2d Casimir space, 2d Einstein cylinder and 4d ($S^1 \times S^3$) Einstein universe. To examine the dependence on the dimensionality of space, for completeness, we have also derived these thermodynamic quantities for the Einstein universes with even-spatial dimensions: $S^1 \times S^2$ and $S^1 \times S^4$. With this array of spacetimes we can investigate the thermodynamic stability of quantum matter fields in them and make some qualitative observations on the compatibility condition for the co-existence between quantum fields and spacetimes, a fundamental issue in the quantum and gravitation conundrum.

Keywords: quantum fields; curved spacetime; heat capacity; vacuum compressibility

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1. Introduction

Three major elements are embedded in the theme explored in this paper: quantum fields, spacetime, and thermodynamics. The first two describe how quantum matter is affected by spacetime, its geometry and topology, and how it steers the dynamics of spacetime, although the latter issue involving backreaction is not explored here. The relation between matter and spacetime dynamics is of course underwritten by Einstein's general relativity theory. With quantum matter as source, one needs a theory of quantum fields in curved spacetime [1,2], which is a test field limit of semiclassical gravity theory [3], where the dynamics of the matter field and the spacetime are treated self-consistently. The first and third elements refer to the thermodynamics of quantum fields, a familiar subject, by way of thermal field theory. In this paper, we shall only consider quantum fields under equilibrium conditions, which can exist for static spacetimes, and be treated by finite temperature quantum field theory. Later, when we tackle cosmological issues, we shall call upon the more challenging subject of nonequilibrium quantum field theory [4] and nonequilibrium quantum thermodynamics [5]. The second and third elements bear on the thermodynamics of spacetime, which is also an old topic, ranging from the thermodynamic properties of classical matter, such as gravitating systems having negative heat capacity [6,7] to black hole mechanics [8] and thermodynamics [9] and, when quantum physics is taken into the consideration, the famous Bekenstein–Hawking (B-H) entropy [10] and its deep physical meaning.[1]

All told, these are the issues one needs to consider behind the grander views of spacetime thermodynamics [15], general relativity as geometro-hydrodynamics [16], and emergent gravity [17–19], where the large scale structure and dynamics of spacetime can be phrased in thermodynamic or hydrodynamic terms. What we want to accomplish in this and a sequel paper [20] is more restricted in scope, aiming at firing some solid bricks toward building this superstructure.

1.1. Energy and Pressure Densities of Quantum Fields and Fluctuations

One fundamental aspect of special interest to us is fluctuation phenomena in quantum matter fields and how they influence the dynamics and the thermodynamics of spacetimes. We describe their importance and plan for two ways to approach them, by calculating two important thermodynamic quantities, the heat capacity and the vacuum compressibility here and later, via noise kernels in quantum field theory in curved spacetime.

Importance of Quantum Field Fluctuations Phenomena

Quantum field fluctuations phenomena have both fundamental theoretical and practical application values. As examples of the fundamental theoretical issues we mention the quantum inequalities and the quantum interest principle [21,22], the highly nonGaussian distribution of the quantum fluctuations of the vacuum stress-energy tensor [23], the role of fluctuations of quantum fields in the 'condensed matter' explanation of the smallness of the cosmological constant today [24]. Toward the other extreme, at the Planck scale, dimensional reduction to effective 2D behavior [25] and the breaking up of space into a large number of causally disconnected regions [26]. For the more tangible, laboratory-accessible phenomena, the simplest example of a vacuum polarization effect is perhaps the Casimir effect [27–30], e.g., an attractive force between two conducting plates, or a repulsive force in a conducting sphere. When the plates are moved rapidly, particles are created from the vacuum. Dynamical Casimir effect [31,32] stems from the amplification of vacuum fluctuations by changing boundaries [33–35]. Cosmological particle creation [36,37] shares

the same physics, the driving agent being the expanding universe. The theoretical basis is quantum fields in curved spacetime [1,2]. Quantum effects in the early universe [38–52] invoking semiclassical and stochastic gravity theories [3] are believed to have played a decisive role in shaping our present universe.

The purpose of our present investigation is to knit a thermodynamic picture of these quantum field fluctuation phenomena. Our plan is to take a two-pronged approach to explore these issues, one pursued in this and a sequel paper uses thermal field theory and quantum field theory in curved spacetimes, via free energy density and partition functions; the other approach invokes the noise kernel in stochastic gravity [3]. The noise kernel is the vacuum expectation value of the stress-energy bitensor [53] or the stress-tensor two point functions (correlators). The 00 component and the ii components give, respectively, the fluctuations in the energy and momentum density. An interesting result from works in the 90s and 00s is the near-unity value of the ratio of fluctuations in the vacuum energy density to the mean. This quantity Δ has been calculated for thermal fields in Minkowski and Casimir geometries, the Einstein universe, de Sitter and anti-deSitter spacetimes and more [53–59]. This is a plain yet intriguing result. It may hold some deeper meanings in how quantum fields (matter contents) co-exist with spacetime (geometry and topology). Possible implications on the condition of our universe is a theme worth further explorations [60].

In this paper, we wish to shed some light on this issue by way of a thermodynamic approach, by computing two essential thermodynamic quantities, the heat capacity and the adiabatic compressibility.

Thermodynamics of quantum fields reflecting the properties of spacetime

One commonly studied thermodynamic quantity is the heat capacity, well-known as a measure of the magnitude of energy fluctuations. When it diverges it spells the breakdown of canonical ensembles in thermodynamics.

In general the heat capacity is negative [6,7] for gravitating systems (see, e.g., [61] and references therein). That is why when one applies thermodynamics in a canonical ensemble setting one needs to add the condition that the system, e.g, a black hole, needs to be placed in a box or in an AdS space. We need to separate two effects simultaneously affecting the heat capacity, one due to quantum fluctuations, the other due to gravity. To explore gravitating systems without an artificially imposed boundary we need to use microcanonical ensembles [62] where the starting point would be the number of accessible states to an isolated system at a certain energy.

The other thermodynamic quantity of equal importance but lesser studied in field theory is compressibililty, isothermal at finite temperature and adiabatic compressibility at zero temperature. As is evident from its definition, it measures how compressible the quantum field is, which depends on the curvature and topology of the space they live in. In fact when we refer to, say, the capacitance of a parallel plate capacitor in electrostatics, we usually just give a formula in terms of geometric measures, such as the area of the plates and the distance between them. It is implicit that we are talking about electric charges and fields. It is in this sense that we refer to the capacity and compressibility of spacetimes in terms of how a quantum field and its fluctuations behave. Since heat capacity is studied widely we shall focus more on the latter. The term 'quantum capacity' in the title of this paper refers to the heat capacity of quantum fields, including high and low temperatures; and when we say 'vacuum compressibility' we refer to adiabatic compressibility which can be defined without any notion of heat.

1.2. Physical Contexts of Quantum Capacitance and Negative Compressibility

To better appreciate the meaning and significance of these thermodynamic quantities it is useful to see how quantum capacitance and vacuum compressibility are defined and used in some more familiar physical systems. We give two examples here: nuclear "liquid drop" model and 2-dim electron gas. Nuclear collective model is a good analog

because the Hamiltonian quadrature has the same mathematical form as the metric of the mixmaster universe [63–65], a compact anisotropic spacetime, where compressibility has some intuitive meaning. 2-dim electron gas has similar thermodynamic behavior as the gravitational field, in that they are both systems of long range interactions and with negative heat capacities (see, e.g., [61] and references therein).

Nuclear compressibility in quantum hadrodynamics

The compressibility of a nucleus is directly related to the surface energy—a high compressibility implies a stiff surface which will have a large contribution to the total energy of the deformed nucleus. For example, Price and Walker [66] developed a self-consistent, relativistic theory of deformed nuclei based on quantum hadrodynamics and the finite Hartree approximation, and have applied this theory to the calculation of deformed orbitals in various light nuclei. They even went into details in explaining why a relativistic mean field calculation based on quantum hadrodynamics [67] gives a larger value of compressibility than non-relativistic calculations. Inasmuch as the nuclear collective model sharing a similar mathematical structure (Hamiltonian quadratic form) as the mixmaster universe (diagonal Bianchi type IX metric), the Hartree approximation in seeking self-consistent solutions are similar to the structure and procedures in semiclassical gravity theory.

What does the negative compressibility of a 2-dim electron gas reveal?

For two dimensional electron systems at high magnetic fields, interaction effects have spectacular transport consequences, e.g., the fractional quantum Hall effect. The importance of exchange and correlation contributions to the total energy, and hence the thermodynamics, of interacting electron systems has long been theoretically appreciated. For example, in the low-density regime, where interactions dominate the kinetic energy, the exchange energy alone is sufficient to produce a negative compressibility for the electron gas – the inverse of the compressibility is given by $\kappa^{-1} = N^2 \partial\mu/\partial n$, where μ is the chemical potential and n is the number density. Eisenstein et al. [68] reported on a new experimental technique whereby they can directly extract both the sign and the magnitude of κ as a function of electron density n. Regions of negative κ are observed at both zero and high magnetic field extreme quantum limit. Observation of these compressibility features constitutes strong thermodynamic evidence for existence of the dilute quasiparticle gases central to theory of the fractional quantum Hall effect. This is one of many examples of how (macroscopic) thermodynamic quantities may reveal some important attributes of the underlying (microscopic) constituents and their interactions.

The above cases exemplify how macroscopic quantities can be judiciously used to reveal the workings of the microscopic constituents, a better understanding of which can serve as an inspiration for the hydrodynamic and thermodynamics approaches to probing the microscopic structures of spacetime.

1.3. Methods, Findings and Organization

Zeta function method is used in our calculations of the thermodynamic quantities for a thermal Bose gas (finite temperature quantum scalar field). The main targets are C_V, C_P related to the energy density fluctuations at constant volume and pressure, and the isothermal κ_T and adiabatic compressibility κ_S related to the momentum density fluctuations at finite and zero temperatures. For completeness we also provide the energy density, entropy and the expansion coefficients α. Thermal fields (with periodic imaginary time) in the following static background spacetimes are considered: (a) 2d Casimir geometry and 2d Einstein Cylinder (periodic in one spatial dimension); (b) 4d Einstein Universe $S^1 \times S^3$, (c) Einstein universes with even-spatial dimensions $S^1 \times S^2$ and $S^1 \times S^4$.

The results for the thermodynamic quantities at low and high temperatures are presented in two tables in the last section. Please refer there for detailed explanations. In Section 2, we derive the thermodynamic quantities in 2d (1-d space) thermal Casimir and 2d Einstein cylinder. In Section 3, in 4d (3-sphere space) Einstein universe, and in Section 4,

in 3d (2-sphere) and 5d (4-sphere) Einstein universes. In Section 5, we conclude with a summary and some discussions. In this paper, we focus more on the technical aspects, aiming to get a complete tally of these thermodynamic quantities. We shall continue to explore their physical significance to better understand how the thermodynamics of quantum fields is governed by the underlying spacetime structure and what we can say about the nature of gravity from the thermodynamics of quantum fields.

2. 2d: Thermal Casimir and Einstein Cylinder

To illustrate the points we would like to make, on the capacities and the compressibilities of quantum fields in spaces with various topologies, we start with the simple case of an Einstein cylinder at finite temperature [69,70]. The topology is basically $S^1 \times S^1$. (The first entry denotes time, the second entry denotes spatial dimensions. Since we shall be working with thermal field in the imaginary time formulation, the first entry will always generically be S^1. Only in certain specified limit will R^1 appear.) The length of the Euclidean time circle is characterized by $\beta = 1/T$ where T is the temperature. The spatial part is in a Casimir setting where the circle is characterized by the length L. In the low temperature limit, $L/\beta \to 0$, one has the $R^1 \times S^1$ topology with the leading contribution coming from the Casimir effect. On the other hand, in the high temperature limit, $\beta/L \to 0$, one has the thermal field in two spacetime with the topology $S^1 \times R^1$.

We shall consider a thermal minimally coupled massless scalar field in the Einstein cylinder with periodic boundary condition. To derive the Helmholtz free energy we start with the partition function, with the notation of Phillips and Hu [53],

$$Z = e^W \tag{1}$$

where using the proper-time zeta function method,

$$\begin{aligned} W &= -\frac{1}{2}\mathrm{Tr}\ln\left(\frac{H}{\mu}\right) \\ &= \lim_{s\to 0}\frac{1}{2}\frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\,\mathrm{Tr}(e^{-tH})\right] \\ &= \lim_{s\to 0}\frac{1}{2}\frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\sum_n e^{-t\lambda_n}\right] \end{aligned} \tag{2}$$

with the operator

$$H = -\Box = -\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial x^2} \tag{3}$$

where τ is the Euclidean time, the eigenvalues

$$\lambda_{n_0,n} = k_0^2 + k^2 \quad ; \quad k_0 = \frac{2\pi n_0}{\beta} \quad ; \quad k = \frac{2\pi n}{L} \tag{4}$$

and the eigenfunctions

$$\phi_{n_0,n}(x) = \left(\frac{1}{\sqrt{\beta}}e^{ik_0\tau}\right)\left(\frac{1}{\sqrt{L}}e^{ikx}\right). \tag{5}$$

With the partition function Z, we proceed to calculate the free energy F.

$$\begin{aligned} Z = e^{-\beta F} \Rightarrow F &= -\frac{W}{\beta} \\ &= \left(-\frac{1}{2\beta}\right)\lim_{s\to 0}\frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\sum_{n_0,n} e^{-tk_0^2}e^{-tk^2}\right] \end{aligned} \tag{6}$$

To proceed we consider the low and the high temperature expansions of this free energy. Then we can discuss the properties of various thermodynamic quantities, notably the capacities and the compressibilities that can be derived from F.

2.1. Low Temperature Expansion: Approaching $R^1 \times S^1$

To develop the low temperature expansion, we first rewrite the free energy F using the Poisson summation formula (see, for example, [71]) on the sum over n_0.

$$\sum_{n_0=-\infty}^{\infty} e^{-t(2\pi n_0/\beta)^2} = \frac{\beta}{2\sqrt{\pi t}} \sum_{n_0=-\infty}^{\infty} e^{-n_0^2\beta^2/4t}, \tag{7}$$

With this replacement, the free energy can then be expressed as

$$F = \left(-\frac{1}{4\sqrt{\pi}}\right) \lim_{s\to 0} \frac{d}{ds} \left[\frac{\mu^s}{\Gamma(s)} \int_0^\infty dt\, t^{s-\frac{3}{2}} \sum_{n_0,n} e^{-n_0^2\beta^2/4t}\, e^{-t\left(\frac{2\pi n}{L}\right)^2} \right] \tag{8}$$

To start with, we look at the $n_0 = n = 0$ term. This term is formally divergent so we regularize it by adding in a mass m.

$$\begin{aligned} F|_{n_0=n=0} &= -\frac{1}{4\sqrt{\pi}} \lim_{m\to 0, s\to 0} \frac{d}{ds} \left[\frac{\mu^s}{\Gamma(s)} \int_0^\infty dt\, t^{s-\frac{3}{2}}\, e^{-tm^2} \right] \\ &= -\frac{1}{4\sqrt{\pi}} \lim_{m\to 0} \left(-2\sqrt{\pi}\, m\right) \\ &= 0. \end{aligned} \tag{9}$$

We shall therefore neglect this term in the following.

For the $n_0 \neq 0$ and $n = 0$ term, we have the finite temperature zero mode contribution to the free energy.

$$\begin{aligned} F|_{n_0\neq 0, n=0} &= \left(-\frac{1}{2\sqrt{\pi}}\right) \lim_{s\to 0} \frac{d}{ds} \left[\frac{\mu^s}{\Gamma(s)} \int_0^\infty dt\, t^{s-\frac{3}{2}} \sum_{n_0=1}^{\infty} e^{-n_0^2\beta^2/4t} \right] \\ &= -\frac{1}{2\sqrt{\pi}} \lim_{s\to 0} \frac{d}{ds} \left[\frac{\mu^s}{\Gamma(s)} 2^{1-2s} \left(\beta^2\right)^{-\frac{1}{2}+s} \Gamma\left(\frac{1}{2} - s\right) \zeta(1-2s) \right] \\ &= \frac{1}{2\beta} \ln(\mu\beta^2), \end{aligned} \tag{10}$$

which is proportional to $\ln\beta/\beta$. Whether this term should be included in evaluating the free energy has been a controversial issue [72–74]. In [72], it was argued that it should be excluded. The main reason is that the entropy derived from this term would go like $\ln\beta$ which will blow up in the zero temperature limit, being at variance with the third law of thermodynamics. On the contrary, the inclusion of the zero mode was advocated in [73], particularly to preserve the inversion relation between free energies at low and high temperatures [75]. In a more mundane argument, the inclusion of zero mode is necessary for the mode function to form a complete set, which in turns enforces causality. When the zero mode is missing, it has been shown that the lightcone structure emerges in massless relativistic field theory can be violated [70,76]. Here, we shall include this term for completeness and from past experiences. The zero mode in the spectrum of an invariant operator governs the infrared behavior of quantum fields in curved spacetimes (see, e.g., [77,78]). The infrared behavior of massless minimally-coupled interacting quantum field in de Sitter universe is an important problem in cosmology (see, e.g., [79] for a review where earlier references can be found). In our subsequent discussions we shall explore its consequences by investigating the properties of the corresponding thermodynamic quantities derived from it. One further remark is that this zero mode contribution existent in all compact spaces we have considered is independent of the details of the spatial geometry. Therefore,

as we shall see in the following, this term will contribute to the free energy in the low temperature expansion for all spatial configurations with a zero mode in the eigenspectrum. However, in contrast to the interacting quantum field case mentioned above, for free fields considered here, there is no zero mode contribution in the zero temperature free energy expression.

For $n_0 = 0$ and $n \neq 0$,

$$\begin{aligned} F|_{n_0=0,n\neq 0} &= \left(-\frac{1}{2\sqrt{\pi}}\right) \lim_{s\to 0} \frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)} \int_0^\infty dt\, t^{s-\frac{3}{2}} \sum_{n=1}^{\infty} e^{-t\left(\frac{2\pi n}{L}\right)^2}\right] \\ &= -\frac{\pi}{6L} \end{aligned} \tag{11}$$

As $n_0 = 0$, we have the zero temperature limit here. The result actually corresponds to the quantum Casimir effect [27–30] due to the spatial circle. This is consistent with the usual derivation of the Casimir energy. With periodic boundary condition, the allowed frequencies are just $\omega_n = 2\pi n/L$. The vacuum energy is thus

$$E = \sum_{n=-\infty}^{\infty} \frac{1}{2}\omega_n = \left(\frac{2\pi}{L}\right)\zeta(-1) = -\frac{\pi}{6L} \tag{12}$$

which is the same as what we had above.

For $n_0 \neq 0$, $n \neq 0$, we have

$$\begin{aligned} F|_{n_0\neq 0,n\neq 0} &= \left(-\frac{1}{\sqrt{\pi}}\right) \lim_{s\to 0} \frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)} \int_0^\infty dt\, t^{s-\frac{3}{2}} \sum_{n_0,n=1}^{\infty} e^{-n_0^2\beta^2/4t}\, e^{-t\left(\frac{2\pi n}{L}\right)^2}\right] \\ &= \sum_{n_0,n=1}^{\infty} \left(-\frac{2}{n_0\beta}\right) e^{-2\pi n_0 n\beta/L} \\ &= -\frac{2}{\beta} e^{-2\pi\beta/L} + \cdots, \end{aligned} \tag{13}$$

where the ellipsis represents terms which are also exponentially small and non-analytic like $e^{-2\pi\beta/L}$.

Hence, combining expressions in Equations (9) to (13), the free energy in the low temperature expansion is given by

$$F = -\frac{\pi}{6L} + \frac{1}{2\beta}\ln(\mu\beta^2) + \cdots. \tag{14}$$

Indeed, when $\beta \to \infty$ or $T \to 0$, we are left with the first term which is just the Casimir energy.

With the free energy we are able to derive various thermodynamic quantities including the heat capacities and the compressibilities [80] in the low temperature limit. First, the entropy

$$S = 1 - \frac{1}{2}\ln(\mu\beta^2) + \cdots. \tag{15}$$

Other than the exponentially small terms, the contribution to the entropy comes solely from the zero mode since the first term corresponding to the Casimir effect in the free energy in Equation (14) is independent of temperature. As we have mentioned above, the $\ln\beta$ term diverges as the temperature $T \to 0$, apparently violating the third law of thermodynamics in the traditional settings, under the assumptions of large spatial volume and high temperatures. This is an important issue which deserves closer examinations, especially in the context of quantum thermodynamics in spacetimes with curvature or

nontrivial topology. We shall therefore leave this term as it is and try to explore more its consequences on the other thermodynamic quantities.

With the entropy S, we have the internal energy

$$E = F + TS = -\frac{\pi}{6L} + \frac{1}{\beta} + \cdots, \tag{16}$$

which is independent of the scale μ. The dominant term of the internal energy comes from the Casimir effect as given by Equation (12). The subdominant term, which is linear in T, is the zero mode contribution. The corresponding energy density would then be

$$\rho = \frac{E}{L} = -\frac{\pi}{6L^2} + \frac{1}{\beta L} + \cdots. \tag{17}$$

From the free energy, pressure P is just

$$P = -\left(\frac{\partial F}{\partial L}\right)_T = -\frac{\pi}{6L^2} + \cdots. \tag{18}$$

We thus have negative pressure coming from the Casimir effect. The magnitude increases as the size L gets smaller. Negative pressure would shrink the spatial circle and this shrinking force would get larger as the size of circle gets smaller. This process is very similar to that of a gravitational collapse. In fact, we shall see in the subsequent discussions, the Casimir effects of spatial spheres with different dimensions would all induce negative pressure and same kind of collpases should therefore occur.

From the entropy, we obtain the heat capacity at constant volume which is given by the second temperature derivative of the Helmholtz free energy:

$$C_V = -\beta\left(\frac{\partial S}{\partial \beta}\right)_L = 1 + \cdots \tag{19}$$

Hence, other than exponentially small terms, C_V is basically a constant.

Furthermore, the second derivative of the free energy with respect to volume L is related to the isothermal compressibility κ_T,

$$\kappa_T = -\frac{1}{L}\left(\frac{\partial L}{\partial P}\right) = -\frac{3L^2}{\pi} + \cdots. \tag{20}$$

κ_T is negative due to the negative pressure of the Casimir effect. This means that when pressure is increased, or the magnitude of the pressure is decreased, the volume L will increase. This is in contrast to the case of a normal gas where the volume would decrease when the pressure is increased with positive compressibility.

There is another second derivative of the free energy

$$\frac{\partial^2 F}{\partial T \partial L} = -\frac{\partial S}{\partial L} = -\frac{\partial P}{\partial T} \tag{21}$$

according to the Maxwell relations. This second derivative is related to the thermal expansion coefficient defined by

$$\alpha = \frac{1}{L}\left(\frac{\partial L}{\partial T}\right)_P. \tag{22}$$

By the cyclic relation

$$\begin{aligned}\left(\frac{\partial L}{\partial T}\right)_P &= -\left(\frac{\partial L}{\partial P}\right)_T\left(\frac{\partial P}{\partial T}\right)_L \\ &= L\kappa_T\left(\frac{\partial P}{\partial T}\right)_L \end{aligned} \tag{23}$$

Hence, the thermal expansion coefficient can be expressed as

$$\alpha = \kappa_T\left(\frac{\partial P}{\partial T}\right)_L. \tag{24}$$

In our present consideration, with the pressure P in Equation (18) having a leading term independent of temperature, we have

$$\left(\frac{\partial P}{\partial T}\right)_L = -\beta^2\left(\frac{\partial P}{\partial \beta}\right)_L \sim e^{-2\pi\beta/L} + \cdots \tag{25}$$

which is exponentially small. Thus, with the isothermal compressibility in Equation (20),

$$\alpha \sim e^{-2\pi\beta/L} + \cdots \tag{26}$$

and it is also exponentially suppressed.

Using the thermodynamic quantities we have obtained, we can also derive the heat capacity at constant pressure C_P. By applying the cyclic and the Maxwell relations, the following relation can be established [81].

$$C_P = C_V + \beta^3 L\kappa_T\left(\frac{\partial P}{\partial \beta}\right)^2 = 1 + \cdots. \tag{27}$$

Since $\partial P/\partial\beta$ is exponentially small, we can see that $C_P \sim C_V$ up to order $e^{-2\pi\beta/L}$.

Finally one can also obtain the adiabatic compressibility

$$\kappa_S = -\frac{1}{L}\left(\frac{\partial L}{\partial P}\right)_S \tag{28}$$

Again using the Maxwell and the cyclic relations, we have the identity [81]

$$\frac{\kappa_S}{\kappa_T} = \frac{C_V}{C_P} \Rightarrow \kappa_S = \left(\frac{C_V}{C_P}\right)\kappa_T. \tag{29}$$

As we have seen that $C_P \sim C_V$ up to order $e^{-2\pi\beta/L}$. Therefore, we also have $\kappa_S \sim \kappa_T$ other than exponentially small terms.

$$\kappa_S = -\frac{1}{L}\left(\frac{\partial L}{\partial P}\right) = -\frac{3L^2}{\pi} + \cdots. \tag{30}$$

Note that for a closed system, the thermodynamic processes would be adiabatic. Here, κ_S being negative would induce the kind of collapses we mentioned above for a closed spatial geometry. In the subsequent sections, we shall see that this is true for the cases of different spatial geometries in the low temperature expansion.

With the considerations above, it is also possible to establish the fluctuations of various thermodynamic quantities. Consider a small part of an equilibrium system. Assume that the small part is still large enough for the thermodynamic limit to hold. According to the fluctuation theory of Landau and Lifshitz [81,82], one can then derive the fluctuations of

the various thermodynamic quantities in this small part. Using the temperature T and the volume L as independent variables, we have the mean square fluctuation of the temperature

$$\langle (\Delta T)^2 \rangle = \frac{T^2}{C_V} = \frac{1}{\beta^2 C_V}, \tag{31}$$

while the fluctuation of the volume

$$\langle (\Delta L)^2 \rangle = LT|\kappa_T| = \frac{L|\kappa_T|}{\beta}. \tag{32}$$

As the fluctuation should be a positive quantity and in our case κ_T is actually negative, we therefore define the fluctuation to be related to the absolute value of κ_T instead.

Then the fluctuation of the internal energy is given by

$$\begin{aligned} \langle (\Delta E)^2 \rangle &= \frac{C_V}{\beta^2} + \frac{L|\kappa_T|}{\beta}\left[-\beta\frac{\partial P}{\partial \beta} - P\right]^2 \\ &= \frac{\pi}{12\beta L} + \frac{1}{\beta^2} + \cdots \end{aligned} \tag{33}$$

which is proportional to T. Note that as C_V is a constant and $\partial P/\partial\beta$ are exponentially small, the leading contribution comes from $L|\kappa_T|P^2/\beta$.

Again according to the Landau–Lifshitz fluctuation theory, the fluctuation of the pressure is given by

$$\langle (\Delta P)^2 \rangle = \frac{1}{\beta L|\kappa_S|} = \frac{\pi}{3\beta L^3} + \cdots \tag{34}$$

which is also proportional to T.

Moreover, the correlated fluctuation is then

$$\langle (\Delta E)(\Delta P) \rangle = \frac{P}{\beta} = -\frac{\pi}{6\beta L^2} + \cdots \tag{35}$$

We can see that the leading behaviors of these fluctuations are all proportional to T. In other words, as $T \to 0$, all the fluctuations will vanish. It is therefore apparent that the fluctuations derived above are all of thermal nature. No quantum fluctuations are included in this theory of fluctuations.

2.2. High Temperature Expansion: Approaching $S^1 \times R^1$

For the high temperature expansion, we take $\beta/L \ll 1$. Actually, we can also view this as a large L expansion or the infinite space limit. In this case, we implement the Poisson summation formula on the spatial sum over n,

$$\sum_{n=-\infty}^{\infty} e^{-t(2\pi n/L)^2} = \frac{L}{2\sqrt{\pi t}} \sum_{n=-\infty}^{\infty} e^{-n^2L^2/4t} \tag{36}$$

Then the free energy becomes

$$F = \left(-\frac{L}{4\sqrt{\pi}\beta}\right) \lim_{s\to 0} \frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)} \int_0^\infty dt\, t^{s-\frac{3}{2}} \sum_{n_0,n} e^{-t\left(\frac{2\pi n_0}{\beta}\right)^2} e^{-n^2L^2/4t}\right]. \tag{37}$$

Again the $n_0 = 0$, $n = 0$ term will be neglected as in the low temperature case. For $n_0 \neq 0$, $n = 0$, we have

$$\begin{aligned} F|_{n_0\neq 0, n=0} &= \left(-\frac{L}{2\sqrt{\pi}\beta}\right) \lim_{s\to 0} \frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)} \int_0^\infty dt\, t^{s-\frac{3}{2}} \sum_{n_0=1}^{\infty} e^{-t\left(\frac{2\pi n_0}{\beta}\right)^2}\right] \\ &= -\frac{\pi L}{6\beta^2} \end{aligned} \tag{38}$$

which represents the flat space limit $L \to \infty$. Note that after implementing the Poisson summation formula in Equation (36), the $n = 0$ term here does not correspond to the zero mode we discussed in the previous subsection.

For $n_0 = 0$, $n \neq 0$,

$$\begin{aligned} F|_{n_0=0, n\neq 0} &= \left(-\frac{L}{2\sqrt{\pi}\beta}\right) \lim_{s\to 0} \frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)} \int_0^\infty dt\, t^{s-\frac{3}{2}} \sum_{n=1}^{\infty} e^{-n^2L^2/4t}\right] \\ &= \frac{1}{2\beta} \ln\left(\mu L^2\right). \end{aligned} \tag{39}$$

Finally, for $n_0 \neq 0$, $n \neq 0$,

$$\begin{aligned} F|_{n_0\neq 0, n\neq 0} &= \left(-\frac{L}{\sqrt{\pi}\beta}\right) \lim_{s\to 0} \frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)} \int_0^\infty dt\, t^{s-\frac{3}{2}} \sum_{n_0,n=1}^{\infty} e^{-t\left(\frac{2\pi n_0}{\beta}\right)^2} e^{-n^2L^2/4t}\right] \\ &= \sum_{n_0,n=1}^{\infty} \left(-\frac{2}{n\beta}\right) e^{-2\pi n_0 nL/\beta} \\ &= -\frac{2}{\beta} e^{-2\pi L/\beta} + \cdots \end{aligned} \tag{40}$$

which consist of exponentially small terms. Hence, for high temperature, the free energy

$$F = -\frac{\pi L}{6\beta^2} + \frac{1}{2\beta} \ln\left(\mu L^2\right) \cdots \tag{41}$$

with the ellipsis indicating terms which are exponentially small like $e^{-2\pi L/\beta}$. The leading term corresponds to the flat space $L \to \infty$ finite temperature result.

It is interesting to note that from this free energy at high temperature in Equation (41), if we exchange $\beta \leftrightarrow L$,

$$\beta F|_{high} \to -\frac{\pi\beta}{6L} + \frac{1}{2}\ln(\mu\beta^2) + \cdots = \beta F|_{low}, \tag{42}$$

including the exponentially small terms, where $F|_{low}$ is the free energy at low temperature in Equation (14). $\beta \leftrightarrow L$ means $\beta/L \leftrightarrow L/\beta$, that is, the free energy in this case possesses an inversion symmetry between the low and the high temperatures. This symmetry is related to the Cardy formula [75] and the contribution of the zero mode is crucial for this relation to hold.

As in the low temperature expansion, various thermodynamic quantities follow from the free energy in Equation (41). The entropy

$$S = \frac{\pi L}{3\beta} - \frac{1}{2}\ln\left(\mu L^2\right) + \cdots \tag{43}$$

which is proportional to T in the leading term.

The internal energy

$$E = \frac{\pi L}{6\beta^2} + \cdots \tag{44}$$

which is independent of the scale μ. Note that the leading term gives the flat space energy density

$$\rho|_{L\to\infty} = \left(\frac{E}{L}\right)_{L\to\infty} = \frac{\pi}{6\beta^2}. \tag{45}$$

The pressure

$$P = \frac{\pi}{6\beta^2} - \frac{1}{\beta L} + \cdots \tag{46}$$

Here the pressure is positive. As $L \to \infty$, $P = \rho$ as it should be for an one-dimensional massless relativistic ideal gas.

It is interesting to note that in the low temperature limit, the dominant contribution to P is the Casimir pressure which is negative. Here in the high temperature limit, the pressure is positive instead. Therefore, there must be a temperature at which the pressure vanishes. Whether the resulting configuration is stable or not depends on the sign of the compressibility. This is what we shall further explore in the following consideration.

The heat capacity at constant volume

$$C_V = \frac{\pi L}{3\beta} + \cdots, \tag{47}$$

is proportional to T in the leading behavior.

The isothermal compressibility

$$\kappa_T = -\beta L + \cdots, \tag{48}$$

where the ellipsis again represents exponentially small terms. Here, although the pressure is positive, the isothermal compressibility is negative. That is, at constant temperature, when the volume L is decreased, the pressure also decreases. Remember κ_T is related to $\partial P/\partial L$. Since the first term in Equation (46) for P is independent of L, the leading contribution of κ_T comes from the second term. Hence, we have the peculiar situation in which the sign of P, that is, whether the pressure is positive or negative, is determined by the first term in the high temperature expansion, while the sign of the isothermal compressibility κ_T is determined by the second term. Therefore, κ_T could be positive or negative no matter what the sign of P is. This is different from what we encounter in the low temperature expansion where P is always negative due to the Casimir effect and the corresponding κ_T is also negative from this negative pressure.

With the isothermal compressibility κ_T, we can derive the thermal expansion coefficient

$$\alpha = -\frac{\pi L}{3} + \beta + \cdots, \tag{49}$$

which is also negative as κ_T. This has the peculiar behavior that, at constant pressure, the volume L will decrease with increase in temperature.

Using the relation in Equation (27), we obtain the heat capacity at constant pressure

$$C_P = -\frac{\pi^2 L^2}{9\beta^2} + \frac{\pi L}{\beta} - 1 + \cdots, \tag{50}$$

which is negative even though C_V is positive.

Finally, from the identity in Equation (29), we have the adiabatic compressibility

$$\kappa_S = \frac{3\beta^2}{\pi}\left[1+\frac{9\beta}{\pi L}+\frac{72\beta^2}{\pi^2 L^2}+O(\frac{\beta^3}{L^3})\right], \tag{51}$$

which is positive in contrast to a negative κ_T. If we view the quantum field in the spatial circle as a closed system, then its evolution satisfies the adiabatic condition. A positive κ_S means that when the volume decreases, the pressure will increase in this adiabatic situation. Therefore, a stable state may be reached inspite of a negative κ_T.

Furthermore, we can also consider the fluctuations of various thermodynamic quantities. First, the fluctuation of the internal energy

$$\langle(\Delta E)^2\rangle \quad = \quad \frac{\pi^2 L^2}{36\beta^4}+\frac{\pi L}{3\beta^3}+\cdots, \tag{52}$$

which is proportional to the T^4. Next, the pressure fluctuation is given by

$$\langle(\Delta P)^2\rangle = \frac{\pi}{3\beta^3 L}-\frac{3}{\beta^2 L^2}+\frac{3}{\pi\beta L^3}+\cdots \tag{53}$$

which is proportional to T^3. Furthermore,

$$\langle(\Delta E)(\Delta P)\rangle = \frac{\pi}{6\beta^3}-\frac{1}{\beta^2 L}+\cdots \tag{54}$$

which is again proportional to T^3. As in the high temperature limit, $E \sim L/\beta^2$ and $P \sim 1/\beta^2$, we have the ratios $\langle(\Delta E)^2\rangle/E^2$ of the order of 1, and both $\langle(\Delta P)^2\rangle/P^2$ and $\langle(\Delta E)(\Delta P)\rangle/EP$ proportional to β/L.

3. 4d: Einstein Universe $S^1 \times S^3$

For thermal fields in the Einstein universe, the topology of spacetime is $S^1 \times S^3$. The metric can be written as

$$ds^2 = d\tau^2 + a^2 d\bar{\Omega}_3^2 \tag{55}$$

where $\bar{\Omega}_3$ is the solid angle of a three sphere, and a is the "radius" characterizing the size of the sphere. The operator H in Equation (2) is now given by

$$H = -\frac{\partial^2}{\partial\tau^2}-\frac{1}{a^2}\bar{\Box}, \tag{56}$$

where $\bar{\Box}$ is the Laplacian on S^3. The eigenvalue of $\bar{\Box}$ on S^3 is [83]

$$\bar{\lambda}_n = -n(n+2), \tag{57}$$

with degeneracy

$$\bar{D}_n = (n+1)^2. \tag{58}$$

Hence, the eigenvalue of H is

$$\lambda = k_0^2 + \left(\frac{1}{a^2}\right)n(n+2). \tag{59}$$

The corresponding eigenfunctions are

$$\phi_{n_0,n}(x) = \left(\frac{1}{\sqrt{\beta}} e^{ik_0\tau}\right)\left(\frac{1}{\sqrt{a^3}} Y_3(\Omega)\right), \tag{60}$$

where $Y_3(\Omega)$ is the hyperspherical harmonics on S^3 [84]. Using the eigenvalues, one can establish the free energy, similar to Equation (6), for the Einstein universe as

$$\begin{aligned} F &= -\frac{1}{2\beta}\lim_{s\to 0}\frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\sum_{n_0=-\infty}^{\infty} e^{-t\left(\frac{2\pi n_0}{\beta}\right)^2}\right. \\ &\qquad \left.\sum_{n=0}^{\infty}(n+1)^2\, e^{-\left(\frac{t}{a^2}\right)n(n+2)}\right] \end{aligned} \tag{61}$$

As in the analysis for the Einstein cylinder case in the last section, we shall consider the low and the high temperature expansions separately [85–87].

3.1. Low Temperature Expansion

In the low temperature expansion, it is appropriate to first rewrite the sum over n_0 using the Poisson summation formula as in Equation (7). The free energy is therefore expressed as

$$\begin{aligned} F &= -\frac{1}{4\sqrt{\pi}}\lim_{s\to 0}\frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)}\int_0^\infty dt\, t^{s-\frac{3}{2}}\sum_{n_0=-\infty}^{\infty} e^{-n_0^2\beta^2/4t}\right. \\ &\qquad \left.\sum_{n=0}^{\infty}(n+1)^2\, e^{-\left(\frac{t}{a^2}\right)n(n+2)}\right]. \end{aligned} \tag{62}$$

To start with, we look at the $n_0 = n = 0$ term. This term is exactly the same as in Equation (9) which is regularized to zero. We shall therefore neglect it in the following. For $n_0 = 0$ and $n \neq 0$ in Equation (62), we evaluate the t-integral to give

$$\begin{aligned} &F|_{n_0=0,n\neq 0} \\ &= \lim_{s\to 0}\frac{d}{ds}\left[\left(-\frac{1}{4\sqrt{\pi}}\right)\left(\frac{\mu^s\Gamma(s-\frac{1}{2})}{\Gamma(s)}\right)\sum_{n=1}^{\infty}(n+1)^2\left(\frac{a^2}{n(n+2)}\right)^{s-\frac{1}{2}}\right] \end{aligned} \tag{63}$$

Here we concentrate on the sum

$$\sum_{n=1}^{\infty}(n+1)^2\left(\frac{1}{n(n+2)}\right)^{s-\frac{1}{2}} = \sum_{n=0}^{\infty}(n+2)^2\left(\frac{1}{(n+3)(n+1)}\right)^{s-\frac{1}{2}} \tag{64}$$

Using the Plana summation formula

$$\sum_{n=0}^{\infty} f(n) = \int_0^\infty dx\, f(x) + \frac{1}{2}f(0) + i\int_0^\infty dy\left(\frac{f(iy)-f(-iy)}{e^{2\pi y}-1}\right), \tag{65}$$

it is possible to evalute this sum as an analytic function of s [88,89].

$$\sum_{n=1}^{\infty}(n+1)^2\left(\frac{1}{n(n+2)}\right)^{s-\frac{1}{2}} = \left(-\frac{1}{16}\right)\frac{1}{s} - 0.411461 + \cdots \tag{66}$$

With these considerations, we have the $n_0 = 0$ part of the free energy,

$$F|_{n_0=0} = -\frac{0.224909}{a} - \frac{1}{32a}\ln\left(\mu a^2\right) \tag{67}$$

which represents the contribution from the Casimir effect. This result is exactly the same as the effective potential obtained in [88]. The dependence on μ in the free energy occurs in all Einstein universes with spatial odd spheres. The corresponding zeta function at $s = 0$ is nonzero and hence contributes to a $\ln\mu$ term.

For $n_0 \neq 0$ and $n = 0$, we evaluate the t-integral to give

$$\begin{aligned} F|_{n_0\neq 0, n=0} &= \left(-\frac{1}{2\sqrt{\pi}}\right) \lim_{s\to 0} \frac{d}{ds}\left[\left(\frac{\mu^s}{2^{2s-1}\Gamma(s)}\right)\Gamma(\frac{1}{2}-s)\zeta(1-2s)\right] \\ &= \frac{1}{2\beta}\ln(\mu\beta^2). \end{aligned} \tag{68}$$

This is the contribution from the zero mode as we have discussed in the last section. This same expression will be present in the low temperature expansions of the free energy in all the cases we consider in this paper.

For $n_0 \neq 0$ and $n \neq 0$, we again evaluate the t-integral first to obtain

$$\begin{aligned} F|_{n_0\neq 0, n\neq 0} &= \sum_{n_0,n=1}^{\infty}\left(-\frac{(n+1)^2}{n_0\beta}\right)e^{\sqrt{n(n+2)}\, n_0\beta/a} \\ &= -\frac{4}{\beta}e^{-\sqrt{3}\beta/a} + \cdots \end{aligned} \tag{69}$$

The $n_0 \neq 0$ and $n \neq 0$ contribution to the free energy in the low energy expansion is exponentially small, and combining the various parts we have

$$F = -\frac{0.224909}{a} - \frac{1}{32a}\ln(\mu a^2) + \frac{1}{2\beta}\ln(\mu\beta^2) + \cdots \tag{70}$$

where the ellipsis represents exponentially small terms.

With the above Helmholtz free energy in the low temperature expansion, we derive the various thermodynamic quantities. For the entropy

$$S = 1 - \frac{1}{2}\ln(\mu\beta^2) + \cdots \tag{71}$$

which is the entropy of the zero mode with the same expression as in Equation (15).

For the internal energy

$$E = -\frac{0.224909}{a} - \frac{1}{32a}\ln(\mu a^2) + \frac{1}{\beta} + \cdots, \tag{72}$$

and the energy density, with the volume of the three sphere $V = 4\pi^2 a^3$,

$$\rho = \frac{E}{V} = -\frac{0.00570}{a^4} - \frac{1}{128\pi^2 a^4}\ln(\mu a^2) + \frac{1}{4\pi^2 a^3\beta} + \cdots. \tag{73}$$

The leading behavior in the low temperature expansion comes from the first two terms. They constitute the Casimir contribution which is μ dependent. Moreover, the pressure

$$P = -\frac{0.00274258}{a^4} - \frac{1}{192\pi^2 a^4}\ln(\mu a^2) + \cdots \tag{74}$$

which is negative due to the Casimir terms and is also dependent on μ.

From the entropy S, we derive the heat capacity at constant volume

$$C_V = 1 + \cdots \tag{75}$$

which is basically a constant with the temperature terms exponentially suppressed. From the pressure P, we obtain the isothermal compressibility

$$\kappa_T = -a^4 \left[0.00330496 + \frac{1}{144\pi^2} \ln(\mu a^2) \right]^{-1} + \cdots \tag{76}$$

which is negative and dependent on μ. Again, the temperature-dependent terms are exponentially suppressed. Since the temperature-dependent terms in P are exponentially small, the thermal expansion coefficient α which depends on $\partial P/\partial T$ is therefore exponentially small too. For the same reason, the heat capacity at constant pressure $C_P \sim C_V$ up to exponentially small terms. The same applies to the two compressibilities $\kappa_S \sim \kappa_T$.

With C_V and κ_T, we can derive the fluctuation of the internal energy

$$\langle (\Delta E)^2 \rangle = \frac{2\pi^2}{\beta a} \left[\frac{(0.00274258 + 0.000527714 \ln(\mu a^2))^2}{0.00330496 + 0.000703619 \ln(\mu a^2)} \right] + \frac{1}{\beta^2} + \cdots \tag{77}$$

which is proportional T. Note that since κ_T is negative and the fluctuation should be a positive quantity, we have taken the absolute value of κ_T in the formula for $\langle (\Delta E)^2 \rangle$. The fluctuation of P is related to the adiabatic compressibility κ_S. Again we take its absolute value in the formula.

$$\langle (\Delta P)^2 \rangle = \frac{1}{\beta a^7} \left[0.000167431 + 0.0000356458 \ln(\mu a^2) \right] + \cdots \tag{78}$$

which is also proportional to T and dependent on the renormalization scale. Moreover, the correlation between the fluctuations of E and P,

$$\langle (\Delta E)(\Delta P) \rangle = -\frac{1}{\beta a^4} \left[0.00274258 + 0.000527714 \ln(\mu a^2) \right] + \cdots \tag{79}$$

It is interesting to see that here all the thermodynamic quantities and their fluctuations depend on the renormalization scale μ. This dependence indicates that the thermodynamics of a quantum field in the Einstein universe involves divergent quantities and a renormalization procedure is needed to define the various physically measureable quantities. As we shall see in the following section, this is not the case with even spatial S^2 and S^4 dimensional spheres. Actually, the dependence on μ comes from the fact that we have an odd-dimensional sphere (S^3) here. We would expect the same kind of dependence to occur for all odd spheres or (when adding in the time dimension) even-dimensional spacetimes [90].

3.2. High Temperature Expansion

In this subsection, we concentrate on the high temperature expansion for the Einstein universe with $\beta/a \ll 1$. In this case, we consider the Helmholtz free energy in Equation (61). Here, we would neglect the $n_0 = n = 0$ term. This term is independent of the temperature T as well as the size a of the spatial three sphere. Although it is formally divergent, it should be subtracted in the renormalization procedure.

For the $n_0 = 0$, $n \neq 0$ contribution to the free energy,

$$\begin{aligned} F_{n_0=0, n\neq 0} &= -\frac{1}{2\beta} \lim_{s\to 0} \frac{d}{ds} \left[\frac{\mu^s}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \sum_{n=1}^{\infty} (n+1)^2 e^{-\left(\frac{t}{a^2}\right) n(n+2)} \right] \\ &= -\frac{1}{2\beta} \lim_{s\to 0} \frac{d}{ds} \left\{ (\mu a^2)^s \sum_{n=1}^{\infty} (n+1)^2 [n(n+2)]^{-s} \right\}. \end{aligned} \tag{80}$$

We apply the Plana summation formula, as in Equation (65), to the sum

$$\begin{aligned}\sum_{n=1}^{\infty}(n+1)^2[n(n+2)]^{-s} &= \sum_{n=0}^{\infty}(n+2)^2[(n+1)(n+3)]^{-s}\\ &= -1-1.20563\,s+\cdots \end{aligned} \tag{81}$$

This shows that the sum as an analytic function of s is well behaved near $s=0$. The $n_0=0$, $n\neq 0$ contribution to the free energy can therefore be written as

$$F_{n_0=0,n\neq 0} = \frac{0.60282}{\beta}+\frac{1}{2\beta}\ln(\mu a^2). \tag{82}$$

Next, we consider the $n_0\neq 0$ part of the free energy

$$\begin{aligned}F|_{n_0\neq 0} = -\frac{1}{\beta}\lim_{s\to 0}\frac{d}{ds}\Bigg[\frac{\mu^s}{\Gamma(s)}\int_0^{\infty}dt\,t^{s-1}\sum_{n_0=1}^{\infty}e^{-t\left(\frac{2\pi n_0}{\beta}\right)^2}\\ \sum_{n=0}^{\infty}(n+1)^2\,e^{-\left(\frac{t}{a^2}\right)n(n+2)}\Bigg]\end{aligned} \tag{83}$$

We concentrate on the sum over n. In the high temperature limit, we are interested in the asymptotic behavior of this sum [91] when a is large or as a power series in t/a. To develop such a series, we again use the Plana summation formula in Equation (65).

$$\begin{aligned}&\sum_{n=0}^{\infty}(n+1)^2\,e^{-\left(\frac{t}{a^2}\right)n(n+2)}\\ &= \left(\frac{a}{\sqrt{t}}\right)^3\left[\frac{\sqrt{\pi}}{4}+\frac{\sqrt{\pi}}{4}\left(\frac{t}{a}\right)+\frac{\sqrt{\pi}}{8}\left(\frac{t}{a}\right)^2+\cdots\right].\end{aligned} \tag{84}$$

With this asymptotic expansion, one can evaluate Equation (83) as a power series in β/a.

$$F|_{n_0\neq 0} = -\frac{\pi^4a^3}{45\beta^4}-\frac{\pi^2a}{12\beta^2}-\frac{1}{32a}\left[2\gamma+\ln\left(\frac{\mu\beta^2}{16\pi^2}\right)\right]+\cdots \tag{85}$$

where γ is the Euler constant.

With Equations (82) and (85), the free energy in the high temperature expansion is then given by

$$\begin{aligned}F = &-\frac{\pi^4a^3}{45\beta^4}-\frac{\pi^2a}{12\beta^2}+\frac{0.60282}{\beta}\\ &+\frac{1}{2\beta}\ln(\mu a^2)-\frac{1}{32a}\left[2\gamma+\ln\left(\frac{\mu\beta^2}{16\pi^2}\right)\right]+\cdots\end{aligned} \tag{86}$$

From this free energy, we again derive the various thermodynamic quantities.

For the entropy,

$$S=\frac{4\pi^4a^3}{45\beta^3}+\frac{\pi^2a}{6\beta}-\frac{1}{2}\ln(\mu a^2)-0.60282-\frac{\beta}{16a}+\cdots \tag{87}$$

which is proportional to T^3 in the leading behavior. Note that it is dependent on the renormalization scale μ but only in the subleading term. For the internal energy,

$$E=\frac{\pi^4a^3}{15\beta^4}+\frac{\pi^2a}{12\beta^2}-\frac{1}{32a}\left[2+2\gamma+\ln\left(\frac{\mu\beta^2}{16\pi^2}\right)\right]+\cdots. \tag{88}$$

With the volume of the three sphere as $V = 2\pi^2 a^3$, the first term of the internal energy E is $\pi^2 V/30\beta^4$ which is again the Stefan's law in flat spatial three dimensions. Indeed, in the infinite space limit, $a \to \infty$, the energy density

$$\rho|_{a\to\infty} = \frac{E|_{a\to\infty}}{V} = \frac{\pi^2}{30\beta^4}. \tag{89}$$

As for the pressure,

$$P = \frac{\pi^2}{90\beta^4} + \frac{1}{72\beta^2 a^2} - \frac{1}{6\pi^2\beta a^3} - \frac{1}{192\pi^2 a^4}\left[2\gamma + \ln\left(\frac{\mu\beta^2}{16\pi^2}\right)\right] + \cdots. \tag{90}$$

The first term gives the pressure in the infinite space limit,

$$P|_{a\to\infty} = \frac{\pi^2}{90\beta^4} = \frac{\rho|_{a\to\infty}}{3} \tag{91}$$

which is the equation of state for a massless relativistic ideal gas in flat three spatial dimensions.

For the heat capacity at constant volume

$$C_V = \frac{4\pi^4 a^3}{15\beta^3} + \frac{\pi^2 a}{6\beta} + \frac{\beta}{16a} + \cdots \tag{92}$$

which is proportional to T^3 in the leading term and is independent of μ. Furthermore, the isothermal compressibility κ_T is

$$\kappa_T = 108\beta^2 a^2 + \left(\frac{1944}{\pi^2}\right)\beta^3 a + \frac{81\beta^4}{\pi^2}\left[\frac{432}{\pi^2} + 2\gamma + \ln\left(\frac{\mu\beta^2}{16\pi^2}\right)\right] + \cdots. \tag{93}$$

From the isothermal compressibility, one can obtain the thermal expansion coefficient α, the heat capacity at constant pressure C_P, and the adiabatic compressibility κ_S. They are, respectively,

$$\alpha = \frac{24\pi^2 a^2}{5\beta} + \frac{432\,a}{5} + \frac{18\beta}{5}\left[\frac{5}{6} + \frac{432}{\pi^2} + 2\gamma + \ln\left(\frac{\mu\beta^2}{16\pi^2}\right)\right] + \cdots. \tag{94}$$

$$C_P = \frac{32\pi^6 a^5}{75\beta^5} + \frac{192\pi^4 a^4}{25\beta^4} + \frac{8\pi^4 a^3}{25\beta^3}\left[\frac{5}{2} + \frac{432}{\pi^2} + 2\gamma + \ln\left(\frac{\mu\beta^2}{16\pi^2}\right)\right] + \cdots. \tag{95}$$

$$\kappa_S = \frac{135\beta^4}{2\pi^2} - \frac{675\beta^6}{8\pi^4 a^2} + \frac{10125\beta^7}{8\pi^6 a^3} + \cdots. \tag{96}$$

The relationship between the heat capacities is $C_P = C_V + \beta^3 V\kappa_T(\partial P/\partial\beta)^2$. In the present high temperature expansion, the second term which is of the order of $(a/\beta)^5$ dominates over the first term which is only of the order $(a/\beta)^3$. Hence, we have $C_P \gg C_V$. For the same reason, since $\kappa_S = (C_V/C_P)\kappa_T$, we have $\kappa_T \gg \kappa_S$.

Next, we give the fluctuations for various thermodynamic quantities in the high temperature expansion. For the internal energy fluctuation,

$$\langle(\Delta E)^2\rangle = \frac{6\pi^6 a^5}{25\beta^7} + \frac{108\pi^4 a^4}{25\beta^6} + \frac{9\pi^4}{50\beta^5}\left[\frac{70}{27} + \frac{432}{\pi^2} + 2\gamma + \ln\left(\frac{\mu\beta^2}{16\pi^2}\right)\right] + \cdots. \tag{97}$$

For the pressure fluctuation,

$$\langle(\Delta P)^2\rangle = \frac{1}{135\beta^5 a^3} + \frac{1}{108\pi^2\beta^3 a^5} - \frac{5}{36\pi^4\beta^2 a^6} + \cdots. \tag{98}$$

Lastly, the correlated fluctuation of E and P is

$$\langle(\Delta E)(\Delta P)\rangle = \frac{\pi^2}{90\beta^5} + \frac{1}{72\beta^3 a^2} - \frac{1}{6\pi^2\beta^2 a^3} + \cdots . \tag{99}$$

We see that in this high temperature expansion, $\langle(\Delta E)^2\rangle$ is proportional to T^7, while both $\langle(\Delta P)^2\rangle$ and $\langle(\Delta E)(\Delta P)\rangle$ are proportional to T^5.

4. Even Spatial Dimensions: $S^1 \times S^2$ and $S^1 \times S^4$

In the last section we have considered the Einstein universe with a spatial three sphere. The consideration there can be extended to Einstein universes with the general topology of $S^1 \times S^{d-1}$ with a spatial $(d-1)$-sphere. The metric is given by

$$ds^2 = d\tau^2 + a^2 d\bar{\Omega}_{d-1}^2 \tag{100}$$

where $\bar{\Omega}_{d-1}$ is the solid angle of the $(d-1)$-sphere. Here the eigenvalue of the Laplacian $\bar{\Box}$ on S^{d-1} is given by [83]

$$\bar{\lambda}_n = -n(n+d-2), \tag{101}$$

and the degeneracy is

$$\bar{D}_n = \frac{(2n+d-2)(n+d-3)!}{n!(d-2)!} \tag{102}$$

Using $\bar{\lambda}_n$ and $\bar{D}_n$, one can write the Helmholtz free energy in this generalized Einstein universe as

$$F = -\frac{1}{2\beta}\lim_{s\to 0}\frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\sum_{n_0=-\infty}^{\infty} e^{-t\left(\frac{2\pi n_0}{\beta}\right)^2}\sum_{n=0}^{\infty}\bar{D}_n e^{-\left(\frac{t}{a^2}\right)\bar{\lambda}_n}\right]. \tag{103}$$

With analyses similar to those in the last section, one can derive the corresponding free energy F in both the low temperature and the high temperature expansions for any value of the dimension d. Then, one can derive the various thermodynamic quantities from the free energy. In the following subsections, we shall work out the $d = 3$ and $d = 5$ cases explicitly. Together with the case $d = 4$ in the last section, we can have a better understanding in how the various thermodynamic quantities depend on the spacetime dimension [86,90].

4.1. Low Temperature Expansion

In the low temperature expansion with $a/\beta \ll 1$, we first rewrite the sum over n_0 in Equation (103) using the Poisson summation formula, as in Equation (62) for the 4d Einstein universe case. Then, following similar considerations there, we obtain the free energies for the $d = 3$ and $d = 5$ Einstein universe cases.

$$\begin{aligned} F|_{d=3} &= -\frac{0.132548}{a} + \frac{1}{2\beta}\ln(\mu\beta^2) + \cdots , \\ F|_{d=5} &= -\frac{0.215872}{a} + \frac{1}{2\beta}\ln(\mu\beta^2) + \cdots , \end{aligned} \tag{104}$$

where the ellipsis represents terms which are exponentially small. Comparing with the free energy of $d = 4$ Einstein universe in Equation (70), we see that in the free energies above, the term $\ln(\mu a^2)/a$ is missing. This can be traced back to the sum in Equation (66), where the sum, as an analytic function of s, has a pole at $s = 0$. This is characteristic for this kind of sums on unit odd dimensional spheres. On the other hand, for even spheres like S^2 and S^4, the corresponding sums would be power series in s without pole singularities at $s = 0$. For this reason, the free energies above for the $d = 3$ and $d = 5$ Einstein universes do not

contain terms like $\ln(\mu a^2)/a$. Subsequently, as we shall see below, the compressibilities κ_T and κ_S, and the thermal expansion coefficient α, together with the various fluctuations, will not depend on the renormalization scale μ.

Using these free energies, we derive the entropies

$$\begin{aligned} S|_{d=3} &= 1 - \frac{1}{2}\ln(\mu\beta^2) + \cdots, \\ S|_{d=5} &= 1 - \frac{1}{2}\ln(\mu\beta^2) + \cdots. \end{aligned} \tag{105}$$

Both have a dependence on the renormalization scale μ. From the entropies, we have the heat capacities at constant volume,

$$\begin{aligned} C_V|_{d=3} &= 1 + \cdots, \\ C_V|_{d=5} &= 1 + \cdots. \end{aligned} \tag{106}$$

which are constant except for exponentially small terms. The corresponding quantities of the Einstein universe in the last section have the same form as the ones above. Therefore, we would expect that this is true for all the Einstein universes with general $(d-1)$ dimensional spatial spheres.

The internal energies

$$\begin{aligned} E|_{d=3} &= -\frac{0.132548}{a} + \frac{1}{\beta} + \cdots, \\ E|_{d=5} &= -\frac{0.215872}{a} + \frac{1}{\beta} + \cdots, \end{aligned} \tag{107}$$

and the corresponding energy densities

$$\rho|_{d=3} = -\frac{0.0105478}{a^3} + \frac{1}{4\pi\beta a^2} + \cdots, \tag{108}$$

$$\rho|_{d=5} = -\frac{0.00820215}{a^5} + \frac{3}{8\pi^2\beta a^5} + \cdots. \tag{109}$$

The first term in both expressions represents the Casimir energy. Other than the exponentially small terms, there is also one term proportional to T. We expect Einstein universes with odd spacetimes dimensions to have the same behaviors for E and ρ as above. Moreover, for Einstein universes with even spacetime dimensions, there should be an extra term proportional to $\ln(\mu a^2)/a$ as shown in Equations (72) and (73) in the last section.

For the pressures in these cases,

$$\begin{aligned} P|_{d=3} &= -\frac{0.00527392}{a^3} + \cdots, \\ P|_{d=5} &= -\frac{0.00205054}{a^5} + \cdots. \end{aligned} \tag{110}$$

The Casimir pressures in both cases are negative. This is true for all the cases we have considered in the previous sections in the low temperature expansion. From the expressions for the pressures, one can derive the isothermal compressibilities

$$\begin{aligned} \kappa_T|_{d=3} &= -126.408\, a^3 + \cdots, \\ \kappa_T|_{d=5} &= -390.141\, a^5 + \cdots. \end{aligned} \tag{111}$$

Both isothermal compressibilities are negative due to the negative pressure. This is also true for all the isothermal compressibilities in all the cases we have considered. However, unlike κ_T in the 4d Einstein universe but similar to the Einstein cylinder case, the isothermal compressibilities here are not dependent on μ. Since the pressures P are dominated by the

Casimir effect which is independent of temperature, the thermal expansion coefficients are exponentially small in these case. Furthermore, for the same reason, we have $C_P \sim C_V$ and $\kappa_S \sim \kappa_T$ up to exponentially small terms.

Finally, we layout the expressions for the various fluctuations. For the internal energy,

$$\begin{aligned} \langle (\Delta E)^2 \rangle|_{d=3} &= 0.0441827\left(\frac{1}{\beta a}\right) + \frac{1}{\beta^2} + \cdots, \\ \langle (\Delta E)^2 \rangle|_{d=5} &= 0.0431744\left(\frac{1}{\beta a}\right) + \frac{1}{\beta^2} + \cdots. \end{aligned} \tag{112}$$

For the pressure,

$$\begin{aligned} \langle (\Delta P)^2 \rangle|_{d=3} &= 0.000629528\left(\frac{1}{\beta a^5}\right) + \cdots, \\ \langle (\Delta P)^2 \rangle|_{d=5} &= 0.0000973889\left(\frac{1}{\beta a^9}\right) + \cdots, \end{aligned} \tag{113}$$

The fluctuations are proportional to T. This is true for all the cases we have considered. In addition, the correlated fluctuations of E and P,

$$\begin{aligned} \langle (\Delta E)(\Delta P) \rangle|_{d=3} &= -0.00527392\left(\frac{1}{\beta a^3}\right) + \cdots, \\ \langle (\Delta E)(\Delta P) \rangle|_{d=5} &= -0.00205054\left(\frac{1}{\beta a^5}\right) + \cdots, \end{aligned} \tag{114}$$

which are also proportional to T.

Since the energies and the pressures in the low temperature limit are all dominated by the Casimir effect, their leading behaviors are independent of T. Therefore, the ratios $\langle (\Delta E)^2 \rangle / E^2$, $\langle (\Delta P)^2 \rangle / P^2$, and $\langle (\Delta E)(\Delta P) \rangle / EP$ go like a/β or aT in the low temperature limit. Fluctuations of all Einstein universes in the low temperature expansion have similar behavior, except that for even spacetimes the fluctuations will have dependences on the renormalization scale μ, while here for odd spacetimes they will not.

4.2. High Temperature Expansion

In this subsection we develop the high temperature expansion with $\beta/a \ll 1$. For the Helmholtz free energies in Equation (103), we again use the same procedure as in the 4d Einstein universe case in Section 3 to obtain

$$\begin{aligned} F|_{d=3} &= -\frac{2\zeta(3)a^2}{\beta^3} + \left(\frac{1}{\beta}\right)\left[0.580842 + \frac{1}{3}\ln(\mu a^2) + \frac{1}{6}\ln(\mu\beta^2)\right] \\ &\quad - \frac{\beta}{360a^2} - \frac{\beta^3}{113400a^4} + \cdots, \\ F|_{d=5} &= -\frac{2\zeta(5)a^4}{\beta^5} - \frac{2\zeta(3)a^2}{3\beta^3} \\ &\quad + \left(\frac{1}{\beta}\right)\left[0.276064 + 0.338889\ln(\mu a^2) + \frac{29}{180}\ln(\mu\beta^2)\right] \\ &\quad - \frac{37\beta}{4536a^2} + \cdots. \end{aligned} \tag{115}$$

The corresponding entropies are

$$
\begin{aligned}
S|_{d=3} &= \frac{6\zeta(3)a^2}{\beta^2} - \left[0.247509 + \frac{1}{3}\ln(\mu a^2) + \frac{1}{6}\ln(\mu\beta^2)\right] \\
&\quad - \frac{\beta^2}{360a^2} - \frac{\beta^4}{37800a^4} + \cdots, \\
S|_{d=5} &= \frac{10\zeta(5)a^4}{\beta^4} + \frac{2\zeta(3)a^2}{\beta^2} \\
&\quad + \left[0.0461582 - 0.338889\ln(\mu a^2) - \frac{29}{180}\ln(\mu\beta^2)\right] - \frac{37\beta^2}{4536a^2} + \cdots.
\end{aligned} \tag{116}
$$

The internal energies are

$$
\begin{aligned}
E|_{d=3} &= \frac{4\zeta(3)a^2}{\beta^3} + \frac{0.333333}{\beta} - \frac{\beta}{180a^2} - \frac{\beta^3}{28350a^4} + \cdots, \\
E|_{d=5} &= \frac{8\zeta(5)a^4}{\beta^5} + \frac{4\zeta(3)a^2}{3\beta^3} + \frac{0.322222}{\beta} - \frac{37\beta}{2268a^2} + \cdots.
\end{aligned} \tag{117}
$$

The first terms again give the flat space limit of the internal energy. Hence, as $a \to \infty$, we have the energy density

$$
\begin{aligned}
\rho|_{d=3,a\to\infty} &= \frac{E|_{d=3,a\to\infty}}{4\pi a^2} = \frac{\zeta(3)}{\pi\beta^3}, \\
\rho|_{d=5,a\to\infty} &= \frac{E|_{d=5,a\to\infty}}{\left(\frac{8}{3}\pi^2 a^4\right)} = \frac{3\zeta(5)}{\pi^2\beta^5}.
\end{aligned} \tag{118}
$$

For the pressures

$$
\begin{aligned}
P|_{d=3} &= \frac{\zeta(3)}{2\pi\beta^3} - \frac{1}{12\pi\beta a^2} - \frac{\beta}{1440\pi a^4} - \frac{\beta^3}{226800\pi a^6} + \cdots, \\
P|_{d=5} &= \frac{3\zeta(5)}{4\pi^2\beta^5} + \frac{\zeta(3)}{8\pi^2\beta^3 a^2} - \frac{0.00643812}{\beta a^4} - \frac{37\beta}{24192\pi^2 a^6} + \cdots.
\end{aligned} \tag{119}
$$

The first terms give the flat space limit. As $a \to \infty$, we see that P will be proportional to the energy density ρ in Equation (118).

$$
\begin{aligned}
P|_{d=3,a\to\infty} &= \frac{\zeta(3)}{2\pi\beta^3} = \frac{1}{2}\rho|_{d=3,a\to\infty}, \\
P|_{d=5,a\to\infty} &= \frac{3\zeta(5)}{4\pi^2\beta^5} = \frac{1}{4}\rho|_{d=5,a\to\infty}.
\end{aligned} \tag{120}
$$

This is consistent with the equation of state of a massless relativistic ideal gas, $P = \rho/n$, in a n-dimensional flat space. From the entropies above, we can also work out the heat capacities at constant volume.

$$
\begin{aligned}
C_V|_{d=3} &= \frac{12\zeta(3)a^2}{\beta^2} + \frac{1}{3} + \frac{\beta^2}{180a^2} + \frac{\beta^4}{9450a^4} + \cdots, \\
C_V|_{d=5} &= \frac{40\zeta(5)a^4}{\beta^4} + \frac{4\zeta(3)a^2}{\beta^2} + \frac{29}{90} + \frac{37\beta^2}{2268a^2} + \cdots.
\end{aligned} \tag{121}
$$

From the pressures, we can derive the isothermal compressibilities,

$$\begin{aligned}\kappa_T|_{d=3} &= -12\pi\beta a^2 + \frac{\pi\beta^3}{5} - \frac{\pi\beta^5}{700a^2} + \cdots, \\ \kappa_T|_{d=5} &= \frac{16\pi^2\beta^3 a^2}{\zeta(3)} + 111.109\beta^5 + \frac{97.9843\beta^7}{a^2} + \cdots. \end{aligned} \tag{122}$$

This result is a little bit surprising as we see that $\kappa_T|_{d=3}$ is negative in the high temperature limit, while the isothermal compressibilities for $d = 4$ and 5 are both positive in the same limit. We have discussed this point briefly in the last section on the isothermal compressiblity of the $d = 4$ Einstein universe in the high temperature expansion. Here, if we look at the expressions for the pressure P above in Equation (119), we see that the leading terms of P are independent of a. Since κ_T is proportional to the inverse of $\partial P/\partial a$, the sign of κ_T would depend on the sign of the second terms in Equation (119). Interestingly, for $d = 3$, the second term of P is negative, while for $d = 5$, this term is positive. In fact, although we have not detailed here, one can show that this term is negative only for $d = 3$, and it is positive for all other dimensions $d \geq 4$. Hence, we have the peculiar result that κ_T is negative only for the $d = 3$ Einstein universe, and positive for all other Einstein universes.

Since the thermal expansion coefficient α is proportional to κ_T, we have a negative α for $d = 3$ and positive α for dimensions $d \geq 4$. Indeed, we have

$$\begin{aligned}\alpha|_{d=3} &= -\frac{18\zeta(3)a^2}{\beta} + 1.36062\beta - \frac{0.0275758\beta^3}{a^2} + \cdots, \\ \alpha|_{d=5} &= \frac{60\zeta(5)a^2}{\zeta(3)\beta} + 49.7752\beta + \frac{42.8332\beta^3}{a^2} + \cdots. \end{aligned} \tag{123}$$

This is also true for the heat capacities at constant pressure,

$$\begin{aligned}C_P|_{d=3} &= -\frac{108\zeta(3)^2a^4}{\beta^4} + \frac{31.4503a^2}{\beta^2} - 0.379195 + \cdots, \\ C_P|_{d=5} &= \frac{600\zeta(5)^2a^6}{\zeta(3)\beta^6} + \frac{619.826a^4}{\beta^4} + \frac{500.021a^2}{\beta^2} + \cdots. \end{aligned} \tag{124}$$

Note that the relationship between C_P and C_V is $C_P = C_V + \beta^2 V\kappa_T(\partial P/\partial\beta)^2$. Here, in the high temperature expansion, the second term, which is proportional to κ, dominates over the first one. Hence, the sign of C_P is determined by κ_T, and we have C_P negative for $d = 3$ and positive for all other dimensions $d \geq 4$.

It is interesting to see that, for the adiabatic compressibilities,

$$\begin{aligned}\kappa_S|_{d=3} &= \frac{4\pi\beta^3}{3\zeta(3)} + \frac{0.724734\beta^5}{a^2} + \frac{0.138007\beta^7}{a^4} + \cdots, \\ \kappa_S|_{d=5} &= \frac{16\pi^2\beta^5}{15\zeta(5)} - \frac{1.96158\beta^7}{a^2} + \frac{1.45329\beta^9}{a^4} + \cdots. \end{aligned} \tag{125}$$

they are both positive. This is due to the relation $\kappa_S = (C_V/C_P)\kappa_T$. For $d = 3$, both C_P and κ_T are negative but the ratio is positive. Therefore, κ_S is positive in both cases.

Lastly, we also list the fluctuations for E and P for completeness. For the energy fluctuations,

$$\begin{aligned}\langle(\Delta E)^2\rangle|_{d=3} &= \frac{69.3572a^4}{\beta^6} + \frac{13.2687a^2}{\beta^4} + \frac{0.501864}{\beta^2} + \cdots, \\ \langle(\Delta E)^2\rangle|_{d=5} &= \frac{343.481a^6}{\beta^8} + \frac{398.348a^4}{\beta^6} + \frac{320.334a^2}{\beta^4} + \cdots. \end{aligned} \tag{126}$$

For the pressure fluctuations,

$$\begin{aligned}\langle(\Delta P)^2\rangle|_{d=3} &= \frac{0.0228363}{\beta^4 a^2} - \frac{0.00474943}{\beta^2 a^4} + \frac{0.0000833667}{a^6} + \cdots, \\ \langle(\Delta P)^2\rangle|_{d=5} &= \frac{0.00374241}{\beta^6 a^4} + \frac{0.000723064}{\beta^4 a^6} - \frac{0.000396002}{\beta^2 a^8} + \cdots. \end{aligned} \tag{127}$$

For the correlated fluctuations of E and P,

$$\begin{aligned}\langle(\Delta E)(\Delta P)\rangle|_{d=3} &= \frac{0.191313}{\beta^4} - \frac{0.0265258}{\beta^2 a^2} - \frac{0.000221049}{a^4} + \cdots, \\ \langle(\Delta E)(\Delta P)\rangle|_{d=5} &= \frac{0.0787971}{\beta^6} + \frac{0.0152242}{\beta^4 a^2} - \frac{0.00643812}{\beta^2 a^4} + \cdots. \end{aligned} \tag{128}$$

From these results and also the ones in the last section for the $d = 4$ Einstein universe, we have in the high temperature expansion, $\langle(\Delta E)^2\rangle$ proportional to T^{d+3}, and both $\langle(\Delta P)^2\rangle$ and $\langle(\Delta E)(\Delta P)\rangle$ proportional to T^{d+1}. We also have the rations $\langle(\Delta E)^2\rangle/E^2$ proportional to $(\beta/a)^{d-3}$, and both $\langle(\Delta P)^2\rangle/P^2$ and $\langle(\Delta E)(\Delta P)\rangle/EP$ proportional to $(\beta/a)^{d-1}$.

5. Conclusions and Discussions

In this paper, towards the loftier goal of exploring the relationship between quantum fields, spacetimes and thermodynamics, we take on a more manageable task, of finding a thermodynamic description of the fluctuations of the stress energy of quantum fields in several generic static spacetimes. We have derived expressions for the heat capacity at constant volume and pressure, as well as the isothermal and adiabatic compressibility of a thermal scalar quantum field in a 2d (1 time and 1 space) Casimir space, a 2d Einstein cylinder, and 3d, 4d, 5d (spatial S^2, S^3, S^4) Einstein universes. We now analyze these results considering these factors: comparison between cases with positive and negative heat capacity and compressibility, effects of curvature, topology and dimensionality. Our next paper [20] will deal with the same issues in dynamical spacetimes relevant to cosmology. No longer is the imaginary-time finite temperature field theory applicable in these cases. Concepts such as squeezing and techniques in nonequilibrium quantum field theories are required for their treatment. We also need to find a good definition of compressibility for noncompact spacetimes such as in the flat and open universes.

Our results are summarized in Tables 1 and 2. In Table 1, the leading behaviors of various thermodynamic quantities of a minimally coupled massless scalar field in the low temperature expansion are tabulated. We can see that the energy density ρ, the pressure P, the compressibilities κ_T and κ_S are all negative. This is because the leading behaviors are mostly dominated by the Casimir effect. We therefore have the universal feature that the compressibilities in the low temperature limit, in the Einstein cylinder and all the Einstein universes studied here, are negative. In fact, the magnitude of the negative pressure is inversely proportional to the size of the spatial geometry, meaning that it would compress the spatial geometry, and while the circle or the sphere gets smaller, the shrinking pressure would get larger. The process is similar to an accelerated gravitational collapse. This is actually the mechanism behind the so-called spontaneous compacification of the extra dimensions in the Kaluza–Klein scenario [92,93].

Table 1. Low temperature leading behaviors of various thermodynamic quantities.

	$d=2$	3	4	5
ρ	$-\pi/6L$	$-0.0105478/a^3$	Equation (73)	$-0.00820215/a^5$
S	$1-\frac{1}{2}\ln(\mu\beta^2)$	$1-\frac{1}{2}\ln(\mu\beta^2)$	$1-\frac{1}{2}\ln(\mu\beta^2)$	$1-\frac{1}{2}\ln(\mu\beta^2)$
C_V	1	1	1	1
C_P	1	1	1	1
P	$-\pi/6L^2$	$-0.0053/a^3$	Equation (74)	$-0.0021/a^5$
κ_T	$-3L^2/\pi$	$-126a^3$	Equation (76)	$-390a^5$
κ_S	$-3L^2/\pi$	$-126a^3$	Equation (76)	$-390a^5$
α	$\sim e^{-2\pi\beta/L}$	$\sim e^{-\sqrt{2}\beta/a}$	$\sim e^{-\sqrt{3}\beta/a}$	$\sim e^{-2\beta/a}$

Since the Casimir effect is solely controlled by the spatial geometries, the leading behavior of the free energy is independent of temperature T or β. The temperature-dependent terms are subdominate. They are related to the entropy S and the heat capacities C_V and C_P as derivatives of the free energy with respect to the temperature. In all cases in the low temperature expansion, the subleading contribution to the free energy comes from the zero mode with $F_{ZM}=\frac{1}{2\beta}\ln(\mu\beta^2)$. From this we obtain the leading behaviors of S, C_V, and C_P as follows:

$$S_{ZM}=1-\frac{1}{2}\ln(\mu\beta^2)+\cdots$$
$$(C_V)_{ZM}=1+\cdots \quad ; \quad (C_P)_{ZM}=1+\cdots \tag{129}$$

As we have mentioned before, S_{ZM} apparently violates the third law of thermodynamics. If the zero mode is ignored in the calculation of the free energy, the temperature-dependent part of the free energy would come from exponentially small terms. Then, S, C_V, and C_P would all be exponentially small.

Another interesting feature in Table 1 is that for the $d=4$ Einstein universe, the various thermodynamic quantities, except α, are explicitly dependent on the renormalization scale μ. In fact, as we have discussed earlier, this feature is also true for all Einstein universes with odd dimensional spatial spheres. This can be traced back to the sum like Equation (66),

$$f(s)=\sum_{n=1}^{\infty}\bar{D}_n\left(\frac{1}{\bar{\lambda}_n}\right)^{s-\frac{1}{2}}, \tag{130}$$

where $\bar{\lambda}_n$ and $\bar{D}_n$ are, respectively, the eigenvalue and the degeneracy of the Laplacian on a sphere. For odd spheres, as in Equation (66), the function $f(s)$ has a pole singularity at $s=0$, and this induces the renormalization scale dependence of the thermodynamic quantities. For even spheres, the function $f(s)$ is analytic at $s=0$ without the pole term, and hence there is no μ dependence for the thermodynamic quantities [89,90].

Table 2. High temperature leading behaviors of various thermodynamic quantities.

	$d=2$	3	4	5
ρ	$\pi/6\beta^2$	$\zeta(3)/\pi\beta^3$	$\pi^2/30\beta^4$	$3\zeta(5)/\pi^2\beta^5$
S	$\pi L/3\beta$	$6\zeta(3)a^2/\beta^2$	$4\pi^4a^3/45\beta^3$	$10\zeta(5)a^4/\beta^4$
C_V	$\pi L/3\beta$	$12\zeta(3)a^2/\beta^2$	$4\pi^4a^3/15\beta^3$	$40\zeta(5)a^4/\beta^4$
C_P	$-\pi^2L^2/9\beta^2$	$-108(\zeta(3))^2a^4/\beta^3$	$32\pi^6a^5/75\beta^5$	$600(\zeta(5))^2a^6/\zeta(3)\beta^5$
P	$\pi/6\beta^2$	$\zeta(3)/2\pi\beta^3$	$\pi^2/90\beta^4$	$3\zeta(5)/4\pi^2\beta^5$
κ_T	$-\beta L$	$-12\pi\beta a^2$	$108\beta^2a^2$	$16\pi^2\beta^3a^2/\zeta(3)$
κ_S	$3\beta^2/\pi$	$4\pi\beta^3/3\zeta(3)$	$135\beta^4/2\pi^2$	$16\pi^2\beta^5/15\zeta(5)$
α	$-\pi L/3$	$-18\zeta(3)a^2/\beta$	$24\pi^2a^2/5\beta$	$60\zeta(5)a^2/\zeta(3)\beta$

In Table 2, the leading behaviors of the thermodynamic quantities in the high temperature expansion are tabulated. One can see that the leading behaviors coincide with those of a massless relativistic ideal thermal gas in d-dimensional flat spacetimes, with the equation of state $P = \rho/(d-1)$. More specifically, the leading behaviors of P and ρ are both of the order of T^d, independent of the size of the spatial sphere.

For $d = 2$ and 3, we have the peculiar results that κ_T, C_P, and α are negative although the pressure P itself is positive. This is related to the subleading terms of P as shown in Equations (46) and (119). We can see that the leading terms are positive but the subleading terms are negative. Therefore, with κ_T being related to the inverse of $\partial P/\partial a$, it is also negative. Due to this negative sign of κ_T, the thermal expansion coefficient α and the heat capacity at constant pressure C_P would both be negative. Note that since $\kappa_S = (C_V/C_P)\kappa_T$ and the signs of κ_T and C_P cancel to give a positive adiabatic compressibility κ_S. On the other hand, for $d \geq 4$, the subleading term of P, as exemplified in Equation(90) for $d = 4$ and Equation (119) for $d = 5$, is again positive so we have all the thermodynamic quantities positive for all dimensions $d \geq 4$. One may wonder if this peculiar behavior of a thermal quantum field in $d = 2$ Einstein cylinder and $d = 3$ Einstein universe would be related to the peculiarity of gravity in $1+1$ and $2+1$ spacetime dimensions [94,95].

In our next paper [20] we shall consider dynamical quantum fields and use the nonequilibrium free energy density discovered in [5] to explore the quantum capacity and vacuum compressibility of the Universe.

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Note

[1] Relevant to the underlying theme of our inquiry is how geometry and topology play a role in determining the thermodynamic properties of spacetimes, and to what extent we can derive these properties from the thermodynamic properties of quantum fields. A beautiful example of the former approach is Wald's proof of the Bekenstein-Hawking (B-H) entropy as the Noether charge of diffeomorphism symmetry [11]. An example of the latter approach (taken by many particle/field theorists) is Frolov and Fursaev's [12] derivation of the B-H entropy the spacetime by working with the thermal field outside of the black hole horizon. Here, we are not saying anything, or implying to say that our work can reveal anything, about the microscopic structrues of spacetime. That belongs to the realm of quantum gravity in its multifarious forms, formulations, approaches and emphasis. To name just two familiar historical proposals. In string theory Strominger and Vafa [13] proposed the B-H entropy can be derived by counting the degeneracy of BPS solition bound states. In loop quantum gravity, Ashtekar et al. [14] showed that the constant of proportionality in the B-H entropy formula is determined by the Immirzi parameter, which fixes the spectrum of the area operator.

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