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Review

Shape Invariant Potentials in Supersymmetric Quantum Cosmology

Shahram Jalalzadeh [1], Seyed Meraj M. Rasouli [2,3] and Paulo Moniz [2,*]

[1] Departamento de Física, Universidade Federal de Pernambuco, Recife 52171-900, PE, Brazil; shahram.jalalzadeh@ufpe.br
[2] Departamento de Física, Centro de Matemática e Aplicações (CMA-UBI), Universidade da Beira Interior, Rua Marquês d'Avila e Bolama, 6200-001 Covilhã, Portugal; mrasouli@ubi.pt
[3] Department of Physics, Qazvin Branch, Islamic Azad University, Qazvin 341851416, Iran
* Correspondence: pmoniz@ubi.pt

Abstract: In this brief review, we comment on the concept of shape invariant potentials, which is an essential feature in many settings of $N = 2$ supersymmetric quantum mechanics. To motivate its application within supersymmetric quantum cosmology, we present a case study to illustrate the value of this promising tool. Concretely, we take a spatially flat FRW model in the presence of a single scalar field, minimally coupled to gravity. Then, we extract the associated Schrödinger–Wheeler–DeWitt equation, allowing for a particular scope of factor ordering. Subsequently, we compute the corresponding supersymmetric partner Hamiltonians, H_1 and H_2. Moreover, we point out how the shape invariance property can be employed to bring a relation among several factor orderings choices for our Schrödinger–Wheeler–DeWitt equation. The ground state is retrieved, and the excited states easily written. Finally, the Hamiltonians, H_1 and H_2, are explicitly presented within a $N = 2$ supersymmetric quantum mechanics framework.

Keywords: supersymmetric quantum mechanics; shape invariant potentials; supersymmetric quantum cosmology

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Citation: Jalalzadeh, S.; Rasouli, S.M.M.; Moniz, P. Shape Invariant Potentials in Supersymmetric Quantum Cosmology. *Universe* **2022**, *8*, 316. https://doi.org/10.3390/universe8060316

Academic Editor: Júlio César Fabris

Received: 11 April 2022
Accepted: 31 May 2022
Published: 5 June 2022

1. Introduction

Shape Invariant Potentials (SIP) constitute one of the hallmarks of supersymmetric quantum mechanics (SQM), in the sense that it enables a prolific framework to be elaborated. Being more specific, the presence of SIP allows us to easily obtain the set of states for a class of quantum systems, suitably based on an elegant algebraic construction. Hence, let us begin by mentioning that there is an algebraic structure associated with the SIP framework. It has gradually been acquiring a twofold relevance and within most of the exactly solvable problems in quantum mechanics [1–7].

On the one hand, such a structure has provided a method to determine eigenvalues and eigenfunctions, by means of which a spectrum is generated. More specifically, a broad set of those exactly solvable cases can be assembled and assigned within concrete classes; very few exceptions are known [1–7]. The distinguishing feature of any of such classes is that any exactly solvable case bears a *shape invariant potential: supersymmetric partners* are of the same shape, and their spectra can be determined entirely by an algebraic procedure comparable to that of the harmonic oscillator. In other words, operators can be defined, namely $A := \frac{d}{dx} + W(x)$, and its Hermitian conjugate $A^\dagger := -\frac{d}{dx} + W(x)$, Hamiltonians H_1 and its superpartner H_2 being expressed as $A^\dagger A$ and $AA^\dagger$, respectively. From this, we can produce and operate with other (more adequate) ladder operators for correspondingly appropriate quantum numbers. These can be maneuvered within a $J_\pm, J_3$ algebra, with comparable features to textbook ladder operators of angular momentum within either $SU(2)$ or $SO(3)$; please see [1–7] for relevant details.

On the other hand, several of these exactly solvable systems also possess a *potential algebra*: the corresponding Hamiltonian can be written as a Casimir operator of an underlying algebra, which in particular cases is of a $SO(2,1)$ nature [2]. Remarkably, there is a close correspondence: shape invariance can be expressed as constraint, which assists in establishing the spectrum; moreover, this shape invariance constraint can be written as an algebraic condition. For a specific set of SIP, the algebraic condition corresponds to the mentioned $SO(2,1)$ potential algebra, where unitary representations become of crucial use. Interestingly, these can be related to that of $SO(3)$; several SIPs are as such [1–7]. On the whole, a connection between SIP and *potential algebra* was attained. Nevertheless, it is also clear that, in spite of the structural similarity between $SO(2,1)$ and $SO(3)$ algebras, there are caveats to be aware of, related to the differences between those unitary representations.

There are also a couple of additional points that we would like to emphasize. To start with, some quantum states can be retrieved by group theoretical methods. This is further endorsed from the connections between shape invariance and potential algebra, wherein the former is translated into a concrete formulation within the latter [2]. As a result, the scope of the class of potentials where that could be applied was made more prominent [1]. Furthermore, other classes have been explored, related to harmonic oscillator induced second order differential equations, bearing group and algebra features, which subsequently allowed more SIP to be found [1–12]. In particular, this was further extended toward graded algebras in [8].

Secondly, these algebraic/group theory procedures (within the concrete use of algebras such as $SO(2,1)$ or $SO(3)$) have a striking resemblance to the approach and descriptive language used in [13–16]; a review of this idea is found in [17]. Therein, it was pointed out that intertwining boundary conditions, the algebra of constraints and hidden symmetries in quantum cosmology could be quite fruitful. Specifically, group/algebraic properties within ladder operators, either from angular momentum or from within the explicit presence of specific matter fields (and their properties), determined a partition of wave functions and boundary conditions, according to the Bargmann index [13–16]. Moreover, we proposed in [17] to extend this framework towards SIP, which could include well known analytically solvable cosmological cases. Being more clear, provided we identify integrability in terms of the shape invariance conditions, we could eventually import those specific features of SQM towards quantum cosmology [18–21]. That was the challenge we laid out in [17], which is still to be addressed: we hope our review paper herein can further enthuse someone to pick this up. A somewhat related and interesting direction to explore is to also consider an elaboration following [22,23]; specifically, accommodating the lines in [13–17] plus supersymmetry (SUSY) [18,19]. In brief, this paragraph conveys our central motivation to produce this review, building from a suggestion advanced in [17].

Thirdly, the interest in the above elements notwithstanding, there are still obstacles that ought to be mentioned, namely, about the scope of the usefulness of SIP in quantum cosmology. In fact, the list of SIP is quite restrictive, and most potentials therein do not emerge naturally within a minisuperspace. A few do, but for very particular case studies [24]. The classification of spatial geometries upon the Bianchi method implies that the potentials extracted from the gravitational degrees of freedom are very specific. Any 'broadness' can be introduced by inserting: (i) very specific matter fields into the minisuperspace (and therein we ought to be using realistic potentials (as indicated by particle physics)) or instead, (ii) try more SIP fitted choices but at the price of being very much *ad hoc,* i.e., an artificial selection. Nevertheless, the list of SIP and similar cases, where the algebraic tools could be adopted, has been extended. Although not a strong positive endorsement, there is work [25–34] that allows us to consider that eventually an extended notion of SIP may be soundly established, such that more cosmological minisuperspaces can be discussed within (see e.g., [24]). For the moment, this is a purpose set in construction and is what this review paper aims to to enthuse about and promote.

Upon this introductory section, this review is structured as follows. In Section 2, we summarize the features of SQM that we will be employing. In particular, a few technicalities about SIP will be presented in Section 2.2. Then, in Section 3, we take a case study, typically a toy model, by means of which we aim to promote work in SUSY quantum cosmology with the novel perspective of SIP. We emphasize that this is a line of investigation that has not yet been attempted before (see Section 3.3 for details). Section 4 conveys the Discussion and suggestions for the outlook for future research work.

2. Supersymmetric Quantum Mechanics

In this section, let us present a brief review of some of the pillars that characterize SQM. Then, we proceed to add a summary of the shape invariance concept. This section contains neither new results nor any innovated procedure, but only a very short overview of the results presented within seminal papers, e.g., [35–37].

2.1. Hamiltonian Formulation of Supersymmetric Quantum Mechanics

In order to describe SQM, let us start with the Schrödinger equation:

$$H\Psi_n(x) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\Psi_n(x) = E_n\Psi_n(x). \tag{1}$$

We assume that the ground state wave function $\Psi_0(x)$ (that has no nodes[1]) associated with a potential $V_1(x)$ is known. Then, by assuming that the ground state energy E_0 to be zero, the Schrödinger equation for this ground state reduces to:

$$H_1\Psi_0(x) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x)\right]\Psi_0(x) = 0, \tag{2}$$

which leads to construct the potential $V_1(x)$:

$$V_1(x) = \frac{\hbar^2}{2m}\frac{\Psi_0''(x)}{\Psi_0(x)}, \tag{3}$$

where a prime denotes differentiation with respect to x. It is straightforward to factorize the Hamiltonian from the operators as follows:

$$A = \frac{\hbar}{\sqrt{2m}}\left[\frac{d}{dx} - \frac{\Psi_0'(x)}{\Psi_0(x)}\right], \qquad A^\dagger = -\frac{\hbar}{\sqrt{2m}}\left[\frac{d}{dx} + \frac{\Psi_0'(x)}{\Psi_0(x)}\right], \tag{4}$$

through the equation

$$H_1 = A^\dagger A. \tag{5}$$

In order to construct the SUSY theory related to the original Hamiltonian H_1, the next step is to define another operator by reversing the order of A and $A^\dagger$, i.e., $H_2 \equiv AA^\dagger$, by which we indeed get the Hamiltonian corresponding to a new potential $V_2(x)$:

$$H_2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2(x), \tag{6}$$

where

$$V_2(x) = -V_1(x) + \frac{\hbar}{m}\left[\frac{\Psi_0'(x)}{\Psi_0(x)}\right]^2. \tag{7}$$

The potentials $V_1(x)$ and $V_2(x)$ have been referred to as the supersymmetric partner potentials. It should be noted that H_2, as the partner Hamiltonian corresponding to H_1 is in general not unique, but there is a class of Hamiltonians $H^{(m)}$ that can be partner Hamiltonians[2]. This point has been specified [35,38,39], conveying a better understanding of the relationship between SQM and the inverse scattering method, established by Gelfand and Levitan [40,41].

In SQM, instead of the ground state wave function $\Psi_0(x)$ associated with H_1, the superpotential $W(x)$ is introduced, being related to $\Psi_0(x)$ and its first derivative (with respect to x) by means of

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\left(\frac{\Psi_0'(x)}{\Psi_0(x)}\right). \tag{8}$$

At this stage, it is worth expressing the operators A and $A^\dagger$ and the supersymmetric partner potentials $V_1(x)$ and $V_2(x)$ in terms of the superpotential. Therefore, using (8), Equations (3), (4) and (7) are rewritten as:

$$A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x), \qquad A^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x), \tag{9}$$

$$V_1(x) = -\frac{\hbar}{\sqrt{2m}}W'(x) + W^2(x), \qquad V_2(x) = \frac{\hbar}{\sqrt{2m}}W'(x) + W^2(x), \tag{10}$$

where the expressions for V_1 and V_2 in (10) constitute Riccati equations. Equations (9) and (10) imply that $W'(x)$ is proportional to the commutator of the operators A and $A^\dagger$, and $W^2(x)$ is the average of the partner potentials.

It is easy to show that the wave functions, the energy eigenvalues and the S-matrices of both the Hamiltonians H_1 and H_2 are related. Let us merely outline the results and abstain from proving them. In this regard, we take $\Psi_n^{(1)}$ and $\Psi_n^{(2)}$ as the eigenfunctions of H_1 and H_2, respectively. Moreover, we denote their corresponding energy eigenvalues with $E_n^{(1)} \geq 0$ and $E_n^{(2)} \geq 0$ where $n = 0, 1, 2, 3, \ldots$ is the number of the nodes in the wave function. It is straightforward to show that supersymmetric partner potentials $V_1(x)$ and $V_2(x)$ possess the same energy spectrum. However, we should note that, for the ground state energy $E_0^{(1)} = 0$ associated with the potential $V_1(x)$, there is no corresponding level for its partner $V_2(x)$. Concretely, it has been shown that:

$$E_n^{(2)} = E_{n+1}^{(1)}, \qquad E_0^{(1)} = 0, \tag{11}$$

$$\Psi_n^{(2)} = \left[E_{n+1}^{(1)}\right]^{-\frac{1}{2}} A\Psi_{n+1}^{(1)}, \tag{12}$$

$$\Psi_{n+1}^{(1)} = \left[E_n^{(2)}\right]^{-\frac{1}{2}} A^\dagger\Psi_n^{(2)}. \tag{13}$$

In what follows, let us express some facts. (i) If the ground-state wave function $\Psi_0^{(1)}$, which is given by ($A\Psi_0^{(1)} = 0$)

$$\Psi_0^{(1)} = N_0 \exp\left[-\int^x W(x')dx'\right], \tag{14}$$

is square integrable, then the ground state of H_1 has zero energy ($E_0 = 0$) [35]. For this case, it can be shown that the SUSY is unbroken; (ii) if the eigenfunction $\Psi_{n+1}^{(1)}$ of H_1 ($\Psi_n^{(2)}$ of H_2) is normalized, then the $\Psi_n^{(2)}$ ($\Psi_{n+1}^{(1)}$), will be also normalized; (iii) Assuming the eigenfunction $\Psi_n^{(1)}$ with eigenvalue $E_n^{(1)}$ ($\Psi_n^{(2)}$ with eigenvalue $E_n^{(2)}$) corresponds to the Hamiltonian H_1 (H_2), it is easy to show that $A\Psi_n^{(1)}$ ($A^\dagger\Psi_n^{(2)}$) will be an eigenfunction of H_2 (H_1) with the same eigenvalue; (iv) In order to destroy (create) an extra node in the eigenfunction as well as convert an eigenfunction of H_1 (H_2) into an eigenfunction of H_2 (H_1) with the same energy, we apply the operator A ($A^\dagger$); (v) The ground state wave function of H_1 has no SUSY partner; (vi) Applying the operator A ($A^\dagger$), all the eigenfunctions of H_2 (H_1, except for the ground state) can be reconstructed from those of H_1 (H_2); please see Figure 1.

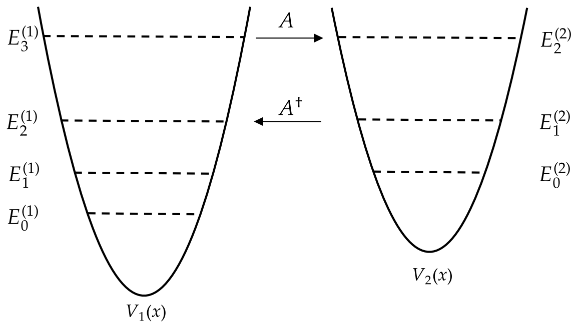


Figure 1. The energy levels of $V_1(x)$ and $V_2(x)$ as two supersymmetric partner potentials. The figure is associated with unbroken SUSY. It is seen that, except an extra state $E_0^{(1)} = 0$, the other energy levels are degenerate. Moreover, in this figure, it is shown that how the operators A and $A^\dagger$ connect eigenfunctions.

It has been believed that this fascinating procedure, which leads to an understanding of the degeneracy of the spectra of H_1 and H_2, can be provided by applying the properties of the SUSY algebra. Therefore, let us consider a matrix SUSY Hamiltonian (which is part of a closed algebra including both bosonic and fermionic operators with commutation and anti-commutation relations) containing both the Hamiltonians H_1 and H_2 [37]:

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2. \end{bmatrix}. \tag{15}$$

Supersymmetric quantum mechanics begins with a set of two matrix operators, Q and $Q^\dagger$, known as supercharges:

$$Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \tag{16}$$

$$Q^\dagger = \begin{bmatrix} 0 & A^\dagger \\ 0 & 0. \end{bmatrix}. \tag{17}$$

The matrix H is part of a closed algebra in which both bosonic and fermionic operators with commutation and anti-commutation relations are included, such that the bosonic degrees of freedom are changed into the fermionic ones and vice versa by the supercharges.

It is straightforward to show that:

$$[H, Q] = [H, Q^\dagger] = 0, \tag{18}$$

$$\{Q, Q^\dagger\} = H, \tag{19}$$

$$\{Q, Q\} = 2Q^2 = \{Q^\dagger, Q^\dagger\} = 2\left(Q^\dagger\right)^2 = 0, \tag{20}$$

by which the closed superalgebra $sl(1,1)$ is described [42] (see also Section 3.3). Note that the relations (18) are responsible for the degeneracy.

In SQM, when the two partner potentials have continuum spectra, it is possible to relate the reflection and transmission coefficients. A necessary condition for providing scattering in both of the partner potentials is that they must be finite when $x \to -\infty$ or $x \to \infty$.

2.2. Shape Invariance and Solvable Potentials

In the context of the non-relativistic quantum mechanics, there are a number of known potentials (e.g., Coulomb, harmonic oscillator, Eckart, Morse, and Pöschl–Teller) for which we can solve the corresponding Schrödinger equation analytically and determine all the energy eigenvalues and eigenfunctions explicitly. In this regard, the following questions naturally arise: Why are just some potentials solvable? Is there any underlying symmetry property? What is this symmetry?

Gendenshtein was the first to answer these questions by introducing the *shape invariance* concept [43]. In fact, for such potentials, all the bound state energy eigenvalues, eigenfunctions and the scattering matrix can be retrieved by applying the generalized operator method, which is essentially equivalent to the Schrödinger's method of factorization [44,45].

In [43], the relationship between SUSY, the hierarchy of Hamiltonians, and solvable potentials has been investigated from an interesting perspective (for detailed discussions see, for instance, [35,37]). In what follows, let us describe briefly the shape invariance concept. "*If the pair of SUSY partner potentials $V_{1,2}(x,b)$ are similar in shape and differ only in the parameters that appear in them, then they are said to be shape invariant*" [46]. Let us be more precise. Consider a pair of SUSY partner potentials, $V_{1,2}(x)$, as defined in (10). If the profiles of these potentials are such that they satisfy the relationship:

$$V_2(x,b) = V_1(x,b_1) + R(b_1), \tag{21}$$

where the parameter b_1 is some function of b, say given by $b_1 = f(b)$, the potentials $V_{1,2}(x)$ are said to bear shape invariance. In other words, to be associated within shape invariance the potentials $V_{1,2}$, while sharing a similar coordinate dependence, can at most differ in the presence of some parameters. To make the definition of shape invariance clear, consider, for example,

$$W = b\tanh\left(\frac{\sqrt{2m}}{\hbar}x\right). \tag{22}$$

Then, inserting this superpotential into (10) gives us:

$$\begin{aligned} V_1(x,b) &= -\frac{b(b+1)}{\cosh^2\left(\frac{\sqrt{2m}}{\hbar}x\right)} + b^2, \\ V_2(x,b) &= -\frac{b(b-1)}{\cosh^2\left(\frac{\sqrt{2m}}{\hbar}x\right)} + b^2. \end{aligned} \tag{23}$$

The above expressions show that one can rewrite V_2 in terms of V_1, as expressed in (10) where, in this example, $b_1 = b-1$, and $R(b_1) := 2b_1 + 1$. Thus, the potentials $V_{1,2}$ bear shape invariance in accordance with the definition (10). Then, to use the shape invariance condition, let us assume that (10) holds for a sequence of parameters, $\{b_k\}_{k=0,1,2,\ldots}$, where,

$$b_k = \underbrace{f\circ f\circ f\circ\ldots\circ f}_{\text{k times}}(b) = f^k(b), \quad k = 0,1,2,\ldots, \quad b_0 := b. \tag{24}$$

Consequently,

$$H_2(x,b_k) = H_1(x,b_{k+1}) + R(b_k). \tag{25}$$

Now, we write $H^{(0)} = H_1(x,b)$, $H^{(1)} = H_2(x,b)$, and we define $H^{(m)}$ as:

$$H^{(m)} := -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x,b_m) + \sum_{k=1}^{m} R(b_k) = H_1(x,b_m) + \sum_{k=1}^{m} R(b_k). \tag{26}$$

Using (25), we can extract $H^{(m+1)}$ as:

$$H^{(m+1)} = H_2(x,b_m) + \sum_{k=1}^{m} R(b_k). \tag{27}$$

Therefore, in this way, we are able to set up a hierarchy of Hamiltonians $H^{(k)}$ for various k values.

Employing condition (21) and the hierarchy of Hamiltonians [37], the energy eigenvalues and eigenfunctions have been obtained for any shape invariant potential when SUSY is unbroken. It should be noted that there is a correspondence between the condition (21) (associated with SQM) and the required mathematical condition applied in the method of the factorization of the Hamiltonian [47]. Although the terminology and ideas associated with these methods are different, they can be considered as the special cases of the procedure employed to handle second-order linear differential equations [48,49]. Notwithstanding the above, it has been believed that a better understanding of analytically solvable potentials could be achieved by SUSY and shape invariance. Let us elaborate more on this aspect.

H_2 contains the lowest state with a zero energy eigenvalue, according to the SQM concepts discussed in Section 2. As a result of (11), the lowest energy level of $H^{(m)}$ has the value of:

$$E_0^{(m)} = \sum_{k=1}^{m} R(b_k). \tag{28}$$

Therefore, it is simple to realize that because of the chain $H^{(m)} \to H^{(m-1)} \ldots \to H^{(1)}(:= H_2) \to H^{(0)}(:= H_1)$, the nth member in this sequence carries the nth level of the energy spectra of $H^{(0)}$ (or H_1), namely [35]:

$$E_n^{(0)} = \sum_{k=1}^{n} R(b_k), \quad E_0^{(0)} = 0. \tag{29}$$

Let us now return to the example (22). We rewrite (21) as:

$$V_1(x,b) = V_2(x,b-1) + b^2 - (b-1)^2. \tag{30}$$

We can generate b_k from $b_0 = b$ as $b_k = b - k$. Hence, the energy spectrum from $V_1(x,b)$ yields:

$$E_n^{(0)} = \sum_{k=1}^{n} R(b_k) = \sum_{k=1}^{n} (b^2 - b_k^2) = b^2 - b_n^2 = b^2 - (b-n)^2. \tag{31}$$

It is worth noting that, according to the requirement (21), the well-known solvable potentials (such as those were listed in the first paragraph of this subsection) are all shape invariant and, therefore, their energy eigenvalue spectra are given by (29). "*In [43], Gendenshtein then conjectured that shape invariance is not only sufficient but may even be necessary for a potential to be solvable*" [35]; In addition, let us mention that new developments have been achieved on this domain, where new approaches (see [50–54]) have either challenged or broaden this assertion. Moreover, by applying SUSY, it is also possible to retrieve the bound-state energy eigenfunctions of H_1 for shape invariant potentials [36]. In particular, in the same paper, by taking $\Psi_0^1(x,b)$ as the ground-state wave function of H_1 (which is given by (14)), and employing relation (13), a relation is obtained for nth-state eigenfunction $\Psi_n^1(x,b)$ as:

$$\Psi_n^1(x,b) = A^\dagger(x,b)A^\dagger(x,b_1)\ldots A^\dagger(x,b_{n-1})\Psi_0^1(x,b). \tag{32}$$

For later convenience, let us concentrate on a specific shape invariance that only involving translation of the parameter b_0 with a translation step η [27] (for other kind of relations between the parameters, see, for instance, [37]):

$$b_1 = b_0 + \eta. \tag{33}$$

It is feasible to introduce a translation operator as:

$$T(b_0) = \exp\left(\eta \frac{\partial}{\partial b_0}\right), \qquad T^{-1}(b_0) = T^\dagger(b_0) = \exp\left(-\eta \frac{\partial}{\partial b_0}\right), \tag{34}$$

which act merely on objects defined on the parameter space.

By composing the translation and bosonic operators, we can construct the following generalized creation and annihilation operators:

$$B_1(b_0) = A^\dagger(b_0)T(b_0), \tag{35}$$

$$B_2(b_0) = T^\dagger(b_0)A(b_0). \tag{36}$$

Applying the shape invariant potentials to solve the Schrödinger equation is similar to the factorization method employed to the case of the harmonic oscillator potential [27]. Therefore, we have:

$$B_2(b_0)A(b_0)\Psi_0(x;b_0) = A(b_0)\Psi_0(x;b_0) = 0. \tag{37}$$

In order to obtain the excited states, the creation operator should repeatedly act on $\Psi_0(x;b_0)$:

$$\Psi_n(x;b_0) = [B_1(b_0)]^n\Psi_0(x;b_0). \tag{38}$$

We should note that the translation operators (T and $T^\dagger$) and ladder operators, A and $A^\dagger$ do not commute with any b_k-dependent and any x-dependent objects, respectively. Therefore, the generalized creation and annihilation operators (B_1 and B_2) act on the objects defined on the dynamical variable space and the objects defined on parameter space via the bosonic operators and the translation operators, respectively. According to (37), we have:

$$\Psi_0(x;b_0) \propto \exp\left(-\int^x W(\tilde{x};b_0)d\tilde{x}\right), \tag{39}$$

which is transformed by a normalization constant (that should, in general, depend on parameters b) into a relation of equality. Concretely, the action of the generalized operators affects in determining such a normalization constant; for more details, see, [55].

Now let us obtain the relations of the energy eigenvalues and energy spectrum. From using (34), we can write:

$$R(b_n) = T(b_0)R(b_{n-1})T^\dagger(b_0), \tag{40}$$

where

$$b_n = b_0 + n\eta \tag{41}$$

is a generalized version of (33). Employing (40), we get:

$$R(b_n)B_1(b_0) = B_1(b_0)R(b_{n-1}). \tag{42}$$

Equations (40) and (42) yield a commutation relation as:

$$[H_1, (B_1)^n] = \left(\sum_{k=1}^{n} R(b_k)\right)(B_1)^n. \tag{43}$$

Applying (43) on the ground state of H_1, i.e., $\Psi_0(x;b_0)$, it is seen that $[B_1(b_0)]^n\Psi_0(x;b_0)$ is also an eigenfunction of H_1 with eigenvalue $E_n^{(1)}$ given by (29). Therefore, the energy spectrum is:

$$E_n = E_0 + E_n^{(1)}, \tag{44}$$

where the ground state energy E_0 is obtained from either:

$$H_1 = H - E_0, \tag{45}$$

or, equivalently,

$$W(x;b) - W'(x;b) = V(x) - E_0 = V_1(x). \tag{46}$$

Finally, it should be noted that the above established algebraic approach is self-consistent. More concretely, by considering supersymmetric and shape invariance properties of the system, it can be applied as an appropriate method for obtaining not only the eigenvalues and eigenfunctions of the bound state of a Schrödinger equation, but also exact resolutions for this equation [27].

3. SUSY Quantum Cosmology

In order to apply the formalism presented in the previous section, let us investigate a homogeneous and isotropic cosmology, in the context of General Relativity (GR) together with a single scalar field, ϕ, minimally coupled to gravity.

3.1. A Case Study: Classical Setting

By considering the Friedmann–Lemaître–Robertson–Walker (FLRW) line element[3]

$$ds^2 = N(t)dt^2 + a(t)^2\left\{\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right\}, \tag{47}$$

the ADM[4] Lagrangian will be[5]

$$L_{\rm ADM} = -\frac{3}{N}a\dot{a}^2 + 3kNa + a^3\left(\frac{\dot{\phi}^2}{2N} - NV(\phi)\right), \tag{48}$$

where an over-dot denotes a differentiation with respect to the cosmic time t; $N(t)$ is a lapse function, $a(t)$ is the scale factor, $V(\phi)$ is a scalar potential and $k = \{-1, 0, 1\}$ is the spatial curvature constant associated with open, flat and closed universes, respectively.

In this work, let us consider the scalar potential $V(\phi)$ to be in the form [65,66]

$$V(\phi) = \lambda + \frac{m^2}{2\alpha^2}\sinh^2(\alpha\phi) + \frac{\vartheta}{2\alpha^2}\sinh(2\alpha\phi), \tag{49}$$

where λ may be related to the cosmological constant; $m^2 = \partial^2 V/\partial\phi^2|_{\phi=0}$ is a mass squared parameter; $\alpha^2 = 3/8$ and ϑ is a coupling parameter. Moreover, we will now investigate only the spatially flat FLRW universe. For this case, it has been shown that an oscillator–ghost–oscillator system is produced [66–71]. More concretely, by applying the following transformations [72–75],

$$X = \frac{a^{\frac{3}{2}}}{\alpha}\cosh(\alpha\phi), \qquad Y = \frac{a^{\frac{3}{2}}}{\alpha}\sinh(\alpha\phi), \tag{50}$$

the Lagrangian (48) transform into

$$L_{\rm ADM} = -\frac{1}{2N}\dot{\xi}^\top J\dot{\xi} + \frac{N}{2}\xi^\top MJ\xi, \tag{51}$$

where

$$\xi := \begin{pmatrix} X \\ Y \end{pmatrix}, \quad M := \begin{pmatrix} 2\lambda\alpha^2 & -\vartheta \\ \vartheta & 2\lambda\alpha^2 - m^2 \end{pmatrix}, \quad J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{52}$$

It is straightforward to decouple (51) into normal modes $\gamma := \Sigma^{-1}\xi$ by means of:

$$\gamma := \begin{pmatrix} u \\ v \end{pmatrix}, \quad \Sigma := \begin{pmatrix} \frac{-m-\sqrt{m^4-4\vartheta^2}}{2\vartheta} & \frac{-m+\sqrt{m^4-4\vartheta^2}}{2\vartheta} \\ 1 & 1 \end{pmatrix}, \tag{53}$$

which diagonalize the matrix M as follows:

$$\Sigma^{-1}M\Sigma = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}, \quad \omega_{1,2}^2 = \frac{3\lambda}{4} + \frac{m^2}{2} \mp \frac{\sqrt{m^4 - 4\vartheta^2}}{2}. \tag{54}$$

Thus, we retrieve the Lagrangian associated with a $2D$ oscillator–ghost–oscillator:

$$\begin{aligned} L_{\mathrm{ADM}}(u,v) &= -\frac{1}{2N}\dot{\gamma}^\top \mathcal{I}\dot{\gamma} + \frac{N}{2}\gamma^\top \mathcal{J}\gamma \\ &= -\frac{1}{2}\left\{ \left(\frac{1}{N}\dot{u}^2 - \omega_1^2 N u^2\right) - \left(\frac{1}{N}\dot{v}^2 - \omega_2^2 N v^2\right) \right\}, \end{aligned} \tag{55}$$

where $\mathcal{I} := \Sigma^\top J \Sigma$, and $\mathcal{J} := \Sigma^\top M J \Sigma$. The conjugate momenta corresponding to u and v are:

$$p_u = \frac{\dot{u}}{N}, \qquad p_v = -\frac{\dot{v}}{N}. \tag{56}$$

Moreover, the classical Euler–Lagrange equations are given by:

$$\frac{d}{dt}\left(\frac{\dot{u}}{N}\right) + N\omega_1^2 u = 0, \quad \frac{d}{dt}\left(\frac{\dot{v}}{N}\right) + N\omega_2^2 v = 0. \tag{57}$$

It is straightforward to show that the Hamiltonian corresponding to the ADM Lagrangian (55) is:

$$H_{\mathrm{ADM}} = -\frac{N}{2}\left\{ \left(p_u^2 + \omega_1^2 u^2\right) - \left(p_v^2 + \omega_2^2 v^2\right) \right\}, \tag{58}$$

which, for the gauge $N = 1$, yields

$$u(t) = u_0 \sin(\omega_1 t - \theta), \qquad v(t) = v_0 \sin(\omega_2 t). \tag{59}$$

In (59), θ is an arbitrary phase factor. From using the Hamiltonian constraint, we obtain $\omega_1 u_0 = \omega_2 v_0$. It is also seen that the classical paths corresponding to solutions (59), in the configuration space (u, v), are the generalized Lissajous ellipsis.

3.2. Quantization

In order to establish a quantum cosmological model corresponding to our model, let us proceed with the Wheeler–DeWitt equation. The canonical quantization of (58) gives

$$\mathcal{H}\Psi(u,v) = \left(-\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \omega_1^2 u^2 - \omega_2^2 v^2\right)\Psi(u,v) = 0. \tag{60}$$

Equation (60) is separable and we can obtain a solution as:

$$\Psi_{n_1,n_2}(u,v) = \alpha_{n_1}(u)\beta_{n_2}(v), \tag{61}$$

where

$$\alpha_n(u) = \left(\frac{\omega_1}{\pi}\right)^{1/4} \frac{H_n(\sqrt{\omega_1}u)}{\sqrt{2^n n!}} \mathrm{e}^{-\omega_1 u^2/2}, \tag{62}$$

$$\beta_n(v) = \left(\frac{\omega_2}{\pi}\right)^{1/4} \frac{H_n(\sqrt{\omega_2}v)}{\sqrt{2^n n!}} \mathrm{e}^{-\omega_2 v^2/2}. \tag{63}$$

In relations (62) and (63), $H_n(x)$ stands for the Hermite polynomials. Moreover, we should note that the Hamiltonian constrain relates the parameters of the model as:

$$\left(n_1 + \frac{1}{2}\right)\omega_1 = \left(n_2 + \frac{1}{2}\right)\omega_2, \qquad n_1, n_2 = 0, 1, 2, \ldots. \tag{64}$$

The recovery of classical solutions from the corresponding quantum model is one of the essential elements of quantum cosmology. For this aim, a coherent wave packet with reasonable asymptotic behavior in the minisuperspace is often constructed, peaking near the classical trajectory. We can herewith produce a widespread wave packet solution,

$$\Psi(u, v) = \sum_{n_1, n_2} C_{n_1 n_2} \alpha_{n_1}(u) \beta_{n_2}(v), \tag{65}$$

where the summing is restricted to overall values of n_1 and n_2 satisfies the relation (64). Let us consider the simplest case which is when $\omega_1 = \omega_2 = \omega$, which means $m^2 = 2\vartheta$ in the definition of scalar field potential (49). Then, the wave packet will be:

$$\Psi(u, v) = \sqrt{\frac{\omega}{\pi}} \sum_{n=0}^{\infty} \frac{C_n}{n! 2^n} \exp\left(-\frac{\omega}{2}(u^2 + v^2)\right) H_n(\sqrt{\omega} u) H_n(\sqrt{\omega} v), \tag{66}$$

where C_n is a complex constant. We apply the following identity to create a coherent wave packet with suitable asymptotic behavior in the minisuperspace, peaking around the classical trajectory:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) H_n(y) = \frac{1}{\sqrt{1 - t^2}} \exp\left(\frac{2txy - t^2(x^2 + y^2)}{2(1 - t^2)}\right). \tag{67}$$

Using this identity and choosing the coefficients C_n in (66) to be $C_n = B 2^n \tanh \xi$, with B and ξ are arbitrary complex constants, we obtain:

$$\begin{aligned} \Psi(u, v) = C \exp\left(-\frac{\omega}{4} \cos(2\beta_2) \cosh(2\beta_1)(u^2 + v^2 - 2\eta \tanh(2\beta_1) uv)\right) \\ \times \exp\left(-\frac{i\omega}{4} \sinh(2\beta_1) \sin(2\beta - 2)(u^2 + v^2 - 2\eta \coth(2\beta_1) uv)\right), \end{aligned} \tag{68}$$

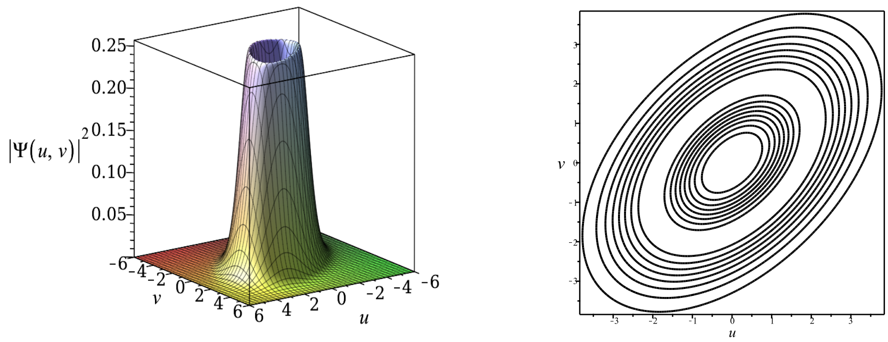
where β_1 and β_2 are the real and imaginary parts of $\xi = \beta_1 + i\beta_2$, respectively, $\eta = \pm 1$, and N is a normalization factor. Figure 2a shows the density plot, and Figure 2b illustrates the contour plot of the wave function for typical values of β_1, β_2, and $\eta = 1$ for the following combination of the solutions:

$$\Psi(u, v) = \Psi_{\beta_1, \beta_2}(u, v) - \Psi_{\beta_1 + \delta\beta_1, \beta_2 + \delta\beta_2}(u, v). \tag{69}$$

The classical solutions (59) can easily be represented as the following trajectories (for $\omega_1 = \omega_2 = \omega$):

$$u^2 + v^2 - 2\eta uv \cos(\theta) - u_0^2 \sin^2(\theta) = 0. \tag{70}$$

This equation describes ellipses whose major axes make angle $\pi/4$ with the positive/negative u axis according to the choices ± 1 for η. Additionally, each trajectory's eccentricity and size are determined by θ and u_0, respectively. It can be seen that the quantum pattern in Figure 2 and the classical paths (70) in configuration space, (u, v), have a high correlation.



(**a**) (**b**)

Figure 2. Density plot- (**a**), and contour plot- (**b**), of a wave packet. These figures are plotted for numerical values $\omega = 1$, $\beta_1 = 1$, $\beta_2 = \pi/6$, $\eta = 1$, $\delta\beta_1 = 0.1$ and $\delta\beta_2 = 3\pi/50$.

Let us point out how we can introduce a time-evolving wave-function. By employing a canonical transformation on the (v, p_v) sector of the Hamiltonian (58), we observe that in the total Hamiltonian, the momentum associated with the the new canonical variable appears linearly. Let us be more precise. Consider the following canonical transformation $(v, p_v) \to (T, p_T)$ given by:

$$v = \sqrt{\frac{2p_T}{\omega_2^2}}\sin(\omega_2 T), \qquad p_v = \sqrt{2p_T}\cos(\omega_2 T), \qquad \{T, p_T\} = 1. \tag{71}$$

It is easy to check out that the inverse map is given by the following relations:

$$p_T = \frac{1}{2}p_v^2 + \frac{1}{2}\omega_1^2 v^2, \quad T = \frac{1}{\omega_2}\tan^{-1}\left(\omega_2 \frac{v}{p_v}\right). \tag{72}$$

In fact, the new set of phase space coordinates (T, p_T) is related to the harmonic oscillator's action-angle variables, (φ, p_φ), by [76,77]:

$$\varphi = \omega_2 T, \quad p_\varphi = \frac{p_T}{\omega_2}. \tag{73}$$

The ADM Hamiltonian (58) simply takes the form:

$$H_{\text{ADM}} = N\left(\frac{1}{2}p_u^2 + \frac{1}{2}\omega_1^2 u^2 - p_{_T}\right). \tag{74}$$

The classical field equations corresponding to (74) are:

$$\begin{cases} \dot{u} = Np_u, & \dot{p}_u = -N\omega_1^2 u, \\ \dot{T} = -N, & \dot{p}_{_T} = 0. \end{cases} \tag{75}$$

For $N = 1$, we find

$$T \;=\; -t, \quad p_{_T} = \text{const.} \tag{76}$$

Thus, the motion in $2D$ phase space (T, p_T) becomes trivial, i.e., flow paths are straight lines with constant p_T. As seen, the second set of solutions for (75) implies that T plays the role of the time parameter. Consequently, the Poisson bracket of the time parameter and super-Hamiltonian does not vanish but instead we have $\{T, \mathcal{H}\} = 1 = \{T, p_{_T}\}$, which implies that T is not a Dirac observable, and therefore, we may consider it as a time variable; see, for instance, [76] and references therein.

3.3. Supersymmetric Quantization

Employing the Hamiltonian constraint upon (74), and then substituting $p_u = -i\frac{d}{du}$ and $p_T = -i\frac{\partial}{\partial T} = i\frac{\partial}{\partial t}$, we get a Schrödinger–Wheeler–WeDitt equation:

$$i\frac{\partial}{\partial t}\Psi(u,t) = \left[-\frac{1}{2}\frac{d^2}{du^2} + \frac{l(l+1)}{2u^2} + \frac{1}{2}\omega_1^2 u^2\right]\Psi(u,t). \tag{77}$$

In the process of obtaining Equation (77), we have further used the following factor ordering procedure:

$$p_u^2 = -\frac{1}{3}\left(u^\alpha \frac{d}{du} u^\beta \frac{d}{du} u^\gamma + u^\gamma \frac{d}{du} u^\alpha \frac{d}{du} x^\beta + u^\beta \frac{d}{du} u^\gamma \frac{d}{du} u^\alpha\right), \tag{78}$$

where the parameters α, β, and γ satisfy the requirement $\alpha+\beta+\gamma = 0$, and we have set $\frac{1}{3}(\beta^2+\gamma^2+\beta\gamma) := l(l+1)$.

The time independent sector of Equation (77) reads:

$$H_l \Psi_n^l(u) = E_n^l \Psi_n^l(u), \tag{79}$$

where

$$H_l := -\frac{1}{2}\frac{d^2}{du^2} + \frac{l(l+1)}{2u^2} + \frac{1}{2}\omega_1^2 u^2. \tag{80}$$

According to the Equation (10), we now introduce first-order differential operators:

$$\begin{cases} A_l := \frac{1}{\sqrt{2\omega_1}}\frac{d}{du} + \sqrt{\frac{\omega_1}{2}}u - \frac{l+1}{\sqrt{2\omega_1}u}\,, \\ A_l^\dagger := -\frac{1}{\sqrt{2\omega_1}}\frac{d}{du} + \sqrt{\frac{\omega_1}{2}}u - \frac{l+1}{\sqrt{2\omega_1}u}\,. \end{cases} \tag{81}$$

For $l \in \mathbb{N}$, we correspondingly obtain the following supersymmetric partner Hamiltonians:

$$\begin{cases} H_1 = \omega_1 A_l^\dagger A_l = -\frac{1}{2}\frac{d^2}{du^2} + \frac{l(l+1)}{2u^2} + \frac{1}{2}\omega_1^2 u^2 + \omega_1(l-\frac{1}{2}) = H_l - \omega_1(l+\frac{3}{2}), \\ H_2 = \omega_1 A_l A_l^\dagger = -\frac{1}{2}\frac{d^2}{du^2} + \frac{(l+1)(l+2)}{2u^2} + \frac{1}{2}\omega_1^2 u^2 - \omega_1(l+\frac{1}{2}) = H_{l+1} - \omega_1(l+\frac{1}{2}). \end{cases} \tag{82}$$

These two Hamiltonians have the same energy spectrum except the ground state of H_2:

$$\begin{cases} H_1\Psi_n^{(l)} = \left[E_n^{(l)} + \omega_1 - (l+\frac{3}{2})\right]\Psi_n^{(l)}, \\ H_2\Psi_{n+1}^{(l+1)} = \left[E_{n+1}^{(l+1)} - \omega_1(l+\frac{1}{2})\right]\Psi_{n+1}^{(l+1)} = \left[E_n^{(l)} - \omega_1(l+\frac{1}{2})\right]\Psi_{n+1}^{(l+1)}. \end{cases} \tag{83}$$

It is seen that these equations refer to the shape-invariance condition, by which we, equivalently, can write:

$$A_{l-1}^\dagger A_{l-1} - A_l A_l^\dagger = \frac{2}{\omega_1}. \tag{84}$$

Thus, altering the sequence of operators A_l and $A_l^\dagger$ causes the value of l to change. This demonstrates how shape-invariance properties link the various factor orderings of the Schrödinger–Wheeler–DeWitt Equation (79). Shape-invariant potentials are well recognized for being simple to deal with when using lowering and raising operators, similar to the harmonic oscillator. However, we should note that the commutator of A_l and $A_l^\dagger$ does not provide a constant value. Namely,

$$[A_l, A_l^\dagger] = 1 + \frac{l+1}{\omega_1 u^2} \,, \tag{85}$$

which implies that these operators are not suitable to proceed with. As the eigenvalue relation (82) shows, the potentials $V_1(u; b_0)$ and $V_2(u; b_1)$ introduced in (9) are given by:

$$\begin{aligned} V_1(u; b_0) &= \frac{b_0(b_0+1)}{2u^2} + \frac{1}{2}\omega_1^2 u^2, \\ V_2(u; b_1) &= \frac{b_1(b_1+1)}{2u^2} + \frac{1}{2}\omega_1^2 u^2, \end{aligned} \tag{86}$$

where $b_1 = l+1$ and $b_0 = l$. Therefore, in relation (33) this corresponds to $\eta = 1$. According to the Section 2.2, we presume that replacing $l+1$ with l in a given operator can be accomplished via a similarity transformation, (34), and so we build an appropriate algebraic structure via translation operator:

$$T(l) = \exp\left(\frac{\partial}{\partial l}\right), \qquad T^{-1}(l) = T^\dagger(l) = \exp\left(-\frac{\partial}{\partial l}\right). \tag{87}$$

Therefore, we introduce the operators:

$$B_l := \frac{1}{\sqrt{2}} T^\dagger(l) A_l, \quad B_l^\dagger := \frac{1}{\sqrt{2}} A_l^\dagger T(l), \quad N_{\mathrm{B}}^l := B_l^\dagger B_l, \tag{88}$$

which lead us to the simple harmonic oscillator (Heisenberg–Weyl) algebra

$$[B_l, B_l^\dagger] = 1, \quad [N_{\mathrm{B}}^l, B_l] = -B_l, \quad [N_{\mathrm{B}}^l, B_l^\dagger] = B_l^\dagger. \tag{89}$$

These commutation relations show that $B_l^\dagger$ and B_l are the appropriate creation and annihilation operators for the spectra of our shape-invariant potentials. The action of these operators on normalized eigenfunctions yields:

$$B_l \Psi_n^l = \sqrt{n}\Psi_{n-1}^l, \quad B_l^\dagger \Psi_n^l = \sqrt{n+1}\Psi_{n+1}^l, \quad N_{\mathrm{B}}^l \Psi_n^l = n\Psi_n^l, \quad n = 0, 1, 2, \ldots. \tag{90}$$

Equation (79) and the last equation of the above set give:

$$E_n^l = \omega_1\left(2n + l + \frac{3}{2}\right). \tag{91}$$

In addition, the condition $B_l \Psi_0^l = 0$ gives us the ground state of the model universe for factor ordering l by:

$$\Psi_0^l = C_l \exp\left(-\frac{\omega_1}{2}u^2 - \frac{l+1}{u^2}\right), \tag{92}$$

where C_l is a normalization constant. The excited states can be easily determined by applying (37).

In what follows, let us complete our procedure by including the Grassmannian variables ψ and $\bar{\psi}$, which satisfy

$$\psi^2 = 0, \qquad \bar{\psi}^2 = 0, \qquad \psi\bar{\psi} + \bar{\psi}\psi = 1, \tag{93}$$

involving them in the Hamiltonian (80). By means of such a procedure, we can subsequently construct a supersymmetric extension of our Hamiltonian:

$$H_{\mathrm{SUSY}} = -\frac{1}{2}\frac{d^2}{du^2} + \frac{l(l+1)}{2u^2} + \frac{1}{2}\omega_1^2 u^2 + \omega_1 \bar{\psi}\psi. \tag{94}$$

In our work herein, the convention of the left derivative for these variables has been adapted. Up to now, we specified the bosonic creation and annihilation operators B and $B^\dagger$ in terms of the dynamical variable u and its conjugate momenta. Here, we can also introduce fermionic creation and annihilation operators $C^\dagger = \bar{\psi}$ and $C = \psi$. Therefore, the Hamiltonian (94) can simply be written as:

$$H_{\text{SUSY}} = 2\omega_1 (B^\dagger B + C^\dagger C). \tag{95}$$

Adapting the basic commutator and anticommutator brackets:

$$[B, B^\dagger] = 1, \qquad \{C, C^\dagger\} = 1, \tag{96}$$

and considering all the others to be zero, it is easy to show that the operators:

$$N_{\text{B}} = B^\dagger B, \qquad N_{\text{F}} = C^\dagger C, \qquad Q = B^\dagger C, \qquad Q^\dagger = C^\dagger B, \tag{97}$$

(where the indices B and F refer to the bosonic and fermionic quantities, respectively) will be conserved quantities. Namely,

$$[Q, H_{\text{SUSY}}] = [Q^\dagger, H_{\text{SUSY}}] = 0, \qquad [N_{\text{B}}, H_{\text{SUSY}}] = [N_{\text{F}}, H_{\text{SUSY}}] = 0. \tag{98}$$

Moreover, we have:

$$\begin{aligned}
[Q, N_{\text{B}}] &= -Q, & [Q, N_{\text{F}}] = Q \\
[Q^\dagger, N_{\text{B}}] &= Q^\dagger, & [Q^\dagger, N_{\text{F}}] = -Q^\dagger, \\
\omega_1\{Q, Q^\dagger\} &= H_{\text{SUSY}}, & Q^2 = \frac{1}{2}\{Q, Q\} = 0, \quad \left(Q^\dagger\right)^2 = \frac{1}{2}\{Q^\dagger, Q^\dagger\} = 0.
\end{aligned} \tag{99}$$

From (98) and (99) an explicit algebra is therefore produced, where the Hamiltonian H_{SUSY} is a Casimir operator for the whole algebra [78]. If we use a matrix representation, then we can write, alternatively,

$$\psi = C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \bar{\psi} = C^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{100}$$

$$H_{\text{SUSY}} = 2\omega_1 \begin{pmatrix} B_l^\dagger B_l + 1 & 0 \\ 0 & B_l B_l^\dagger \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}. \tag{101}$$

4. Discussion

This paper is a review that embraces a twofold endorsement. On the one hand, it imports SUSY features (in a quantum mechanical setting). However, the current fact is that SUSY has not (yet . . .) been found in nature; searches prevail for any evidence, being it directly or indirectly. On the other hand, this review refers to quantum cosmology as a phenomenological domain regarding the full quantization of gravity. Likewise, there is as yet no clear-cut observational evidence of such a stage in the very early universe. Whereas the latter is fairly expected as the cosmos is further probed, proceeding gradually to prior times, the former, although alluringly elegant, may just be a formal framework. So, producing a review on a topic involving these two ideas may seem twice likely to raise discomfort. However, maybe not; perhaps SUSY quantum cosmology deserves to be kept nearby, just over an arm's length, so as to say[6], if the occasion (or data) emerges to either support it or at least enthuse more research about it. There are still open aspects to appraise and the one brought up in this review is among them [18,19].

In what concerns SUSY quantum cosmology, there have been a few books in the past 30 years or so [18,19,79] plus selected reviews on the different procedures that were constructed and subsequently promoted [80–90]. Likewise, there are chapters (and sections) about SUSY quantum cosmology in well known textbooks concerning quantum gravity [91–94]. In particular, the direction and extension of $N = 2$ SQM to SUSY quantum cosmology was led by [95–98] and subsequently by [99,100], referring to conformal issues.

Therefore, the opportunity to produce this review enthused us to refer to and explore a specific particular aspect that was indicated as a concrete open problem in SUSY quantum cosmology (Nb. we emphasize many more still remain; cf. in [18,19]); concretely, investigating the setting of SIP, fairly present in $N = 2$ SQM. This framework has been previously developed and independently from SUSY quantum cosmology, to explore issues in supersymmetric quantum field theory, namely SUSY breaking (which the seminal paper [101] has made possible).

Hence, a necessary and yet to be performed analysis remains to be elaborated: explicitly considering SIP as we just mentioned within quantum cosmology, i.e., bringing up this possibility and using it intertwined within SUSY quantum cosmology. This is a new idea for other researchers to pick up and evolve forward, producing their own assertions. The content of our review is thus very open, conveying a direction for further exploration. It involves algebraic quantum-mechanical aspects that are present when SIP characterizes particular models. It also deals with integrability, which SUSY seems to bring so elegantly.

In this paper, besides contributing a topical review towards this Special Issue, we also provided a constructive example to illustrate how promising the framework can be. Concretely, we provided a case study, consisting of a spatially flat FRW model in the presence of a single scalar field, minimally coupled to gravity. We extracted the Schrödinger–Wheeler–DeWitt equation containing a particular set of possible factor ordering. Next, we computed the corresponding supersymmetric partner Hamiltonians. Intriguingly, the shape invariance properties can be related to the several factor orderings of our Schrödinger–Wheeler–DeWitt equation. The ground state was computed and the excited states as well. Consistently, the partner Hamiltonians, were explicitly presented within an $N = 2$ SQM framework.

We implicitly made another suggestion in Section 1 (Introduction). In more detail, we suggested building a twofold framework; on the one hand, importing the ideas employed in references [13–17], where the presence of constraints, their algebra plus a natural integrability induces separability in the Hilbert space of solutions for the Wheeler–DeWitt equation. On the other hand, exploring, at least to begin with on formal terms, whether any such algebra of the constraints generators for a minisuperspace would bear any similarity to an algebra of supersymmetry generators. In other words, perhaps producing a sequence of *new* operators A_l and $A_l^\dagger$, assisting SIP properties but also related to SUSY constraints of a Schrödinger–Wheeler–DeWitt equation similar to (79). SIP are well recognized for being simple to deal with. In essence, our suggestion is to explore (*i*) if there is any relation between SIP and the descriptive report in [13–17] and, if positive, (*ii*) apply it within SUSY quantum cosmology.

Author Contributions: Conceptualization, S.J., S.M.M.R. and P.M.; Formal analysis, S.J., S.M.M.R. and P.M.; Methodology, S.J., S.M.M.R. and P.M.; Writing–original draft, S.J., S.M.M.R. and P.M.; Writing–review & editing, S.J., S.M.M.R. and P.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: PVM and SMMR acknowledge the FCT grants UID-B-MAT/00212/2020 and UID-P-MAT/00212/2020 at CMA-UBI plus the COST Action CA18108 (Quantum gravity phenomenology in the multi-messenger approach).

Conflicts of Interest: The authors declare no conflict of interest.

Notes

1 In [37], the excited wave functions have also been studied. More concretely, instead of the choice of a nonsingular superpotential that is based on the ground state wave function $\Psi_0(x)$, a generalized procedure was presented to construct all possible superpotentials.

2 Cf. next subsection, concretely about Equation (28).

3 Throughout this paper we work in natural units where $\hbar = c = k_B = 1$.

4 Adler–Deser–Misner (ADM); see [56] for more details.

5 In this work, we consider a framework in which the scalar field is minimally coupled to gravity, see also [57–59]. Instead, one can choose other interesting gravitational models where ϕ is non-minimally coupled, see for instance [60–64].

6 "Ah, but a man's reach should exceed his grasp, Or what's a heaven for?", Robert Browning (in 'Andrea del Sarto' l. 97 (1855)).

References

1. Cooper, F.; Ginocchio, J.N.; Wipf, A. Supersymmetry, operator transformations and exactly solvable potentials. *J. Phys. A* **1989**, *22*, 3707–3716. [CrossRef]
2. Gangopadhyaya, A.; Mallow, J.V.; Sukhatme, U.P. Shape invariance and its connection to potential algebra. In *Supersymmetry and Integrable Models*; Aratyn, H., Imbo, T.D., Keung, W.Y., Sukhatme, U., Eds.; Springer: Berlin/Heidelberg, Germany, 1998; Volume 502, pp. 341–350. [CrossRef]
3. Balantekin, A.B. Algebraic approach to shape invariance. *Phys. Rev. A* **1998**, *57*, 4188–4191. [CrossRef]
4. Gangopadhyaya, A.; Mallow, J.V.; Sukhatme, U.P. Broken supersymmetric shape invariant systems and their potential algebras. *Phys. Lett. A* **2001**, *283*, 279–284. [CrossRef]
5. Chen, G.; Chen, Z.D.; Xuan, P.C. Exactly solvable potentials of the Klein Gordon equation with the supersymmetry method. *Phys. Lett. A* **2006**, *352*, 317–320. [CrossRef]
6. Khare, A.; Bhaduri, R.K. Supersymmetry, shape invariance and exactly solvable noncentral potentials. *Am. J. Phys.* **1994**, *62*, 1008–1014. [CrossRef]
7. Quesne, C. Deformed Shape Invariant Superpotentials in Quantum Mechanics and Expansions in Powers of $\hbar$. *Symmetry* **2020**, *12*, 1853. [CrossRef]
8. Oikonomou, V.K. A relation between Z_3-graded symmetry and shape invariant supersymmetric systems. *J. Phys. A* **2014**, *47*, 435304. [CrossRef]
9. Bazeia, D.; Das, A. Supersymmetry, shape invariance and the Legendre equations. *Phys. Lett. B* **2012**, *715*, 256–259. [CrossRef]
10. Stahlhofen, A. Remarks on the equivalence between the shape-invariance condition and the factorisation condition. *J. Phys. A* **1989**, *22*, 1053–1058. [CrossRef]
11. Jafarizadeh, M.A.; Fakhri, H. Supersymmetry and shape invariance in differential equations of mathematical physics. *Phys. Lett. A* **1997**, *230*, 164–170. [CrossRef]
12. Amani, A.; Ghorbanpour, H. Supersymmetry Approach and Shape Invariance for Pseudo-harmonic Potential. *Acta Phys. Pol. B* **2012**, *43*, 1795–1803. [CrossRef]
13. Fathi, M.; Jalalzadeh, S.; Moniz, P.V. Classical Universe emerging from quantum cosmology without horizon and flatness problems. *Eur. Phys. J. C* **2016**, *76*, 527. [CrossRef]
14. Jalalzadeh, S.; Rostami, T.; Moniz, P.V. On the relation between boundary proposals and hidden symmetries of the extended pre-big bang quantum cosmology. *Eur. Phys. J. C* **2015**, *75*, 38. [CrossRef]
15. Rostami, T.; Jalalzadeh, S.; Moniz, P.V. Quantum cosmological intertwining: Factor ordering and boundary conditions from hidden symmetries. *Phys. Rev. D* **2015**, *92*, 023526. [CrossRef]
16. Jalalzadeh, S.; Moniz, P.V. Dirac observables and boundary proposals in quantum cosmology. *Phys. Rev. D* **2014**, *89*, 083504. [CrossRef]
17. Jalalzadeh, S.; Rostami, T.; Moniz, P.V. Quantum cosmology: From hidden symmetries towards a new (supersymmetric) perspective. *Int. J. Mod. Phys. D* **2016**, *25*, 1630009. [CrossRef]
18. Moniz, P.V. *Quantum Cosmology—The Supersymmetric Perspective—Vol. 1*; Lecture Notes in Physics; Springer: Berlin/Heidelberg, Germany, 2010; Volume 803. [CrossRef]
19. Moniz, P.V. *Quantum Cosmology—The Supersymmetric Perspective—Vol. 2*; Lecture Notes in Physics; Springer: Berlin/Heidelberg, Germany, 2010; Volume 804. [CrossRef]
20. Moniz, P.V. Origin of structure in supersymmetric quantum cosmology. *Phys. Rev. D* **1998**, *57*, R7071. [CrossRef]
21. Kiefer, C.; Lück, T.; Moniz, P.V. Semiclassical approximation to supersymmetric quantum gravity. *Phys. Rev. D* **2005**, *72*, 045006. [CrossRef]
22. Cordero, R.; Granados, V.D.; Mota, R.D. Novel Complete Non-Compact Symmetries for the Wheeler–DeWitt Equation in a Wormhole Scalar Model and Axion-Dilaton String Cosmology. *Class. Quant. Grav.* **2011**, *28*, 185002. [CrossRef]
23. Cordero, R.; Mota, R.D. New exact supersymmetric wave functions for a massless scalar field and axion–dilaton string cosmology in a FRWL metric. *Eur. Phys. J. Plus* **2020**, *135*, 78. [CrossRef]

24. Díaz, J.S.G.; Reyes, M.A.; Mora, C.V.; Pozo, E.C. Supersymmetric Quantum Mechanics: Two Factorization Schemes and Quasi-Exactly Solvable Potentials. In *Panorama of Contemporary Quantum Mechanics-Concepts and Applications*; IntechOpen: London, UK 2019. [CrossRef]
25. Bhaduri, R.K.; Sakhr, J.; Sprung, D.W.L.; Dutt, R.; Suzuki, A. Shape invariant potentials in SUSY quantum mechanics and periodic orbit theory. *J. Phys. A* **2005**, *38*, L183–L189. [CrossRef]
26. Bougie, J.; Gangopadhyaya, A.; Mallow, J.V. Method for generating additive shape-invariant potentials from an Euler equation. *J. Phys. A* **2011**, *44*, 275307. [CrossRef]
27. Filho, E.D.; Ribeiro, M.A.C. Generalized Ladder Operators for Shape-invariant Potentials. *Phys. Scripta* **2001**, *64*, 548–552. [CrossRef]
28. Gangopadhyaya, A.; Mallow, J.V.; Rasinariu, C.; Bougie, J. Exactness of SWKB for shape invariant potentials. *Phys. Lett. A* **2020**, *384*, 126722. [CrossRef]
29. Bougie, J.; Gangopadhyaya, A.; Mallow, J.; Rasinariu, C. Supersymmetric Quantum Mechanics and Solvable Models. *Symmetry* **2012**, *4*, 452–473. [CrossRef]
30. Cariñena, J.F.; Ramos, A. Shape-invariant potentials depending on n parameters transformed by translation. *J. Phys. A Math. Gen.* **2000**, *33*, 3467–3481. [CrossRef]
31. Su, W.C. Shape invariant potentials in second-order supersymmetric quantum mechanics. *J. Phys. Math.* **2011**, *3*, 1–12. [CrossRef]
32. Dong, S.H. *Factorization Method in Quantum Mechanics*; Springer: Berlin/Heidelberg, Germany, 2007. [CrossRef]
33. Nasuda, Y.; Sawado, N. SWKB Quantization Condition for Conditionally Exactly Solvable Systems and the Residual Corrections. *arXiv* **2021**, arXiv:2108.12567.
34. Gangopadhyaya, A.; Mallow, J.V.; Rasinariu, C. *Supersymmetric Quantum Mechanics: An Introduction*; World Scientific: Singapore, 2017. [CrossRef]
35. Cooper, F.; Ginocchio, J.N.; Khare, A. Relationship Between Supersymmetry and Solvable Potentials. *Phys. Rev. D* **1987**, *36*, 2458–2473. [CrossRef] [PubMed]
36. Dutt, R.; Khare, A.; Sukhatme, U.P. Supersymmetry, shape invariance, and exactly solvable potentials. *Am. J. Phys.* **1988**, *56*, 163–168. [CrossRef]
37. Cooper, F.; Khare, A.; Sukhatme, U. Supersymmetry and quantum mechanics. *Phys. Rept.* **1995**, *251*, 267–385. [CrossRef]
38. Nieto, M.M. Relationship Between Supersymmetry and the Inverse Method in Quantum Mechanics. *Phys. Lett. B* **1984**, *145*, 208–210. [CrossRef]
39. Pursey, D.L. Isometric operators, isospectral Hamiltonians, and supersymmetric quantum mechanics. *Phys. Rev. D* **1986**, *33*, 2267–2279. [CrossRef] [PubMed]
40. Gel'fand, I.M.; Levitan, B.M. On the determination of a differential equation from its spectral function. *Izv. Akad. Nauk SSSR Ser. Mat.* **1951**, *15*, 309–360.
41. Abraham, P.B.; Moses, H.E. Changes in potentials due to changes in the point spectrum: Anharmonic oscillators with exact solutions. *Phys. Rev. A* **1980**, *22*, 1333–1340. [CrossRef]
42. Kac, V.G. A sketch of Lie superalgebra theory. *Comm. Math. Phys.* **1977**, *53*, 31–64. [CrossRef]
43. Gendenshtein, L.E. Derivation of Exact Spectra of the Schrodinger Equation by Means of Supersymmetry. *JETP Lett.* **1983**, *38*, 356–359.
44. Schrödinger, E. A Method of Determining Quantum-Mechanical Eigenvalues and Eigenfunctions. *Proc. R. Ir. Acad. A Math. Phys. Sci.* **1940**, *46*, 9–16.
45. Infeld, L.; Hull, T.E. The Factorization Method. *Rev. Mod. Phys.* **1951**, *23*, 21–68. [CrossRef]
46. Khare, A. Supersymmetry in quantum mechanics. *Pramana-J. Phys.* **1997**, *49*, 41–64. [CrossRef]
47. Schrödinger, E. Further Studies on Solving Eigenvalue Problems by Factorization. *Proc. R. Ir. Acad., A Math. phys. sci.* **1940**, *46*, 183–206.
48. Darboux, G. On a proposition relative to linear equations. *C. R. Acad. Sci. Paris* **1882**, *94*, 1456–1459.
49. Luban, M.; Pursey, D.L. New Schrödinger equations for old: Inequivalence of the Darboux and Abraham-Moses constructions. *Phys. Rev. D* **1986**, *33*, 431–436. [CrossRef] [PubMed]
50. Arancibia, A.; Plyushchay, M.S.; Nieto, L.-M. Exotic supersymmetry of the kink-antikink crystal, and the infinite period limit. *Phys. Rev. D* **2016**, *83*, 065025. [CrossRef]
51. Sonnenschein, J.; Tsulaia, M. A Note on Shape Invariant Potentials for Discretized Hamiltonians *arXiv* **2022**, arXiv:2205.10100.
52. Cariñena, J.F.; Plyushchay, M.S. Ground-state isolation and discrete flows in a rationally extended quantum harmonic oscillator. *Phys. Rev. D* **2016**, *94*, 105022. [CrossRef]
53. Cariñena, J.F.; Inzunza, L.; Plyushchay, M.S. Rational deformations of conformal mechanics. *Phys. Rev. D* **2018**, *98*, 026017. [CrossRef]
54. Arancibia, A.; Plyushchay, M.S. Chiral asymmetry in propagation of soliton defects in crystalline backgrounds. *Phys. Rev. D* **2015**, *92*, 105009. [CrossRef]
55. Fukui, T.; Aizawa, N. Shape-invariant potentials and an associated coherent state. *Phys. Lett. A* **1993**, *180*, 308–313. [CrossRef]
56. Jalalzadeh, S.; Moniz, P.V. *Challenging Routes in Quantum Cosmology*; World Scientific: Singapore, 2022. [CrossRef]
57. Rasouli, S.; Saba, N.; Farhoudi, M.; Marto, J.; Moniz, P. Inflationary universe in deformed phase space scenario. *Ann. Phys.* **2018**, *393*, 288–307. [CrossRef]

58. Rasouli, S.; Pacheco, R.; Sakellariadou, M.; Moniz, P. Late time cosmic acceleration in modified Sáez–Ballester theory. *Phys. Dark Univ.* **2020**, *27*, 100446. [CrossRef]
59. Rasouli, S.M.M. Noncommutativity, Saez-Ballester Theory and Kinetic Inflation. *Universe* **2022**, *8*, 165. [CrossRef]
60. Rasouli, S.; Farhoudi, M.; Khosravi, N. Horizon problem remediation via deformed phase space. *Gen. Rel. Grav.* **2011**, *43*, 2895–2910. [CrossRef]
61. Rasouli, S.M.M.; Moniz, P.V. Noncommutative minisuperspace, gravity-driven acceleration, and kinetic inflation. *Phys. Rev. D* **2014**, *90*, 083533. [CrossRef]
62. Rasouli, S.; Ziaie, A.; Jalalzadeh, S.; Moniz, P. Non-singular Brans–Dicke collapse in deformed phase space. *Ann. Phys.* **2016**, *375*, 154–178. [CrossRef]
63. Rasouli, S.M.M.; Vargas Moniz, P. Gravity-Driven Acceleration and Kinetic Inflation in Noncommutative Brans-Dicke Setting. *Odessa Astron. Pub.* **2016**, *29*, 19. [CrossRef]
64. Rasouli, S.; Marto, J.; Moniz, P. Kinetic inflation in deformed phase space Brans–Dicke cosmology. *Phys. Dark Univ.* **2019**, *24*, 100269. [CrossRef]
65. Dereli, T.; Tucker, R.W. Signature dynamics in general relativity. *Class. Quant. Grav.* **1993**, *10*, 365–374. [CrossRef]
66. Dereli, T.; Onder, M.; Tucker, R.W. Signature transitions in quantum cosmology. *Class. Quant. Grav.* **1993**, *10*, 1425–1434. [CrossRef]
67. Jalalzadeh, S.; Ahmadi, F.; Sepangi, H.R. Multidimensional classical and quantum cosmology: Exact solutions, signature transition and stabilization. *J. High Energy Phys.* **2003**, *2003*, 012. [CrossRef]
68. Khosravi, N.; Jalalzadeh, S.; Sepangi, H.R. Quantum noncommutative multidimensional cosmology. *Gen. Rel. Grav.* **2007**, *39*, 899–911. [CrossRef]
69. Pedram, P.; Jalalzadeh, S. Quantum cosmology with varying speed of light: Canonical approach. *Phys. Lett. B* **2008**, *660*, 1–6. [CrossRef]
70. Bina, A.; Atazadeh, K.; Jalalzadeh, S. Noncommutativity, generalized uncertainty principle and FRW cosmology. *Int. J. Theor. Phys.* **2008**, *47*, 1354–1362. [CrossRef]
71. Jalalzadeh, S.; Vakili, B. Quantization of the interior Schwarzschild black hole. *Int. J. Theor. Phys.* **2012**, *51*, 263–275. [CrossRef]
72. Khosravi, N.; Jalalzadeh, S.; Sepangi, H.R. Non-commutative multi-dimensional cosmology. *J. High Energy Phys.* **2006**, *2006*, 134. [CrossRef]
73. Vakili, B.; Pedram, P.; Jalalzadeh, S. Late time acceleration in a deformed phase space model of dilaton cosmology. *Phys. Lett. B* **2010**, *687*, 119–123. [CrossRef]
74. Darabi, F. Large scale - small scale duality and cosmological constant. *Phys. Lett. A* **1999**, *259*, 97–103. [CrossRef]
75. Darabi, F.; Rastkar, A. A quantum cosmology and discontinuous signature changing classical solutions. *Gen. Rel. Grav.* **2006**, *38*, 1355–1366. [CrossRef]
76. Jalalzadeh, S.; Rashki, M.; Abarghouei Nejad, S. Classical universe arising from quantum cosmology. *Phys. Dark Univ.* **2020**, *30*, 100741. [CrossRef]
77. Kastrup, H.A. A new look at the quantum mechanics of the harmonic oscillator. *Ann. Phys.* **2007**, *519*, 439–528. [CrossRef]
78. Kumar, R.; Malik, R.P. Supersymmetric Oscillator: Novel Symmetries. *EPL* **2012**, *98*, 11002. [CrossRef]
79. D'Eath, P.D. *Supersymmetric Quantum Cosmology*; Cambridge Monographs on Mathematical Physics; Cambridge University Press: Cambridge, UK, 1996. [CrossRef]
80. Moniz, P.V. Supersymmetric quantum cosmology: Shaken, not stirred. *Int. J. Mod. Phys. A* **1996**, *11*, 4321–4382. [CrossRef]
81. Moniz, P.V. Quantum Cosmology: Meeting SUSY. In *Progress in Mathematical Relativity, Gravitation and Cosmology*; García-Parrado, A., Mena, F.C., Moura, F., Vaz, E., Eds.; Springer: Berlin/Heidelberg, Germany, 2014; pp. 117–125. [CrossRef]
82. García-Compeán, H.; Obregón, O.; Ramírez, C. Topics in Supersymmetric and Noncommutative Quantum Cosmology. *Universe* **2021**, *7*, 434. [CrossRef]
83. Moniz, P.V. Supersymmetric quantum cosmology: A 'Socratic' guide. *Gen. Rel. Grav.* **2014**, *46*, 1618. [CrossRef]
84. López, J.; Obregón, O. Supersymmetric quantum matrix cosmology. *Class. quant. grav.* **2015**, *32*, 235014. [CrossRef]
85. Obregon, O.; Ramirez, C. Dirac-like formulation of quantum supersymmetric cosmology. *Phys. Rev. D* **1998**, *57*, 1015. [CrossRef]
86. Bene, J.; Graham, R. Supersymmetric homogeneous quantum cosmologies coupled to a scalar field. *Phys. Rev. D* **1994**, *49*, 799. [CrossRef] [PubMed]
87. Csordas, A.; Graham, R. Supersymmetric minisuperspace with nonvanishing fermion number. *Phys. Rev. Lett.* **1995**, *74*, 4129. [CrossRef]
88. Kleinschmidt, A.; Koehn, M.; Nicolai, H. Supersymmetric quantum cosmological billiards. *Phys. Rev. D* **2009**, *80*, 061701. [CrossRef]
89. Macías, A.; Mielke, E.W.; Socorro, J. Supersymmetric quantum cosmology for Bianchi class A models. *Int. J. Mod. Phys. D* **1998**, *7*, 701–712. [CrossRef]
90. Damour, T.; Spindel, P. Quantum supersymmetric cosmology and its hidden Kac–Moody structure. *Class. Quant. Grav.* **2013**, *30*, 162001. [CrossRef]
91. Kiefer, C. *Quantum Gravity*; International Series of Monographs on Physics; Clarendon: Oxford, UK, 2004; Volume 124. [CrossRef]
92. Esposito, G. *Quantum Gravity, Quantum Cosmology and Lorentzian Geometries*; Lecture Notes in Physics Monographs; Springer: Berlin/Heidelberg, Germany, 2009; Volume 12. [CrossRef]

93. Calcagni, G. *Classical and Quantum Cosmology*; Springer: Cham, Switzerland, 2017. [CrossRef]
94. Bojowald, M. *Quantum Cosmology: A Fundamental Description of the Universe*; Lecture Notes in Physics; Springer: New York, NY, USA, 2011; Volume 835. [CrossRef]
95. Lidsey, J.E. Quantum cosmology of generalized two-dimensional dilaton-gravity models. *Phys. Rev. D* **1995**, *51*, 6829. [CrossRef] [PubMed]
96. Lidsey, J.E.; Moniz, P.V. Supersymmetric quantization of anisotropic scalar-tensor cosmologies. *Class. Quant. Grav.* **2000**, *17*, 4823. [CrossRef]
97. Graham, R. Supersymmetric Bianchi type IX cosmology. *Phys. Rev. Lett.* **1991**, *67*, 1381. [CrossRef]
98. Lidsey, J.E. Scale factor duality and hidden supersymmetry in scalar-tensor cosmology. *Phys. Rev. D* **1995**, *52*, R5407. [CrossRef] [PubMed]
99. Tkach, V.; Rosales, J.; Obregón, O. Supersymmetric action for Bianchi type models. *Class. Quant. Grav.* **1996**, *13*, 2349. [CrossRef]
100. Obregón, O.; Rosales, J.; Tkach, V. Superfield description of the FRW universe. *Phys. Rev. D* **1996**, *53*, R1750. [CrossRef]
101. Witten, E. Dynamical breaking of supersymmetry. *Nucl. Phys. B* **1981**, *188*, 513–554. [CrossRef]