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Noncommutative Reduction of Nonlinear Schrödinger Equation on Lie Groups

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Abstract: We propose a new approach that allows one to reduce nonlinear equations on Lie groups to equations with a fewer number of independent variables for finding particular solutions of the nonlinear equations. The main idea is to apply the method of noncommutative integration to the linear part of a nonlinear equation, which allows one to find bases in the space of solutions of linear partial differential equations with a set of noncommuting symmetry operators. The approach is implemented for the generalized nonlinear Schrödinger equation on a Lie group in curved space with local cubic nonlinearity. General formalism is illustrated by the example of the noncommutative reduction of the nonstationary nonlinear Schrödinger equation on the motion group $E(2)$ of the two-dimensional plane $\mathbb{R}^2$. In this particular case, we come to the usual $(1+1)$-dimensional nonlinear Schrödinger equation with the soliton solution. Another example provides the noncommutative reduction of the stationary multidimensional nonlinear Schrödinger equation on the four-dimensional exponential solvable group.

Keywords: nonlinear Schrödinger equation; noncommutative integration; Lie groups; induced representations; orbit method

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1. Introduction

The Lie group theory provides powerful methods for studying linear and nonlinear differential equations in mathematical physics. Generally, for the equation with a symmetry group, one can efficiently find and classify group invariant solutions and conservation laws, and generate new solutions from those already found (see, for example, the well-known books of Ovsyannikov, Ibragimov, Olver [1–3], and many others).

Remarkable potentialities for finding explicit solutions are opened up when an equation can be represented directly in terms of the coordinates of a Lie group. For example, equations on a curved space with a simply transitive motion group can be represented as equations on a Lie group manifold. We call such an equation the equation on the Lie group. Some aspects of the integrability of nonlinear equations on Lie groups are the subject of the present work.

Here, we propose a new approach based on the Lie group theory that allows one to reduce a nonlinear equation presented in terms of a Lie group to an equation with a fewer number of independent variables using the noncommutative ansatz of the work [4], which is determined by the linear part of the nonlinear equation. The noncommutative integration method (NIM) has been proposed for linear partial differential equations (PDEs) in Ref. [4]. Following this method, one can find a basis for the solution space of the linear equation admitting a set of noncommuting symmetry operators related to the Lie group of the invariance of the equation. Then, the noncommutative reduction of a nonlinear equation on a Lie group yields families of particular solutions containing the parameters ("quantum numbers") of the basis of solutions to the corresponding linear equation. We describe the proposed noncommutative reduction for the nonlinear Schrödinger equation (NLSE) in curved space with local cubic nonlinearity and the simply transitive motion group written in terms of the Lie group. The general formalism is illustrated by the examples of the noncommutative reduction of the multidimensional NLSE on the Lie group $E(2)$ of the two-dimensional plane $\mathbb{R}^2$, and on the four-dimensional exponential solvable group. A family of particular solutions of the NLSE on the Lie group obtained within the framework of our approch contains the parameters of solutions of the corresponding linear Schrödinger equation.

The nonlinear Schrödinger equation is one of the fundamental equations in nonlinear theoretical physics and mathematics. It arises in a number of nonlinear models of various physical phenomena and in a wide range of applications. As an example, we recall the theory of optical pulse propagation in nonlinear media [5,6]. In the theory of Bose–Einstein condensates, the NLSE is referred to as the Gross–Pitaevskii equation (GPE) [7–9]. The $(1+1)$ dimensional NLSE is integrable within the framework of the soliton theory (see, e.g., Ref. [10] and references therein).

The approach proposed here expands the possibilities of constructing exact solutions of field equations in curved spaces in addition to the method of separation of variables, which is widely used in general relativity (see, e.g., recent papers [11,12] and references therein) and cosmology [13–16].

We also emphasize that here we consider the noncommutative reduction of nonlinear equations with *local nonlinearity* in contrast to the papers [4,17,18] where NIM was applied to equations with a nonlocal nonlinearity of convolution type.

The paper is structured as follows. In Section 2, we present the required concepts and definitions from the theory of Lie groups, and introduce notations and the problem setup. In Section 3, we describe a special representation of the Lie algebra, which is constructed using the orbit method. Then, we apply an ansatz for the noncommutative reduction of the nonlinear Schrödinger equation on the Lie group. Section 4 illustrates a general approach using the example of the noncommutative reduction of the nonstationary nonlinear Schrödinger Equation (8) on the motion group $E(2)$ of the two-dimensional plane $\mathbb{R}^2$. In this particular case, we come to the usual $(1+1)$-dimensional NLSE with the soliton solution. In Section 5, the noncommutative reduction of the stationary multidimensional NLSE is studied in the case of the four-dimensional exponential solvable group. In Section 6, our concluding remarks are given.

2. Notations and the Problem Setup

In this section, we briefly review the required concepts and definitions from Lie group theory and introduce the technical notations.

Let G be an n-dimensional Lie group, its Lie algebra $\mathfrak{g}$ be the tangent space at the group unity $e \in G$, and $\{e_a\}$ be a fixed basis in the linear space $\mathfrak{g}$ $(a, b, c = 1, \dots, n)$. The Lie group G acts on itself as the left, $L_{\tilde{g}}(g) = \varphi(\tilde{g}, g)-$, and the right, $R_{\tilde{g}}(g) = \varphi(g, \tilde{g})-$, is a translation, where $\varphi(g, \tilde{g})$ is a composition function, and $g, \tilde{g} \in G$. The differentials of the left and right translations determine the left-invariant, $\xi_X(g) = (L_g)_* X$, the right-invariant, $\eta_X(g) = -(R_g)_* X$, and vector fields on the Lie group G $(X \in \mathfrak{g})$. Additionally, we have

$$[\xi_X, \xi_Y] = \xi_{[X,Y]}, \quad [\eta_X, \eta_Y] = \eta_{[X,Y]}, \quad [\xi_X, \eta_Y] = 0, \quad X, Y \in \mathfrak{g}, \tag{1}$$

where $[X, Y]$ is the commutator of $X, Y \in \mathfrak{g}$.

Let $\{e^b\}$ be the dual basis to $\{e_a\}$ in the Lie algebra $\mathfrak{g}$, $\langle e^b, e_a \rangle = \delta^b_a$, and the brackets $\langle \cdot, \cdot \rangle$ denote the natural pairing of a 1-form and a vector. Then, the left-invariant, $\omega^X(g) = (L_g)^* X$, the right-invariant, $\sigma^X(g) = -(R_g)^* X$, and the Maurer–Cartan 1-forms satisfy the equations

$$d\omega^a = -\frac{1}{2} C^a_{bc} \omega^b \wedge \omega^c, \quad d\sigma^a = -\frac{1}{2} C^a_{bc} \sigma^b \wedge \sigma^c, \quad C^a_{bc} = [e_b, e_c]^a. \tag{2}$$

The implicit summation over repeated indices is assumed.

We take as the basis the right-invariant vector fields $\eta_a(g) = \eta_{e_a}(g)$, and the dual right-invariant 1-forms $\sigma^a(g) = \sigma^{e^a}(g)$ as the moving frame on G, and introduce the right-invariant metric

$$ds^2 = g_{\mu\nu}(g) dg^\mu dg^\nu, \quad \mu, \nu = 1, \ldots, n, \tag{3}$$

where g^μ are local coordinates on G. The metric tensor $g_{\mu\nu}(g)$ of the right-invariant metric (3) is expanded over a moving frame with a constant symmetric matrix G_{ab}:

$$g_{\mu\nu}(g) = G_{ab} \sigma^a_\mu(g) \sigma^b_\nu(g), \quad g^{\mu\nu}(g) = G^{ab} \eta^\mu_a(g) \eta^\nu_b(g), \quad G^{ac} G_{cb} = \delta^a_b. \tag{4}$$

The Christoffel symbols of the symmetric connection consistent with the metric ds^2 on the Lie group G are defined in terms of the metric tensor (4) as

$$\Gamma^\rho_{\nu\mu}(g) = \frac{1}{2} g^{\rho\tau}(g) \big(\partial_\nu g_{\tau\mu}(g) + \partial_\mu g_{\mu\tau}(g) - \partial_\tau g_{\nu\mu}(g) \big), \quad \partial_\nu \equiv \frac{\partial}{\partial g^\nu}. \tag{5}$$

Substituting (4) in (5) and taking into account the Maurer–Cartan Equation (2), we get (see Ref. [19]):

$$\Gamma^\rho_{\nu\mu}(g) = \Gamma^a_{bd} \sigma^b_\nu(g) \sigma^d_\mu(g) \eta^\rho_a(g) + \eta^\rho_a(g) \frac{\partial \sigma^a_\nu(g)}{\partial g^\mu},$$
$$\Gamma^a_{bd}(g) = -\frac{1}{2} C^a_{bd} - \frac{1}{2} G^{ac} (G_{eb} C^e_{dc} + G_{ed} C^e_{bc}).$$

To simplify the presentation, we consider unimodular Lie groups when the left Haar measure $d\mu_L(g)$ coincides with the right Haar measure on the Lie group G: $d\mu_R(g) = d\mu_L(g) = d\mu(g)$.

Now, we can consider differential equations on Lie groups. The Schrödinger equation on a unimodular Lie group G with the metric (3) for the wave function $\psi = \psi(t, g)$ has the form

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta_G \psi, \tag{6}$$

where $\hbar$ is the Planck constant, $m\,(> 0)$ is the mass of the particle, and t is the time. The Laplace operator Δ_G on the Lie group G is a quadratic polynomial in the right-invariant vector fields:

$$-\hbar^2 \Delta_G = H(-i\hbar\eta), \quad H(f) = G^{ab} f_a f_b. \tag{7}$$

The operator Δ is a symmetric operator with respect to the Riemannian measure

$$d\mu(g) = \sqrt{\det g_{\mu\nu}} dg = \sqrt{\mathbf{G}} d\mu(g), \quad \mathbf{G} = \det(G_{ab}).$$

A linear differential operator $X(g) = X(g, \partial_g)$ commuting with the operator $H(\eta)$ on some space of functions,

$$[X(g), H(\eta)] = 0,$$

leaves invariant the set of solutions to the equation and it is the *symmetry operator* of Equation (6). From Equation (1), one can easily see that the linear Equation (6) admits a set of left-invariant vector fields ξ_a as symmetry operators. It can be shown that the Laplace operator on an n-dimensional manifold admitting a set of n linearly independent symmetry operators of the first order can always be represented locally in the form (7) up to a constant factor for some Lie groups G with a right-invariant metric [20].

In this paper, we consider the following nonlinear Schrödinger Equation (6) on the Lie group G:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta_G\psi + U(g,\psi)\psi. \tag{8}$$

Note that the nonlinearity $U(g,\psi)$ does not admit ξ_a as symmetry operators of Equation (8). When $G=\mathbb{R}^3$, we have $U(g,\psi)=|\psi|^2$ and (8) is the well-known nonlinear Schrödinger equation (see, e.g., refs. [7–10], and references therein).

We will show that the NIM is effective for solving Equation (8) under some restrictions on the Lie group G.

3. Noncommutative Reduction of the Nonlinear Schrödinger Equation

The approach for the noncommutative reduction of Equation (8) is based on a special representation of the Lie algebra $\mathfrak{g}$ constructed in terms of the orbit method. We also need a suitable direct and inverse Fourier transform on the Lie group G.

First, we recall some necessary definitions from the orbit method that will be used hereinafter.

The degenerate Poisson–Lie bracket,

$$\{\phi,\psi\}(f) = \langle f,[d\phi(f),d\psi(f)]\rangle = C^c_{ab}f_c\frac{\partial\phi(f)}{\partial f_a}\frac{\partial\psi(f)}{\partial f_b}, \quad \phi,\psi\in C^\infty(\mathfrak{g}^*), \tag{9}$$

endows the space $\mathfrak{g}^*$ with a Poisson structure [21]. Here, f_a are the coordinates of a linear functional $f=f_ae^a\in\mathfrak{g}^*$ relative to the dual basis $\{e^a\}$. The number $\mathrm{ind}\mathfrak{g}$ of functionally independent Casimir functions $K_\mu(f)$ with respect to the bracket (9) is called the index of the Lie algebra $\mathfrak{g}$, $\mu=1,\dots,\mathrm{ind}\mathfrak{g}$.

A coadjoint representation $\mathrm{Ad}^*\colon G\times\mathfrak{g}^*\to\mathfrak{g}^*$ splits $\mathfrak{g}^*$ into coadjoint orbits (K-orbits). The restriction of the bracket (9) to an orbit is nondegenerate and coincides with the Poisson bracket generated by the Kirillov symplectic form ω_λ [21]. Orbits of maximum dimension $\mathrm{dim}\mathcal{O}^{(0)}=\mathrm{dim}\mathfrak{g}-\mathrm{ind}\mathfrak{g}$ are called *nondegenerate* [21,22].

Let $\mathcal{O}_\lambda$ be a nondegenerate K-orbit passing through the covector $\lambda\in\mathfrak{g}^*$. Using Kirillov's orbit method [22], we construct a unitary irreducible representation of the Lie group G with respect to a given orbit. This representation can be constructed if for the functional λ there exists a subalgebra $\mathfrak{h}\subset\mathfrak{g}^{\mathbb{C}}$ in the complex extension $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ satisfying the conditions:

$$\langle\lambda,[\mathfrak{h},\mathfrak{h}]\rangle = 0, \quad \mathrm{dim}\mathfrak{h} = \mathrm{dim}\mathfrak{g}-\frac{1}{2}\mathrm{dim}\mathcal{O}_\lambda. \tag{10}$$

The subalgebra $\mathfrak{h}$ is called the *polarization* of the functional λ. Equation (10) assumes that the functionals from $\mathfrak{g}^*$ can be prolonged to $\mathfrak{g}^{\mathbb{C}}$ by linearity. In this paper, to simplify the presentation, we restrict ourselves to the case when $\mathfrak{h}$ is the real polarization.

Next, we introduce a special coordinate system on the Lie group G compatible with nondegenerate K-orbits of G. Let H be a closed subgroup in a Lie group G, and $\mathfrak{h}$ be the Lie algebra of H. The Lie group acts on the right homogeneous space $Q\simeq G/H: q'=qg$ and defines a principal bundle with the base Q, fibers H, and the canonical projection $\pi: G\to Q$. Choose a basis $\{e_{\overline{\alpha}}\}$ in the subalgebra $\mathfrak{h}$ and a basis $\{e'_{\overline{a}}\}$ in the complementary subspace $\mathfrak{m}=\mathfrak{h}^\perp$. In some trivializing neighborhood V_0 of the unit of the Lie group G, we introduce the local coordinates of the second kind

$$g(q,h) = \left(e^{h^{\dim\mathfrak{h}}e_{\overline{\dim\mathfrak{h}}}}e^{h^{\dim\mathfrak{h}-1}e_{\overline{\dim\mathfrak{h}-1}}}\dots e^{h^1e_{\overline{1}}}\right)\left(e^{q^{\dim Q}e'_{\overline{\dim Q}}}e^{q^{\dim Q-1}e'_{\overline{\dim Q-1}}}\dots e^{q^1e'_{\overline{1}}}\right).$$

We fix a section $s: Q\to G$ of the principal bundle of G by the equality

$$g(q,h) \quad = \quad hs(q).$$

The left-invariant vector fields on G in local coordinates (q,h) have the form

$$\xi_X(q,h) = \xi_X^{\bar{a}}(q)\partial_{q^{\bar{a}}} + \xi_X^{\bar{\alpha}}(q,h)\partial_{h^{\bar{\alpha}}},$$

where $\alpha_X(q) = \xi_X^{\bar{a}}(q)\partial_{q^{\bar{a}}}$ are the generators of the group action on the homogeneous space Q.

According to the orbit method [21], we introduce a unitary one-dimensional irreducible representation of the Lie group G, which in a neighborhood of V_0 is given by

$$U^\lambda(e^X) = \exp\left(\frac{i}{\hbar}\langle\lambda, X\rangle\right), \quad X \in \mathfrak{h}. \tag{11}$$

The representation of the Lie group G corresponding to the orbit $\mathcal{O}_\lambda$ is induced by the one-dimensional representation

$$(T_g^\lambda\psi)(q) = \Delta_H^{-1/2}(h(q,g))U^\lambda(h(q,g))\psi(qg) = U^{\lambda+i\hbar\beta}(h(q,g))\psi(qg), \tag{12}$$
$$\beta_{\bar{\alpha}} = -\frac{1}{2}\mathrm{Tr}\left(\mathrm{ad}_{\bar{\mathfrak{f}}}\big|_{\mathfrak{h}}\right),$$

where $\Delta_H(g) = \det\mathrm{Ad}_h$ is the module of the subgroup H, $h \in H; e_H$ is the unit element in the Lie group H. The function $h(q,g)$ in (12) is a factor of the homogeneous space Q:

$$s(q)g = h(q,g)s(qg), \quad h(q,e) = 1.$$

Let $L_2(Q,\mathfrak{h},\lambda)$ denote the space of functions defined on Q where the representation (12) acts. Restriction of the left-invariant vector fields $\xi_X(g)$ on the homogeneous space Q, which reads

$$\ell_X(q,\partial_q,\lambda) = \left(\left[U^{\lambda+i\hbar\beta}(h)\right]^{-1}\xi_X(g)U^{\lambda+i\hbar\beta}(h)\right)\bigg|_{h=e_H}, \tag{13}$$
$$[\ell_X(q,\partial_q,\lambda), \ell_Y(q,\partial_q,\lambda)] = \ell_{[X,Y]}(q,\partial_q,\lambda), \quad X, Y \in \mathfrak{g}.$$

The representation (12) is unitary with respect to the scalar product in the space of functions $L_2(Q,\mathfrak{h},\lambda)$:

$$(\psi_1,\psi_2) = \int_Q \overline{\psi_1(q)}\psi_2(q)d\mu(q), \quad d\mu(q) = \rho(q)dq^1\dots dq^l. \tag{14}$$

The function $\rho(q)$ is determined from the condition that the operators $-i\ell_X(q,\partial_q,\lambda)$ are Hermitian with respect to the given scalar product (14).

The irreducible representation of the Lie algebra $\mathfrak{g}$ by linear operators of the first order (13) depending on $\dim Q = \dim\mathcal{O}_\lambda/2 = (\dim\mathfrak{g} - \mathrm{ind}\mathfrak{g})/2$ variables is called the *λ-representation* of the Lie algebra $\mathfrak{g}$. It was introduced in Ref. [4].

The explicit form of the λ-representation operators is determined by left-invariant vector fields in the trivialization domain V_0 of the principal bundle G:

$$\ell_X(q,\partial_q,\lambda) = \xi_X^{\bar{a}}(q)\partial_{q^{\bar{a}}} + \frac{i}{\hbar}\xi_X^{\bar{\alpha}}(q,e_H)(\lambda_{\bar{\alpha}} + i\hbar\beta_{\bar{\alpha}}).$$

Let us introduce the direct and inverse generalized Fourier transform, which is the essential point of the noncommutative integration method. The representation operators (12) can be rewritten in the integral form as

$$(T_g^\lambda\psi)(q) = \textstyle\int_Q \psi(q')\mathscr{D}_{qq'}^\lambda(g)d\mu(q),$$
$$\mathscr{D}_{qq'}^\lambda(g) = \Delta_H^{-1/2}(h(q,g))U^\lambda(h(q,g))\delta(qg,q'),$$

where $\delta(q, q')$ is a generalized delta function with respect to the measure $d\mu(q)$. The generalized kernels $\mathscr{D}^{\lambda}_{qq'}(g)$ of this representation have the properties

$$\mathscr{D}^{\lambda}_{qq'}(g_1 g_2) = \int_Q \mathscr{D}^{\lambda}_{qq''}(g_1)\mathscr{D}^{\lambda}_{q''q'}(g_2)d\mu(q''), \quad g_1, g_2 \in G,$$
$$\mathscr{D}^{\lambda}_{qq'}(g) = \overline{\mathscr{D}^{\lambda}_{q'q}(g^{-1})}, \quad \mathscr{D}^{\lambda}_{qq'}(e) = \delta(q, q'),$$

and satisfy the system of equations

$$(\eta_X(g) + \ell_X(q, \partial_q, \lambda))\mathscr{D}^{\lambda}_{qq'}(g) = 0, \quad \left(\xi_X(g) + \overline{\ell_X(q', \partial_{q'}, \lambda)}\right)\mathscr{D}^{\lambda}_{qq'}(g) = 0. \tag{15}$$

Note that the functions $\mathscr{D}^{\lambda}_{qq'}(g)$ are defined globally on the whole Lie group G if the K-orbit $\mathcal{O}_\lambda$ is integer in the sense of Kirillov's definition [22].

The set of generalized functions $\mathscr{D}^{\lambda}_{qq'}(g)$ satisfying the system of Equation (15) has the properties of completeness and orthogonality for a certain choice of the measure $d\mu(\lambda)$ in parameter space J:

$$\int_G \overline{\mathscr{D}^{\tilde{\lambda}}_{\tilde{q}\tilde{q}'}(g)}\mathscr{D}^{\lambda}_{qq'}(g)d\mu(g) = \delta(q, \tilde{q})\delta(\tilde{q}', q')\delta(\tilde{\lambda}, \lambda), \tag{16}$$
$$\int_{Q\times Q\times J} \overline{\mathscr{D}^{\lambda}_{qq'}(\tilde{g})}\mathscr{D}^{\lambda}_{qq'}(g)d\mu(q)d\mu(\lambda) = \delta(\tilde{g}, g), \tag{17}$$

where $\delta(g)$ is the generalized Dirac delta function with respect to the right Haar measure $d\mu(g)$ on the Lie group G.

Consider the function space $L(G, d\mu(g))$ of functions of the form

$$\psi(g) = \int_Q \psi(q, q', \lambda)\mathscr{D}^{\lambda}_{qq'}\left(g^{-1}\right)d\mu(q')d\mu(q)d\mu(\lambda), \tag{18}$$

where the function $\psi(q, q', \lambda)$ with respect to the variables q and q' belongs to the space $L_2(Q, \mathfrak{h}, \lambda)$. From (16) and (17), we can write the inverse transform as

$$\psi(q, q', \lambda) = \int_G \psi^{\lambda}(g)\overline{\mathscr{D}^{\lambda}_{qq'}(g^{-1})}d\mu(g). \tag{19}$$

It follows from (18) and (19) that the action of the operators $\xi_X(g)$ and $\eta_X(g)$ on the function $\psi^{\lambda}(g)$ from $L_2(G, \lambda, d\mu(g))$ corresponds to the action of the operators $\overline{\ell^{\dagger}_X(q, \partial_q, \lambda)}$ and $\ell_X(q', \partial_{q'}, \lambda)$ on the function $\psi(q, q', \lambda)$:

$$\xi_X(g)\psi^{\lambda}(g) \Longleftrightarrow \overline{\ell^{\dagger}_X(q, \partial_q, \lambda)}\psi(q, q', \lambda),$$
$$\eta_X(g)\psi^{\lambda}(g) \Longleftrightarrow \ell_X(q', \partial_{q'}, \lambda)\psi(q, q', \lambda). \tag{20}$$

The functions (18) are eigenfunctions for the Casimir operators $K^{(s)}_{\mu}(i\hbar\xi) = K^{(s)}_{\mu}(-i\hbar\eta)$:

$$K^{(s)}_{\mu}(i\hbar\xi)\psi^{\lambda}(g) \Longleftrightarrow \kappa^{(s)}_{\mu}(\lambda)\psi(q, q', \lambda),$$
$$K^{(s)}_{\mu}(-i\hbar\ell(q', \partial_{q'}, \lambda)) = \kappa^{(s)}_{\mu}(\lambda), \quad \overline{\kappa^{(s)}_{\mu}(\lambda)} = \kappa^{(s)}_{\mu}(\lambda), \quad \lim_{\hbar\to 0}\kappa^{(s)}_{\mu}(\lambda) = \omega^{(s)}_{\mu}(\lambda).$$

As a result of the generalized Fourier transform (18), the left and right fields are converted to λ-representations, and the Casimir operators become constants.

This fact is core to the method of noncommutative integration of linear differential equations on Lie groups. The method allows one to reduce the original linear differential equation

$$-\hbar^2\Delta_G\psi(g; q, \lambda) = \Lambda^2\psi(g; q, \lambda), \quad \Lambda = \text{const} \tag{21}$$

with the number of independent variables g equal to $\dim\mathfrak{g}$ to the equation

$$H(-i\hbar\ell(q',\partial_{q'},\lambda))\psi(q';q,\lambda) = \Lambda^2\psi(q';q,\lambda)$$

with a fewer number of independent variables q' that is equal to $(\dim\mathfrak{g} - \mathrm{ind}\mathfrak{g})/2$ using the ansatz

$$\psi^\lambda(g;q,\lambda) = U^{\lambda+i\hbar\beta}(h(q,g^{-1}))\psi(qg^{-1};q,\lambda) \tag{22}$$

parameterized by q and λ. In view of (17), the set of functions (22) parameterized by q, λ and Λ forms a complete set of solutions to the Equation (21).

Then, we apply the ansatz of the form (22) to the noncommutative reduction of the nonlinear Schrödinger Equation (8). Let us look for a solution of (8) in the form

$$\psi^\lambda(t,g;q) = U^{\lambda+i\hbar\beta}(h(q,g^{-1}))\psi(t,qg^{-1};q,\lambda).$$

In view of the relations (20), the linear part of the Equation (8) can be written as

$$\begin{aligned}&\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\Delta_G\right)\psi^\lambda(t,g;q) = \\ &U^{\lambda+i\hbar\beta}(h(q,g^{-1}))\times \\ &\times\frac{1}{2m}\left[i\hbar\frac{\partial}{\partial t} + H(-i\hbar\ell(q',\partial_{q'},\lambda))\right]\psi(t,q';q,\lambda)\bigg|_{q'=qg^{-1}},\end{aligned}$$

and $\left|\psi^\lambda(t,g;q)\right|^2$ reads

$$\begin{aligned}\left|\psi^\lambda(t,g;q)\right|^2 &= \left|U^{\lambda+i\hbar\beta}h(q,g^{-1})\right|^2\left|\psi(t,qg^{-1};q,\lambda)\right|^2 = \\ &= e^{-2h(q,g)\beta_{\overline{\alpha}}}\left|U^\lambda h(q,g^{-1})\right|^2\left|\psi(t,qg^{-1};q,\lambda)\right|^2.\end{aligned}$$

For the real polarization $\mathfrak{h}$, in view of the Formula (11), $|U^\lambda(h(q,g^{-1}))| = 1$. Then, we have

$$\left|\psi^\lambda(t,g;q)\right|^2 = e^{-2h(q,g)\beta_{\overline{\alpha}}}\left|\psi(t,qg^{-1};q,\lambda)\right|^2.$$

We only consider the Lie groups G for which

$$e^{-2h(q,g)\beta_{\overline{\alpha}}} = \kappa^2(q). \tag{23}$$

The condition (23) is satisfied if the covector β is zero. Thus, under the condition (23), we obtain the reduced nonlinear Schrödinger equation

$$\begin{aligned}&\left[i\hbar\frac{\partial}{\partial t} + \frac{1}{2m}H(-i\hbar\ell(q',\partial_{q'},\lambda))\right]\psi(t,q';q,\lambda) + \\ &+ U\left(\kappa^2(q)\left|\psi(t,q';q,\lambda)\right|^2\right)\psi(t,q';q,\lambda) = 0\end{aligned}$$

with the fewer number of independent variables q'.

4. The Three-Dimensional Group $E(2)$

Here, we consider an example of noncommutative reduction of the nonlinear Schrödinger Equation (8) on the motion group $E(2)$ of the two-dimensional plane $\mathbb{R}^2$. The three-dimensional Lie algebra $\mathfrak{e}(2)$ of $E(2)$ is determined by the commutation relations $[e_1,e_3] = -e_2$, $[e_2,e_3] = e_1$ relative to the fixed basis $\{e_1,e_2,e_3\}$.

The left-invariant and the right-invariant vector fields on a $E(2)$ have the form

$$\begin{aligned}
&\xi_1 = \partial_x, \quad \xi_2 = \partial_y, \quad \xi_3 = y\partial_x - x\partial_y + \partial_\alpha, \\
&\eta_1 = -\cos\alpha\partial_x + \sin\alpha\partial_y, \\
&\eta_2 = -\sin\alpha\partial_x - \cos\alpha\partial_y, \quad \eta_3 = -\partial_\alpha
\end{aligned}$$

with respect to the canonical coordinates (x, y, α) of the second kind:

$$g = (x, y, \alpha) = e^{\alpha e_3} e^{y e_2} e^{x e_1}, \quad (x, y) \in \mathbb{R}^2, \quad \alpha \in [0, 2\pi).$$

The invariant measure on the group coincides with the Lebesgue measure $d\mu(g) = dxdyd\alpha$. The composition law of the group is

$$\begin{aligned}
&g_1 g_2 = (x_2 + x_1\cos\alpha_2 + y_1\sin\alpha_2, y_2 - x_1\sin\alpha_2 + y_1\cos\alpha_2, \alpha_1 + \alpha_2), \\
&g_1 = (x_1, y_1, \alpha_1), \quad g_2 = (x_2, y_2, \alpha_2).
\end{aligned}$$

Each nondegenerate orbit is determined by the Casimir function $K(f) = f_1^2 + f_2^2$ on the dual space $\mathfrak{e}^*(2) \simeq \mathbb{R}^3$ and passes through the covector $\lambda(j) = (j, 0, 0)$, $j > 0$, i.e.,

$$\begin{aligned}
&\mathcal{O}_j = \{f \in \mathbb{R}^3 \mid K(f) = j^2, \neg(f_1 = f_2 = 0)\}, \\
&\dim \mathcal{O}_j = 2.
\end{aligned}$$

The λ-representation operators corresponding to the real polarization $\mathfrak{h} = \{e_1, e_2\}$ have the form

$$\ell_1 = i\frac{j}{\hbar}\cos q, \quad \ell_2 = -i\frac{j}{\hbar}\sin q, \quad \ell_3 = \partial_q, \quad q \in [0; 2\pi).$$

The operators $-i\hbar\ell_a$ are symmetric with respect to the measure $d\mu(q) = dq$, and all nondegenerate orbits are integer. Solving the system of Equation (15), we find the functions $\mathscr{D}^\lambda_{qq'}(g^{-1})$, and the completeness and orthogonality conditions for them yield the following measure $d\mu(\lambda)$:

$$\begin{aligned}
&\mathscr{D}^\lambda_{qq'}(g^{-1}) = \exp\left[\frac{ij_1}{\hbar}(y\sin q - x\cos q)\right]\delta(q' - q + \alpha), \\
&d\mu(\lambda) = \frac{1}{(2\pi)^2} jdj.
\end{aligned}$$

Let us introduce the right-invariant metric given by the matrix $(G^{ab}) = \mathrm{diag}(\delta_1, \delta_2, \delta_3)$. In local coordinates, this metric can be written as

$$\begin{aligned}
ds^2 = &\left(\delta_1^{-1}\cos^2\alpha + \delta_2^{-1}\sin^2\alpha\right)dx^2 + \\
&+ \left(\delta_1^{-1}\sin^2\alpha + \delta_2^{-1}\cos^2\alpha\right)dy^2 + \delta_3^{-1}d\alpha^2.
\end{aligned} \tag{24}$$

The metric (24) has the nonzero scalar curvature $R = \delta_3(\delta_1 - \delta_2)^2/(2\delta_1\delta_2)$, and the corresponding Laplace operator reads

$$\begin{aligned}
\Delta_{E(2)} = &\left(\delta_1\cos^2\alpha + \delta_2\sin^2\alpha\right)\partial^2_{xx} + \\
&\left(\delta_1\sin^2\alpha + \delta_2\cos^2\alpha\right)\partial^2_{yy} + (\delta_2 - \delta_1)\sin 2\alpha\partial^2_{xy} + \delta_3\partial^2_{\alpha\alpha}.
\end{aligned}$$

For the nonlinear Schrödinger equation with the Laplace operator $\Delta_{E(2)}$ and potential $V = V(\alpha)$,

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta_{E(2)} + V(\alpha) - \varepsilon|\psi|^2\right)\psi, \tag{25}$$

$$\psi(t,g;q,j) = \exp\left[\frac{ij}{\hbar}(y\sin q - x\cos q)\right]\psi(t,q-\alpha).$$

Then, for the function $\psi(t,q')$, Equation (25) yields the following reduced equation:

$$i\hbar\frac{\partial\psi(t,q')}{\partial t} + \frac{\hbar^2}{2m}\delta_3\frac{\partial^2\psi(t,q')}{\partial q'^2} - \\ - \left[\frac{\hbar^2 j}{2m}\left(\delta_1\cos^2 q' + \delta_2\sin^2 q'\right) + V(q-q') - \varepsilon\left|\psi(t,q')\right|^2\right]\psi(t,q') = 0. \tag{26}$$

It can be seen that in the particular case $V(\alpha) = 0$, $\delta_1 = \delta_2$, $\delta_3 = 1$, Equation (26) takes the form of the usual nonlinear Schrödinger equation and has the soliton solution

$$\psi(t,q') = \frac{\hbar a}{\sqrt{\varepsilon m}}\cosh^{-1}[(q'-vt)]\exp\left[\frac{im}{\hbar}\left(q' - \frac{v}{2}\right)v - \frac{i\hbar}{2m}\left(a^2 - \delta_1 n'^2\right)\right], \\ j = \hbar n', \quad \varepsilon > 0.$$

The solution to the original Equation (25) has the form

$$\psi(t,g;q,n') = \frac{\hbar a}{\sqrt{\varepsilon m}}\cosh^{-1}[(q-\alpha-vt)]\times \\ \times\exp\left[i(y\sin q - x\cos q)n' + \frac{im}{\hbar}\left(q-\alpha-\frac{v}{2}\right)v - \frac{i\hbar}{2m}\left(a^2-\delta_1 n'^2\right)\right].$$

Concluding this section, we note that the nonlinear Equation (8) on the Lie groups includes as a particular case the well-known classical $(1+1)$-dimensional nonlinear Schrödinger equation integrable by the Inverse Scattering Transform method (e.g., Ref. [10]), and the noncommutative reduction method proposed in this paper yields the one-soliton solution. This case follows from the more general Equation (25) with a potential $V(\alpha)$, which can be regarded as an example of the Gross–Pitaevskii equation [7].

5. The Four-Dimensional Solvable Exponential Group

Consider a four-dimensional solvable exponential group G. The Lie algebra $\mathfrak{g}$ of G, with respect to a fixed basis $\{e_1, e_2, e_3, e_4\}$, is defined by the commutation relations $[e_2, e_3] = e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = -e_3$. The algebra index equals 2 and there are two Casimir functions

$$K_1(f) = f_1, \quad K_2(f) = f_1 f_4 - f_3 f_2, \quad f \in \mathfrak{g}^* \simeq \mathbb{R}^4.$$

In canonical coordinates of the second kind

$$g(x_1,x_2,x_3,x_4) = e^{x_4 e_4}e^{x_3 e_3}e^{x_2 e_2}e^{x_1 e_1}, \quad x_1 \in [0,2\pi), (x_2,x_3,x_4) \in \mathbb{R}^3,$$

the left-invariant and the right-invariant vector fields are given by

$$\xi_1 = \partial_{x_1}, \quad \xi_2 = \partial_{x_3}, \quad \xi_3 = x_2\partial_{x_1} + \partial_{x_3}, \quad \xi_4 = x_2\partial_{x_2} - x_3\partial_{x_3}, \\ \eta_1 = \partial_{x_1}, \quad \eta_2 = -e^{x_4}(x_3\partial_{x_1} + \partial_{x_2}) \quad \eta_3 = -e^{-x_4}\partial_{x_3}, \quad \eta_4 = -\partial_{x_4}.$$

The invariant measure on the group coincides with the Lebesgue measure and is of the form $d\mu(g) = dx_1 dx_2 dx_3 dx_4$. The subgroup $G_1 = \{\exp(e_1 x_1)\}$ of the Lie group G can be either compact ($x_1 \in [0;2\pi)$) or noncompact ($x_1 \in \mathbb{R}^1$). Let us choose the right-invariant metric on the group as follows:

$$ds^2 = \delta_1^{-1} dx_1 dx_4 + \left(\delta_2^{-1} dx_3 - \delta_1^{-1} x_3 dx_4\right) dx_2, \tag{27}$$
$$\left(g^{ab}\right) = 2\text{antidiag}(\delta_1, \delta_2, \delta_2, \delta_1),$$
$$\delta_2 \neq -\delta_1, \quad \delta_1, \delta_2 = \text{const}.$$

The metric (27) is not flat because there is a nonzero component of the Ricci tensor $R_{\mu\nu}(g) : R_{44}(g) = (\delta_2/\delta_1)^2/2$. The Laplace operator of the metric (27) reads

$$\Delta_G = 4\delta_1 \partial^2_{x_1 x_4} + 2\delta_2 \left(2\partial^2_{x_2 x_3} + 2x_3 \partial^2_{x_1 x_3} + \partial_{x_1}\right).$$

In this section, we will consider a stationary nonlinear Schrödinger equation of the form

$$-\frac{\hbar^2}{2m} \Delta_G \psi(g) + \varepsilon e^{x_4} |\psi(g)|^2 \psi(g) = E\psi(g), \quad E > 0. \tag{28}$$

There is a complete set of commuting symmetry operators $\{-i\hbar\xi_1, -i\hbar\xi_2, K_2(-i\hbar\xi)\}$ that allows one to perform a complete separation of variables in the linear Equation (28) with $\varepsilon = 0$:

$$\psi_{p_1 p_2 j_2}(g) = e^{\frac{i}{\hbar}(p_1 x_1 + p_2 x_2)} \left(\frac{\hbar}{p_2 + p_1 x_3}\right)^{\frac{1}{2} + \frac{i j_2}{\hbar p_1}} \varphi_{p_1 p_2 j_2}\left(x_4 + \ln \frac{p_2 + p_1 x_3}{\hbar}\right), \tag{29}$$

$$-i\hbar\xi_1 \psi_{p_1 p_2 j_2}(g) = p_1 \psi_{p_1 p_2 j_2}(g),$$
$$-i\hbar\xi_2 \psi_{p_1 p_2 j_2}(g) = p_2 \psi_{p_1 p_2 j_2}(g),$$
$$K_2(-i\hbar\xi) \psi_{p_1 p_2 j_2}(g) = j_2 \psi_{p_1 p_2 j_2}(g).$$

Substituting the ansatz (29) into the Equation (28) with $\varepsilon = 0$, we get the ordinary differential equation

$$2(\delta_1 + \delta_2) p_1 \frac{d\varphi_{p_1 p_2 j_2}(z)}{dz} - \frac{i}{\hbar}(2\delta_2 j_2 + mE) \varphi_{p_1 p_2 j_2}(z) = 0.$$

Nevertheless, it is not possible to reduce the nonlinear Equation (28) (when $\varepsilon \neq 0$) because

$$e^{x_4} \left|\psi_{p_1 p_2 j_2}(g)\right|^2 \psi_{p_1 p_2 j_2}(g) = \frac{e^z}{(p_2 + p_1 x_3)^2} \left|\varphi_{p_1 p_2 j_2}(z)\right|^2 \varphi_{p_1 p_2 j_2}(z)$$

and the expression $e^z/(p_2 + p_1 x_3)^2$ depends on the variable x_3.

Let us now carry out the noncommutative reduction. Each nondegenerate K-orbit passes through the parameterized covector $\lambda(j) = (j_1, 0, 0, j_2), j = (j_1, j_2) \in \mathbb{R}^2$:

$$\mathcal{O}_j = \{f \in \mathbb{R}^4 \mid K(f) = j_1, K(f) = j_1 j_2, \neg(f_1 = f_2 = f = 0)\},$$
$$\dim \mathcal{O}_j = 2.$$

The λ-representation operators corresponding to nondegenerate K-orbits and real polarization $\mathfrak{h} = \{e_1, e_3, e_4\}$ have the form

$$\ell_1 = i\frac{j_1}{\hbar}, \quad \ell_2 = \partial_q, \quad \ell_3 = i\frac{j_1}{\hbar} q, \quad \ell_4 = q\partial_q + \frac{i}{\hbar}\left(j_2 - i\hbar\frac{1}{2}\right),$$
$$K_1(-i\hbar\ell) = j_1, \quad K_2(-i\hbar\ell) = j_1 j_2,$$

where the covector $\beta = (0, 0, 0, -1/2)$. The operators $-i\hbar\ell_a$ are symmetric with respect to the measure $d\mu(q) = dq, q \in Q \simeq \mathbb{R}^1$.

Solving the system of Equation (15), we obtain the functions $\mathscr{D}^{\lambda}_{qq'}(g^{-1})$, and the completeness and orthogonality conditions for them yield the following measure $d\mu(\lambda)$:

$$\mathscr{D}^{\lambda}_{qq'}(g^{-1}) = \exp\left(-\frac{1}{2}x_4 - \frac{ij_1}{\hbar}(x_3(q-x_2)+x_1) - \frac{ij_2}{\hbar}x_4\right)\delta(q' + e^{-x_4}(x_2 - q)),$$
$$d\mu(\lambda) = \frac{1}{(2\pi)^3}j_1 dj_1 dj_2.$$

Then, the noncommutative ansatz has the form

$$\psi(g;q,j_1,j_2) = e^{-x_4/2}\exp\left(-\frac{ij_1}{\hbar}(x_3(q-x_2)+x_1) - \frac{ij_2}{\hbar}x_4\right)\times \times\,\psi(e^{-x_4}(q-x_2)). \tag{30}$$

Substituting (30) into (25), we obtain the ordinary differential equation

$$-\frac{n_1\hbar^2}{m}\left[i(\delta_1+\delta_2)\left(2q'\frac{d}{dq'}+1\right) - 2\hbar\delta_2 n_2\right]\psi(q') + +\varepsilon|\psi(q')|^2\psi(q') = E\psi(q'). \tag{31}$$

In the linear case $\varepsilon = 0$, we have a solution

$$\psi(q') = \frac{1}{\sqrt{q'}}\exp\left(i\frac{mE/(2\hbar^2) - \delta_1 n_1 n_2}{(\delta_1+\delta_2)n_1}\ln q'\right), \quad \varepsilon = 0.$$

We seek a solution of the Equation (31) in the form

$$\psi(q') = f(q')\exp(i\Phi(q')), \tag{32}$$

where $f(q')$ and $\Phi(q')$ are real functions. Substituting (32) in (31), we get the ODE system:

$$2\frac{\hbar^2}{m}(\delta_1+\delta_2)j_1 q' f'(q') + \varepsilon f(q')^3\cot\phi(q') +$$
$$\left[2\frac{\hbar^2}{m}(\delta_1+\delta_2)j_1(2q'\phi'(q')\cot\phi(q')+1) - E\cot\phi(q')\right]f(q') = 0,$$
$$2q'f'(q') + f(q') = 0.$$

The solution of this system yields

$$\psi(q') = \sqrt{\frac{\hbar^2}{\varepsilon m}\frac{2(\delta_1+\delta_2)n_1}{q'}}\exp\left\{i\left[\frac{c_1}{q'} + \frac{mE/(2n_1\hbar^2) - \delta_1 n_2}{\delta_1+\delta_2}\right]\ln q' + \frac{\ln c_1}{2}\right\}. \tag{33}$$

Substituting (33) into the expression (30), we obtain a set of particular solutions $\psi(g)$ of the nonlinear equation (25) that are parameterized by $\{q', n_1, n_2\}$ and c_1. For this set of solutions, the following equality holds:

$$|\psi(g)|^2 = \frac{2\hbar^2}{\varepsilon m}(\delta_1+\delta_2)\left|\frac{n_1 c_1}{q - x_2}\right|. \tag{34}$$

Thus, the noncommutative reduction of the Equations (25)–(31) made it possible to find a family of particular solutions of the original Equation (25). The solutions obtained tend to infinity on the plane $x_2 = q$ and tend to zero as $x_2 \to \pm\infty$ that can be seen from (34).

6. Conclusions

In this article, we considered an approach in which the noncommutative integration method developed in Ref. [4] for finding bases for solution spaces of linear PDEs with symmetries can be applied to constructing families of particular solutions of nonlinear equations on Lie groups by reducing the nonlinear equation to an equation with a fewer number of independent variables. In terms of this approach, we study the generalized nonlinear Schrödinger equation in curved space with local cubic nonlinearity on a Lie group.

The application of the noncommutative integration method to nonlinear equations on Lie groups (under certain restrictions), allows finding families of particular solutions parameterized by the eigenvalues of the noncommutative set of symmetry operators for the *linear part* of the nonlinear equation under consideration. The nonlinear term in the original nonlinear equation does not admit those symmetry operators that its linear part admits. On the other hand, the noncommutative ansatz is determined only by the algebra of symmetry operators of the linear part of the nonlinear equation. The special form of the ansatz (22), because of its algebraic properties, allows us in a number of cases to carry out a noncommutative reduction of the original nonlinear equation.

The parameters q and λ in the noncommutative ansatz (22) acquire a physical meaning when comparing the solution of a nonlinear equation with the solution of its linear counterpart, as was considered in Ref. [23].

In some cases, it is possible to carry out the noncommutative reduction to a nonlinear equation with an external potential. In the case of the NLSE with a potential, we arrive at the Gross–Pitaevskii equation, which is the model mean field equation in BEC theory [7–9]. This case is demonstrated by the example of the NLSE with the external potential (25) on the three-dimensional Lie group $E(2)$ in Section 4. With the special choice of the right-invariant metric on the group $E(2)$, we obtained the classical $(1+1)$-dimensional NLSE as a result of noncommutative reduction. This made it possible to obtain a soliton-type solution for the NLSE on the group $E(2)$.

We also note that in this paper we consider the NLSE with local nonlinearity in contrast to papers [24,25], where the noncommutative reduction was applied to nonlinear equations with a nonlocal term of the convolution type. In those papers, the original nonlocal nonlinear equation was reduced to a nonlocal nonlinear equation with a fewer number of independent variables using the generalized Fourier transform.

The broad implication of the present research is that the noncommutative reduction of the NLSE considered in this paper expands the possibilities of the exact integration of nonlinear equations on Lie groups and, importantly, in multidimensional cases. The proposed approach is more limited by the symmetries of the equation than by its specific form. Therefore, our proposed version of noncommutative reduction can be applied to other equations, among which the nonlinear relativistic equations are of particular interest, for example, the nonlinear Dirac, sine–Gordon, and reaction–diffusion-type equations. In addition, the problem of the search of nonlinear equations admitting a noncommutative reduction naturally arises.

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