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Optical Solitons and Conservation Laws for the Concatenation Model: Undetermined Coefficients and Multipliers Approach

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Abstract: This paper retrieves an optical 1–soliton solution to a model that is written as a concatenation of the Lakshmanan–Porsezian–Daniel model and Sasa–Satsuma equation. The method of undetermined coefficients obtains a full spectrum of 1–soliton solutions. The multiplier approach yields the conserved densities, which subsequently lead to the conserved quantities from the bright 1–soliton solution.

Keywords: optical solitons; dispersive; gratings; Kudryashov

1. Introduction

The concept and applications of optical solitons has been strong and steady ever since its inception in 1973. For the last five decades, the concept has paved the way for various advanced concepts, and many performance enhancements have been achieved. This gave way to the concepts of non-Kerr law solitons, dispersion-managed solitons, quiescent solitons, cubic–quartic solitons, highly dispersive solitons, various new forms of self-phase modulations and many others. This emerged into an ocean of novel concepts that is consistently being studied and analyzed [1–20]. It is worth mentioning that topological solitons are a very wide class of solutions to non-linear Klein–Gordon equations (KGE) [21–25]. KGE appears in the study of relativistic Quantum Mechanics, while the study model in the current paper appears in the study of Quantum Optics and Optoelectronics.

Another form of new concept that has lately been proposed is the concatenation model formulated of a combination of two familiar models. These are the Lakshmanan–Porsezian–Daniel (LPD) model [1] and the Sasa–Satsuma equation (SSE) [13], while the linear temporal evolution, chromatic dispersion and the Kerr law of nonlinear refractive index change is the same as the familiar, nonlinear Schrödinger's equation (NLSE). This model was first proposed in 2014 [2,3]. Later, it was revisited by Triki et al. during 2022 [16]. A Painleve analysis of this model was carried out very recently by Kudryashov et al. [6]. The current paper again considers the model to retrieve a full spectrum of 1–soliton solutions using the technique of undetermined coefficients. This is followed by the location of the conservation laws using the multipliers method. The soliton solutions, together with the parameter constraints for their existence, as well as the conservation laws, are displayed after a brief and concise introduction to the model.

Governing Model

The concatenation model given by (1) is true to its name. For $c_1 = 0$, (1) reduces to the familiar SSE, while for $c_2 = 0$, Equation (1) reduces to LPD model. However, for $c_1 = c_2 = 0$, (1) collapses to the familiar NLSE. The bright 1–soliton solution, obtained using the indeterminate coefficients approach, will reveal the conserved quantities. The soliton existence criteria will also be guaranteed by the several constraint conditions that will naturally emerge from the scheme.

$$\begin{aligned} iq_t + aq_{xx} + b|q|^2 q &+ c_1\left[\sigma_1 q_{xxxx} + \sigma_2 (q_x)^2 q^* + \sigma_3 |q_x|^2 q + \sigma_4 |q|^2 q_{xx} + \sigma_5 q^2 q^*_{xx} + \sigma_6 |q|^4 q\right] \\ &+ ic_2\left[\sigma_7 q_{xxx} + \sigma_8 |q|^2 q_x + \sigma_9 q^2 q^*_x\right] = 0. \end{aligned} \tag{1}$$

This model is an extended version of the familiar NLSE, SSE and LPD models. Thus, the current model describes the propagation of solitons through an optical fiber in an accurate manner. The presence of CD and self–phase modulation provides the basic ingredients needed for the solitons to exist. In case CD runs low, this is compensated by the third-order dispersion and fourth-order dispersion effects that stem from the SSE and LPD components of the concatenation model. The effect of the soliton self-frequency shift is also provided by the SSE component. The effect of nonlinear dispersion as well as the two-photon absorption effect are emerging from the LPD component of the model. Thus, this model carries an "universal" structure that can more accurately describe the soliton propagation dynamics through optical fibers and other form of waveguides. The details of the derivation of the soliton solutions and its conservation laws are enumerated and displayed in the subsequent sections.

2. Undetermined Coefficients

To construct soliton solutions for the system (1), we allow for it to have solutions in the form

$$q(x,t) = P(x,t)e^{i\Psi(x,t)} = P(\xi)e^{i\Psi(x,t)} \tag{2}$$

for which the arguments of the amplitude factors and the phase factor, respectively, are:

$$\xi(x,t) = x - vt, \quad \text{and} \quad \Psi(x,t) = -\kappa x + \omega t + \theta_0, \tag{3}$$

where $P(x,t)$ denotes the waveform according to each type of nonlinear wave, κ represents the soliton frequency, while ω and θ_0 represent the wave number and phase constant, respectively. By substituting (2) into (1) and separating it into real and imaginary components, one obtains, for the real component,

$$\begin{aligned} &\left(c_1\sigma_1\kappa^4 - c_2\sigma_7\kappa^3 - a\kappa^2 - \omega\right)P + \left\{b - c_1(\sigma_2 - \sigma_3 + \sigma_4 + \sigma_5)\kappa^2 + c_2(\sigma_8 - \sigma_9)\kappa\right\}P^3 \\ &+ c_1\sigma_6 P^5 + \left(a - 6c_1\sigma_1\kappa^2 + 3c_2\sigma_7\kappa\right)\frac{\partial^2 P}{\partial x^2} + c_1\sigma_1\frac{\partial^4 P}{\partial x^4} \\ &+ c_1(\sigma_4 + \sigma_5)P^2\frac{\partial^2 P}{\partial x^2} + c_1(\sigma_2 + \sigma_3)P\left(\frac{\partial P}{\partial x}\right)^2 = 0. \end{aligned} \tag{4}$$

while the imaginary counterpart was

$$\begin{aligned}&\frac{\partial P}{\partial t}-\left(2a\kappa+3c_2\sigma_7\kappa^2-4c_1\sigma_1\kappa^3\right)\frac{\partial P}{\partial x}\\+\quad&\{c_2(\sigma_8+\sigma_9)-2\kappa c_1(\sigma_2+\sigma_4-\sigma_5)\}P^2\frac{\partial P}{\partial x}+(c_2\sigma_7-4\kappa c_1\sigma_1)\frac{\partial^3 P}{\partial x^3}=0.\end{aligned} \tag{5}$$

From the imaginary part, the soliton speed is,

$$v=-2\kappa\left(a+4c_1\sigma_1\kappa^2\right) \tag{6}$$

whenever

$$c_2(\sigma_8+\sigma_9)=2\kappa c_1(\sigma_2+\sigma_4-\sigma_5), \tag{7}$$

and

$$c_2\sigma_7=4\kappa c_1\sigma_1 \tag{8}$$

hold. In view of the above two conditions (7) and (8), the real part of the equation reduces to

$$\begin{aligned}-\quad&\left(3c_1\sigma_1\kappa^4+a\kappa^2+\omega\right)P+\left\{b-c_1(\sigma_2-\sigma_3+\sigma_4+\sigma_5)\kappa^2+c_2(\sigma_8-\sigma_9)\kappa\right\}P^3+c_1\sigma_6P^5\\+\quad&\left(a+6c_1\sigma_1\kappa^2\right)\frac{\partial^2 P}{\partial x^2}+c_1\sigma_1\frac{\partial^4 P}{\partial x^4}+c_1(\sigma_4+\sigma_5)P^2\frac{\partial^2 P}{\partial x^2}+c_1(\sigma_2+\sigma_3)P\left(\frac{\partial P}{\partial x}\right)^2=0.\end{aligned} \tag{9}$$

In the following four subsections, the last real component Equation (9) will be employed to generate four distinct soliton solutions for the concatenation model (1).

2.1. Bright Soliton

In this subsection, a bright soliton solution is constructed with the assistance of the balancing principle. First, we assume the following form for the wave profile:

$$P(x,t)=A\,\mathrm{sech}^p\tau,\qquad \tau=B(x-vt), \tag{10}$$

where A is the soliton's amplitude, p is a parameter to be determined, and B is the corresponding inverse width, with v representing the soliton speed. The substitution of (10) into (9) leads to

$$\begin{aligned}&\left[p^2Z_3B^2+c_1\sigma_1p^4B^4-\left(\omega+a\kappa^2+3c_1\sigma_1\kappa^4\right)\right]A\,\mathrm{sech}^p\,\tau\\&+\left[Z_2+p^2(Z_4+Z_5)B^2\right]A^3\,\mathrm{sech}^{3p}\,\tau+c_1\sigma_6A^5\,\mathrm{sech}^{5p}\,\tau\\&-p(1+p)\left[Z_3+2\{2+p(2+p)\}c_1\sigma_1B^2\right]AB^2\,\mathrm{sech}^{p+2}\,\tau\\&+p(1+p)(2+p)(3+p)c_1\sigma_1AB^4\,\mathrm{sech}^{p+4}\,\tau\\&-p[(1+p)Z_4+pZ_5]A^3B^2\,\mathrm{sech}^{3p+2}\,\tau=0\end{aligned} \tag{11}$$

where, for convenience, we adopted the following notation:

$$Z_2=b-c_1\kappa^2(\sigma_2-\sigma_3+\sigma_4+\sigma_5)+\kappa c_2(\sigma_8-\sigma_9) \tag{12}$$

$$Z_3=a+6c_1\sigma_1\kappa^2 \tag{13}$$

$$Z_4=c_1(\sigma_4+\sigma_5) \tag{14}$$

$$Z_5=c_1(\sigma_2+\sigma_3) \tag{15}$$

The balance between nonlinearity and dispersion leads to equating $3p$ and $p+2$, from which the value of the unknown parameter p becomes

$$p=1. \tag{16}$$

Substituting (16) into (11), by setting the coefficients of the linearly independent set of functions $\operatorname{sech}^j[\tau]$ for $j = 1, 3, 5$ to zero, we obtain:

$$\omega = -a\kappa^2 + Z_3 B^2 - c_1\sigma_1\left(3\kappa^4 - B^4\right) \tag{17}$$

and

$$A = \sqrt{\frac{2(Z_3 + 10c_1\sigma_1 B^2)B^2}{Z_2 + (Z_4 + Z_5)B^2}}, \tag{18}$$

subject to

$$\begin{aligned} &\left\{a + 2c_1\sigma_1\left(3\kappa^2 + 5B^2\right)\right\} \\ \times\ &\left\{b - c_1\kappa^2(\sigma_2 - \sigma_3 + \sigma_4 + \sigma_5) + c_2\kappa(\sigma_8 - \sigma_9) + c_1(\sigma_2 + \sigma_3 + \sigma_4 + \sigma_5)B^2\right\} > 0 \end{aligned} \tag{19}$$

where the soliton width can be written as

$$\begin{aligned} B \;=\; &\frac{1}{2}\left[\frac{Z_3\{(Z_4 + Z_5)(2Z_4 + Z_5) - 40c_1^2\sigma_1\sigma_6\} - 2c_1\sigma_1(2Z_4 + 7Z_5)Z_2}{c_1\sigma_1\left(100c_1^2\sigma_1\sigma_6 - 4Z_4^2 - 3Z_4Z_5 + Z_5^2\right)}\right. \\ &\pm \left.\frac{\sqrt{\{Z_3(Z_4 + Z_5) - 10c_1\sigma_1 Z_2\}^2\left\{(2Z_4 + Z_5)^2 - 96c_1^2\sigma_1\sigma_6\right\}}}{c_1\sigma_1\left(100c_1^2\sigma_1\sigma_6 - 4Z_4^2 - 3Z_4Z_5 + Z_5^2\right)}\right]^{\frac{1}{2}} \end{aligned} \tag{20}$$

whenever the conditions

$$\sigma_1 c_1^3[100\sigma_1\sigma_6 + (\sigma_2 + \sigma_3 + \sigma_4 + \sigma_5)\{\sigma_2 + \sigma_3 - 4(\sigma_4 + \sigma_5)\}] \neq 0 \tag{21}$$

and

$$\{2c_1(\sigma_2 + \sigma_3 + \sigma_4 + \sigma_5)\}^2 > 96c_1^2\sigma_1\sigma_6 \tag{22}$$

hold. Therefore, a bright soliton solution for the system (1) is

$$q(x,t) = A\,\operatorname{sech}[B(x - vt)]e^{i(\kappa x + \omega t + \theta_0)} \tag{23}$$

where the soliton speed v was computed early on (6), the amplitude A and wave numbers ω are expressed on (18) and (17) in terms of the soliton width, while the width B is given in (20) with corresponding constraints (21) and (22). The other solvability conditions needed for the bright soliton to exist are (7), (8), and the inequality given in (19).

Figure 1 shows the profile of a bright soliton, where the parameters are chosen as: $a = 0.3$, $b = 0.8$, $c_1 = 1.2$, $c_2 = 1.1$, $\sigma_1 = 0.5$, $\sigma_2 = 1.0$, $\sigma_3 = 2.1$, $\sigma_4 = 2.0$, $\sigma_5 = -1.0$, $\sigma_6 = 3.2$, $\sigma_7 = 0.1$, $\sigma_8 = 0.4$, $\sigma_9 = 0.2$.

2.2. Dark Soliton

Then, we created a dark soliton solution following a similar procedure to the one used to construct the bright solution on the previous part. First, we assume the following form for the wave profile:

$$P(x,t) = A\,\tanh^p\tau, \qquad \tau = B(x - vt), \tag{24}$$

where, in this case, A and B are free parameters for the soliton, while p is a parameter whose value will be calculated. The rest of the parameters have the same meaning as for the bright soliton. Now, substituting (24) into (9), and after adopting the notation (12)–(15), the real part of the Equation (9) emerges as:

$$
\begin{aligned}
&-\Big[3c_1\sigma_1\kappa^4+a\kappa^2+\omega+2p^2(Z_3-c_1\sigma_1(5+3p^2)B^2)B^2\Big]A\tanh^p\tau\\
&+\Big[Z_2-2p^2(Z_4+Z_5)B^2\Big]A^3\tanh^{3p}\tau+c_1\sigma_6A^5\tanh^{5p}\tau\\
&+p(1+p)\Big[Z_3-4c_1\sigma_1\{2+p(2+p)\}B^2\Big]AB^2\tanh^{p+2}\tau\\
&+p(1+p)(2+p)(3+p)c_1\sigma_1AB^4\tanh^{p+4}\tau\\
&+p\{(1+p)Z_4+pZ_5\}A^3B^2\tanh^{3p+2}\tau\\
&+p\{(p-1)Z_4+pZ_5\}A^3B^2\tanh^{3p-2}\tau\\
&+p(p-1)\Big\{Z_3-4c_1\sigma_1(2+p(p-2))B^2\Big\}AB^2\tanh^{p-2}\tau\\
&+p(p-1)(p-2)(p-3)c_1\sigma_1AB^4\tanh^{p-4}\tau=0.
\end{aligned}
\tag{25}
$$

The appropriate balance leads to $p=1$, as for the case of the bright soliton. Substituting the resulting value of p into the Equation (25), and taking the set of linearly independent functions $\tanh^p\tau$ for $p=1,3,5$ and putting all their coefficients zero, we can obtain the following three equalities:

$$\omega+a\kappa^2+\Big(2Z_3-Z_5A^2\Big)B^2+c_1\sigma_1\Big(3\kappa^4-16B^4\Big)=0 \tag{26}$$

$$\Big\{Z_2-2(Z_4+Z_5)B^2\Big\}A^2+2\Big(Z_3-20c_1\sigma_1B^2\Big)B^2=0 \tag{27}$$

$$c_1\sigma_6A^4+24c_1\sigma_1B^4+(2Z_4+Z_5)A^2B^2=0 \tag{28}$$

From (27), the free parameter A becomes

$$A=\sqrt{-\frac{2(Z_3+10c_1\sigma_1B^2)B^2}{Z_2+(Z_4+Z_5)B^2}} \tag{29}$$

while from (26), the wave number is

$$\omega=-\frac{2Z_5(Z_3-20c_1\sigma_1B^2)B^4+\{Z_2-2(Z_4+Z_5)B^2\}\{a\kappa^2+2Z_3B^2+c_1\sigma_1(3\kappa^4-16B^4)\}}{Z_2-2(Z_4+Z_5)B^2} \tag{30}$$

in view of (29), and as long as,

$$\sigma_1c_1^3[100\sigma_1\sigma_6+(\sigma_2+\sigma_3+\sigma_4+\sigma_5)\{\sigma_2+\sigma_3-4(\sigma_4+\sigma_5)\}]<0 \tag{31}$$

and the constraint from ω:

$$\Big\{b-c_1(\sigma_2-\sigma_3+\sigma_4+\sigma_5)\kappa^2+c_2(\sigma_8-\sigma_9)\kappa\Big\}\neq 2c_1(\sigma_2+\sigma_3+\sigma_4+\sigma_5)B^2 \tag{32}$$

are satisfied. From (27) and (28), the free parameter B is

$$
\begin{aligned}
B \;=\; &\pm\frac{1}{2}\Bigg[\frac{2c_1\sigma_1(2Z_4+7Z_5)Z_2-Z_3\{(Z_4+Z_5)(2Z_4+Z_5)-40c_1^2\sigma_1\sigma_6\}}{2c_1\sigma_1(100c_1^2\sigma_1\sigma_6-4Z_4^2-3Z_4Z_5+Z_5^2)}\\
&\pm\frac{\sqrt{\{Z_3(Z_4+Z_5)-10c_1\sigma_1Z_2\}^2\Big\{(2Z_4+Z_5)^2-96c_1^2\sigma_1\sigma_6\Big\}}}{2c_1\sigma_1(100c_1^2\sigma_1\sigma_6-4Z_4^2-3Z_4Z_5+Z_5^2)}\Bigg]^{\frac{1}{2}}
\end{aligned}
\tag{33}
$$

where both constraints (21) and (22) are also required for dark soliton. Therefore, the single dark soliton solution for the concatenation model (1) is

$$q(x,t) = A\tanh[B(x - vt)]e^{i(\kappa x + \omega t + \theta_0)} \tag{34}$$

where the parameters and associated restrictions are addressed previously.

Figure 2 shows the profile of a dark soliton, where the parameters are chosen as: $a = 0.5$, $b = 0.1$, $c_1 = 0.5$, $c_2 = 3.2$, $\sigma_1 = 1.2$, $\sigma_2 = 0.2$, $\sigma_3 = 2.3$, $\sigma_4 = 1.1$, $\sigma_5 = 1.7$, $\sigma_6 = 2.0$, $\sigma_7 = -1.1$, $\sigma_8 = 1.3$, $\sigma_9 = 0.8$.

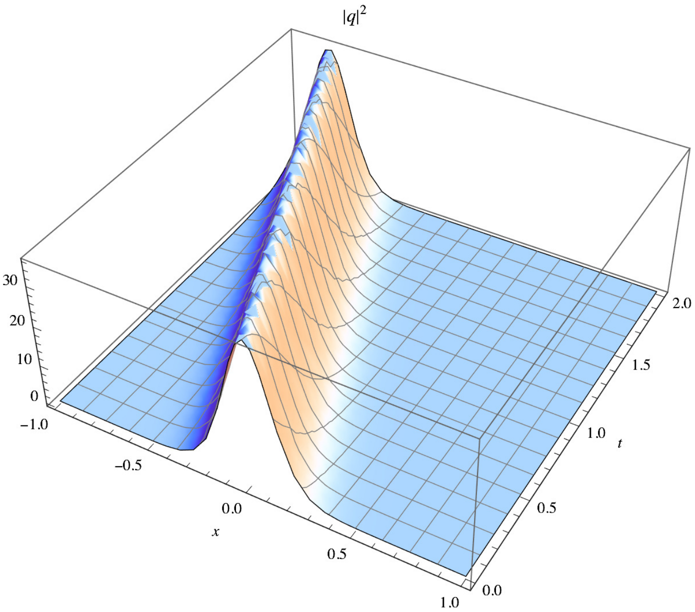

Figure 1. Bright soliton for concatenation model.

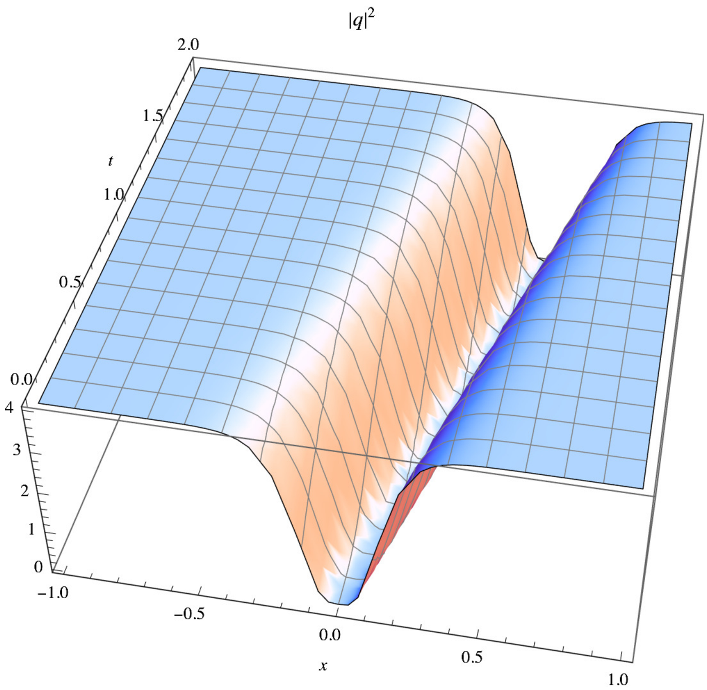

Figure 2. Dark soliton for concatenation model.

2.3. Singular Soliton (Type-I)

We accept the following wave form for the first type of singular soliton:

$$P(x,t) = A \operatorname{csch}^p \tau, \qquad \tau = B(x - vt), \tag{35}$$

where A and B, are considered free parameters for the soliton. Substituting the chosen type-I form into (9) reduces the last into

$$\begin{aligned}
&\left[p^2 Z_3 B^2 + c_1\sigma_1 p^4 B^4 - \left(\omega + a\kappa^2 + 3c_1\sigma_1\kappa^4\right)\right] A \operatorname{csch}^p \tau \\
&+\left[Z_2 + p^2(Z_4 + Z_5)B^2\right] A^3 \operatorname{csch}^{3p} \tau + c_1\sigma_6 A^5 \operatorname{csch}^{5p} \tau \\
&+p(1+p)\left[Z_3 + 2\{2 + p(2+p)\}c_1\sigma_1 B^2\right] AB^2 \operatorname{csch}^{p+2} \tau \\
&+p(1+p)(2+p)(3+p)c_1\sigma_1 AB^4 \operatorname{csch}^{p+4} \tau \\
&+p[(1+p)Z_4 + pZ_5]A^3B^2 \operatorname{csch}^{3p+2} \tau = 0
\end{aligned} \tag{36}$$

where the $Z's$ were defined above on (12)–(15). Balancing yields, as expected $p = 1$. Substituting this into the Equation (36), and taking the set of linearly independent functions $\operatorname{csch}^p[\tau]$ for $p = 1, 3, 5$ and setting all their coefficients as zero, we can obtain the following three equalities:

$$\omega + a\kappa^2 + 3c_1\sigma_1\kappa^4 - \left(Z_3 + c_1\sigma_1 B^2\right)B^2 = 0 \tag{37}$$

$$\left[Z_2 + (Z_4 + Z_5)B^2\right]A^2 + 2\left(Z_3 + 10c_1\sigma_1 B^2\right)B^2 = 0 \tag{38}$$

$$c_1\sigma_6 A^4 + 24c_1\sigma_1 B^4 + (2Z_4 + Z_5)A^2B^2 = 0. \tag{39}$$

Isolating ω on (37) leads to the same expression for the wave number as in the bright soliton (17), while solving for A (38) yields the same result as in (29), as long as the constraint (31) is satisfied. In view of the result for the parameter A, the free parameter B is the same as for the bright soliton given in (20), along with corresponding constraints. Therefore, the type-I singular soliton solution supported by the system (1) is

$$q(x,t) = A \operatorname{csch}[B(x - vt)]e^{i(\kappa x + \omega t + \theta_0)} \tag{40}$$

where the free parameters A and B are expressed in (29) and (20), respectively, as long as the solvability conditions (21), (22) and (31) are satisfied. The soliton speed was obtained earlier (6), while the wave number ω was the same as for bright soliton (17).

2.4. Singular Soliton (Type-II)

The second type of singular soliton solution will be the last type of soliton discussed in this work. To retrieve type-I singular solitons from our concatenation system (1), a wave form with the following structure is assumed:

$$P(x,t) = A \coth^p \tau, \qquad \tau = B(x - vt), \tag{41}$$

where A and B are considered free parameters, p is a parameter whose value is to be determined, and v represents the soliton speed. The substitution of the hypothesis (41) into the real part of Equation (9) brings:

$$
\begin{aligned}
&-\Big[3c_1\sigma_1\kappa^4+a\kappa^2+\omega+2p^2\Big\{Z_3-c_1\sigma_1\Big(5+3p^2\Big)B^2\Big\}B^2\Big]A\coth^p\tau\\
&+\Big[Z_2-2p^2(Z_4+Z_5)B^2\Big]A^3\coth^{3p}\tau+c_1\sigma_6A^5\coth^{5p}\tau\\
&+p(1+p)\Big[Z_3-4c_1\sigma_1\{2+p(2+p)\}B^2\Big]AB^2\coth^{p+2}\tau\\
&+p(1+p)(2+p)(3+p)c_1\sigma_1AB^4\coth^{p+4}\tau\\
&+p[(1+p)Z_4+pZ_5]A^3B^2\coth^{3p+2}\tau\\
&+p[(p-1)Z_4+pZ_5]A^3B^2\coth^{3p-2}\tau\\
&+p(p-1)\Big[Z_3-4c_1\sigma_1\{2+p(p-2)\}B^2\Big]AB^2\coth^{p-2}\tau\\
&+p(p-1)(p-2)(p-3)c_1\sigma_1AB^4\coth^{p-4}\tau=0.
\end{aligned}
\tag{42}
$$

Proper balancing leads to the same value of p as in the previous three cases. Substituting the resulting value of p into (43) and setting the coefficients of the resulting linearly independent functions as equal to zero yields the same three identities (26)–(28). Thus, the parameter findings for this subsection will be identical to those for the dark soliton case (29)–(33). Finally, the type-II soliton solution supported by the concatenation model is

$$
q(x,t)=A\coth[B(x-vt)]e^{i(\kappa x+\omega t+\theta_0)}
\tag{43}
$$

where all parameter values are the same as the case of dark soliton, along with the corresponding solvability constraints.

3. Conservation Laws

It is important and imperative to investigate the conservation laws for any model to obtain a better understanding of its physical features. This gives a deeper perspective to the model and will lead to any future avenues that could be explored. For the conserved flow that renders a closed form of the respective nonlinear evolution equations (NLEEs), we let $q=u+iv$ and split the NLEE into a real system with conserved vectors; (T^t,T^x) satisfies the $D_tT^t+D_xT^x=0$ along the solutions of the NLEEs. We have

(a) conserved **power density** for $\sigma_2=0$ and $\sigma_4=\sigma_5$, viz.,

$$
\Phi^t_{Pr}=\frac{1}{2}|q|^2
\tag{44}
$$

and, additionally, for $\sigma_3=2\sigma_5$ and $\sigma_9=0$, we get

(b) **linear momentum density** given by

$$
\Phi^t_M=a\mathcal{I}(q^*q_x)
\tag{45}
$$

and

(c) **Hamiltonian density** given by,

$$
\begin{aligned}
\Phi^t_H&=\frac{1}{2}a\mathcal{R}(qq^*_{xx})+\frac{1}{4}b|q|^4\\
&+c_1\left[\frac{1}{2}\sigma_1\mathcal{R}(qq^*_{xxxx})+\frac{1}{2}(\sigma_4+\sigma_5)\Big\{\mathcal{R}(qq^*_{xx})+|q|^2|q_x|^2\Big\}+\frac{1}{6}\sigma_6|q|^6\right]\\
&+c_2\left[\frac{1}{2}\sigma_7\mathcal{I}(q^*q_{xxx})+\frac{1}{4}\sigma_8\mathcal{I}(q^*q_x)\right].
\end{aligned}
\tag{46}
$$

The conserved quantity are therefore, respectively, given as:

$$Pr = \int_{-\infty}^{\infty} \Phi^t_{Pr} dx = \int_{-\infty}^{\infty} |q|^2 dx = \frac{2A^2}{B}, \tag{47}$$

$$M = a\int_{-\infty}^{\infty} \Phi^t_M dx = a\int_{-\infty}^{\infty} \mathcal{I}(q^* q_x) dx = \frac{a}{2i}\int_{-\infty}^{\infty} (q^* q_x - q q_x^*) dx = \frac{2a\kappa A^2}{B}, \tag{48}$$

and

$$\begin{aligned} H &= \int_{-\infty}^{\infty} \Phi^t_H dx \\ &= \frac{aA^2}{3B}\left(B^2 + 3\kappa^2\right) + \frac{bA^4}{3B} \\ &+ c_1\left[\frac{\sigma_1 A^2}{15B}\left(7B^4 + 30\kappa^2 A^2 B^2 + 15\kappa^4\right) + \frac{(\sigma_4+\sigma_5)A^2}{15B}\left\{5\left(B^2+3\kappa^2\right) + 2A^2\left(B^2+5\kappa^2\right)\right\}\right] \\ &+ \frac{8c_1\sigma_6 A^6}{45B} + c_2\left[\frac{\sigma_7\kappa A^2}{B}\left(B^2+\kappa^2\right) - \frac{\sigma_8\kappa A^2}{2B}\right] \end{aligned} \tag{49}$$

4. Conclusions

Using the method of indeterminate coefficients, the current paper recovers the 1-soliton solution to the recently constructed concatenation model. Later, the bright 1-soliton solution is utilized to establish the conserved quantities derived for the model using the multipliers method. Power, linear momentum, and the Hamiltonian are the three conserved quantities that were derived. These results open up a floodgate of opportunities for further studies of this model. The perturbation terms will be incorporated and, thus, the extended version of the concatenation model will be addressed by various means, including the semi-inverse variational principle. The variational principle will be addressed to recover the soliton parameter dynamics. The established conservation laws will next be implemented to obtain the adiabatic parameter dynamics of the solitons. The stochasticity effect will also be addressed by studying the corresponding Langevin equations to give way to the mean free velocity of the soliton. The collective variables approach would also lead to the parameter dynamics of the solitons. From a numerical standpoint, the model will later be examined numerically with the variational iteration method, Adomian decomposition scheme and Laplace–Adomian decomposition scheme. Evidently, a lot of work lies ahead and the results should subsequently be revealed.

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