

CKM Matrix Parameters from the Exceptional Jordan Algebra

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Abstract: We report a theoretical derivation of the Cabibbo–Kobayashi–Maskawa (CKM) matrix parameters and the accompanying mixing angles. These results are arrived at from the exceptional Jordan algebra applied to quark states, and from expressing flavor eigenstates (i.e., left chiral states) as a superposition of mass eigenstates (i.e., the right chiral states) weighted by the square root of mass. Flavor mixing for quarks is mediated by the square root mass eigenstates, and the mass ratios used are derived from earlier work from a left–right symmetric extension of the standard model. This permits a construction of the CKM matrix from first principles. There exist only four normed division algebras, and they can be listed as follows: the real numbers \mathbb{R} , the complex numbers \mathbb{C} , the quaternions \mathbb{H} and the octonions \mathbb{O} . The first three algebras are fairly well known; however, octonions as algebra are less studied. Recent research has pointed towards the importance of octonions in the study of high-energy physics. Clifford algebras and the standard model are being studied closely. The main advantage of this approach is that the spinor representations of the fundamental fermions can be constructed easily here as the left ideals of the algebra. Also, the action of various spin groups on these representations can also be studied easily. In this work, we build on some recent advances in the field and try to determine the CKM angles from an algebraic framework. We obtain the mixing angle values as $\theta_{12} = 11.093^\circ$, $\theta_{13} = 0.172^\circ$, $\theta_{23} = 4.054^\circ$. In comparison, the corresponding experimentally measured values for these angles are $13.04^\circ \pm 0.05^\circ$, $0.201^\circ \pm 0.011^\circ$, $2.38^\circ \pm 0.06^\circ$. The agreement of theory with experiment is likely to improve when the running of quark masses is taken into account.

Keywords: Weak interaction; CKM matrix; quark mixing; exceptional Jordan algebra; Clifford algebras; octonions



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1. Introduction

There has been occasional interest in the last few decades regarding the significance of octonions for understanding the standard model of particle physics [1]. Research on this topic has picked up significant pace in the last seven years or so since the publication of Furey’s Ph.D. thesis [2], and also the discovery by Todorov and Dubois-Violette [3] that the exceptional groups G_2 , F_4 , E_6 contain symmetries of the standard model as maximal sub-groups. This has given rise to the hope that octonions could play a significant role in the unification of electroweak and strong interactions and, in turn, their unification with gravitation. Octonionic chains can be used to generate a Clifford algebra, and spinors made as minimal left ideals of Clifford algebras possess symmetries observed in the standard model [2,4].

We propose a left–right symmetric extension of the standard model, based on complex split bioctonions, which incorporates gravitation [5]. This is consistent with unification based on an $E_8 \times E_8$ symmetry, and the breaking of this symmetry reveals the standard model [6,7]. Chiral fermions arise after symmetry breaking; left-handed fermions are eigenstates of electric charge, and right-handed fermions are eigenstates of the newly introduced $U(1)$ quantum number, square root of mass. By expressing the charge eigenstates

as superpositions of square root mass eigenstates, one is able to theoretically derive the observed mass ratios of quarks and charged leptons [8–12].

In the present paper, we extend these methods to provide a theoretical derivation of the CKM matrix parameters for quark mixing and the accompanying mixing angles. Also, we show that the complex Clifford algebra $Cl(9)$ is the algebra of unification. Further, we conclude from our investigations that our universe possesses a second 4D spacetime with its own distinct light cone structure. Distances in this spacetime are invariably microscopic and only quantum systems can access this second spacetime.

This paper is organized as follows. Sections 2–4 review a few basics of group representations, Clifford algebras, and the octonions. Sections 5 and 6 briefly recall earlier work on particle representations made from octonions, and our own work on the derivation of mass ratios from the exceptional Jordan algebra. Section 7 is the heart of the paper; the space of minimal ideals is constructed, and the role of $SU(2)_L$ and $SU(2)_R$ symmetry is elucidated. The triality property of the spinor and vector reps of $SO(8)$ is used to motivate the methodology for the theoretical derivation of the CKM matrix parameters. The calculation of these matrix parameters and mixing angles is then carried out in Section 8. Conclusions are in Section 9.

The CKM matrix plays a central role in the understanding of weak interactions of quarks and provides a quantitative measure of the flavor change brought about by these interactions. It plays a key role in the understanding of CP violation, and a possible violation of the unitarity condition might be an indication of physics beyond the standard model. What is important is to note that to date, our knowledge of the CKM matrix parameters comes exclusively from experiments. The CKM angles are free parameters of the standard model, and there is no generally accepted theory which explains why these angles should have the values measured in experiments. To the best of our knowledge, the present paper is the first to provide a first-principles derivation of the CKM angles, starting from a theory of unification of the standard model with gravitation. Based on the spontaneous breaking of the unified $E_8 \times E_8$ symmetry, a new $U(1)$ symmetry arises, which we name $U(1)_{grav}$. Its associated charge is square root of mass $\pm\sqrt{m}$, which can have either sign (analogous to electric charge): positive sign for matter, and negative sign for anti-matter. Left-handed fermion states are eigenstates of electric charge, and right-handed fermion states are eigenstates of the square root mass. These characteristics enable us to construct the CKM matrix, and the fact that mass eigenstates are labeled by the square root mass and not by the mass plays a very important role in correctly determining the values of the CKM angles. An earlier paper on CKM angles which foresaw the significance of square root mass is the one by Nishida [13] and is titled “Phenomenological formula for CKM matrix and its physical interpretation”. An even earlier interesting work is by Fritzsch [14,15], who also aimed to derive the mixing angles in terms of quark mass ratios. While these important works bear some interesting similarity to ours, they take quark masses and their ratios as inputs from experiments. On the other hand, we first derived mass ratios from an underlying theory of unification, and in the present work, these mass ratios are used to derive the weak mixing angles. Thus, the octonionic theory of unification provides strong evidence that the fundamental constants of the standard model are derivable from a coherent framework and are not free parameters of nature.

2. A Few Basics

To engage in the study of the Clifford algebras, mass ratios and their application to the standard model itself, we first need a basic introduction to some mathematical concepts. A basic review is given in the following sections about some of the required concepts.

2.1. Algebra

An algebra $(A, +, \cdot, F)$ over a field F is defined to be a vector space over the field, equipped with a bi-linear operation that follows the following properties:

$$m : A \times A \longrightarrow A \tag{1}$$

$$(a, b) \longrightarrow a \cdot b \quad a, b, a \cdot b \in A \tag{2}$$

- $(\alpha a) \cdot (\beta b) = \alpha \beta (a \cdot b) \quad \alpha, \beta \in F; a, b \in A$
- $(a + b) \cdot c = (a \cdot c) + (b \cdot c) \quad a, b, c \in A.$
- $((a \cdot b) \cdot c) = (a \cdot (b \cdot c))$

An ideal I is defined as a subspace of A which survives multiplication by any element of A . A left ideal is defined as

$$a \in I, \forall b \in A \implies (b \cdot a) \in I \tag{3}$$

2.2. Group Representations

We recall a few essential basics about group theory.

- If there is a homomorphism from a group G to a group of operators $U(G)$ on a vector space V , then $U(G)$ forms a representation of group G on V .
- The dimension of the representation is the same as the dimension of the vector space:

$$g \in G \xrightarrow{U} U(g) \tag{4}$$

$$U(g)e_i = D(g)_i^j e_j \quad i, j = 1, 2 - \dots - \dim(V) \tag{5}$$

Here, the D is the matrix representation of G on the vector space V . As a representation is a homomorphism, it must preserve the group operation, so we have

$$U(g_1)U(g_2) = U(g_1 \cdot g_2) \tag{6}$$

$$D(g_1)D(g_2) = D(g_1 \cdot g_2) \tag{7}$$

If for a representation $U(G)$ of G on V , there exists a subspace V_1 in V such that

$$U(g)|x_1\rangle \in V_1 \quad \forall x_1 \in V_1 \tag{8}$$

then such a subspace is called an invariant subspace of V with respect to the group representation $U(G)$. The trivial invariant subspaces of V are V itself, and the space of null vectors. A subspace which does not have any non-trivial invariant subspace is called minimal or proper. The representation $U(G)$ on V is called irreducible if there is no non-trivial invariant subspace in V ; otherwise, the representation is reducible [16].

2.3. The Standard Model

The gauge group of the standard model is given below:

$$G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y \tag{9}$$

Also, the forces and their respective carriers are presented in Table 1.

- A representation of the gauge group G acts on a finite-dimensional Hilbert space V .
- Particles then live in the irreducible invariant subspace of V as their basis vectors.

Table 1. Forces and force carriers.

Force	Gauge Boson	Symbol
Electromagnetism	Photon	γ
Weak Force	W and Z bosons	W^+, W^-, Z
Strong Force	Gluons	g

3. Clifford Algebras

A Clifford algebra $Cl(p, q)$ over \mathbb{R} is defined to be an associative algebra, generated by n elements e_i . These n generators exhibit the following properties:

$$\{e_i, e_j\} = e_i e_j + e_j e_i = 2\eta_{ij} \tag{10}$$

$$e_i^2 = 1 \quad e_j^2 = -1 \tag{11}$$

Here i runs from 1 to p , and j runs from 1 to q . The multiplication, also called the Clifford product, can be realized in terms of dot product and wedge product of vectors. An example is

$$xy = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \wedge \mathbf{y} \tag{12}$$

The signature becomes irrelevant when we form the algebra over \mathbb{C} as the field. For a vector v (a linear combination of generators), we have

$$v^2 = -\|v\|^2 \implies v^{-1} = \frac{-v}{\|v\|^2} \tag{13}$$

3.1. Pin and Spin Groups

There is a natural automorphism in the Clifford algebra for all vectors in the Clifford algebra, given by

$$v \longrightarrow \tilde{v} = -v \tag{14}$$

Let us denote this automorphism as α . It partitions the algebra into two parts. Firstly, we have the part that is the product of even number of vectors, given as

$$Cl^{even}(n) = \left\{ \alpha(x) = x; \forall x \in Cl(n) \right\} \tag{15}$$

The other part contains an odd number of vectors as the product

$$Cl^{odd}(n) = \left\{ \alpha(x) = -x; \forall x \in Cl(n) \right\} \tag{16}$$

For a non-null vector u , we can define an inverse given by

$$\forall u \in V \subset Cl(V) \tag{17}$$

$$\exists u^{-1} \in Cl^*(V) : u^{-1} = -\frac{u}{\|u\|^2} \tag{18}$$

Here, $Cl^*(V)$ is the group of elements that have inverses. The definition of the inverse of the vector can be extended to the inverse of the product of the vectors. Thus, we can define two groups as given below [17,18]:

$$\mathbf{Pin} = \left\{ a \in Cl^*(V) : a = u_1 u_2 \dots u_r, u_j \in V, |u_j| = 1 \right\} \tag{19}$$

$$\mathbf{Spin} = \mathbf{Pin} \cap Cl^{even}(n) = \left\{ a \in Cl(n) : a = u_1 - u_{2r}; u_j \in V, |u_j| = 1 \right\} \tag{20}$$

The action of both these groups on V can be defined as the twisted adjoint action:

$$\tilde{A}d_a x = \alpha(a) x a^{-1} \in V \quad \forall x \in V. \tag{21}$$

$$(\alpha(a) v a^{-1})^2 = v^2 \quad \forall v \in V \tag{22}$$

As both of these group preserve the magnitude of the vectors, they are orthogonal and special orthogonal transformations:

$$\mathbf{Pin} \longrightarrow O(n) \tag{23}$$

$$\mathbf{Spin} \longrightarrow SO(n) \tag{24}$$

3.2. Representations of Clifford Algebras

The real and complex Clifford algebras have matrix representations. Here, however, we will focus on representations of complex Clifford algebras. The representations of the even subalgebra can be similarly obtained by the identity [19,20]

$$Cl^{even}(n) \cong Cl(n-1) \quad ; n \geq 1 \tag{25}$$

The matrix representations are given below. Here, $M_p(\mathbb{C})$ represents a $p \times p$ matrix with complex entries:

$$Cl(n) \cong M_p(\mathbb{C}) \quad p = 2^{\frac{n}{2}} \quad ; n = \text{even} \tag{26}$$

$$Cl(n) \cong M_p(\mathbb{C}) \oplus M_p(\mathbb{C}) \quad p = 2^{\frac{n-1}{2}} \quad ; n = \text{odd} \tag{27}$$

Again, notice that for the odd case, the total representation is reduced to two irreducible representations. In particular, look at the case of $n = 3, 7 \pmod 8$. The irreducible subspace on which matrices act is represented by P . These $M_n(F)$ act on the n -dimensional irreducible space. The choice of the volume element can split the algebra into two parts [20,21]; total space also becomes partitioned into two irreducible subspaces. For dimensions 3 and 7, there are two choices of irreducible spaces, positive spinor space (P_+) and negative spinor space (P_-).

$$Cl(n) \cong Cl^+(n) \oplus Cl^-(n) \cong \text{End}_{\mathbb{C}}(P_+) \oplus \text{End}_{\mathbb{C}}(P_-) \tag{28}$$

$$P = P_+ \oplus P_- \tag{29}$$

Now, look at the case for the complexified Dirac algebra $\mathbb{C} \otimes Cl(1,3)$. It is equivalent to complex Clifford algebra $Cl(4)$. We need to study the usual spinors, so we look at the matrix representations of $Cl^{even}(4)$. We know that $Cl^{even}(4) \cong Cl(3)$. For those cases, where the even subalgebra is partitioned into two, we similarly obtain positive spinor space (S_+) and negative spinor space (S_-):

$$Cl^{even}(4) \cong Cl(3) \cong M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \tag{30}$$

$$S = S_+ \oplus S_- = S_L \oplus S_R \tag{31}$$

These are the matrix representations of the spin groups that act on the spinor space. The total spinor space is the vector sum of the positive and negative spinor spaces. Both spaces are two-dimensional, and indeed these spaces are interpreted as the left-handed Weyl spinor and right-handed Weyl spinor. Keeping this information in mind, we construct

two irreducible subspaces in higher dimensions, with the brief outline discussed below. For the $Cl(8)$ algebra, we look at its even subalgebra:

$$Cl^{even}(8) = Cl(7) = M_8(\mathbb{C}) \oplus M_8(\mathbb{C}) \tag{32}$$

As $n = 7$, the representation space can be decomposed into two irreducible subspaces. This fact can be used later to include spin and other things in the analysis.

4. Octonions

A generic complex octonion can be represented as

$$\mathbb{C} \otimes \mathbb{O} = \sum_{n=0}^7 A_n e_n \tag{33}$$

Here, A_n are complex coefficients, and e_n are octonionic units, with properties $e_0^2 = 1$ and $e_i^2 = -1$. So, $e_0 = 1$ and the rest are imaginary octonionic units. In general, octonionic multiplication is non-associative. An example is given:

$$e_3(e_4(e_6 + ie_2)) = -1 + ie_7 \tag{34}$$

$$(e_3e_4)(e_6 + ie_2) = -1 - ie_7 \tag{35}$$

To tackle this problem of the octonions, we need to define an order of multiplication on a product of octonions. It leads to a chain of octonions made from maps:

$$e_1(e_2(e_3(e_4))) \longrightarrow \overleftarrow{e_1e_2e_3e_4} \tag{36}$$

$$\overleftarrow{-e_ie_j - f} = -(\overleftarrow{-e_ie_i - f}) \tag{37}$$

We will work with octonionic chains only. Octonionic multiplication is represented by the Fano plane given below. A multiplication example is given by

$$e_7e_1 = e_3 \quad \text{and} \quad e_1e_7 = -e_3 \tag{38}$$

$$e_ie_j + e_je_i = 0 \tag{39}$$

The octonionic chains form a representation of the Clifford algebras, and hence, we are interested in their study. They form a representation of $Cl(6)$ [2]. The generators of the Clifford algebra can be constructed from the octonionic imaginary units as shown in Furey’s work [2]. The Fano plane in Figure 1 lists the methods to multiply octonionic units.

$$Cl(6) \cong \mathbb{C} \otimes \overleftarrow{\mathbb{O}} \tag{40}$$

The 64 dim $Cl(6)$ algebra is fully generated by the set $\{ \overleftarrow{ie_1}, \overleftarrow{ie_2}, \overleftarrow{ie_3}, \overleftarrow{ie_4}, \overleftarrow{ie_5}, \overleftarrow{ie_6} \}$. These are the generators of the Clifford algebra and act as the underlying vector space structure:

$$\overleftarrow{e_1e_2e_3e_4e_5e_6}f = \overleftarrow{e_7}f \tag{41}$$

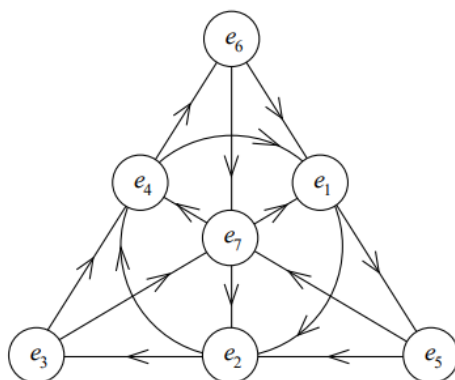


Figure 1. The Fano plane [19].

5. Minimal Left Ideals

The generators of $Cl(6)$ can be used to make elements of maximally totally isotropic space (MTIS). An element of maximally totally isotropic space has a quadratic norm equal to zero [22]. This space for the maximally isotropic subspaces follows the algebraic structure given below:

$$\{q_i, q_j\}f = q_i(q_jf) + q_j(q_if) = 0 \tag{42}$$

$$\{q_i^\dagger, q_j^\dagger\}f = q_i^\dagger(q_j^\dagger f) + q_j^\dagger(q_i^\dagger f) = 0 \tag{43}$$

$$\{q_i, q_j^\dagger\}f = \delta_{ij}f \tag{44}$$

The a^\dagger represents the Hermitian conjugation. It is basically the complex conjugation a^* and octonionic conjugation \tilde{a} performed simultaneously. The elements of the MTIS can be constructed from the generators of $Cl(6)$. One choice is given below [2,5]. The six generators give rise to six elements with a quadratic norm equal to zero. There can be other equivalent choices also [22]:

$$q_1 = \frac{1}{2}(-e_5 + ie_4) \quad q_1^\dagger = \frac{1}{2}(e_5 + ie_4) \tag{45}$$

$$q_2 = \frac{1}{2}(-e_3 + ie_1) \quad q_2^\dagger = \frac{1}{2}(e_3 + ie_1) \tag{46}$$

$$q_3 = \frac{1}{2}(-e_6 + ie_2) \quad q_3^\dagger = \frac{1}{2}(e_6 + ie_2) \tag{47}$$

We construct quantities out of these isotropic vectors, with the nilpotent given as [2]

$$q = q_1q_2q_3 \quad q^\dagger = q_3^\dagger q_2^\dagger q_1^\dagger \tag{48}$$

$$q^2 = 0 \quad (q^\dagger)^2 = 0 \tag{49}$$

We also have the idempotent given as

$$p = qq^\dagger \quad p' = q^\dagger q \tag{50}$$

$$p^2 = p \quad (p')^2 = p' \tag{51}$$

We act on the idempotent by the q and q^\dagger operators and obtain various algebraic states and the minimal left ideals. These states are later classified according to the transformations they undergo [2,23].

5.1. Symmetry Transformations

We first look at such transformations for which the maximally isotropic space is closed. Operator transforms of the type

$$e^{i\phi_k g_k} | e^{-i\phi_k g_k} \quad \phi_k \in R \tag{52}$$

$$[g_k, \sum_i b_i q_i] = \sum_j c_j q_j \quad [g_k, \sum_i b'_i q_i^\dagger] = \sum_j c'_j q_j^\dagger \tag{53}$$

We can make Hermitian operators by the following procedures:

$$q = c_1 q_1 + c_2 q_2 + c_3 q_3 \quad \text{and} \quad q' = c'_1 q_1 + c'_2 q_2 + c'_3 q_3 \tag{54}$$

The charge operator has a $U(1)$ symmetry $Q = \frac{1}{3} \sum_i q_i^\dagger q_i$ and $SU(3)$ generators:

$$\Lambda_1 = -q_2^\dagger q_1 - q_1^\dagger q_2 \quad \Lambda_2 = iq_2^\dagger q_1 - iq_1^\dagger q_2 \tag{55}$$

$$\Lambda_3 = q_2^\dagger q_2 - q_1^\dagger q_1 \quad \Lambda_4 = -q_1^\dagger q_3 - q_3^\dagger q_1 \tag{56}$$

$$\Lambda_5 = -iq_1^\dagger q_3 + iq_3^\dagger q_1 \quad \Lambda_6 = -q_3^\dagger q_2 - q_2^\dagger q_3 \tag{57}$$

$$\Lambda_7 = iq_3^\dagger q_2 - iq_2^\dagger q_3 \quad \Lambda_8 = -\frac{1}{\sqrt{3}}(q_1^\dagger q_1 + q_2^\dagger q_2 - 2q_3^\dagger q_3) \tag{58}$$

A general Hermitian operator can be written as

$$\sum_H H = r_0 Q + r_i \sum_{i=1}^8 \Lambda_i \tag{59}$$

We see that the idempotent remains unaffected by these operations:

$$e^{i\sum H} q q^\dagger e^{-i\sum H} = (1 + i \sum H + \dots) q q^\dagger (1 - i \sum H - \dots) = q q^\dagger = p \tag{60}$$

Hence, it is identified as a neutrino. The down isospin family can be obtained via complex conjugation of all the particles. Operators for that family also become complex conjugated and then are used to identify the particles.

5.2. Particle Representations

We have the symmetry groups $SU(3)$ and $U(1)$ of the standard model; we now look at the action of these groups on the elements of the minimal left ideals and see how they transform. Depending upon their transformations and eigenvalues, we label them accordingly [2] as shown in Table 2. We look at their charges obtained by the action of the Q operator and also observe the action of $SU(3)$ generators to classify them.

The \bar{d}_i and u_j have indices running from one to three, representing the three colored up and anti-down quarks. The left ideal present above gives another left ideal after the complex conjugation. This time, it gives the isospin down family. Observe that the transition from one family to other can be performed by complex conjugation. Now, the creation operator and the annihilation operator reverse their roles, and we also obtain a new idempotent.

Hence, we have a representation of one generation of standard model particles under the unbroken symmetry $SU(3)_c \times U(1)_{em}$ [2].

Table 2. (a) Up-isospin particles; (b) down-isospin particles.

(a)			
Q	Λ	P^u	Particle
0	1	p	ν
$\frac{1}{3}$	$\bar{3}$	$q_i^\dagger p$	\bar{d}_i
$\frac{2}{3}$	3	$q_i^\dagger q_j^\dagger p$	u_i
1	1	$q_i^\dagger q_j^\dagger q_k^\dagger p$	e^+
(b)			
$-Q^*$	$-\Lambda^*$	P^d	Particle
0	1	p'	$\bar{\nu}$
$-\frac{1}{3}$	3	$q_i p'$	d_i
$-\frac{2}{3}$	$\bar{3}$	$q_i q_j p'$	\bar{u}_i
-1	1	$q_i q_j q_k p'$	e^-

6. Split Bioctonions and Mass Ratios

Split bioctonions are simply two copies of octonions in the same algebra. They can be constructed from the generators in the $Cl(7)$ algebra [5]:

$$Cl(7) \cong Cl(6) \oplus Cl(6) \tag{61}$$

Observe that the spinor representations of $Cl^{even}(8)$ again give us the positive and negative spinor spaces:

$$Cl^{even}(8) \cong Cl(7) \cong M_8(\mathbb{C}) \oplus M_8(\mathbb{C}) \tag{62}$$

6.1. Construction

The seven generators of $Cl(7)$, given as $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, can be arranged in the manner given below. Keeping in mind the non-associativity of the octonions, we use the octonionic chains [5]:

$$\omega = \overleftarrow{e_1 e_2 e_3 e_4 e_5 e_6 e_7} \tag{63}$$

$$e_8 = \overleftarrow{e_1 e_2 e_3 e_4 e_5 e_6} \tag{64}$$

$$(1, e_1, e_2, e_3, e_4, e_5, e_6, e_8) \oplus \omega(1, -e_1, -e_2, -e_3, -e_4, -e_5, -e_6, -e_8) \tag{65}$$

$$\omega^2 = 1 \tag{66}$$

Here, e_8 acts as an octonionic unit, and ω as a pseudoscalar that commutes with every octonionic unit and hence with every element of the $Cl(7)$ algebra. It is the analog of the split complex number that squares to one but is neither one nor minus one. To generate the system with opposite parity, look at an example given below:

$$\overleftarrow{e_1 e_2 e_4 e_5 e_6 e_7} = -\overleftarrow{e_1 e_2 e_3 e_4 e_5 e_7 e_6 e_8} = -\overleftarrow{e_3 e_1 e_2 e_3 e_4 e_5 e_6 e_7} = -e_3 \omega = -\omega e_3 \tag{67}$$

A $Cl(6)$ algebra can be used to construct a left-sided ideal. It is similar to an irreducible space; the action (left multiplication) of various elements of algebra on the elements in the ideal keeps the space of the ideal closed, similar to the working of the irreducible space. The two sets of octonions can now be used to construct ideals that represent states of opposite chirality, similar to positive and negative spinor states. By the complex conjugation of the two chiral families, we can also construct the antiparticle states. We can do so by defining

the idempotents and nilpotents as done earlier, and perform our analysis. But notice this time that for the second copy of the octonions, the generators have a negative sign. This helps us to introduce chirality into the problem. From the first copy of the octonions, we obtain the left-handed neutrino family and its right-handed anti-particle’s family. Similarly, from the second copy we can obtain the right-handed neutrino family and its left-handed anti-particle’s family [5].

6.2. Mass Ratios

We construct all three families from a single real octonionic family by a set of transformations. Both cases for Dirac and Majorana neutrinos are analyzed [9]. The solution of the Dirac equation in (9, 1) spacetime is connected with the eigenvalue problem of the Hermitian octonionic matrices as explained in [24,25]. The eigenvalues thus calculated give us the square root mass ratios of various fundamental fermions.

6.2.1. Hermitian Octonionic Matrices

The quarks have different representations for different colors. Octonions are difficult to work with, while quaternions are much easier to deal with. To make the problem simpler, we take the representations of neutrino and electron and choose the color state of the quarks accordingly such that only one quaternionic copy is used for one family of fermions. Now, this complex quaternionic representation is mapped to real octonionic representation by the mapping given below [9]:

$$C \otimes H \longrightarrow R \otimes O \tag{68}$$

$$(a_0 + ia_1) + (a_2 + ia_3)e_4 + (a_4 + ia_5)e_5 + (a_6 + ia_7)e_7 \tag{69}$$

↓

$$a_0 + a_1e_1 + a_5e_2 + a_3e_3 + a_2e_4 + a_4e_5 + a_7e_6 + a_6e_7 \tag{70}$$

Once we have the real representation for one family, we perform an internal rotation about some axis and obtain the real octonionic representation for all three families. We can use these representations to fill the entries in 3×3 octonionic Hermitian matrices. The uniqueness of the axis used for transformation and similar matters have already been discussed [9]. It is observed that the ratios of the square root mass of the positron, the up quark and the down quark is 1:2:3. Motivated by this information, we can define a new quantity as the gravi charge. The ratio of gravi charges will then be

$$e^+ : u : d = \frac{1}{3} : \frac{2}{3} : 1 \tag{71}$$

The gravi charges can be negative also. These gravi charges are then used on the diagonals of these octonionic Hermitian matrices. These 3×3 octonionic Hermitian matrices are referred to as exceptional Jordan matrices, and they form the exceptional Jordan algebra, with a specified Jordan product [26]

$$A \circ B = \frac{1}{2}(AB + BA) \tag{72}$$

We fill the entries in the matrices accordingly with the diagonals filled with the gravi charge

$$X_\nu = \begin{bmatrix} 0 & V_\tau & \bar{V}_\mu \\ \bar{V}_\tau & 0 & V_\nu \\ V_\mu & \bar{V}_\nu & 0 \end{bmatrix} \quad X_e = \begin{bmatrix} \frac{1}{3} & V_\tau & \bar{V}_\mu \\ \bar{V}_\tau & \frac{1}{3} & V_{e^+} \\ V_\mu & \bar{V}_{e^+} & \frac{1}{3} \end{bmatrix} \tag{73}$$

$$X_u = \begin{bmatrix} \frac{2}{3} & V_t & \bar{V}_c \\ \bar{V}_t & \frac{2}{3} & V_u \\ V_c & \bar{V}_u & \frac{2}{3} \end{bmatrix} \quad X_d = \begin{bmatrix} 1 & V_{\bar{b}} & \bar{V}_{\bar{s}} \\ \bar{V}_{\bar{b}} & 1 & V_{\bar{d}} \\ V_{\bar{s}} & \bar{V}_{\bar{d}} & 1 \end{bmatrix} \tag{74}$$

These matrices satisfy the characteristic equation given as [26]

$$A^3 - (trA)A^2 + \sigma(A)A - (detA)I = 0 \tag{75}$$

The definition and explanation for each quantity are presented in Appendix A. The exact nature of these matrices in the context of standard model is still not completely understood. However, some results do suggest that the $\mathbb{O}\mathbb{P}^2$ space is crucial for our understanding of the spinors, and these spaces are closely related to these Hermitian matrices [27]. These matrices with real octonionic entries can be further decomposed as given [24]:

$$A = \sum_i^3 \lambda_i P_{\lambda_i} \tag{76}$$

$$P_{\lambda_i} \circ P_{\lambda_j} = 0 = \frac{1}{2}(P_{\lambda_i} P_{\lambda_j} + P_{\lambda_j} P_{\lambda_i}) \tag{77}$$

$$\implies A \circ P_{\lambda} = \lambda P_{\lambda} \tag{78}$$

It gives us an eigenmatrix equation. These eigenvalues are used to calculate the square root masses of various fundamental fermions [9] as shown in Figure 2 below.

Mass ratios: Square root of the mass of a charged fermion with respect to the down quark

Down quark 1	Strange quark $\frac{1 + \sqrt{3/8}}{1 - \sqrt{3/8}}$	Bottom quark $\frac{(1 + \sqrt{3/8})^3}{(1 - \sqrt{3/8})^2}$
Up quark 2/3	Charm quark $\frac{2}{3} \times \frac{\frac{2}{3} + \sqrt{3/8}}{\frac{2}{3} - \sqrt{3/8}}$	Top quark $\frac{4}{9} \times \frac{\frac{2}{3} + \sqrt{3/8}}{(\frac{2}{3} - \sqrt{3/8})^2}$
Positron 1/3	Muon $\frac{1}{3} \times \frac{1 + \sqrt{3/8}}{1 - \sqrt{3/8}} \times \frac{1/3 + \sqrt{3/8}}{1/3 - \sqrt{3/8}}$	Tau lepton $\frac{1}{3} \times \frac{(1 + \sqrt{3/8})^2 \times (1/3 + \sqrt{3/8})}{(1 - \sqrt{3/8})^2 \times (1/3 - \sqrt{3/8})}$

Figure 2. The square root of mass of fermions with respect to the down quark [9].

6.2.2. Inclusion of Gravity

The mass ratios of the up quark, down quark, and positron motivated us to extend the gauge group to $SU(3)_{grav} \times SU(2)_R \times U_g(1)$. This U(1) symmetry is similar to the usual U(1), with the gravi charge as the quantity analogous to the electric charge. We can group the particles with up isospin together as was done earlier and proceed as follows:

$$e^+ : u : \bar{d} = \frac{1}{3} : \frac{2}{3} : 1 \tag{79}$$

We have the following families that are expected to observe the $SU(2)_R$ symmetry:

$$\begin{pmatrix} u \\ e^- \end{pmatrix} \quad \begin{pmatrix} \nu_e \\ d \end{pmatrix} \tag{80}$$

Notice the swapping of the down quark and electron. This structure can be extended to all three generations. Now, we are working in the $Cl(7)$ algebra; it has two copies of the $Cl(6)$ algebra. One copy can be used to construct the octonionic representations of the gravitationally inactive particles that transform according to the normal standard model gauge group. The second copy of the $Cl(6)$ can be used to construct a new minimal left ideal for this new extension to the gauge group, which will then have the following octonionic representation for the various gravitationally active particles. The minimal left ideal and the right-handed nilpotents and the idempotent for these spinors that are gravitationally active are then given below:

$$q_1 = \frac{-\omega}{2}(-e_5 + ie_4) \quad q_1^\dagger = \frac{-\omega}{2}(e_5 + ie_4) \tag{81}$$

$$q_2 = \frac{-\omega}{2}(-e_3 + ie_1) \quad q_2^\dagger = \frac{-\omega}{2}(e_3 + ie_1) \tag{82}$$

$$q_3 = \frac{-\omega}{2}(-e_6 + ie_2) \quad q_3^\dagger = \frac{-\omega}{2}(e_6 + ie_2) \tag{83}$$

$$q_R = q_1q_2q_3 \quad q_R^\dagger = q_3^\dagger q_2^\dagger q_1^\dagger \tag{84}$$

$$p_R = q_R q_R^\dagger \tag{85}$$

This helps us to generate the following particle eigenstates:

$$v_{e,R} = \frac{ie_8 + 1}{2} \tag{86}$$

$$V_{e+1} = \omega \frac{(-e_5 - ie_4)}{2} \tag{87}$$

$$V_{e+2} = \omega \frac{(-e_3 - ie_1)}{2} \tag{88}$$

$$V_{e+3} = \omega \frac{(-e_6 - ie_2)}{2} \tag{89}$$

$$V_{u_1} = \frac{e_4 + ie_5}{2} \tag{90}$$

$$V_{u_2} = \frac{e_1 + ie_3}{2} \tag{91}$$

$$V_{u_3} = \frac{e_2 + ie_6}{2} \tag{92}$$

$$V_{\bar{d}} = \omega \frac{(i + e_8)}{2} \tag{93}$$

7. Space of Minimal Left Ideals

The complete space related to minimal left ideals is not used in the $Cl(6)$ algebra. We intend to use it fully. We already have information about the square root mass ratios. We know that $p = qq^\dagger$ is idempotent, and q_i^\dagger are the ladder operators. By using this, we can construct a left ideal, and by the right multiplication on this space of the left ideal, we can span the whole space of the algebra [23,28].

7.1. Patterns in the Standard Model

To study the standard model, the first thing to do is to introduce vector spaces (or the Hilbert space) which are later made into an algebra. The underlying complex vector space (V, h) establishes a natural isomorphism between the vector space dual and its conjugate. h here is the inner product on the vector space. We, therefore, have the following relations [23]:

$$V^{-1} \cong V^\dagger \cong \bar{V} \tag{94}$$

Table 3 represents the vector space required to explain the appropriate symmetries [23]. The space χ_{em} represents one vector that corresponds to a charge of $\frac{e}{3}$ and the space χ_c represents a three-dimensional complex vector space that has three basis vectors given as $\{r, g, b\}$. For the electromagnetic space, the charges add up for the tensor product of such spaces; they appear as numbers in the exponential associated with the U(1) symmetry. By the above relations, we then have information about the dual space or the conjugate space. We have the space $\bar{\chi}_{em}$, which has the charge equal to $\frac{-e}{3}$, and the dual color space, which now has the vectors as $\{\bar{r}, \bar{g}, \bar{b}\}$. We can use our knowledge of how particles transform under various symmetry transformations and define the internal electro-color space for various particles as given in Table 4. This will later help us to develop isomorphisms between the exterior algebra related to the internal space and the elements of the $Cl(6)$ algebra. For the color space of fermions, we can use the exterior powers of the χ_c to represent different fermions. The color space χ_c and its dual (or conjugate) $\bar{\chi}_c$ have the basis as given below:

$$\chi_c = \{r, g, b\} \quad \bar{\chi}_c = \{\bar{r}, \bar{g}, \bar{b}\} \tag{95}$$

For the exterior algebra of the vectors of the color space and its dual, we have the following relation:

$$\Lambda^{-k} \chi_c = \Lambda^k \bar{\chi}_c \tag{96}$$

With this knowledge, we have the following isomorphisms:

$$\Lambda^0 \chi_c \cong \mathbb{C} \tag{97}$$

$$\Lambda^1 \chi_c \cong \chi_c \tag{98}$$

$$\Lambda^2 \chi_c \cong \bar{\chi}_c : \quad \{r \wedge g \rightarrow \bar{b} \quad , \quad g \wedge b \rightarrow \bar{r} \quad , \quad r \wedge b \rightarrow \bar{g}\} \tag{99}$$

$$\Lambda^3 \chi_c \cong \mathbb{C} : \quad r \wedge g \wedge b \tag{100}$$

The representations of particles in exterior algebra are given in Table 5. For the simplification of the notation, define

$$\chi = \bar{\chi}_{em} \otimes \bar{\chi}_c \tag{101}$$

Note that the Hilbert space is equipped with $h = h_{em} \otimes h_c$, and the space is three-dimensional.

Table 3. Internal space for various symmetries.

Force/Charge	Internal Space	Dimension	Symmetry
Electromagnetism	χ_{em}	1	U(1)
Strong	χ_c	3	SU(3)
Weak Hypercharge	χ_Y	1	U(1)
Weak- Electromagnetism	χ_{ew}	2	U(2)

Table 4. Internal space of particles.

Particle	Internal Space
e^-	χ_{em}^3
\bar{u}	$\chi_{em}^2 \bar{\chi}_c$
d	$\chi_{em} \chi_c$
ν	C
$\bar{\nu}$	C
\bar{d}	$\chi_{em}^{-1} \bar{\chi}_c$
u	$\chi_{em}^{-2} \bar{\chi}_c$
e^+	χ_{em}^{-3}

We choose a basis of the isotropic vectors for the newly defined space χ as $\{q_1, q_2, q_3\}$, and its dual basis for the space $\bar{\chi}$ as $\{q_1^\dagger, q_2^\dagger, q_3^\dagger\}$. So the total Hilbert space can be seen as $\chi^\dagger \oplus \chi$, and other particles are the elements of the exterior algebra defined by this space. These vectors are the Grassmann numbers; they indeed define a basis for the exterior powers of the χ (wedging replaced by the Clifford product).

Table 5. Particles as the representations of the exterior algebra [23].

Particles	Vectors in Exterior Space
e^-	$\Lambda^3 \bar{\chi}$
\bar{u}	$\Lambda^2 \bar{\chi}$
d	$\Lambda^1 \bar{\chi}$
ν	$\Lambda^0 \bar{\chi}$
$\bar{\nu}$	$\Lambda^0 \chi$
\bar{d}	$\Lambda^1 \chi$
u	$\Lambda^2 \chi$
e^+	$\Lambda^3 \chi$

7.2. Algebra for the Standard Model

We construct an algebra over the space $\chi^\dagger \oplus \chi$ and generate a basis of null vectors [23]. The two chiral spaces are the maximally isotropic subspaces for the inner product. So from our previous knowledge and definitions in the earlier section, we have the following:

$$\chi = \{q_1, q_2, q_3\} \quad \chi^\dagger = \{q_1^\dagger, q_2^\dagger, q_3^\dagger\} \tag{102}$$

$$\{q_i^\dagger, q_j^\dagger\} = 0 \quad \{q_i, q_j\} = 0 \quad \{q_i, q_j^\dagger\} = \delta_{ij} \tag{103}$$

$$q = q_1 q_2 q_3 \quad q^\dagger = q_3^\dagger q_2^\dagger q_1^\dagger \tag{104}$$

$$p = q q^\dagger \quad p' = q^\dagger q \tag{105}$$

Here, p and p' are the idempotents; q and q^\dagger are the nilpotents as defined earlier. We can now define an orthonormal basis using these null vectors by the following construction:

$$\chi^\dagger \oplus \chi = \{e_1, e_2, e_3, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\} \tag{106}$$

$$e_j = q_j + q_j^\dagger \tag{107}$$

$$\tilde{e}_j = i(q_j^\dagger - q_j) \tag{108}$$

$$e = e_1 e_2 e_3 \quad \tilde{e} = \tilde{e}_1 \tilde{e}_2 \tilde{e}_3 \tag{109}$$

$$e_i^2 = \tilde{e}_i^2 = 1 \tag{110}$$

$$e\tilde{e} = -\tilde{e}e \tag{111}$$

Observe that we could have chosen $-\tilde{e}_j$ as the orthonormal vector instead of e_j ; this will change the definition of null vectors in terms of the orthonormal vectors. Here, however, we choose the above given definitions.

7.3. Ideals and Representations

We recall that

$$Cl^{even}(7) \cong Cl(6) \cong M_8(\mathbb{C}) \tag{112}$$

We know that the $Cl(7)$ spinors have representations as the elements of the $Cl(6)$ algebra. We construct left ideals in the $Cl(6)$ algebra and now left multiply various elements of the $Cl(6)$ algebra with the elements of the left ideal; as the space is closed, the resulting space is invariant. It gives us the matrix representations of the elements of $Cl(6)$. Following the earlier framework [2], we act with the creation operators on the idempotents to create the particles and thus obtain the representation of particles in the algebra. A basis of the minimal left ideal or the action of all creation operators on one idempotent can be written as [23]

$$\{p, q_{23}^\dagger p, q_{31}^\dagger p, q_{12}^\dagger p, q_{321}^\dagger p, q_1^\dagger p, q_2^\dagger p, q_3^\dagger p\} \tag{113}$$

Upon simplification of the above given basis in terms of the q_i s, we have

$$\{qq^\dagger, -q_1 q^\dagger, -q_2 q^\dagger, -q_3 q^\dagger, q^\dagger, q_{23} q^\dagger, q_{31} q^\dagger, q_{12} q^\dagger\} \tag{114}$$

We act on this algebraic basis using various creation and annihilation operators. It gives us the representations of the algebra as the endomorphisms on the underlying vector space. For the algebraic ideal A , we have

$$\rho : A \longrightarrow \rho(A) \tag{115}$$

$$\rho(A) : b \in A \longrightarrow \rho(A)(b) \in A \tag{116}$$

$$\rho(A) \cong \text{End}_C(A \cong_{vec} \mathbb{C}^8) \tag{117}$$

Using the above information, we have

$$\begin{bmatrix} qq^\dagger \\ -q_1 q^\dagger \\ -q_2 q^\dagger \\ -q_3 q^\dagger \\ q^\dagger \\ q_{23} q^\dagger \\ q_{31} q^\dagger \\ q_{12} q^\dagger \end{bmatrix} \xrightarrow{q_1^\dagger} \begin{bmatrix} 0 \\ 0 \\ -q_1 q_2 q^\dagger \\ -q_1 q_3 q^\dagger \\ q_1 q^\dagger \\ q q^\dagger \\ 0 \\ 0 \end{bmatrix} \tag{118}$$

So the action of q_1^\dagger can be represented as

$$\begin{bmatrix} 0 \\ 0 \\ -q_1q_2q^\dagger \\ -q_1q_3q^\dagger \\ q_1q^\dagger \\ qq^\dagger \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} qq^\dagger \\ -q_1q^\dagger \\ -q_2q^\dagger \\ -q_3q^\dagger \\ q^\dagger \\ q_{23}q^\dagger \\ q_{31}q^\dagger \\ q_{12}q^\dagger \end{bmatrix} \tag{119}$$

The matrix representation of the q_1^\dagger and other null vectors is therefore given below:

$$q_1^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sigma_2 \\ -i\sigma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad q_2^\dagger = \begin{bmatrix} 0 & 0 & 0 & \sigma_3^- \\ 0 & 0 & -\sigma_3^- & 0 \\ 0 & -\sigma_3^+ & 0 & 0 \\ \sigma_3^+ & 0 & 0 & 0 \end{bmatrix} \tag{120}$$

$$q_3^\dagger = \begin{bmatrix} 0 & 0 & 0 & -\sigma_- \\ 0 & 0 & \sigma_+ & 0 \\ 0 & -\sigma_+ & 0 & 0 \\ \sigma_- & 0 & 0 & 0 \end{bmatrix} \tag{121}$$

$$\sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \sigma_3^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \sigma_3^- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{122}$$

With the matrix definitions of the null vectors, we have matrix representations for other defined elements as given below, the nilpotents, idempotents and the orthonormal vectors, respectively:

$$q^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma_3^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad q = \begin{bmatrix} 0 & 0 & \sigma_3^+ & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{123}$$

$$p = \begin{bmatrix} \sigma_3^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad p' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3^+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{124}$$

The orthonormal vectors are given below:

$$e_1 = \begin{bmatrix} 0 & 0 & i\sigma_2 & 0 \\ 0 & 0 & 0 & -i\sigma_2 \\ -i\sigma_2 & 0 & 0 & 0 \\ 0 & i\sigma_2 & 0 & 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & 1_2 \\ 0 & 0 & -1_2 & 0 \\ 0 & -1_2 & 0 & 0 \\ 1_2 & 0 & 0 & 0 \end{bmatrix} \tag{125}$$

$$e_3 = \begin{bmatrix} 0 & 0 & 0 & i\sigma_2 \\ 0 & 0 & i\sigma_2 & 0 \\ 0 & -i\sigma_2 & 0 & 0 \\ -i\sigma_2 & 0 & 0 & 0 \end{bmatrix} \quad \tilde{e}_1 = \begin{bmatrix} 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_2 \\ \sigma_2 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \end{bmatrix} \tag{126}$$

$$\tilde{e}_2 = \begin{bmatrix} 0 & 0 & 0 & -i\sigma_3 \\ 0 & 0 & i\sigma_3 & 0 \\ 0 & -i\sigma_3 & 0 & 0 \\ i\sigma_3 & 0 & 0 & 0 \end{bmatrix} \quad \tilde{e}_3 = \begin{bmatrix} 0 & 0 & 0 & -i\sigma_1 \\ 0 & 0 & i\sigma_1 & 0 \\ 0 & -i\sigma_1 & 0 & 0 \\ i\sigma_1 & 0 & 0 & 0 \end{bmatrix} \tag{127}$$

To compute the inner product between various orthonormal vectors, we use the matrix multiplication:

$$\vec{a} \cdot \vec{b} = \frac{1}{2}(ab + ba) \tag{128}$$

7.3.1. SU(2) Symmetry

We will first partition this eight-dimensional space into a vector sum of two irreducible spaces of dimension four. Then these four-dimensional spaces have to be further decomposed into irreducible subspaces, defined to be of different chirality. To proceed, we need to define new matrix operators; for the weak isospin $\frac{1}{2}$ and $-\frac{1}{2}$, we use an isospin operator (it decomposes the space into two irreducible representations):

$$e = \begin{bmatrix} 0 & 1_4 \\ -1_4 & 0 \end{bmatrix} \quad \tilde{e} = i \begin{bmatrix} 0 & 1_4 \\ 1_4 & 0 \end{bmatrix} \tag{129}$$

$$e\tilde{e} = i \begin{bmatrix} 1_4 & 0 \\ 0 & -1_4 \end{bmatrix} \tag{130}$$

The $\frac{1 \pm e\tilde{e}}{2}$ operator partitions the \mathbb{C}^8 space into two \mathbb{C}^4 spaces. We have the chirality operator given below:

$$\Gamma^5 = -ie_1\tilde{e}_1 = \begin{bmatrix} 1_2 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \\ 0 & 0 & -1_2 & 0 \\ 0 & 0 & 0 & 1_2 \end{bmatrix} \tag{131}$$

This operator can be used to define projectors on left and right chiral subspaces of two irreducible representations. The minus sign of the chirality operator represents the left chiral subspace. We need to mix the left chiral subspace of the particles for a given $SU_L(2)$ doublet. We can define a new basis of null vectors for the excited weak iso-spin states as given below [23]:

$$w_u = \begin{bmatrix} 0 & 1_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad w_d = \begin{bmatrix} 0 & 0 & -1_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{132}$$

$$w_o = \begin{bmatrix} \sigma_+ & 0 & 0 & 0 \\ 0 & -\sigma_+ & 0 & 0 \\ 0 & 0 & -\sigma_+ & 0 \\ 0 & 0 & 0 & \sigma_+ \end{bmatrix} \tag{133}$$

$$\{w_i, w_j\} = 0 \quad \{w_i^\dagger, w_j^\dagger\} = 0 \quad \{w_i, w_j^\dagger\} = \delta_{ij} \tag{134}$$

We have the matrix representations of the various elements. We can make the following identifications:

$$\begin{bmatrix} p \\ w_o^\dagger p \\ w_u^\dagger p \\ w_u^\dagger w_o^\dagger p \\ w_d^\dagger p \\ w_d^\dagger w_o^\dagger p \\ w_d^\dagger w_u^\dagger p \\ w_d^\dagger w_u^\dagger w_o^\dagger p \end{bmatrix} = \begin{bmatrix} p \\ q_{23}^\dagger p \\ q_{31}^\dagger p \\ q_{12}^\dagger p \\ q_{321}^\dagger p \\ q_1^\dagger p \\ q_2^\dagger p \\ q_3^\dagger p \end{bmatrix} \tag{135}$$

w_u^\dagger represents the creation of the left chiral subspace for an up-isospin particle from the idempotent; similarly, w_d^\dagger represents the creation of the left chiral subspace for a down-

isospin particle. $w_d^\dagger w_u^\dagger$ represents the creation of the right chiral subspace of a down-isospin particle [23]. Observe the following decomposition. Due to isospin projectors and later projections due to the chirality operator, W_{C^j} represents the j -dimensional complex space:

$$W_{\mathbb{C}^8} = W_{\mathbb{C}^4}^1 \oplus W_{\mathbb{C}^4}^2 \tag{136}$$

$$W_{\mathbb{C}^8} = W_{\mathbb{C}^2,R}^1 \oplus W_{\mathbb{C}^2,L}^1 \oplus W_{\mathbb{C}^2,L}^2 \oplus W_{\mathbb{C}^2,R}^2 \tag{137}$$

$$= \{p, w_o^\dagger p\} \oplus \{w_u^\dagger p, w_u^\dagger w_o^\dagger p\} \oplus \{w_d^\dagger p, w_d^\dagger w_o^\dagger p\} \oplus \{w_d^\dagger w_u^\dagger p, w_d^\dagger w_u^\dagger w_o^\dagger p\} \tag{138}$$

$$= \{p, q_{23}^\dagger p\} \oplus \{q_{31}^\dagger p, q_{12}^\dagger p\} \oplus \{q_{321}^\dagger p, q_1^\dagger p\} \oplus \{q_2^\dagger p, q_3^\dagger p\} \tag{139}$$

We now define $SU(2)$ symmetry generators. These will only mix the left chiral space for both fermions:

$$T_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1_2 & 0 \\ 0 & 1_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad T_2 = \frac{-i}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1_2 & 0 \\ 0 & 1_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{140}$$

$$T_3 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \\ 0 & 0 & 1_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{141}$$

$$[T_i, T_j] = i\epsilon_{ijk} T_k \tag{142}$$

Observe that no mixing takes place for the right chiral space.

7.3.2. Complete Space of Ideals

The complete basis of the algebra $Cl(6)$ in terms of the minimal left ideal can be written as given below, where the initial basis is expanded via the right multiplication on that ideal:

$$\begin{bmatrix} p & pq_{23} & pq_{31} & pq_{12} & pq_{321} & pq_1 & pq_2 & pq_3 \\ q_{23}^\dagger p & q_{23}^\dagger pq_{23} & q_{23}^\dagger pq_{31} & q_{23}^\dagger pq_{12} & q_{23}^\dagger pq_{321} & q_{23}^\dagger pq_1 & q_{23}^\dagger pq_2 & q_{23}^\dagger pq_3 \\ q_{31}^\dagger p & q_{31}^\dagger pq_{23} & q_{31}^\dagger pq_{31} & q_{31}^\dagger pq_{12} & q_{31}^\dagger pq_{321} & q_{31}^\dagger pq_1 & q_{31}^\dagger pq_2 & q_{31}^\dagger pq_3 \\ q_{12}^\dagger p & q_{12}^\dagger pq_{23} & q_{12}^\dagger pq_{31} & q_{12}^\dagger pq_{12} & q_{12}^\dagger pq_{321} & q_{12}^\dagger pq_1 & q_{12}^\dagger pq_2 & q_{12}^\dagger pq_3 \\ q_{321}^\dagger p & q_{321}^\dagger pq_{23} & q_{321}^\dagger pq_{31} & q_{321}^\dagger pq_{12} & q_{321}^\dagger pq_{321} & q_{321}^\dagger pq_1 & q_{321}^\dagger pq_2 & q_{321}^\dagger pq_3 \\ q_1^\dagger p & q_1^\dagger pq_{23} & q_1^\dagger pq_{31} & q_1^\dagger pq_{12} & q_1^\dagger pq_{321} & q_1^\dagger pq_1 & q_1^\dagger pq_2 & q_1^\dagger pq_3 \\ q_2^\dagger p & q_2^\dagger pq_{23} & q_2^\dagger pq_{31} & q_2^\dagger pq_{12} & q_2^\dagger pq_{321} & q_2^\dagger pq_1 & q_2^\dagger pq_2 & q_2^\dagger pq_3 \\ q_3^\dagger p & q_3^\dagger pq_{23} & q_3^\dagger pq_{31} & q_3^\dagger pq_{12} & q_3^\dagger pq_{321} & q_3^\dagger pq_1 & q_3^\dagger pq_2 & q_3^\dagger pq_3 \end{bmatrix}_{8 \times 8} \tag{143}$$

Now we can identify four-dimensional spaces using the classifier spaces, isospin spaces and spinor chiral spaces with various particles [28]. We use the elements from row 1 and row 5 to assign the electric charge to the two four-dimensional column spinors present in a column by calculating the total electric charge from the product of the creation

and the annihilation operators. We identify these bases with the following particle spaces, remembering that these are now complex numbers on which $M_8(C)$ can act from the left:

$$\begin{bmatrix}
 \nu_{R_1} & u_{R_1}^r & u_{R_1}^b & u_{R_1}^g & \bar{e}_{L_1} & \bar{d}_{L_1}^r & \bar{d}_{L_1}^b & \bar{d}_{L_1}^g \\
 \nu_{R_2} & u_{R_2}^r & u_{R_2}^b & u_{R_2}^g & \bar{e}_{L_2} & \bar{d}_{L_2}^r & \bar{d}_{L_2}^b & \bar{d}_{L_2}^g \\
 \nu_{L_1} & u_{L_1}^r & u_{L_1}^b & u_{L_1}^g & \bar{e}_{R_1} & \bar{d}_{R_1}^r & \bar{d}_{R_1}^b & \bar{d}_{R_1}^g \\
 \nu_{L_2} & u_{L_2}^r & u_{L_2}^b & u_{L_2}^g & \bar{e}_{R_2} & \bar{d}_{R_2}^r & \bar{d}_{R_2}^b & \bar{d}_{R_2}^g \\
 e_{L_1} & d_{L_1}^r & d_{L_1}^g & d_{L_1}^b & \bar{\nu}_{R_1} & \bar{u}_{R_1}^r & \bar{u}_{R_1}^b & \bar{u}_{R_1}^g \\
 e_{L_2} & d_{L_2}^r & d_{L_2}^g & d_{L_2}^b & \bar{\nu}_{R_2} & \bar{u}_{R_2}^r & \bar{u}_{R_2}^b & \bar{u}_{R_2}^g \\
 e_{R_1} & d_{R_1}^r & d_{R_1}^g & d_{R_1}^b & \bar{\nu}_{L_1} & \bar{u}_{L_1}^r & \bar{u}_{L_1}^b & \bar{u}_{L_1}^g \\
 e_{R_2} & d_{R_2}^r & d_{R_2}^g & d_{R_2}^b & \bar{\nu}_{L_2} & \bar{u}_{L_2}^r & \bar{u}_{L_2}^b & \bar{u}_{L_2}^g
 \end{bmatrix} \text{Charge} \tag{144}$$

7.3.3. Left Action on the Space of Ideals

Now, we have arranged our total complex ideal space in such a manner that left multiplication will only cause transformation within an ideal. We already showed our $SU(2)$ generators and their intended action on an ideal (a C^8 column, basically). It is important to notice that for $Cl(6) \cong Cl(4) \otimes Cl(2)$, now $Cl(4)$ represents the Dirac algebra and $Cl(2)$ represents the spin algebra. Essential transformations are basically Lorentzian in nature and $SU(2)$ transformations. Thus, if we want to include spin in our analysis, we can do so by looking at the algebra $Cl(4) \otimes Cl(2) \otimes Cl(2) \cong Cl(4)_{Dirac} \otimes Cl(2)_{Iso-spin} \otimes Cl(2)_{Spin}$ and the left action of various elements of $Cl(8)$ algebra on the ideals of the $Cl(8)$.

7.3.4. Right Action on the Space of Ideals

Looking at the total space of ideals, we see that a right multiplication by $M_8(C)$ will permute the columns. It can basically change the color space of various quarks. So, here, essential transformations for us will be $SU(3)$ transformations. The matrices that can do so will form one-to-one correspondence with Gell-Mann $SU(3)$ matrices [28].

7.4. $Cl(7)$ Algebra

We have

$$Cl(7) = \mathbb{C} \times \mathbb{O} \oplus \omega(\mathbb{C} \times \mathbb{O}) = Cl(6) \oplus Cl(6) \tag{145}$$

With the above information, we proceed for the extended gauge group $SU(3)_{grav} \times SU(2)_R \times U_g(1)$. With this, we can define a new internal space, as performed earlier for all the particles. It is given in Table 6.

Table 6. New symmetry group.

Force/Charge	Internal Space	Dimension	Symmetry
Gravi Electromagnetism	χ_{gem}	1	U(1)
Gravi Strong	χ_{grav}	3	SU(3)
Gravi-Weak Hypercharge	χ_g	1	U(1)
Gravi-Weak Electromagnetism	χ_{g-ew}	2	U(2)

As done earlier, we again define a space χ as given below:

$$\chi \cong \bar{\chi}_{gem} \otimes \bar{\chi}_{grav} \tag{146}$$

The space χ_{gem} assigns $-\frac{1}{3}$ units of the gravi charge to the particles. We again have three null basis vectors for this tensor product space. Every basis represents a **gravi charge** of $\frac{1}{3}$, and each of the three anti-colors is related to $SU(3)_{grav}$. The **gravi charge** is additive in

nature and it will add up for a product of the null basis vectors. For χ space, we denote the basis as $\{q_i^\dagger\}_{i=1,2,3}$. Each basis vector has a gravi charge equal to $\frac{1}{3}$ and one gravi anti-color. We then have the total space as $\chi \oplus \chi^\dagger$, with their basis vectors as given below:

$$\chi = \{q_1^\dagger, q_2^\dagger, q_3^\dagger\} \quad \chi^\dagger = \{q_1, q_2, q_3\} \tag{147}$$

With this notation, we can proceed further and classify particles according to the representations of the exterior algebra. This is shown in Tables 7 and 8.

Table 7. Internal space due to extended symmetry group.

Particle	Internal Space
d	χ_{gem}^3
\bar{u}	$\chi_{gem}^2 \bar{\chi}_{grav}$
e^-	$\chi_{gem} \chi_{grav}$
\bar{v}	\mathbb{C}
v	\mathbb{C}
e^+	$\chi_{gem}^{-1} \bar{\chi}_{grav}$
u	$\chi_{gem}^{-2} \chi_{grav}$
\bar{d}	χ_{gem}^{-3}

Table 8. Particles in exterior algebra.

Particle	Vectors in Exterior Space
d	$\Lambda^3 \bar{\chi}$
\bar{u}	$\Lambda^2 \bar{\chi}$
e^-	$\Lambda^1 \bar{\chi}$
\bar{v}	$\Lambda^0 \bar{\chi} \cong \mathbb{C}$
v	$\Lambda^0 \chi \cong \mathbb{C}$
e^+	$\Lambda^1 \chi$
u	$\Lambda^2 \chi$
\bar{d}	$\Lambda^3 \chi$

Now for the other copy of $Cl(6)$, we can use the complex conjugated vector space and, similarly, p' as the idempotent. The new basis will then be

$$\left\{ p', q_{23} p', q_{31} p', q_{12} p', q_{321} p', q_1 p', q_2 p', q_3 p' \right\} \tag{148}$$

$$\left\{ q^\dagger q, q_1^\dagger q, q_2^\dagger q, q_3^\dagger q, -q, q_{32}^\dagger q, q_{13}^\dagger q, q_{21}^\dagger q \right\} \tag{149}$$

Similarly, we can define the complete space of ideals as defined earlier:

$$\begin{bmatrix} p' & p'q_{32}^\dagger & p'q_{13}^\dagger & p'q_{21}^\dagger & p'q_{321}^\dagger & p'q_1^\dagger & p'q_2^\dagger & p'q_3^\dagger \\ q_{23}p' & q_{23}p'q_{32}^\dagger & q_{23}p'q_{13}^\dagger & q_{23}p'q_{21}^\dagger & q_{23}p'q_{321}^\dagger & q_{23}p'q_1^\dagger & q_{23}p'q_2^\dagger & q_{23}p'q_3^\dagger \\ q_{31}p' & q_{31}p'q_{32}^\dagger & q_{31}p'q_{13}^\dagger & q_{31}p'q_{21}^\dagger & q_{31}p'q_{321}^\dagger & q_{31}p'q_1^\dagger & q_{31}p'q_2^\dagger & q_{31}p'q_3^\dagger \\ q_{12}p' & q_{12}p'q_{32}^\dagger & q_{12}p'q_{13}^\dagger & q_{12}p'q_{21}^\dagger & q_{12}p'q_{321}^\dagger & q_{12}p'q_1^\dagger & q_{12}p'q_2^\dagger & q_{12}p'q_3^\dagger \\ q_{321}p' & q_{321}p'q_{32}^\dagger & q_{321}p'q_{13}^\dagger & q_{321}p'q_{21}^\dagger & q_{321}p'q_{321}^\dagger & q_{321}p'q_1^\dagger & q_{321}p'q_2^\dagger & q_{321}p'q_3^\dagger \\ q_1p' & q_1p'q_{32}^\dagger & q_1p'q_{13}^\dagger & q_1p'q_{21}^\dagger & q_1p'q_{321}^\dagger & q_1p'q_1^\dagger & q_1p'q_2^\dagger & q_1p'q_3^\dagger \\ q_2p' & q_2p'q_{32}^\dagger & q_2p'q_{13}^\dagger & q_2p'q_{21}^\dagger & q_2p'q_{321}^\dagger & q_2p'q_1^\dagger & q_2p'q_2^\dagger & q_2p'q_3^\dagger \\ q_3p' & q_3p'q_{32}^\dagger & q_3p'q_{13}^\dagger & q_3p'q_{21}^\dagger & q_3p'q_{321}^\dagger & q_3p'q_1^\dagger & q_3p'q_2^\dagger & q_3p'q_3^\dagger \end{bmatrix}_{8 \times 8} \tag{150}$$

As done earlier, we can obtain a matrix representation of the elements of $Cl(6)$ by the left action of various elements on the left ideal. We have a method to compute the $U(1)$ charges using the classifier space. We employed this method to assign electric charges to four-dimensional column vectors and hence classify the various subspaces of the complete space of ideals as particles. We use the same method and classify particles according to the gravi-charges.

7.4.1. Right Adjoint Action

The right action has a similar working. $M_8(C)$ acting from the right can permute the columns and hence can cause color changes for colored particles. We have similar matrices for such a transformation as we defined earlier for $SU(3)$. Here, too, we can do the same for $SU(3)_{grav}$, the gravi color symmetry.

7.4.2. Left Adjoint Action

For the left action of the elements of the algebra, the space of ideals is closed. This gives us the matrix representations of the algebraic elements. But now, we want our spinors such that they are $SU(2)_R$ active, which means that their right chiral space mixes due to $SU(2)_R$. We can define a new basis of gravi weak isospin null vectors and similarly a set of $SU(2)$ generators:

$$\bar{w}_u = \begin{bmatrix} 0 & -1_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \bar{w}_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1_2 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \end{bmatrix} \tag{151}$$

$$\bar{w}_o = - \begin{bmatrix} \sigma_+ & 0 & 0 & 0 \\ 0 & -\sigma_+ & 0 & 0 \\ 0 & 0 & -\sigma_+ & 0 \\ 0 & 0 & 0 & \sigma_+ \end{bmatrix} \tag{152}$$

$$\{\bar{w}_i, \bar{w}_j\} = 0 \quad \{\bar{w}_i^\dagger, \bar{w}_j^\dagger\} = 0 \quad \{\bar{w}_i, \bar{w}_j^\dagger\} = \delta_{ij} \tag{153}$$

$$\begin{bmatrix} p' \\ q_{23}p' \\ q_{31}p' \\ q_{12}p' \\ q_{321}p' \\ q_1p' \\ q_2p' \\ q_3p' \end{bmatrix} \longrightarrow \begin{bmatrix} p' \\ \bar{w}_o^\dagger p' \\ \bar{w}_u^\dagger p' \\ \bar{w}_u^\dagger \bar{w}_o^\dagger p' \\ \bar{w}_d^\dagger p' \\ \bar{w}_d^\dagger \bar{w}_o^\dagger p' \\ \bar{w}_d^\dagger \bar{w}_u^\dagger p' \\ \bar{w}_d^\dagger \bar{w}_u^\dagger \bar{w}_o^\dagger p' \end{bmatrix} \longrightarrow \begin{bmatrix} W_{1R} \\ W_{1L} \\ W_{2L} \\ W_{2R} \end{bmatrix} \tag{154}$$

Interpret these new null vectors as follows: \bar{w}_u^\dagger as the creation operator of the left chiral subspace of the gravi weak up-isospin particle, and \bar{w}_d^\dagger as the creation operator to generate the left chiral subspace of the gravi weak down-isospin particle. Similarly, $\bar{w}_d^\dagger \bar{w}_u^\dagger$

generates the right chiral subspace for the gravi weak down-isospin particle. With these definitions for the null basis, we can define an orthonormal basis too, as defined earlier:

$$\bar{u}_j = \bar{\omega}_j + \bar{\omega}_j^\dagger \tag{155}$$

$$\bar{u}'_j = i(\bar{\omega}_j^\dagger - \bar{\omega}_j) \tag{156}$$

We have the following set of orthonormal vectors:

$$\left\{ \bar{u}_u, \bar{u}_d, \bar{u}_o, \bar{u}'_u, \bar{u}'_d, \bar{u}'_o \right\} \tag{157}$$

We now check the action of the SU(2) operator constructed from the \bar{u}_i and \bar{u}'_i . Define the new SU(2) generators as the following:

$$\bar{T}_1 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1_2 & 0 & 0 & 0 \end{bmatrix} \quad \bar{T}_2 = \frac{-i}{2} \begin{bmatrix} 0 & 0 & 0 & 1_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1_2 & 0 & 0 & 0 \end{bmatrix} \tag{158}$$

$$\bar{T}_3 = \frac{1}{2} \begin{bmatrix} 1_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_2 \end{bmatrix} \tag{159}$$

$$[\bar{T}_i, \bar{T}_j] = i\epsilon_{ijk}\bar{T}_k \tag{160}$$

Look carefully; it does not mix the left chiral components of the spinors from the two irreducible representations of different chirality. Hence, it gives us the gravitationally active right chiral spinors.

7.4.3. Particle Identification

Now, we can proceed further and identify the various particles in the new complete space of ideals:

$$\begin{bmatrix} \nu_{R_1} & u^r_{R_1} & u^b_{R_1} & u^s_{R_1} & \bar{d}_{L_1} & \bar{e}^r_{L_1} & \bar{e}^b_{L_1} & \bar{e}^s_{L_1} \\ \nu_{R_2} & u^r_{R_2} & u^b_{R_2} & u^s_{R_2} & \bar{d}_{L_2} & \bar{e}^r_{L_2} & \bar{e}^b_{L_2} & \bar{e}^s_{L_2} \\ \nu_{L_1} & u^r_{L_1} & u^b_{L_1} & u^s_{L_1} & \bar{d}_{R_1} & \bar{e}^r_{R_1} & \bar{e}^b_{R_1} & \bar{e}^s_{R_1} \\ \nu_{L_2} & u^r_{L_2} & u^b_{L_2} & u^s_{L_2} & \bar{d}_{R_2} & \bar{e}^r_{R_2} & \bar{e}^b_{R_2} & \bar{e}^s_{R_2} \\ d_{L_1} & e^r_{L_1} & e^s_{L_1} & e^b_{L_1} & \bar{\nu}_{R_1} & \bar{u}^r_{R_1} & \bar{u}^b_{R_1} & \bar{u}^s_{R_1} \\ d_{L_2} & e^r_{L_2} & e^s_{L_2} & e^b_{L_2} & \bar{\nu}_{R_2} & \bar{u}^r_{R_2} & \bar{u}^b_{R_2} & \bar{u}^s_{R_2} \\ d_{R_1} & e^r_{R_1} & e^s_{R_1} & e^b_{R_1} & \bar{\nu}_{L_1} & \bar{u}^r_{L_1} & \bar{u}^b_{L_1} & \bar{u}^s_{L_1} \\ d_{R_2} & e^r_{R_2} & e^s_{R_2} & e^b_{R_2} & \bar{\nu}_{L_2} & \bar{u}^r_{L_2} & \bar{u}^b_{L_2} & \bar{u}^s_{L_2} \end{bmatrix}_{Mass} \tag{161}$$

This is performed using the classifier space, weak force generators and SU(3) operations. This gives us the following gravi weak isospin doublets SU(2)_R:

First generation

$$\begin{pmatrix} u \\ e^- \end{pmatrix}_R \quad \begin{pmatrix} \nu \\ d \end{pmatrix}_R \tag{162}$$

Second generation

$$\begin{pmatrix} t \\ \mu^- \end{pmatrix}_R \quad \begin{pmatrix} \nu_\mu \\ b \end{pmatrix}_R \tag{163}$$

Third generation

$$\begin{pmatrix} c \\ \tau^- \end{pmatrix}_R \quad \begin{pmatrix} \nu_\tau \\ s \end{pmatrix}_R \tag{164}$$

7.5. Triality and $Cl(8)$ Algebra

The basic reason to look into the $Cl(8)$ algebra is to use triality mapping. Triality mapping is generally a very interesting object to study. Some authors have pointed towards its importance in studying three generations [2,19]:

$$Cl^{even}(8) \cong Cl(7) \cong M_8(C) \oplus M_8(C) \tag{165}$$

As explained earlier, $M_8(C) \oplus M_8(C)$ acts on a spinor space $S_8^+ \oplus S_8^-$. Both S_8^+ and S_8^- are eight-dimensional complex spinor spaces. The eight generators of the $Cl(8)$ algebra give us the vector representation denoted by V_8 . These can be considered the basis vectors of the underlying vector space. Triality denoted by t_8 is defined as the following mapping [19]:

$$t_8 : S_8^+ \times S_8^- \times V_8 \longrightarrow \mathbb{C} \tag{166}$$

So it basically takes three complex vector spaces and gives us a number as an output. Now, focus on the space of the ideals for the $Cl(8)$ algebra. We saw earlier that even the subalgebra of $Cl(8)$ is the same as $Cl(7)$, and we know that $Cl(7) \cong Cl(6) \oplus Cl(6)$, so the subspace, the even subalgebra of $Cl(8)$, is the same as the direct sum of the left ideal space of the two copies of $Cl(6)$.

7.5.1. Space of Ideals in $Cl(8)$

We require an eight-dimensional null basis to obtain the complete maximally totally isotropic subspace of the null vectors. To the six-dimensional vector space of $\chi \oplus \chi^\dagger$, add a two-dimensional space S for the two spin vectors. Our final underlying space will then be $\chi \oplus \chi^\dagger \oplus S$. To describe this new space, we also add $\{q_4, q_4^\dagger\}$ to the pre-existing set of null vectors. Now, any element in ideal will be a product from these eight vectors, then we have [28]

$$Cl(8) \cong Cl(4) \otimes Cl(2) \otimes Cl(2) \cong Cl(4)_{Dirac} \otimes Cl(2)_{Iso-spin} \otimes Cl(2)_{Spin} \tag{167}$$

$$\{q_1, q_2, q_3, q_4, q_1^\dagger, q_2^\dagger, q_3^\dagger, q_4^\dagger\} \tag{168}$$

$$q = q_1 q_2 q_3 q_4 \quad q^\dagger = q_4^\dagger q_3^\dagger q_2^\dagger q_1^\dagger \tag{169}$$

$$p = q q^\dagger \quad p' = q^\dagger q \tag{170}$$

Here, q and q^\dagger are the nilpotents, and p and p' are the idempotents. We use p as the idempotent; from the previous information, we know the importance of $Cl^{even}(8)$, so we can write the $Cl^{even}(8)$ ideal subspace as [28]

$$Cl(8) = \begin{bmatrix} Even_1 & Odd \\ Odd & Even_2 \end{bmatrix} \implies Cl^{even}(8) = \begin{bmatrix} Even_1 & 0 \\ 0 & Even_2 \end{bmatrix} \tag{171}$$

The $Even_1$ part of the complete space of the ideal of $Cl(8)$ is given below:

$$\begin{bmatrix}
 p & pq_{32} & pq_{13} & pq_{21} & pq_{4321} & pq_{41} & pq_{42} & pq_{43} \\
 q_{23}^\dagger p & q_{23}^\dagger pq_{32} & q_{23}^\dagger pq_{13} & q_{23}^\dagger pq_{21} & q_{23}^\dagger pq_{4321} & q_{23}^\dagger pq_{41} & q_{23}^\dagger pq_{42} & q_{23}^\dagger pq_{43} \\
 q_{31}^\dagger p & q_{31}^\dagger pq_{32} & q_{31}^\dagger pq_{13} & q_{31}^\dagger pq_{21} & q_{31}^\dagger pq_{4321} & q_{31}^\dagger pq_{41} & q_{31}^\dagger pq_{42} & q_{31}^\dagger pq_{43} \\
 q_{12}^\dagger p & q_{12}^\dagger pq_{32} & q_{12}^\dagger pq_{13} & q_{12}^\dagger pq_{21} & q_{12}^\dagger pq_{4321} & q_{12}^\dagger pq_{41} & q_{12}^\dagger pq_{42} & q_{12}^\dagger pq_{43} \\
 q_{1234}^\dagger p & q_{1234}^\dagger pq_{32} & q_{1234}^\dagger pq_{13} & q_{1234}^\dagger pq_{21} & q_{1234}^\dagger pq_{4321} & q_{1234}^\dagger pq_{41} & q_{1234}^\dagger pq_{42} & q_{1234}^\dagger pq_{43} \\
 q_{14}^\dagger p & q_{14}^\dagger pq_{32} & q_{14}^\dagger pq_{13} & q_{14}^\dagger pq_{21} & q_{14}^\dagger pq_{4321} & q_{14}^\dagger pq_{41} & q_{14}^\dagger pq_{42} & q_{14}^\dagger pq_{43} \\
 q_{24}^\dagger p & q_{24}^\dagger pq_{32} & q_{24}^\dagger pq_{13} & q_{24}^\dagger pq_{21} & q_{24}^\dagger pq_{4321} & q_{24}^\dagger pq_{41} & q_{24}^\dagger pq_{42} & q_{24}^\dagger pq_{43} \\
 q_{34}^\dagger p & q_{34}^\dagger pq_{32} & q_{34}^\dagger pq_{13} & q_{34}^\dagger pq_{21} & q_{34}^\dagger pq_{4321} & q_{34}^\dagger pq_{41} & q_{34}^\dagger pq_{42} & q_{34}^\dagger pq_{43}
 \end{bmatrix} \tag{172}$$

The $Even_2$ part of the complete space of the ideal of $Cl(8)$ is given below:

$$\begin{bmatrix}
 q_4^\dagger pq_4 & q_4^\dagger pq_{432} & q_4^\dagger pq_{413} & q_4^\dagger pq_{421} & q_4^\dagger pq_{321} & q_4^\dagger pq_1 & q_4^\dagger pq_2 & q_4^\dagger pq_3 \\
 q_{234}^\dagger pq_4 & q_{234}^\dagger pq_{432} & q_{234}^\dagger pq_{413} & q_{234}^\dagger pq_{421} & q_{234}^\dagger pq_{321} & q_{234}^\dagger pq_1 & q_{234}^\dagger pq_2 & q_{234}^\dagger pq_3 \\
 q_{314}^\dagger pq_4 & q_{314}^\dagger pq_{432} & q_{314}^\dagger pq_{413} & q_{314}^\dagger pq_{421} & q_{314}^\dagger pq_{321} & q_{314}^\dagger pq_1 & q_{314}^\dagger pq_2 & q_{314}^\dagger pq_3 \\
 q_{124}^\dagger pq_4 & q_{124}^\dagger pq_{432} & q_{124}^\dagger pq_{413} & q_{124}^\dagger pq_{421} & q_{124}^\dagger pq_{321} & q_{124}^\dagger pq_1 & q_{124}^\dagger pq_2 & q_{124}^\dagger pq_3 \\
 q_{123}^\dagger pq_4 & q_{123}^\dagger pq_{432} & q_{123}^\dagger pq_{413} & q_{123}^\dagger pq_{421} & q_{123}^\dagger pq_{321} & q_{123}^\dagger pq_1 & q_{123}^\dagger pq_2 & q_{123}^\dagger pq_3 \\
 q_1^\dagger pq_4 & q_1^\dagger pq_{432} & q_1^\dagger pq_{413} & q_1^\dagger pq_{421} & q_1^\dagger pq_{321} & q_1^\dagger pq_1 & q_1^\dagger pq_2 & q_1^\dagger pq_3 \\
 q_2^\dagger pq_4 & q_2^\dagger pq_{432} & q_2^\dagger pq_{413} & q_2^\dagger pq_{421} & q_2^\dagger pq_{321} & q_2^\dagger pq_1 & q_2^\dagger pq_2 & q_2^\dagger pq_3 \\
 q_3^\dagger pq_4 & q_3^\dagger pq_{432} & q_3^\dagger pq_{413} & q_3^\dagger pq_{421} & q_3^\dagger pq_{321} & q_3^\dagger pq_1 & q_3^\dagger pq_2 & q_3^\dagger pq_3
 \end{bmatrix} \tag{173}$$

We know that there is a volume element in $Cl(7)$ algebra that can partition the algebra into two parts. Here, the $Cl^{even}(8)$ algebra gets partitioned into two parts depending upon whether the element is self-dual or not. So by this, we can assign different spins to both the even parts. Let us assign spin up to $Even_1$ and spin down to $Even_2$ parts of the $Cl^{even}(8)$. By our previous arguments, we know that a correspondence can be established between each even part of the $Cl(8)$ algebra and two copies of $Cl(6)$, so we can identify a given subspace of even subalgebra by particles from one generation with two different definite spins. Let us make some identifications; for example, for the $SU(2)_L$ active particles, we can identify the $Even_1$ part as

$$A^\dagger = \begin{bmatrix}
 \uparrow_{R_1} & \uparrow_{R_1} & \uparrow_{R_1} & \uparrow_{R_1} & \uparrow_{L_1} & \uparrow_{L_1} & \uparrow_{L_1} & \uparrow_{L_1} \\
 \uparrow_{R_2} & \uparrow_{R_2} & \uparrow_{R_2} & \uparrow_{R_2} & \uparrow_{L_2} & \uparrow_{L_2} & \uparrow_{L_2} & \uparrow_{L_2} \\
 \uparrow_{L_1} & \uparrow_{L_1} & \uparrow_{L_1} & \uparrow_{L_1} & \uparrow_{R_1} & \uparrow_{R_1} & \uparrow_{R_1} & \uparrow_{R_1} \\
 \uparrow_{L_2} & \uparrow_{L_2} & \uparrow_{L_2} & \uparrow_{L_2} & \uparrow_{R_2} & \uparrow_{R_2} & \uparrow_{R_2} & \uparrow_{R_2} \\
 \uparrow_{L_1} & \uparrow_{L_1} & \uparrow_{L_1} & \uparrow_{L_1} & \uparrow_{R_1} & \uparrow_{R_1} & \uparrow_{R_1} & \uparrow_{R_1} \\
 \uparrow_{L_2} & \uparrow_{L_2} & \uparrow_{L_2} & \uparrow_{L_2} & \uparrow_{R_2} & \uparrow_{R_2} & \uparrow_{R_2} & \uparrow_{R_2} \\
 \uparrow_{R_1} & \uparrow_{R_1} & \uparrow_{R_1} & \uparrow_{R_1} & \uparrow_{L_1} & \uparrow_{L_1} & \uparrow_{L_1} & \uparrow_{L_1} \\
 \uparrow_{R_2} & \uparrow_{R_2} & \uparrow_{R_2} & \uparrow_{R_2} & \uparrow_{L_2} & \uparrow_{L_2} & \uparrow_{L_2} & \uparrow_{L_2}
 \end{bmatrix} \text{Charge} \tag{174}$$

Similarly, the $Even_2$ part can be identified by the second-generation $SU(2)_L$ particle eigenstates. We replace the particles with the corresponding second-generation particles:

$$\{v, \bar{v}\} \longrightarrow \{v_\mu, \bar{v}_\mu\} \tag{175}$$

$$\{e, \bar{e}\} \longrightarrow \{\mu, \bar{\mu}\} \tag{176}$$

$$\{u, d\} \longrightarrow \{c, s\} \tag{177}$$

$$\{\bar{u}, \bar{d}\} \longrightarrow \{\bar{c}, \bar{s}\} \tag{178}$$

However, this family will have the opposite spin sign; let us denote the second-generation $SU(2)_L$ active family with down spin as B^\downarrow . Similarly, the third-generation family with up spin can be represented as C^\uparrow . So the total $SU(2)_L$ active vector spaces ($\mathbb{C}^8 \times \mathbb{C}^8$) with different spins available to us can be listed as follows:

$$\{A^\uparrow, A^\downarrow, B^\uparrow, B^\downarrow, C^\uparrow, C^\downarrow\} \tag{179}$$

Now, observe the following:

$$Cl(9) = Cl(7) \otimes Cl(2) = (C \times O \oplus \omega(C \times O)) \otimes Cl(2) = Cl(8) \oplus Cl(8) \tag{180}$$

Now we can use one copy of $Cl(8)$ to construct the representations for left active $SU(2)_L$ particles. The other copy of $Cl(8)$ can be used to construct the right active $SU(2)_R$ particles. Both copies will give us the spin-up and spin-down particles. For $SU(2)_R$ active particles, we can use the complexified space of ideals and use the p' as the idempotent. We perform a similar procedure; now, again, the $Cl^{even}(8)$ algebra will be partitioned into two subalgebras denoting different spins. An example of the particle identification of different gravi charges and the $SU(2)_R$ active first generation is present below:

$$P^\uparrow = \begin{bmatrix} v_{R_1}^\uparrow & u_{R_1}^\uparrow & u_{R_1}^{b\uparrow} & u_{R_1}^{s\uparrow} & \bar{d}_{L_1} & \bar{e}_{L_1}^\uparrow & \bar{e}_{L_1}^{b\uparrow} & \bar{e}_{L_1}^{s\uparrow} \\ v_{R_2}^\uparrow & u_{R_2}^\uparrow & u_{R_2}^{b\uparrow} & u_{R_2}^{s\uparrow} & \bar{d}_{L_2} & \bar{e}_{L_2}^\uparrow & \bar{e}_{L_2}^{b\uparrow} & \bar{e}_{L_2}^{s\uparrow} \\ v_{L_1}^\uparrow & u_{L_1}^\uparrow & u_{L_1}^{b\uparrow} & u_{L_1}^{s\uparrow} & \bar{d}_{R_1} & \bar{e}_{R_1}^\uparrow & \bar{e}_{R_1}^{b\uparrow} & \bar{e}_{R_1}^{s\uparrow} \\ v_{L_2}^\uparrow & u_{L_2}^\uparrow & u_{L_2}^{b\uparrow} & u_{L_2}^{s\uparrow} & \bar{d}_{R_2} & \bar{e}_{R_2}^\uparrow & \bar{e}_{R_2}^{b\uparrow} & \bar{e}_{R_2}^{s\uparrow} \\ d_{L_1}^\uparrow & e_{L_1}^\uparrow & e_{L_1}^{s\uparrow} & e_{L_1}^{b\uparrow} & \bar{v}_{R_1} & \bar{u}_{R_1}^\uparrow & \bar{u}_{R_1}^{b\uparrow} & \bar{u}_{R_1}^{s\uparrow} \\ d_{L_2}^\uparrow & e_{L_2}^\uparrow & e_{L_2}^{s\uparrow} & e_{L_2}^{b\uparrow} & \bar{v}_{R_2} & \bar{u}_{R_2}^\uparrow & \bar{u}_{R_2}^{b\uparrow} & \bar{u}_{R_2}^{s\uparrow} \\ d_{R_1}^\uparrow & e_{R_1}^\uparrow & e_{R_1}^{s\uparrow} & e_{R_1}^{b\uparrow} & \bar{v}_{L_1} & \bar{u}_{L_1}^\uparrow & \bar{u}_{L_1}^{b\uparrow} & \bar{u}_{L_1}^{s\uparrow} \\ d_{R_2}^\uparrow & e_{R_2}^\uparrow & e_{R_2}^{s\uparrow} & e_{R_2}^{b\uparrow} & \bar{v}_{L_2} & \bar{u}_{L_2}^\uparrow & \bar{u}_{L_2}^{b\uparrow} & \bar{u}_{L_2}^{s\uparrow} \end{bmatrix}_{Mass} \tag{181}$$

Similarly, the second family will be represented by Q and the third family by R , both presenting as spin up and spin down. The three mass families with different spins that transform according to $SU(3)_{grav} \times SU(2)_R \times U(1)_g$ can be represented as

$$\{P^\uparrow, P^\downarrow, Q^\uparrow, Q^\downarrow, R^\uparrow, R^\downarrow\} \tag{182}$$

7.5.2. Triality Operator

The action of the triality operator on $Cl(8)$ representations [19,23,29] can be seen as

$$Trial : \{V_8, S_8^+, S_8^-\} \longrightarrow \{S_8^+, S_8^-, V_8\} \tag{183}$$

$$Trial : \begin{pmatrix} A^\uparrow & 0 \\ 0 & B^\downarrow \end{pmatrix} = \begin{pmatrix} C^\uparrow & 0 \\ 0 & A^\downarrow \end{pmatrix} \tag{184}$$

where $\{A, B, C\}$ represents the usual $SU(2)_L$ active generations. Now look at the $Cl(9)$ algebra. It gives us the spin up and spin down for both flavors as well as mass eigenstates; one transforms according to $SU(2)_L$ and the other transforms according to $SU(2)_R$. If $\{P, Q, R\}$ are the three generations that transform according to $SU(2)_R$, then the total space for us is

$$\begin{pmatrix} A^\uparrow \oplus A^\downarrow \\ B^\uparrow \oplus B^\downarrow \\ C^\uparrow \oplus C^\downarrow \end{pmatrix} \oplus \begin{pmatrix} P^\uparrow \oplus P^\downarrow \\ Q^\uparrow \oplus Q^\downarrow \\ R^\uparrow \oplus R^\downarrow \end{pmatrix} \tag{185}$$

$$O = 1 \otimes Trial \tag{186}$$

Now, if we operate the operator O on our total space, we can group various mass and flavor families in a given $Cl(9)$ algebra by permuting the rows. This gives us a theoretical framework to construct the CKM matrix.

8. CKM Matrix Parameters

Let us focus our attention on one generation that transforms according to $SU(3)_{grav} \times SU(2)_R \times U(1)_g$. Here, we have eight mass eigenstates or the particles in one generation, considering the two particles that transform according to $SU(3)_{grav}$. Here, we develop some isomorphisms to make further progress. As we already had octonionic representations of various particles and quaternionic representations of particles from one generation, it was natural to proceed with them. However, those methods did not yield any significant progress, which forced us to adopt the method given below.

8.1. Gravi-Charge Operator

We can develop an isomorphism from the space of representations (space of ideals) of one generation of mass eigenstates to an eight-dimensional complex vector space. For some definite spin, suppressing the spin, we can write the above argument of isomorphism for all the particles for three generations as given below:

$$\{P, Q, R\} \longrightarrow \mathbb{C}^8 \oplus \mathbb{C}^8 \oplus \mathbb{C}^8 \tag{187}$$

We can now act on this space of $\mathbb{C}^8 \oplus \mathbb{C}^8 \oplus \mathbb{C}^8$ with an operator G —the gravi charge operator—to assign the gravi charges to various particles:

$$G = M_8(\mathbb{C}) \oplus M_8(\mathbb{C}) \oplus M_8(\mathbb{C}) \tag{188}$$

$$M_8(\mathbb{C}) = \begin{bmatrix} g_1 & 0 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 & 0 \\ 0 & 0 & g_3 & 0 & 0 \\ 0 & 0 & 0 & g_4 & 0 \\ 0 & 0 & 0 & 0 & \dots \end{bmatrix} \tag{189}$$

This matrix $M_8(\mathbb{C})$ will be used three times for three mass families. So the gravi charge operator only has diagonal entries. It acts on linear column vectors that are $SU(2)_R$ mass eigenstates and assigns them a gravi charge.

8.2. Mass and Gravi-Charge

Now before moving further, we make some assumptions:

- Mass is a derived quantity. The gravi-charge is more fundamental.
- The mass operator will be constructed from the gravi-charge operator, and the gravi-charge eigenvectors are weighed accordingly by the value of the square root of the mass of respective particles to make them massive eigenvectors.

8.3. Left Handed Quarks

We now look only at a part of the operator G and its action on down, charm and strange quarks, and similarly, the action on up, charm and top quarks. The operator G can be reduced to a small matrix representation as given below:

$$G = \begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{pmatrix} \tag{190}$$

It acts on $SU(2)_R$ active mass eigenstates and gives the gravi charges:

$$\{u, c, t\} \in \mathbb{C}^3 \tag{191}$$

$$\{d, s, b\} \in \mathbb{C}^3 \tag{192}$$

Observe that the \mathbb{C}^3 vector space is needed for both families of quarks Figure 3.

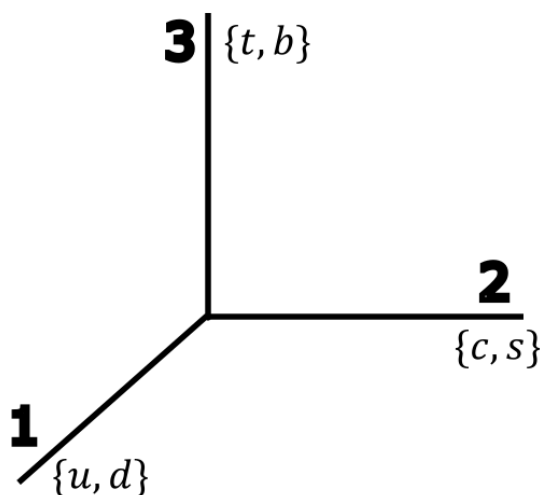


Figure 3. Basis vectors of reduced vector space \mathbb{C}^3 act as $SU(2)_R$ active quarks. The space is used for $SU(2)_L$ active quarks.

We will use this later, when one axis represents one quark from the up-isospin family and one from the down-isospin family. This is done to observe the transformation between quark states. The right-handed up quarks (eigenstates of the gravi charge operators) are given by

$$u_{g,R} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad c_{g,R} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad t_{g,R} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{193}$$

We can define massive quark vectors as

$$u_{m,R} = \begin{pmatrix} \sqrt{m_u} \\ 0 \\ 0 \end{pmatrix} \quad c_{m,R} = \begin{pmatrix} 0 \\ \sqrt{m_c} \\ 0 \end{pmatrix} \quad t_{m,R} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{m_t} \end{pmatrix} \tag{194}$$

Now in nature, we see a left-handed quark, an $SU(2)_L$ active left-handed quark vector is present. We propose that it is a linear combination of massive quark vectors. So a normalized left-handed vector can be represented by

$$e'_q = \frac{1}{\sqrt{m_u + \alpha^2 m_c + \beta^2 m_t}} \begin{pmatrix} \sqrt{m_u} \\ \alpha \sqrt{m_c} \\ \beta \sqrt{m_t} \end{pmatrix} \tag{195}$$

By varying α and β , we can change the contribution of various massive vectors to the given $SU(2)_L$ active left-handed quark vector. The same can be done for the down-quark family. However, it should be kept in mind that only the integer linear combination of massive quark vectors can be performed.

8.4. CKM Matrix

Now observe these two left-handed vectors:

$$e'_1 = \frac{1}{\sqrt{m_u + \alpha^2 m_c + \beta^2 m_t}} \begin{pmatrix} \sqrt{m_u} \\ \alpha \sqrt{m_c} \\ \beta \sqrt{m_t} \end{pmatrix} \tag{196}$$

$$e'_2 = \frac{1}{\sqrt{m_d + a^2 m_s + b^2 m_b}} \begin{pmatrix} \sqrt{m_d} \\ a \sqrt{m_s} \\ b \sqrt{m_b} \end{pmatrix} \tag{197}$$

We try a set of values $\alpha = \beta = a = b = 1$. With this choice, for e_1 , the probability of it being in a top quark gravi eigenstate is 99.33%. Similarly, for e_2 , the probability of it being in bottom quark gravi eigenstate will then be equal to 97.7%. So let us identify e_1 and e_2 as the left-handed top quark (e_t) and a left-handed bottom quark (e_b), respectively. Now let us see the decay of the flavor eigenstate of the bottom quark to a flavor eigenstate of the top quark $e'_b \rightarrow e'_t$:

$$e'_t = \frac{1}{\sqrt{m_u + m_c + m_t}} \begin{pmatrix} \sqrt{m_u} \\ \sqrt{m_c} \\ \sqrt{m_t} \end{pmatrix} \quad e'_b = \frac{1}{\sqrt{m_d + m_s + m_b}} \begin{pmatrix} \sqrt{m_d} \\ \sqrt{m_s} \\ \sqrt{m_b} \end{pmatrix} \tag{198}$$

These vectors can be rotated into each other by the application of normal rotation matrices. Here, u represents the matrices acting on vectors in the space of the up-isospin particles, and similarly, d represents the matrices acting on the space of the down-isospin particles Figure 4:

$$e'_t = R_{12}^u(-\beta)R_{23}^u(-\alpha)R_{23}^d(\rho)R_{12}^d(\delta)e'_b = Ve'_b \tag{199}$$

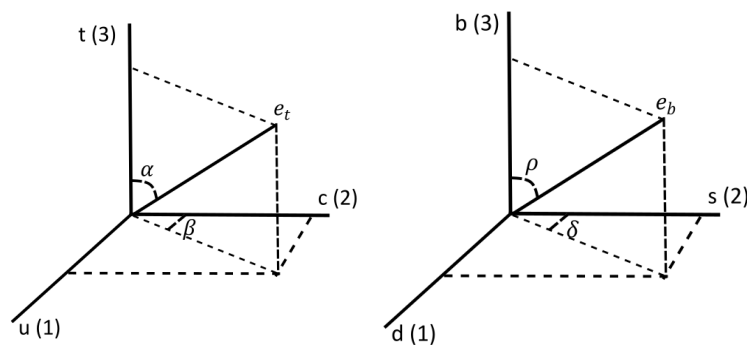


Figure 4. $SU(2)_L$ active particles and their projections.

$$R_{12}^d(\delta) = \begin{pmatrix} \cos(\delta) & -\sin(\delta) & 0 \\ \sin(\delta) & \cos(\delta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \cos(\delta) = \frac{\sqrt{m_s}}{\sqrt{m_s + m_d}} \tag{200}$$

$$R_{23}^d(\rho) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\rho) & -\sin(\rho) \\ 0 & \sin(\rho) & \cos(\rho) \end{pmatrix} \quad \cos(\rho) = \frac{\sqrt{m_b}}{\sqrt{m_b + m_s + m_d}} \tag{201}$$

$$R_{12}^u(-\beta) = (R_{12}^u(\beta))^T = \begin{pmatrix} \cos(\beta) & \sin(\beta) & 0 \\ -\sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \cos(\beta) = \frac{\sqrt{m_c}}{\sqrt{m_u + m_c}} \tag{202}$$

$$R_{23}^u(-\alpha) = (R_{23}^u(\alpha))^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad \cos(\alpha) = \frac{\sqrt{m_t}}{\sqrt{m_u + m_c + m_t}} \quad (203)$$

Now we use the numerical values of the square root masses of various quarks obtained from the eigenvalues of 3×3 octonionic Hermitian matrices as shown in Figure 2. By that substitution, we obtain

$$V_{ij} = \begin{pmatrix} 0.9813 & -0.1924 & -0.0030 \\ 0.1917 & 0.9789 & -0.0707 \\ 0.0165 & 0.0688 & 0.9975 \end{pmatrix} \quad (204)$$

$$|V_{ij}| = \begin{pmatrix} 0.9813 & 0.1924 & 0.0030 \\ 0.1917 & 0.9789 & 0.0707 \\ 0.0165 & 0.0688 & 0.9975 \end{pmatrix} \quad (205)$$

The code used to obtain the above CKM matrix using the square root mass as projections is presented in Appendix B. Every element of V_{ij} represents a projection of quark j on quark i . Its square represents the probability of transitioning from quark j to quark i in standard particle physics.

8.4.1. Standard CKM Matrix

In standard QFT textbooks [30], it is given that the CKM matrix is just a unitary transformation from mass eigenstates to states that are weak iso-spin doublets. The weak isospin doublets are $SU(2)_L$ active. The weak interaction doublets are given below:

$$\begin{pmatrix} u \\ d' \end{pmatrix} \quad \begin{pmatrix} c \\ s' \end{pmatrix} \quad \begin{pmatrix} t \\ b' \end{pmatrix} \quad (206)$$

The CKM matrix can then be written as

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \quad (207)$$

The $\{d, s, b\}$ represents the mass eigenstates. Each entry in the CKM matrix written as V_{ij} represents the transition of the j quark to i quark by weak interactions. The CKM matrix is parameterized using three Euler angles $\{\theta_{12}, \theta_{13}, \theta_{23}\}$ and a phase factor δ_{13} [31] as given below:

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \quad (208)$$

The experimental determination of the entries of the CKM matrix gives the values [20]

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} = \begin{pmatrix} 0.97370 \pm 0.00014 & 0.2245 \pm 0.0008 & 0.00382 \pm 0.00024 \\ 0.221 \pm 0.004 & 0.987 \pm 0.011 & 0.0410 \pm 0.0014 \\ 0.0080 \pm 0.0003 & 0.0388 \pm 0.0011 & 1.013 \pm 0.030 \end{pmatrix} \quad (209)$$

This yields the following experimentally determined values of the angles and the complex phase [32]

$$\theta_{12} = 13.04^\circ \pm 0.05^\circ \quad (210)$$

$$\theta_{13} = 0.201^\circ \pm 0.011^\circ \quad (211)$$

$$\theta_{23} = 2.38^\circ \pm 0.06^\circ \tag{212}$$

$$\delta_{13} = 68.8^\circ \pm 4.5^\circ \tag{213}$$

8.4.2. Theoretical Determination of CKM Matrix Angles

With the values of the CKM matrix obtained from the theoretical considerations, we calculate the following values of the CKM Euler angles:

$$\theta_{12} = 11.093^\circ \tag{214}$$

$$\theta_{13} = 0.172^\circ \tag{215}$$

$$\theta_{23} = 4.054^\circ \tag{216}$$

We have no information about phase in our analysis so far. Further assumptions and research are required in this direction. The values obtained are in reasonable agreement with the measured values. Basically, the off-diagonal matrix elements are different from the experimentally determined values and hence are the reason for these values of the angles. A correction to the mass matrices and hence to the masses of particles itself is required to obtain better values. This is because we used mass ratios derived in the asymptotically free limit, whereas mixing angles are likely impacted by the running of masses.

8.4.3. CKM Parameters Using Mass as Projections

Instead of using the square root mass as the projections, we tried using mass. With this new definition, our $SU(2)_L$ active particles are given by

$$e'_b \longrightarrow e'_t \tag{217}$$

$$e'_t = \frac{1}{\sqrt{m_u^2 + m_c^2 + m_t^2}} \begin{pmatrix} m_u \\ m_c \\ m_t \end{pmatrix} \tag{218}$$

$$e'_b = \frac{1}{\sqrt{m_d^2 + m_s^2 + m_b^2}} \begin{pmatrix} m_d \\ m_s \\ m_b \end{pmatrix} \tag{219}$$

$$e'_t = R_{12}^u(-\beta_1)R_{23}^u(-\beta_2)R_{23}^d(\alpha_2)R_{12}^d(\alpha_1)e'_b = Ve'_b \tag{220}$$

We use the same machinery, and rotate the vectors into each other by the application of rotation matrices. It gives us the following matrix required for the transformation:

$$V_{ij} = \begin{pmatrix} 0.9984 & -0.0559 & 0.2228 \times 10^{-5} \\ 0.0559 & 0.9982 & 0.1236 \times 10^{-2} \\ -0.7134 \times 10^{-5} & -0.1234 \times 10^{-2} & 0.9998 \end{pmatrix} \tag{221}$$

The code to obtain the above given CKM matrix is presented in Appendix C. With the above values of the various CKM matrix elements, we obtain the following values of the CKM parameters:

$$\theta_{12} = 3.205^\circ \tag{222}$$

$$\theta_{13} = 0.00013^\circ \tag{223}$$

$$\theta_{23} = 0.071^\circ \tag{224}$$

The above values are very different from the experimentally obtained values. This thus provides us with additional justification for using the square root mass values over the mass values while constructing the massive and the $SU(2)_L$ active left-handed vectors.

8.4.4. Connection between Mass Eigenstates and Weak-Isospin Doublets

Observe that the physically massive vectors used in the above calculations are a linear combination of gravi-charge eigenstates of the right-handed quarks. Also, observe that as we developed an isomorphism between the vector space of ideals to this new vector space $\mathbb{C}^8 \oplus \mathbb{C}^8 \oplus \mathbb{C}^8$, for $SU(2)_R$ active mass eigenstates, we can perform a similar mapping for the space of the $SU(2)_L$ active flavour eigenstates. So for the three left-handed quarks of the same color of $SU_c(3)$, we need the following space to describe them $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, just as for mass eigenstates. This time however, instead of the gravi charge operator, another diagonal operator corresponding to the electric charge will act on this space. Let us use the same \mathbb{C}^3 for both left and right active states (suppressing the color for both $SU(3)_c$ and $SU(3)_{grav}$). Then we can interpret the CKM matrix as a transformation that rotates the normalized mass eigenstates of the gravi charge vectors to the normalized left-handed flavor eigenstates. This connection can be made because of the triality. Triality allows for the mixing of various families in the spinor representations of the $Cl(8)$ algebra:

$$\begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d_{g,R} \\ s_{g,R} \\ b_{g,R} \end{pmatrix} \tag{225}$$

We have to use the normalized mass eigenstates and hence the gravi charge eigenvectors.

9. Summary and Discussion

As is evident from the analysis in the previous sections, the complex Clifford algebra $Cl(9)$ is one of great significance. It is the algebra of unification of the standard model with gravitation, via a left–right symmetric extension of the standard model. We also note that $Cl(9)$ has dimension 512, and its irrep is 16×16 matrices with complex number entries. If we assume the diagonal entries of these matrices to be real, their dimensionality is reduced to $512 - 16 = 496$, which is precisely the dimension of the $E_8 \times E_8$ symmetry group ($248 + 248$) proposed by us earlier for unification [6]. Hence, there is consistency between $E_8 \times E_8$ symmetry and the algebra $Cl(9)$ vis a vis unification. Prior to left–right symmetry breaking, which breaks unification in this theory, the coupling constant is simply unity, and the role of the emergent $U(1)$ charge is played by this coupling constant divided by 3. Thus, the fundamental entities prior to symmetry breaking are lepto-quark states, which all have an associated charge $1/3$: these are neither bosonic nor fermionic in nature, and the charge value $1/3$ is evident when one finds the eigenmatrices corresponding to the Jordan eigenvalues in the exceptional Jordan eigenvalue problem. For these eigenmatrices, see the appendix in [9]. The neutrino family, the up quark family, the down quark family and the electron family all are expressed as different superpositions of three basis states, which all have an associated charge $1/3$. This means that the left-chiral families are electric charge eigenstates expressed as the superposition of pre-unification basis states, and right-chiral families are square root mass eigenstates expressed as the superposition of pre-unification basis states. This fact permits electric charge eigenstates to be expressed as superpositions of square root mass eigenstates, which in turn allows mass ratios to be determined theoretically [8].

We recall from the above that the unification algebra $Cl(9)$ is written as a direct sum of two copies of $Cl(8)$. On the other hand, $Cl(9)$ can also be written as $Cl(9) = Cl(7) \otimes Cl(2) = [Cl(6) \otimes Cl(2)] \oplus [Cl(6) \otimes Cl(2)]$. This last expression has profound implications for our understanding of spacetime structure in quantum field theory. Recall that each of the two $Cl(6)$ represents one generation of standard model chiral quarks and leptons: the first $Cl(6)$ for left-chiral particles and the second $Cl(6)$ for right-chiral particles. In so far as the $Cl(2)$ are concerned, the second $Cl(2)$ (associated with right chiral fermions) is used to generate the Lorentz algebra $SL(2, C)$ of 4D spacetime (via complex quaternions with one quaternionic imaginary kept fixed), which includes the Lorentz boosts and the three-dimensional $SU(2)_R$ rotations. The gauging of this $SU(2)_R$ symmetry can be used to achieve Einstein's general relativity on a 4D spacetime manifold [33]. As for the first $Cl(2)$, the one associated with left-chiral fermions, the $SU(2)_L$ rotations describe weak isospin. However, undoubtedly, this $Cl(2)$ has its own set of Lorentz boosts, which, along with the weak isospin rotations, generate a second 4D spacetime algebra $SL(2, C)$ distinct from the first, familiar 4D spacetime. In spite of its counterintuitive nature, this second spacetime is also an element of physical reality, and there is definitive evidence for it in our earlier work [7,11,12]. In this second spacetime, distances are at most of the order of the range of the weak force, and only microscopic quantum systems access this second spacetime. Classical systems do not access it—their penetration depth into this spacetime is much less than one Planck length. Our universe thus has two 4D spacetimes, which have resulted from the symmetry breaking of a 6D spacetime, consistent with the equivalence $SL(2, \mathbb{H}) \simeq SO(1, 5)$. See also [34–37]. The second spacetime also obeys the laws of special relativity, and has a causal light cone structure. A quantum system travels from a spacetime point A to another spacetime point B through both space-times but gets to B much faster through the second spacetime, on a time scale of the order $L/c \sim 10^{-26}$ s, where $L \sim 10^{-16}$ cm is the range of the weak force. This is true even if B is located billions of light years away from A , and this offers a convincing resolution of the EPR paradox as to how quantum influences manage to arise nonlocally. These influences are local through the second spacetime. In spirit, our resolution could be compared to the ER=EPR proposal, but unlike the latter, our resolution has a sound mathematical basis. Moreover, our resolution was not invented with the express purpose of understanding quantum nonlocality, but is an indirect implication of the algebraic unification of the standard model with gravitation. The weak force is seen as the geometry of this second spacetime.

How is the Coleman–Mandula theorem evaded by our proposed unification of spacetime and internal symmetries? The Coleman–Mandula theorem [38] is a no-go theorem that states that the spacetime symmetry (Lorentz invariance) and internal symmetry of the S-matrix can only be combined in a trivial way, i.e., as a direct product. However, this does not prevent the $E_8 \times E_8$ unification of gravitation and the standard model, on which the analysis of the present paper is based. This is because, as pointed out, for instance, in Section 7 of the work on gravi weak unification [39] the theorem applies only to the spontaneously broken phase, in which the Minkowski metric is present. The unified phase does not have a metric, and hence the Minkowski metric does not either; therefore, the Coleman–Mandula theorem does not apply to the unified symmetry.

Interpreting the theoretically derived mass ratios: In the first paragraph of this section, we explain how the eigenvalues and eigenmatrices of the exceptional Jordan algebra determine the quantization of mass and charge. Furthermore, the expression of charge eigenstates as a superposition of mass eigenstates permits derivation of the mass ratios because mass measurements are eventually carried out using electric charge eigenstates. This explains the strange observed mass ratios of elementary particles. Nonetheless, it is known that masses run with the energy scale, and one can legitimately ask how the derived mass ratios are to be interpreted. The answer is straightforward: the ratio is of those mass values which are obtained in the no-interaction (asymptotically free) limit. Thus, the ratio of the muon to electron mass is derived in the low-energy limit, whereas the ratio of, say, the down quark to the electron mass is obtained by comparing the down quark mass at the relatively high

energy at which the quark asymptotic freedom is achieved, to the electron mass at the low energy free limit. These two compared masses (down quark and electron) are *not* at the same energy. Moreover, all these mass ratios will run with energy—that running is not part of the present derivation and is left for future work.

Evidence for a second 4D spacetime: The Clifford algebra associated with the complex quaternions (when none of the quaternionic imaginary directions is kept fixed) is $Cl(3)$, and is a direct sum of two $Cl(2)$ algebras, which together correspond to complex split biquaternions [5]. The spacetime associated with $Cl(3)$ is the 6D spacetime $SO(1,5)$ because of the homomorphism $SL(2, H) \sim SO(1,5)$ whereas each of the $Cl(2)$ is individually associated with a 4D spacetime each, because $Cl(2)$ generates the Lorentz algebra $SL(2, C)$. See also the related work of Kritov [40]. The construction of two copies of such a spacetime is made explicit in Equation (13) and the subsequent discussion in [12] and also in [11]. The presence of a second spacetime is also fully evident in [7], where we discussed in detail the bosonic content of the spontaneously broken $E_8 \times E_8$ symmetry.

Implications for fundamental physics in the early universe/high-energy regime: In our algebraic approach to unification, Clifford algebras and the standard model are studied, with dynamics given by the theory of trace dynamics. The main advantage of this approach is that the spinor representations of the fundamental fermions can be constructed easily here as the left ideals of the algebra. This formalism makes unique predictions for fundamental physics, including new particle content which should be looked for in experiments. The predicted particles include three right-handed sterile neutrinos (the only new fermions predicted beyond the standard model), a second (electrically charged) Higgs, eight gravitons associated with the newly predicted $SU(3)_{grav}$ symmetry, and the dark photon associated with the new $U(1)_{grav}$ symmetry, which possibly underlies Milgrom's MOND as an alternative to dark matter. We predict that the Higgs bosons are composites of those very fermions to which they are said to assign mass. Prior to electroweak symmetry breaking, the universe obeys the unified $E_8 \times E_8$ symmetry, which combines the standard model forces with gravitation. In this phase, there is no distinction between spacetime and matter, and the fundamental degrees of freedom are the so-called atoms of spacetime matter.

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Conflicts of Interest: The authors declare they have no conflict of interest.

Appendix A

The 3×3 Hermitian octonionic matrices, known as the exceptional Jordan algebra, satisfy the characteristic equation given as [18,26]

$$A^3 - (trA)A^2 + \sigma(A)A - (detA)I = 0 \quad (A1)$$

For the definition of each part, look at the example shown here:

$$A = \begin{pmatrix} p & a & \bar{b} \\ \bar{a} & m & c \\ b & \bar{c} & n \end{pmatrix} \tag{A2}$$

$$p, m, n \in \mathbb{R} \quad a, b, c \in \mathbb{O} \tag{A3}$$

$$\text{tr}A = p + m + n \tag{A4}$$

$$\sigma(A) = pm + pn + mn - |a|^2 - |b|^2 - |c|^2 \tag{A5}$$

$$\det A = pmn + b(ac) + \bar{b}(\bar{a}c) - n|a|^2 - m|b|^2 - p|c|^2 \tag{A6}$$

The real eigenvalues of the 3×3 Hermitian octonionic matrix satisfy a modified characteristic equation given by

$$\det(\lambda I - A) = \lambda^3 - (\text{tr}A)\lambda^2 + \sigma(A)\lambda - \det(A) = r \tag{A7}$$

$$r^2 + 4\Phi(a, b, c)r - |[a, b, c]|^2 = 0 \tag{A8}$$

$$\Phi(a, b, c) = \frac{1}{2}\text{Re}([a, \bar{b}]c) \tag{A9}$$

$$[a, b, c] = (ab)c - a(bc) \tag{A10}$$

The $[a, b, c]$ is the associator. It is a measure of the associativity of the algebra involved. Now, for our case, the mass matrix has only quaternionic entries. In that case, $r = 0$, and we have the usual characteristic equation that gives us real roots. These real roots are then used to calculate the mass ratios [9].

Appendix B

Here, in this code, we use mass eigenstates weighted by the square root of mass. The method is explained in Section 8.4. The identifications used in the code are written below:

$$e'_i = R_{12}^u(-\beta)R_{23}^u(-\alpha)R_{23}^d(\rho)R_{12}^d(\delta)e'_b = Ve'_b \tag{A11}$$

$$A12T \longrightarrow R_{12}^u(-\beta) \tag{A12}$$

$$A23T \longrightarrow R_{23}^u(-\alpha) \tag{A13}$$

$$B23 \longrightarrow R_{23}^d(\rho) \tag{A14}$$

$$B12 \longrightarrow R_{12}^d(\delta) \tag{A15}$$

```
[1]: import numpy as np
```

```
[2]: #values of square root masses :- mhsq means square root mass of h quark wrt down,
      ↳quark
      mdsq = 1
      mssq = (1+np.sqrt(3/8))/(1-np.sqrt(3/8))
      mbsq = (1+np.sqrt(3/8))**3/(1-np.sqrt(3/8))**2
      musq = 2/3
      mcsq = 2/3*((2/3 + np.sqrt(3/8))/(2/3-np.sqrt(3/8)))
      mtsq = 4/9*((2/3 + np.sqrt(3/8))/(2/3-np.sqrt(3/8))**2)
```

```
[3]: #cos(delta) mssq**2 will make it equal to mass of strange quarks
      a = mssq**2/(mdsq**2 + mssq**2) # we calculate ratio m_s/(m_s + m_d) , we will,
      ↳later take square root.
      print(np.sqrt(a))
```

0.9722971620935255

```
[4]: #sin(delta); calculate ratio m_d/(m_s + m_d), we will later take square root
      a1 = mdsq**2/(mdsq**2 + mssq**2)
      print(np.sqrt(a1))
```

0.23374821621752906

```
[6]: #cos(rho); calculate ratio m_b/(m_d + m_s + m_b), we will later take square root.
      ↳
      b = mbsq**2/(mbsq**2 + mdsq**2 + mssq**2)
      print(np.sqrt(b))
```

0.9884452276601956

```
[7]: #sin(rho); calculate ratio (m_d + m_s)/(m_d + m_s + m_b), we will take square,
      ↳root
      b1 = (mdsq**2 + mssq**2)/(mdsq**2 + mssq**2 + mbsq**2)
      print(np.sqrt(b1))
```

0.15157846785010118

```
[9]: #cos(beta)
h = mcsq**2/(mcsq**2 + musq**2) # we calculate ratio m_c/(m_c + m_u)
print(np.sqrt(h))

0.999100247030756

[10]: #sin(beta)
h1 = musq**2/(mcsq**2 + musq**2) # ratio= m_u/(m_u + m_c)
print(np.sqrt(h1))

0.042411040815834994

[12]: #cos(alpha)
k = mtsq**2/(mtsq**2 + mcsq**2 + musq**2) #we calculate m_t/(m_t + m_c + m_u)
print(np.sqrt(k))

0.9966941432846482

[13]: #sin(alpha)
k1 = (musq**2 + mcsq**2)/(musq**2 + mcsq**2 + mtsq**2) # ratio is (m_u + m_c)/
    ->(m_u + m_c + m_t)
print(np.sqrt(k1))

0.0812452136564426

[16]: A12T = np.array([[0.999100,0.0424110,0],[-0.0424110,0.999100,0],[0,0,1]])
A23T = np.array([[1,0,0],[0,0.996694,0.081245],[0,-0.081245,0.996694]])
B12 = np.array([[0.972297,-0.233748,0],[0.233748,0.972297,0],[0,0,1]])
B23 = np.array([[1,0,0],[0,0.988445,-0.151579],[0,0.151579,0.988445]])

[17]: print(A12T) #R_{12}~u

[[ 0.9991    0.042411  0.         ]
 [-0.042411  0.9991    0.         ]
 [ 0.         0.         1.         ]]

[18]: print(A23T) #R_{23}~u

[[ 1.         0.         0.         ]
 [ 0.         0.996694  0.081245]
 [ 0.         -0.081245  0.996694]]

[19]: print(B23) #R_{23}~d

[[ 1.         0.         0.         ]
 [ 0.         0.988445 -0.151579]
 [ 0.         0.151579  0.988445]]

[20]: print(B12) #R_{12}~d
```

```
[[ 0.972297 -0.233748  0.      ]
 [ 0.233748  0.972297  0.      ]
 [ 0.         0.         1.       ]]
```

```
[21]: result = [[0, 0, 0],
               [0, 0, 0],
               [0, 0, 0]]
for i in range(3):
    for j in range(3):
        s=0
        for m in range(3):
            for n in range(3):
                for k in range(3):
                    s = s + (A12T[i][m])*(A23T[m][n])*(B23[n][k])*(B12[k][j])
                result[i][j] = result[i][j] + s
print(np.array(result))
```

```
[[ 0.98131056 -0.19240495 -0.0030015 ]
 [ 0.19171588  0.97889932 -0.07070797]
 [ 0.01654274  0.06881108  0.99749224]]
```

```
[3]: #For calculating the CKM angles in degrees
r1 = 0.19240495/np.sqrt(0.19240495**2 + 0.98131056**2)
r2 = 0.07070797/0.19240495
r3 = 0.0030015

A12 = np.arcsin(r1)*(180/np.pi)
A23 = np.arcsin(r1*r2)*(180/np.pi)
A13 = np.arcsin(r3)*(180/np.pi)

print(A12)
print(A23)
print(A13)
```

```
11.093223432207658
4.054671819831707
0.17197354042751115
```

These values are reported in the earlier section.

Appendix C

Here is a code for computing CKM matrix parameters and mixing angles with mass eigenstates weighted by mass (instead of square root of mass). The definitions of rotation matrices and the mass vectors correspondingly get changed:

$$e'_t = R_{12}^u(-\beta_1)R_{23}^u(-\beta_2)R_{23}^d(\alpha_2)R_{12}^d(\alpha_1)e'_b = Ve'_b \quad (\text{A16})$$

$$P_{12T} \longrightarrow R_{12}^u(-\beta_1) \quad (\text{A17})$$

$$P_{23T} \longrightarrow R_{23}^u(-\beta_2) \quad (\text{A18})$$

$$Q_{23} \longrightarrow R_{23}^d(\alpha_2) \quad (\text{A19})$$

$$Q_{12} \rightarrow R_{12}^d(\alpha_1)$$

(A20)

```
[27]: import numpy as np
```

```
[28]: # trying m as the projection and not the square root
```

```
[29]: #values of square root masses :- mhsq means square root mass of h quark
mdsq = 1
mssq = (1+np.sqrt(3/8))/(1-np.sqrt(3/8))
mbsq = (1+np.sqrt(3/8))**3/(1-np.sqrt(3/8))**2
musq = 2/3
mcsq = 2/3*((2/3 + np.sqrt(3/8))/(2/3-np.sqrt(3/8)))
mtsq = 4/9*((2/3 + np.sqrt(3/8))/(2/3-np.sqrt(3/8))**2)
```

```
[30]: #cos(alpha1) m_s/(m_d**2 + m_s**2)*0.5
a = mssq**2/np.sqrt(mdsq**4 + mssq**4)
print(a)
```

0.9983339778128235

```
[31]: #sin(alpha1) m_d/(m_d**2 + m_s**2)*0.5
a1 = mdsq**2/np.sqrt(mdsq**4 + mssq**4)
print(a1)
```

0.057699815809278965

```
[32]: #cos(alpha2) m_b/(m_d**2 + m_s**2 + m_b**2)*0.5
b = mbsq**2/np.sqrt(mbsq**4 + mdsq**4 + mssq**4)
print(b)
```

0.9997521479566492

```
[33]: #sin(alpha2) (m_d**2 + m_s**2)*0.5/(m_d**2 + m_s**2 + m_b**2)*0.5
b1 = np.sqrt(mdsq**4 + mssq**4)/np.sqrt(mdsq**4 + mssq**4 + mbsq**4)
print(b1)
```

0.00539495658506901

```
[34]: #cos(beta1) m_c/(m_c**2 + m_u**2)*0.5
c = mcsq**2/np.sqrt(mcsq**4 + musq**4)
print(c)
```

0.9999983765145377

```
[35]: #sin(beta1) m_u/(m_c**2 + m_u**2)*0.5
c1 = musq**2/np.sqrt(mcsq**4 + musq**4)
print(c1)
```

0.001801934596165667

```
[36]: #cos(beta2) m_t/(m_t**2 + m_u**2 + m_c**2)*0.5
d = mtsq**2/np.sqrt(mtsq**4 + musq**4 + mcsq**4)
print(d)
```

0.9999780043466765

```
[37]: #sin(beta2) (m_u**2 + m_c**2)*0.5/(m_u**2 + m_c**2 + m_t**2)*0.5
d1 = np.sqrt(musq**4 + mcsq**4)/np.sqrt(musq**4 + mcsq**4 + mtsq**4)
print(d1)
```

0.0066325577900636715

```
[38]: P12T = np.array([[0.999998,0.001802,0],[-0.001802,0.999998,0],[0,0,1]])
P23T = np.array([[1,0,0],[0,0.999978,0.006633],[0,-0.006633,0.999978]])
Q23 = np.array([[1,0,0],[0,0.999752,-0.005395],[0,0.005395,0.999752]])
Q12 = np.array([[0.998334,-0.057700,0],[0.057700,0.998334,0],[0,0,1]])
```

```
[39]: print(P12T)
```

```
[[ 0.999998  0.001802  0.         ]
 [-0.001802  0.999998  0.         ]
 [ 0.         0.         1.         ]]
```

```
[40]: print(P23T)
```

```
[[ 1.         0.         0.         ]
 [ 0.         0.999978  0.006633]
 [ 0.        -0.006633  0.999978]]
```

```
[41]: print(Q23)
```

```
[[ 1.         0.         0.         ]
 [ 0.         0.999752 -0.005395]
 [ 0.         0.005395  0.999752]]
```

```
[42]: print(Q12)
```

```
[[ 0.998334 -0.0577  0.         ]
 [ 0.0577   0.998334 0.         ]
 [ 0.         0.         1.         ]]
```

```
[43]: result = [[0, 0, 0],#Attempting ckm with the mass as a projection
               [0, 0, 0],
```

```

    [0, 0, 0]
for i in range(3):
    for j in range(3):
        s=0
        for m in range(3):
            for n in range(3):
                for k in range(3):
                    s = s + (P12T[i][m])*(P23T[m][n])*(Q23[n][k])*(Q12[k][j])
                result[i][j] = result[i][j] + s
print(np.array(result))

```

```

[[ 9.98435954e-01 -5.59013081e-02  2.22812562e-06]
 [ 5.58873729e-02  9.98202160e-01  1.23647123e-03]
 [-7.13445328e-05 -1.23441374e-03  9.99765790e-01]]

```

```

[44]: #For calculating the CKM angles in degrees
r1 = 5.59013081e-02/np.sqrt(5.59013081e-02**2 + 9.98435954e-01**2)
r2 = 1.23647123e-03/5.59013081e-02
r3 = 2.22812562e-06

A12 = np.arcsin(r1)*(180/np.pi)
A23 = np.arcsin(r1*r2)*(180/np.pi)
A13 = np.arcsin(r3)*(180/np.pi)

print(A12)
print(A23)
print(A13)

```

```

3.204580637958553
0.07084462544443867
0.00012766219425107547

```

Here, it can be seen that the values obtained for the CKM parameters are very different from the experimentally seen values. It justifies our choice of using the square root mass as a more fundamental quantity over the mass of the fermions.

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