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The Secular Dressed Diffusion Equation

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Abstract: The secular dressed diffusion equation describes the long-term evolution of collisionless systems of particles with long-range interactions, such as self-gravitating systems submitted to a weak external stochastic perturbation. We successively consider nonrotating spatially homogeneous systems, rotating spatially homogeneous systems, and spatially inhomogeneous systems. We contrast the secular dressed diffusion equation applying to collisionless systems perturbed by an externally imposed stochastic field from the Lenard–Balescu equation applying to isolated systems evolving because of discreteness effects ("collisions"). We discuss the connection between these two equations when the external noise is produced by a random distribution of field particles.

Keywords: self-gravitating systems; kinetic theory; Fokker–Planck equation; angle-action variables; collective effects

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1. Introduction

The dynamics and thermodynamics of systems with long-range interactions has been a topic of active research in the last 20 years [1]. In a first regime, provided that the number of particles N is sufficiently large, the evolution of the system is collisionless and a mean field approximation can be implemented. In that case, the evolution of the distribution function of the particles $f(\mathbf{r}, \mathbf{v}, t)$ is governed by the collisionless Boltzmann equation [2], also known as the Vlasov equation [3] (see Hénon [4] for a discussion about the name that should be given to that equation). Following a process of (possibly incomplete) violent relaxation [5,6], the system generically reaches a virialized state (quasistationary state) after a few dynamical times. The slow evolution of the system on a longer (secular) timescale may be due to the intrinsic internal noise arising from the graininess of the system (finite N effects) if the system is isolated, or to the influence of an external stochastic potential if the system is not isolated. In the first case, the system evolves because of "collisions" (encounters) between the particles. In the second case, it evolves because of the action of the external forcing. We briefly review the kinetic theories that have been developed in the past in these two situations, starting with the case of isolated systems (see Refs. [7,8] for more detailed reviews on this subject).

The first kinetic equation applicable to systems with long-range interactions was derived by Landau [9] for a spatially homogeneous neutral Coulombian plasma. Starting from the Boltzmann equation [10] and expanding this equation in terms of a weak deflection parameter, he obtained a kinetic equation governing the evolution of the velocity distribution $f(\mathbf{v}, t)$. His approach ignores collective effects (Debye shielding) and exhibits a logarithmic divergence at large scales that he cured heuristically by introducing a cut-off at the Debye length [11]. There is also a logarithmic divergence at small scales due to the neglect of strong collisions that Landau cured heuristically by introducing a cut-off at the distance at which the particles are deflected by about 90° (Landau length). A more exact kinetic equation taking into account collective effects was derived by Lenard [12] and Balescu [13] from the Liouville equation.[1] Their treatment removes the divergence at large scales present in the Landau equation. As a result, the Debye length appears naturally in their formalism. A kinetic theory of spatially homogeneous plasmas was independently developed by Hubbard [15,16]. He started from the Fokker–Planck equation [17–20] and derived the diffusion tensor and the friction force by a direct calculation, taking collective effects into account (see also Rostoker and Rosenbluth [21] and Thompson and Hubbard [22] in the thermal bath approximation). When the diffusion tensor and the friction force are substituted in the Fokker–Planck equation and a simple transformation is carried out (a step that Hubbard did not make explicitly), one obtains an equation identical to the Lenard–Balescu equation. Therefore, the Fokker–Planck equation derived by Hubbard is equivalent to the Lenard–Balescu equation. The Landau and the Lenard–Balescu equations can be applied to the whole class of systems with long-range interactions in various dimensions of space provided that they are spatially homogeneous [7,23,24]. These kinetic equations are valid at the order $1/N$ (since they neglect three-body and higher correlations) so they describe the evolution of the system of a timescale of the order $N\,t_D$, where t_D is the dynamical time. In $d \geq 2$ dimensions, they relax towards the Boltzmann distribution of statistical equilibrium. However, when the kinetic theory is applied to one-dimensional (1D) systems, such as 1D plasmas [25,26] and the Hamiltonian Mean Field (HMF) model [27,28], it is found that the Landau and the Lenard–Balescu collision operators vanish identically for spatially homogeneous systems. This is a situation of kinetic blocking at the order $1/N$. This implies that the relaxation towards statistical equilibrium takes more time that $N\,t_D$ since it is due to higher order correlations [23]. Recently, a new kinetic equation has been obtained by Fouvry et al. [29,30] for 1D homogeneous systems with long-range interactions. This equation, which takes three-body correlations into account, is valid at the order $1/N^2$. It satisfies an H-theorem and relaxes towards the Boltzmann distribution of statistical equilibrium on a timescale of the order $N^2\,t_D$ whatever the potential of interaction (i.e., there is no kinetic blocking at the order $1/N^2$).

The kinetic theory of stellar systems was initiated by Chandrasekhar [31–43] and further developed by Rosenbluth et al. [44]. They proceeded as if stellar systems were spatially homogeneous and neglected collective effects. Using the Fokker–Planck approach, they obtained a kinetic equation which is essentially equivalent to the Landau equation although written in a different—less symmetric—form (see Ref. [8] for a detailed comparison of the two approaches).[2] This equation exhibits a logarithmic divergence at large scales that they cured heuristically by introducing a cut-off at the size of the system (Jeans length) [45] which is the presumed analogue of the Debye length in plasma physics.[3] If one tries to account for collective effects in homogeneous self-gravitating systems by using the Lenard–Balescu equation, one obtains a strong (algebraic) divergence at large scales associated with the Jeans instability of a uniform self-gravitating medium. This reveals the inadequacy of kinetic theories of self-gravitating systems based on the homogeneity assumption to correctly describe collective effects. If we consider finite homogeneous systems and truncate the integration over wavelengths at a maximum wavelength $\lambda_{\rm max}$ smaller than the Jeans length λ_J, we find that the diffusion and friction coefficients diverge algebraically when $\lambda_{\rm max} \to \lambda_J$ (see Ref. [50] and Appendix E of Ref. [8]). This suggests that, if we were able to account for spatial inhomogeneity rigorously, collective effects (Debye anti-shielding) would tend to decrease the relaxation time of stellar systems.[4]

There is a situation where a stellar system can be rigorously spatially homogeneous. This is when it is uniformly rotating at a specific angular velocity $\Omega = (2\pi G\rho)^{1/2}$ such that the mean gravitational force is exactly balanced by the centrifugal force.[5] In that case, a spatially homogeneous system can be in static equilibrium in the rotating frame and no Jeans swindle is necessary. The Landau and Lenard–Balescu equations for spatially homogeneous rotating stellar systems were derived by Wu [53]. Following the earlier work of Lynden-Bell [54], he also investigated their dynamical stability with respect to the Vlasov–Poisson equations by studying the dispersion relation [55]. He found that the equilibrium state is always linearly unstable so that it evolves on a dynamical timescale. Indeed, a spatially homogeneous collisionless stellar system suffers from the Jeans instability above a critical wavelength λ_J [54]. As a result, the whole kinetic theory which implicitly assumes that the system is dynamically (Vlasov) stable breaks down. At a formal level, the

Lenard–Balescu equation which takes into account collective effects is ill-posed because it suffers from an algebraic divergence at large scales associated with the Jeans instability. By contrast, the Landau equation which ignores collective effects just suffers from a logarithmic divergence. This is like in the case of nonrotating spatially homogeneous self-gravitating systems when we make the Jeans swindle. As a result, Wu [53] had to truncate artificially the integration over the wavelength at a maximum wavelength smaller than the Jeans length in order to obtain sensible results. In other words, he considered a "very large but finite" system characterized by a dimension smaller than the Jeans wavelength.

To solve these difficulties rigorously, it is necessary to develop an appropriate kinetic theory for spatially inhomogeneous stellar systems. This can be performed most conveniently by using angle-action variables [46]. The inhomogeneous Landau equation was obtained in Refs. [8,56,57] and the inhomogeneous Lenard–Balescu equation was obtained in Refs. [58,59].[6] The inhomogeneous Landau and Lenard–Balescu equations have been recently studied in relation to stellar discs [62–64], globular clusters [65,66], galactic nuclei [67–70], and 1D gravity [71]. The inhomogeneous Landau and Lenard–Balescu equations have also been applied to other systems with long-range interactions such as the inhomogeneous phase of the HMF model [72]. In Ref. [73] they have been used to study the dynamics of spins with long-range interactions moving on a sphere (recovering and generalizing previous works on the subject [74–76]) in relation to the process of vector resonant relaxation (VRR) of stars in galactic nuclei [77–79]. In some cases, collective effects can considerably accelerate the relaxation. For example, the work [63] demonstrated that collective effects decrease the relaxation time of stellar discs by three orders of magnitude. A similar acceleration is obtained for the periodic stellar cube as its mass approaches the Jeans mass [8,50,80] and for the homogeneous HMF model close to the critical temperature T_c [24]. In other cases, such as a spherical galaxy modeled by an $n = 3$ polytrope [81], the inhomogeneous phase of the HMF model [72], globular clusters represented by a spherical isotropic isochrone distribution function [66], and 1D self-gravitating systems [71], collective effects unexpectedly reduce the diffusion and the friction and increase the relaxation time. The inhomogeneous Landau and Lenard–Balescu equations also share many analogies with the kinetic equations obtained in the context of point vortices in 2D hydrodynamics [82–88] and in pure electron plasmas under a strong magnetic field [89,90]. The numerous analogies between stellar systems and 2D vortices are discussed in Refs. [57,91–93].

The case of collisionless self-gravitating systems submitted to an external stochastic forcing was first studied by Binney and Lacey [94] following the suggestion of Spitzer and Schwarzschild [95,96] that inhomogeneities in a galactic disc accelerate stars (see also the works of Lacey [97] and Lacey and Ostriker [98] for the problem of disc heating by giant molecular clouds or massive black holes in galactic halos). They considered spatially inhomogeneous systems (using angle-action variables) and neglected collective effects. They showed that the long-term evolution of the system is described by a nonlinear diffusion equation in which the diffusion tensor is sourced by the power spectrum of the stochastic force evaluated at the resonance frequencies. This formalism can be applied to a galactic disc undergoing (cosmic) perturbations from its surrounding dark matter halo or to the secular diffusion of accretion streams within the Galactic halo. This problem was also treated independently by Weinberg [99] who stressed the importance of collective effects. In the same spirit, Ma and Bertschinger [100] used a quasilinear approach to investigate dark matter diffusion induced by cosmological fluctuations while Pichon and Aubert [101] sketched a time-decoupling approach to solve the Vlasov equation and applied it to the statistical study of dynamical flows through dark matter halos. The same types of problems were studied independently in the context of systems with long-range interactions, such as the HMF model, by Nardini et al. [102,103] and Chavanis [7,23]. Specifically, they considered the evolution of spatially homogeneous collisionless systems of particles with long-range interactions driven by external stochastic forces and derived a nonlinear diffusion equation. These calculations were generalized to the case of inhomogeneous systems by Fouvry et al. [64,104–106] with specific applications to stellar discs. More recently, Fouvry and Bar-Or [107] developed an approach based on the η-formalism [108] that allowed them to treat both graininess effects and external stochastic perturbations.[7] The η-formalism also offers a numerical method to compute the long-term diffusion and friction coefficients in inhomogeneous systems.

In summary, there are two main kinetic equations describing the secular evolution of systems with long-range interactions. The first one is the Lenard–Balescu equation (or the Landau equation when collective effects are neglected) which describes the "collisional" evolution of an isolated system of particles undergoing perturbations arising from its own discreteness (finite N effects). The second one is the secular dressed diffusion (SDD) equation (or the secular bare diffusion (SBD) equation when collective effects are neglected) which describes the long-term evolution of a collisionless system of particles ($N \to +\infty$ with $m \sim 1/N$) undergoing external perturbations. In the first case, the perturbations are intrinsic and self-induced, whereas in the second case the perturbations are exterior and the system responds to them. The Lenard–Balescu equation has been discussed and reviewed in several papers and books. The SDD equation is less well-known. Therefore, in this paper, we offer a brief review of the formalism leading to that equation. We successively consider nonrotating spatially homogeneous systems (Section 2), rotating spatially homogeneous systems (Section 3), and spatially inhomogeneous systems (Sections 4 and 5). We adopt the same presentation in each case in order to show the unity of the formalism. At the end of each section, we specifically consider the case where the external noise is produced by a random distribution of field particles and discuss the connection between the SDD equation and the multi-species Lenard–Balescu equation.

2. Spatially Homogeneous Systems

2.1. Derivation of the SDD Equation from the Klimontovich Equation

We first derive the SDD equation from the Klimontovich equation. We follow an approach similar to the one developed in Section 6 of [23].

2.1.1. Quasilinear Theory and Bogoliubov Ansatz

We consider a system of particles of mass m interacting via a long-range binary potential $u(|\mathbf{r}-\mathbf{r}'|)$ decreasing more slowly than $r^{-\gamma}$ with $\gamma \leq d$ in a space of dimension d. We assume that the particles are subjected to an external stochastic force (exterior perturbation) of zero mean arising from a fluctuating potential $\Phi_e(\mathbf{r},t)$. The equations of motion of the particles are

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = -\nabla\Phi_d(\mathbf{r}_i) - \nabla\Phi_e(\mathbf{r}_i,t), \tag{1}$$

where $\Phi_d(\mathbf{r}) = m\sum_j u(|\mathbf{r}-\mathbf{r}_j|)$ is the exact potential produced by the particles. They can be written in Hamiltonian form as $m d\mathbf{r}_i/dt = \partial(H_d+H_e)/\partial\mathbf{v}_i$ and $m d\mathbf{v}_i/dt = -\partial(H_d+H_e)/\partial\mathbf{r}_i$, where $H_d = (1/2)\sum_i m v_i^2 + \sum_{i<j} m^2 u(|\mathbf{r}_i-\mathbf{r}_j|)$ is the Hamiltonian of the particles and $H_e = \sum_i m\Phi_e(\mathbf{r}_i,t)$ is the Hamiltonian associated with the external force. The discrete distribution function of the particles, $f_d(\mathbf{r},\mathbf{v},t) = m\sum_i \delta(\mathbf{r}-\mathbf{r}_i(t))\delta(\mathbf{v}-\mathbf{v}_i(t))$, satisfies the Klimontovich equation

$$\frac{\partial f_d}{\partial t} + \mathbf{v}\cdot\frac{\partial f_d}{\partial \mathbf{r}} - \nabla(\Phi_d+\Phi_e)\cdot\frac{\partial f_d}{\partial \mathbf{v}} = 0, \tag{2}$$

where

$$\Phi_d(\mathbf{r},t) = \int u(|\mathbf{r}-\mathbf{r}'|)\rho_d(\mathbf{r}',t)\, d\mathbf{r}' \tag{3}$$

is the potential produced by the discrete density of particles $\rho_d(\mathbf{r},t) = \int f_d(\mathbf{r},\mathbf{v},t)\,d\mathbf{v} = m\sum_i \delta(\mathbf{r}-\mathbf{r}_i(t))$. The Klimontovich Equation (2) can be written under the form

$$\frac{\partial f_d}{\partial t} + \{f_d, H_d + H_e\} = 0, \tag{4}$$

where $H_d + H_e = p^2/2m + m(\Phi_d + \Phi_e)$ denotes here the total Hamiltonian of a particle and $\{f,g\} = \partial_{\mathbf{r}} f \partial_{\mathbf{p}} g - \partial_{\mathbf{p}} f \partial_{\mathbf{r}} g$ is the usual Poisson bracket ($\mathbf{p} = m\mathbf{v}$ is the impulse). This Poisson structure is valid for all canonical coordinates and is particularly useful to treat spatially inhomogeneous systems (see Section 4 and Refs. [59,99,104]).

We introduce the mean distribution function $f(\mathbf{r},\mathbf{v},t) = \langle f_d(\mathbf{r},\mathbf{v},t)\rangle$ corresponding to an ensemble average of $f_d(\mathbf{r},\mathbf{v},t)$. We then write $f_d(\mathbf{r},\mathbf{v},t) = f(\mathbf{r},\mathbf{v},t) + \delta f(\mathbf{r},\mathbf{v},t)$, where $\delta f(\mathbf{r},\mathbf{v},t)$ denotes the fluctuations about the mean distribution. Similarly, we write $\Phi_d(\mathbf{r},t) = \Phi(\mathbf{r},t) + \delta\Phi(\mathbf{r},t)$, where $\delta\Phi(\mathbf{r},t)$ denotes the fluctuations about the mean potential $\Phi(\mathbf{r},t)$. Substituting these decompositions into Equation (2), we obtain

$$\frac{\partial f}{\partial t} + \frac{\partial \delta f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \mathbf{v}\cdot\frac{\partial \delta f}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} - \nabla\Phi\cdot\frac{\partial \delta f}{\partial \mathbf{v}} \\ -\nabla(\delta\Phi + \Phi_e)\cdot\frac{\partial f}{\partial \mathbf{v}} - \nabla(\delta\Phi + \Phi_e)\cdot\frac{\partial \delta f}{\partial \mathbf{v}} = 0. \tag{5}$$

Taking the ensemble average of this equation, we find that

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\cdot\langle \delta f \nabla(\delta\Phi + \Phi_e)\rangle, \tag{6}$$

where the right hand side is the "collision" term arising from the granularity of the system (finite N effects) and the external stochastic force.[8] Subtracting this expression from Equation (5), we obtain the equation for the fluctuations of the distribution function

$$\frac{\partial \delta f}{\partial t} + \mathbf{v}\cdot\frac{\partial \delta f}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial \delta f}{\partial \mathbf{v}} - \nabla(\delta\Phi + \Phi_e)\cdot\frac{\partial f}{\partial \mathbf{v}} \\ = \frac{\partial}{\partial \mathbf{v}}\cdot(\delta f \nabla(\delta\Phi + \Phi_e)) - \frac{\partial}{\partial \mathbf{v}}\cdot\langle \delta f \nabla(\delta\Phi + \Phi_e)\rangle. \tag{7}$$

The foregoing equations are exact since no approximation has been made for the moment.

We now assume that the external force is weak and treat the stochastic potential $\Phi_e(\mathbf{r},t)$ as a small perturbation to the mean field dynamics. We also assume that the fluctuations $\delta\Phi(\mathbf{r},t)$ of the long-range potential are weak. Since the mass scales as $m \sim 1/N$ this approximation is valid when $N \gg 1$. If we ignore the external stochastic force and the fluctuations of the potential due to finite N effects altogether, the collision term vanishes and Equation (6) reduces to the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = 0, \tag{8}$$

where

$$\Phi(\mathbf{r},t) = \int u(|\mathbf{r}-\mathbf{r}'|)\rho(\mathbf{r}',t)\,d\mathbf{r}' \tag{9}$$

is the potential produced by the mean density of particles $\rho(\mathbf{r},t) = \int f(\mathbf{r},\mathbf{v},t)\,d\mathbf{v} = \langle \rho_d(\mathbf{r},t)\rangle$. The Vlasov equation describes a self-consistent mean field dynamics corresponding to a collisionless and unperturbed evolution. It is valid in the limit $\Phi_e \to 0$ and $N \to +\infty$ with $m \sim 1/N$, or for sufficiently "short" times.

We now take into account a small correction to the Vlasov equation obtained by keeping the collision term on the right hand side of Equation (6) but neglecting the quadratic terms on the right hand side of Equation (7). We therefore obtain a set of two coupled equations

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \langle \delta f \nabla (\delta \Phi + \Phi_e) \rangle \tag{10}$$

and

$$\frac{\partial \delta f}{\partial t} + \mathbf{v} \cdot \frac{\partial \delta f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial \delta f}{\partial \mathbf{v}} - \nabla (\delta \Phi + \Phi_e) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \tag{11}$$

These equations form the starting point of the quasilinear theory which is valid in a weak coupling approximation ($m \sim 1/N \ll 1$) and for a weak external stochastic force ($\Phi_e \ll \Phi$). Equation (10) describes the evolution of the mean distribution function sourced by the correlations of the fluctuations and Equation (11) describes the evolution of the fluctuations due to the external stochastic potential and the granularities of the system (finite N effects). These equations are valid at the order $1/N$ and to leading order in Φ_e.[9]

If we restrict ourselves to spatially homogeneous systems, the mean field force $-\nabla\Phi$ vanishes. In that case, the unperturbed equations of motion of the particles are simply straight lines traveled at constant velocity: $\mathbf{v}(t) = \mathbf{v}$ and $\mathbf{r}(t) = \mathbf{r} + \mathbf{v}t$. Any distribution function $f(\mathbf{v})$ is a steady state of the Vlasov equation. On the other hand, the fundamental Equations (10) and (11) of the quasilinear theory reduce to[10]

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \langle \delta f \nabla (\delta \Phi + \Phi_e) \rangle, \tag{12}$$

$$\frac{\partial \delta f}{\partial t} + \mathbf{v} \cdot \frac{\partial \delta f}{\partial \mathbf{r}} - \nabla (\delta \Phi + \Phi_e) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \tag{13}$$

Our goal is to solve Equation (13) for the fluctuations $\delta f(\mathbf{r}, \mathbf{v}, t)$ and substitute the result back into Equation (12) in order to obtain a closed kinetic equation for the mean distribution function $f(\mathbf{v}, t)$. Assuming a timescale separation, we will regard $\partial f/\partial \mathbf{v}$ in Equation (13) as being independent of time (Bogoliubov ansatz). Indeed, the mean distribution function evolves on a secular timescale which is long compared to the dynamical time over which the correlations of the fluctuations have their essential support. In order to derive the Lenard–Balescu equation describing the mean evolution of the system under purely discreteness effects ("collisions") at the order $1/N$ we take $\Phi_e = 0$ and consider an initial value problem as described in, e.g., Section 2 of [23]. In that case, Equation (13) must be solved by introducing a Fourier transform in space and a Laplace transform in time. This involves the initial perturbed distribution function $\delta \hat{f}(\mathbf{k}, \mathbf{v}, 0)$ which accounts for the granularities of the system (finite N effects). Here, we neglect discreteness effects (assuming $N \to +\infty$) and focus on the effect of the external stochastic potential Φ_e as in Section 6 of [23]. In that case, Equation (13) can be solved by introducing Fourier transforms in space and time.[11] The Fourier transform of the fluctuations of the distribution function $\delta f(\mathbf{r}, \mathbf{v}, t)$ is defined by

$$\delta \hat{f}(\mathbf{k}, \mathbf{v}, \omega) = \int \frac{d\mathbf{r}}{(2\pi)^d} \int dt \, e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \delta f(\mathbf{r}, \mathbf{v}, t) \tag{14}$$

and its inverse Fourier transform is given by

$$\delta f(\mathbf{r}, \mathbf{v}, t) = \int d\mathbf{k} \int \frac{d\omega}{2\pi} \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \delta \hat{f}(\mathbf{k}, \mathbf{v}, \omega). \tag{15}$$

Similar expressions hold for the fluctuating potentials $\delta\Phi(\mathbf{r}, t)$ and $\Phi_e(\mathbf{r}, t)$. We note that, for periodic potentials, the integral over $\mathbf{k}$ must be replaced by a discrete summation over the different modes. For future reference, we recall the Fourier representation of the Dirac δ-function

$$\delta(t) = \int_{-\infty}^{+\infty} e^{-i\omega t}\,\frac{d\omega}{2\pi}, \qquad \delta(\mathbf{k}) = \int e^{-i\mathbf{k}\cdot\mathbf{r}}\,\frac{d\mathbf{r}}{(2\pi)^d}. \tag{16}$$

2.1.2. Dielectric Function

We are now ready to solve the equation for the fluctuations for a given external perturbation. Taking the Fourier transform of Equation (13), we find that

$$\delta\hat{f}(\mathbf{k},\mathbf{v},\omega) = \frac{\mathbf{k}\cdot\frac{\partial f}{\partial\mathbf{v}}}{\mathbf{k}\cdot\mathbf{v}-\omega}\left[\delta\hat{\Phi}(\mathbf{k},\omega)+\hat{\Phi}_e(\mathbf{k},\omega)\right]. \tag{17}$$

The fluctuations of the potential are related to the fluctuations of the density $\delta\rho(\mathbf{r},t) = \int \delta f(\mathbf{r},\mathbf{v},t)\,d\mathbf{v}$ by a convolution

$$\delta\Phi(\mathbf{r},t) = \int u(|\mathbf{r}-\mathbf{r}'|)\delta\rho(\mathbf{r}',t)\,d\mathbf{r}'. \tag{18}$$

Taking the Fourier transform of this equation, we obtain

$$\delta\hat{\Phi}(\mathbf{k},\omega) = (2\pi)^d\hat{u}(k)\delta\hat{\rho}(\mathbf{k},\omega) \tag{19}$$

with $\delta\hat{\rho}(\mathbf{k},\omega) = \int \delta\hat{f}(\mathbf{k},\mathbf{v},\omega)\,d\mathbf{v}$, where $\hat{u}(k)$ is the Fourier transform of the potential of interaction. Integrating Equation (17) over $\mathbf{v}$ and using Equation (19), we find that the Fourier transform of the fluctuations of the potential is related to the Fourier transform of the external stochastic potential by

$$\delta\hat{\Phi}(\mathbf{k},\omega) = \frac{1-\epsilon(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}\hat{\Phi}_e(\mathbf{k},\omega), \tag{20}$$

where the "dielectric" function is defined by[12]

$$\epsilon(\mathbf{k},\omega) = 1-(2\pi)^d\hat{u}(k)\int\frac{\mathbf{k}\cdot\frac{\partial f}{\partial\mathbf{v}}}{\mathbf{k}\cdot\mathbf{v}-\omega}\,d\mathbf{v}. \tag{21}$$

Without external perturbation, the system would be rigorously spatially homogeneous (in the limit $N\to+\infty$ that we consider here) and described by the distribution function $f(\mathbf{v})$. The external perturbation $\Phi_e(\mathbf{r},t)$ creates a small inhomogeneity $\delta\rho(\mathbf{r},t) = \int \delta f(\mathbf{r},\mathbf{v},t)\,d\mathbf{v}$ (via Equation (13)) producing in turn a weak potential $\delta\Phi(\mathbf{r},t)$ (via Equation (18)). This is like a polarization process. As a result, the total potential acting on a particle, which is sometimes called the dressed or effective potential, is $\delta\Phi_{\rm tot}(\mathbf{r},t) = \Phi_e(\mathbf{r},t)+\delta\Phi(\mathbf{r},t)$. This is the sum of the external perturbation plus the potential fluctuation induced by the system itself (i.e., the system's own response). Equations (13) and (18) are coupled together. One manner to solve this loop is to use Fourier transforms as we have shown above. Using Equation (20), the Fourier transform of the total potential $\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\omega) = \hat{\Phi}_e(\mathbf{k},\omega)+\delta\hat{\Phi}(\mathbf{k},\omega)$ is related to the Fourier transform of the external stochastic potential $\hat{\Phi}_e(\mathbf{k},\omega)$ by

$$\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\omega) = \frac{\hat{\Phi}_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}. \tag{22}$$

Although not explicitly written, we must use the Landau prescription $\omega\to\omega+i0^+$ in Equation (21).[13] As a result, $1/\epsilon(\mathbf{k},\omega)$ is a complex function which plays the role of the response function in plasma physics. It determines the response of the system $\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\omega)$ to an external perturbation $\hat{\Phi}_e(\mathbf{k},\omega)$ through Equation (22). The dielectric function takes into account the polarization of the medium caused by the self-interaction of the particles. This corresponds to the so-called "collective effects". Depending on the form of the self-interaction (attractive or repulsive), the polarization cloud can amplify or shield the effect of the imposed external potential. This amounts to replacing the bare potential $\hat{\Phi}_e(\mathbf{k},\omega)$ by the dressed potential $\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\omega) = \hat{\Phi}_e(\mathbf{k},\omega)/\epsilon(\mathbf{k},\omega)$ which is the potential $\hat{\Phi}_e(\mathbf{k},\omega)$ dressed by the polarization cloud. Without self-interaction, or if we neglect collective effects, we just have $\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\omega) = \hat{\Phi}_e(\mathbf{k},\omega)$, corresponding to $\epsilon(\mathbf{k},\omega) = 1$, and $\delta\hat{\Phi}(\mathbf{k},\omega) = (1-\epsilon(\mathbf{k},\omega))\hat{\Phi}_e(\mathbf{k},\omega) \simeq 0$. The notion of "dressed" particles (or quasiparticles) has been introduced and developed by Hubbard [15,16,22] and Rostoker [115–117] in plasma physics.

Remark 1. *The dielectric function can be written as* $\epsilon(\mathbf{k},\omega) = 1 - M(\mathbf{k},\omega)$*, where* $M(\mathbf{k},\omega)$ *is the polarization function of the system such that* $\delta\hat{\Phi}(\mathbf{k},\omega) = M(\mathbf{k},\omega)\hat{\Phi}_{\rm tot}(\mathbf{k},\omega)$*. On the other hand,* $R(\mathbf{k},\omega) = (1-\epsilon(\mathbf{k},\omega))/\epsilon(\mathbf{k},\omega)$ *is the potential response function of the system such that* $\delta\hat{\Phi}(\mathbf{k},\omega) = R(\mathbf{k},\omega)\hat{\Phi}_e(\mathbf{k},\omega)$ *(see, e.g., Refs. [46,118,119]). In the absence of external perturbation (*$\Phi_e = 0$*), we obtain the dispersion relation* $\epsilon(\mathbf{k},\omega) = 0$ *determining the proper complex pulsations* $\omega(\mathbf{k}) = \omega_r(\mathbf{k}) + i\omega_i(\mathbf{k})$ *of the system for each wavenumber* $\mathbf{k}$*. A distribution function* $f(\mathbf{v})$ *is linearly stable with respect to the Vlasov equation if, and only if,* $\omega_i(\mathbf{k}) < 0$ *for all modes* $\mathbf{k}$ *[114]. The vanishing of* $\omega_i(\mathbf{k})$ *at some* $\mathbf{k}_c$ *signals the onset of an instability. This implies the vanishing of* $\epsilon(\mathbf{k}_c,\omega)$*, or the divergence of* $R(\mathbf{k}_c,\omega)$*, when the real pulsation* $\omega \to \omega(\mathbf{k}_c)$*.*

2.1.3. Power Spectrum

We now assume that the time evolution of the perturbing potential is a stationary stochastic process so that the auto-correlation function depends on t and t' only in the combination $\tau = t - t'$. On the other hand, since the system is spatially homogeneous, the auto-correlation function depends on $\mathbf{r}$ and $\mathbf{r}'$ only in the combination $\mathbf{x} = \mathbf{r} - \mathbf{r}'$. As a result, the auto-correlation function of the external potential can be written as

$$\langle \Phi_e(\mathbf{r},t)\Phi_e(\mathbf{r}',t')\rangle = C(\mathbf{r}-\mathbf{r}',t-t'). \tag{23}$$

The spectral auto-correlation function of the external stochastic potential is

$$\langle \hat{\Phi}_e(\mathbf{k},\omega)\hat{\Phi}_e(\mathbf{k}',\omega')^*\rangle = 2\pi\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')\hat{C}(\mathbf{k},\omega), \tag{24}$$

where $\hat{C}(\mathbf{k},\omega)$ is the Fourier transform in time of $C(\mathbf{x},\tau)$ (Wiener–Khinchin theorem [120,121]). The function $C(\mathbf{r}-\mathbf{r}',t-t')$ describes a possibly colored noise. A white noise corresponds to a correlation function $C(\mathbf{r}-\mathbf{r}',t-t')$ which is δ-correlated in time. In that case, its Fourier transform $\hat{C}(\mathbf{k},\omega)$ is independent of ω.

The auto-correlation function of the total fluctuating potential in Fourier space is given by

$$\langle \delta\hat{\Phi}_{\rm tot}(\mathbf{k},\omega)\delta\hat{\Phi}^*_{\rm tot}(\mathbf{k}',\omega')\rangle = 2\pi\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')P(\mathbf{k},\omega), \tag{25}$$

where $P(\mathbf{k},\omega)$ is the power spectrum. Combining Equations (22), (24), and (25), we obtain[14]

$$P(\mathbf{k},\omega) = \frac{\hat{C}(\mathbf{k},\omega)}{|\epsilon(\mathbf{k},\omega)|^2}. \tag{26}$$

This equation relates the power spectrum $P(\mathbf{k},\omega)$ of the total fluctuating potential acting on the particles to the spectral auto-correlation function $\hat{C}(\mathbf{k},\omega)$ of the external stochastic potential. We note that $\hat{C}(\mathbf{k},\omega)$ and $P(\mathbf{k},\omega)$ are real and positive. The power spectrum $P(\mathbf{k},\omega)$ takes into account collective effects through the dielectric function $\epsilon(\mathbf{k},\omega)$. It can be seen as a dressed correlation function. If we neglect collective effects ($\epsilon = 1$) we obtain the bare power spectrum (or the bare correlation function)

$$P_{\rm bare}(\mathbf{k},\omega) = \hat{C}(\mathbf{k},\omega). \tag{27}$$

It can be directly obtained from Equations (24) and (25) with $\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\omega) = \hat{\Phi}_e(\mathbf{k},\omega)$.

Remark 2. *If the system has a weakly damped mode ω_M, i.e., $\epsilon(\mathbf{k},\omega_M)=0$ with $\omega_M=\omega_0+i\eta$ where $\eta<0$ and $|\eta|\ll\omega_0$, then the power spectrum (for ω real) is close to a Lorentzian $P(\mathbf{k},\omega)\sim 1/[(\omega-\omega_0)^2+\eta^2]$ (we have used $\epsilon(\mathbf{k},\omega)\simeq\epsilon[\mathbf{k},\omega_M+(\omega-\omega_M)]\simeq\epsilon'(\mathbf{k},\omega_M)(\omega-\omega_M)$ implying $|\epsilon(\mathbf{k},\omega)|^2\simeq|\epsilon'(\mathbf{k},\omega_M)|^2[(\omega-\omega_0)^2+\eta^2]$). This is because the (real) pulsation is close to the singularities of $R(\mathbf{k},\omega)$ or $1/\epsilon(\mathbf{k},\omega)$. The fluctuations become very large as $\eta\to 0^-$, i.e., as the mode approaches neutral stability. Indeed, $P(\mathbf{k},\omega)\sim\frac{\pi}{\eta}\delta(\omega-\omega_0)$ when $\eta\to 0^-$ (i.e., $\mathbf{k}\to\mathbf{k}_c$). This is similar to the phenomenon of critical opalescence [122,123]. Large fluctuations also occur in self-gravitating systems close to the Jeans length and in the HMF model close to the critical temperature [8,50,80,124].*

2.1.4. SDD Equation

The basic equations governing the evolution of the mean distribution function $f(\mathbf{v},t)$ of a spatially homogeneous system of particles with long-range interactions forced by an external perturbation are given by Equations (12) and (13). Introducing the total fluctuating potential $\delta\Phi_{\rm tot}=\delta\Phi+\Phi_e$, we can rewrite Equation (12) as

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial\mathbf{v}}\cdot\langle\delta f\nabla\delta\Phi_{\rm tot}\rangle. \tag{28}$$

We now compute the effective collision term[15] appearing on the right hand side of this equation. Introducing the Fourier transforms of the fluctuations of the distribution function and potential, we obtain

$$\frac{\partial f}{\partial t}=-i\int d\mathbf{k}\int\frac{d\omega}{2\pi}\int d\mathbf{k}'\int\frac{d\omega'}{2\pi}\left(\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}}\right)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}e^{-i(\mathbf{k}'\cdot\mathbf{r}-\omega' t)}\langle\delta\hat{f}(\mathbf{k},\mathbf{v},\omega)\delta\hat{\Phi}^*_{\rm tot}(\mathbf{k}',\omega')\rangle. \tag{29}$$

Equation (13) can be written in Fourier space as (see Equation (17))

$$\delta\hat{f}(\mathbf{k},\mathbf{v},\omega)=\frac{\mathbf{k}\cdot\frac{\partial f}{\partial\mathbf{v}}}{\mathbf{k}\cdot\mathbf{v}-\omega}\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\omega). \tag{30}$$

Substituting Equation (30) into Equation (29), we obtain

$$\frac{\partial f}{\partial t}=-i\int d\mathbf{k}\int\frac{d\omega}{2\pi}\int d\mathbf{k}'\int\frac{d\omega'}{2\pi}\left(\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}}\right)e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}e^{-i(\omega-\omega')t}\times\frac{\mathbf{k}\cdot\frac{\partial f}{\partial\mathbf{v}}}{\mathbf{k}\cdot\mathbf{v}-\omega}\langle\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\omega)\delta\hat{\Phi}^*_{\rm tot}(\mathbf{k}',\omega')\rangle. \tag{31}$$

Introducing the power spectrum from Equation (25) and performing the integration over ω' and $\mathbf{k}'$, we find that

$$\frac{\partial f}{\partial t}=-i\int d\mathbf{k}\int\frac{d\omega}{2\pi}\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\frac{\mathbf{k}\cdot\frac{\partial f}{\partial\mathbf{v}}}{\mathbf{k}\cdot\mathbf{v}-\omega}P(\mathbf{k},\omega). \tag{32}$$

Recalling the Landau prescription $\omega\to\omega+i0^+$ and using the Sokhotski–Plemelj [125,126] formula

$$\frac{1}{x\pm i0^+}=\mathrm{P}\left(\frac{1}{x}\right)\mp i\pi\delta(x), \tag{33}$$

where P denotes the principal value, we can replace $1/(\mathbf{k}\cdot\mathbf{v}-\omega-i0^+)$ by $+i\pi\delta(\mathbf{k}\cdot\mathbf{v}-\omega)$ in Equation (32). Accordingly,

$$\frac{\partial f}{\partial t}=\pi\int d\mathbf{k}\int\frac{d\omega}{2\pi}\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\delta(\omega-\mathbf{k}\cdot\mathbf{v})P(\mathbf{k},\omega)\left(\mathbf{k}\cdot\frac{\partial f}{\partial\mathbf{v}}\right). \tag{34}$$

The integration over ω, which corresponds to a condition of resonance at $\omega = \mathbf{k} \cdot \mathbf{v}$, leads to the kinetic equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} \int d\mathbf{k} \left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} \right) P(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}) \left(\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}} \right). \tag{35}$$

Therefore, for spatially homogeneous systems with long-range interactions, the secular evolution of the mean distribution function $f(\mathbf{v}, t)$ of the particles sourced by an external stochastic force is governed by a nonlinear diffusion equation of the form[16]

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left(D_{ij}[f, \mathbf{v}] \frac{\partial f}{\partial v_j} \right), \tag{36}$$

with an anisotropic diffusion tensor

$$D_{ij} = \frac{1}{2} \int d\mathbf{k}\, k_i k_j P(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}). \tag{37}$$

This equation expresses the diffusion tensor in terms of the power spectrum of the total fluctuating potential at the resonance frequencies $\omega = \mathbf{k} \cdot \mathbf{v}$.

Using Equation (26), we can express the diffusion tensor in terms of the spectral auto-correlation function of the external perturbation $\hat{C}(\mathbf{k}, \omega)$ as

$$D_{ij}[f, \mathbf{v}] = \frac{1}{2} \int d\mathbf{k}\, k_i k_j \frac{\hat{C}(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2}. \tag{38}$$

Therefore, the diffusion tensor involves the ratio of the power spectrum of the external perturbation $\langle |\hat{\Phi}_e(\mathbf{k}, \omega)|^2 \rangle$ divided by the squared susceptibility $|\epsilon(\mathbf{k}, \omega)|^2$ of the system both evaluated at the resonance frequencies $\omega = \mathbf{k} \cdot \mathbf{v}$.[17] Equation (36) implies that the secular response of the system is due to the combined effect of the external noise and its susceptibility. The diffusion tensor $D_{ij}[f, \mathbf{v}]$ depends on the velocity $\mathbf{v}$ and on the velocity distribution $f(\mathbf{v}, t)$ itself through the dielectric function $\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})$ defined by Equation (21). As a result, $D_{ij}[f, \mathbf{v}]$ is a functional of f. Equation (36) with the diffusion tensor from Equation (38) is therefore a complicated integrodifferential equation called the SDD equation.

For noninteracting systems or when collective effects are neglected, i.e., when we make $\epsilon = 1$ in Equation (38), the diffusion tensor reduces to

$$D_{ij}(\mathbf{v}) = \frac{1}{2} \int d\mathbf{k}\, k_i k_j \hat{C}(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}). \tag{39}$$

In that case, it depends only on the velocity $\mathbf{v}$. Equation (36) with Equation (39) is called the SBD equation. If $\Phi_e(\mathbf{r}, t)$ is a white noise in time so that $\hat{C}(\mathbf{k}, \omega)$ does not depend on ω, we obtain a pure diffusion equation with a constant diffusion tensor $D_{ij} = \frac{1}{2} \int d\mathbf{k}\, k_i k_j \hat{C}(\mathbf{k})$. If $\hat{C}(\mathbf{k})$ depends only on $k = |\mathbf{k}|$, the diffusion tensor is isotropic: $D_{ij} = D\delta_{ij}$ with $D = \frac{1}{2d} \int \hat{C}(k) k^2\, d\mathbf{k}$. For a colored noise, the spectral auto-correlation function of the external stochastic potential $\hat{C}(\mathbf{k}, \omega)$ is a function of ω that has to be evaluated at the resonance frequencies $\omega = \mathbf{k} \cdot \mathbf{v}$.

Remark 3. *The SDD Equation (36) with Equation (38) was first obtained in Section 6 of Chavanis [23]. A related equation was obtained by Nardini et al. [102] and studied in detail in [103]. However, the diffusion tensor has a form different from Equation (38). Some connections between the two equations have been discussed in [7].*

2.1.5. Properties of the SDD Equation

Some general properties of the SDD Equation (36) can be given. First of all, the total mass of the system $M = \int f\, d\mathbf{v}$ is conserved since the right hand side of Equation (36) is the divergence of a current. By contrast, the energy of the system is not conserved, contrary to the case of the Lenard–Balescu Equation [23], since the system is forced by an external medium. Taking the time derivative of the energy which, for a spatially homogeneous system, reduces to the kinetic energy

$$E = \frac{1}{2} \int f v^2 \, d\mathbf{v}, \tag{40}$$

using Equation (36), and integrating by parts, we obtain

$$\dot{E} = -\int D_{ij}[f, \mathbf{v}] \frac{\partial f}{\partial v_j} v_i \, d\mathbf{v}. \tag{41}$$

In general, $\dot{E}$ has not a definite sign. However, when $f = f(H)$ with $f'(H) \leq 0$ where $H = (1/2)mv^2$, we obtain $\dot{E} = -\int f'(H) D_{ij} v_i v_j \, d\mathbf{v} \geq 0$ (since $D_{ij} v_i v_j \geq 0$) so that energy is injected in the system. Finally, introducing the H-functions

$$S = -\int C(f) \, d\mathbf{v}, \tag{42}$$

where $C(f)$ is any convex function, we obtain

$$\dot{S} = \int C''(f) \frac{\partial f}{\partial v_i} D_{ij}[f, \mathbf{v}] \frac{\partial f}{\partial v_j} \, d\mathbf{v}. \tag{43}$$

Because of the convexity condition $C'' \geq 0$ and the fact that the quadratic form $x_i D_{ij} x_j$ is definite positive (see Section 2.1.4), we find that $\dot{S} \geq 0$. Therefore, all the H-functions increase monotonically with time. This is different from the case of the Lenard–Balescu equation in dimension $d > 1$ where only the Boltzmann entropy increases monotonically [23].

If we heuristically introduce a dissipative term in the SDD equation by analogy with the homogeneous Kramers equation (see Section 4.1 of [23]), we get an equation of the form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left(D_{ij}[f, \mathbf{v}] \frac{\partial f}{\partial v_j} + \xi f v_i \right), \tag{44}$$

where $\xi > 0$ is the friction coefficient. The general behavior of this nonlinear equation is difficult to predict as it depends on the potential of interaction of the particles and on the correlation function of the external potential. Furthermore, the diffusion tensor is a functional of f. Equation (44) may relax towards a non-Boltzmannian steady state determined by the equation

$$D_{ij}[f, \mathbf{v}] \frac{\partial f}{\partial v_j} + \xi f v_i = 0, \tag{45}$$

or exhibit a complicated (e.g., periodic) dynamics. Since the diffusion tensor is a functional of f, the nonlinear SDD equation presents a rich and complex behavior.

2.1.6. Stochastic SDD Equation

The SDD Equation (44), which is a deterministic partial differential equation, describes the evolution of the mean distribution function $f(\mathbf{v}, t)$. If we take fluctuations into account, by analogy with the results presented in [124], we expect that the mesoscopic distribution function $\overline{f}(\mathbf{r},\mathbf{v},t)$ will satisfy a stochastic partial differential equation of the form[18]

$$\frac{\partial \overline{f}}{\partial t}+\mathbf{v}\cdot\frac{\partial \overline{f}}{\partial \mathbf{r}}-\nabla\overline{\Phi}\cdot\frac{\partial \overline{f}}{\partial \mathbf{v}}=\frac{\partial}{\partial v_i}\left(D_{ij}[\overline{f},\mathbf{v}]\frac{\partial \overline{f}}{\partial v_j}+\xi\overline{f}v_i\right)+\zeta(\mathbf{r},\mathbf{v},t), \tag{46}$$

where $\zeta(\mathbf{r},\mathbf{v},t)$ is a noise term with zero mean that generally depends on $\overline{f}(\mathbf{r},\mathbf{v},t)$. When $D_{ij}=D\delta_{ij}$ is constant and isotropic and when the fluctuation–dissipation theorem is fulfilled so that $\xi=D\beta m$, as in the case of Brownian particles with long-range interactions, the noise term is given by [124]

$$\zeta(\mathbf{r},\mathbf{v},t)=\frac{\partial}{\partial \mathbf{v}}\cdot\left[\sqrt{2Dm\overline{f}}\,\mathbf{Q}(\mathbf{r},\mathbf{v},t)\right], \tag{47}$$

where $\mathbf{Q}(\mathbf{r},\mathbf{v},t)$ is a Gaussian white noise satisfying $\langle Q_i(\mathbf{r},\mathbf{v},t)\rangle=0$ and $\langle Q_i(\mathbf{r},\mathbf{v},t)Q_j(\mathbf{r}',\mathbf{v}',t')\rangle=\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{v}-\mathbf{v}')\delta(t-t')$. This expression can be obtained from an adaptation of the theory of fluctuating hydrodynamics [124]. In that case, the stochastic partial differential equation reads

$$\frac{\partial \overline{f}}{\partial t}+\mathbf{v}\cdot\frac{\partial \overline{f}}{\partial \mathbf{r}}-\nabla\overline{\Phi}\cdot\frac{\partial \overline{f}}{\partial \mathbf{v}}=\frac{\partial}{\partial \mathbf{v}}\cdot\left[D\left(\frac{\partial \overline{f}}{\partial \mathbf{v}}+\beta m\overline{f}\mathbf{v}\right)\right]+\frac{\partial}{\partial \mathbf{v}}\cdot\left[\sqrt{2Dm\overline{f}}\,\mathbf{Q}(\mathbf{r},\mathbf{v},t)\right], \tag{48}$$

and the deterministic equation for the mean distribution function (i.e., Equation (48) without the noise term) relaxes towards the Boltzmann distribution $f=Ae^{-\beta m(v^2/2+\Phi(\mathbf{r}))}$. When $D[\overline{f}]$ is a functional of $\overline{f}$, the noise term may be more complicated.[19] When the deterministic Equation (44) admits several equilibrium states, the noise term in Equation (46) can trigger random transitions from one state to the other (see, e.g., [128,129]).

Remark 4. *We may alternatively consider a stochastic partial differential equation of the form*

$$\frac{\partial \overline{f}}{\partial t}+\mathbf{v}\cdot\frac{\partial \overline{f}}{\partial \mathbf{r}}-\nabla\overline{\Phi}\cdot\frac{\partial \overline{f}}{\partial \mathbf{v}}=\frac{\partial}{\partial v_i}\left(D_{ij}[\overline{f},\mathbf{v}]\frac{\partial \overline{f}}{\partial v_j}\right)-\nu\overline{f}+\zeta(\mathbf{r},\mathbf{v},t), \tag{49}$$

where $-\nu\overline{f}$ is a linear damping term with $0<\nu\ll 1$. The stationary solutions of the corresponding deterministic equation (Equation (49) with $\zeta=0$) are determined by

$$\frac{\partial}{\partial v_i}\left(D_{ij}[f,\mathbf{v}]\frac{\partial f}{\partial v_j}\right)-\nu f=0. \tag{50}$$

Since $D_{ij}[f]$ is a functional of f, Equation (50) is a very nonlinear equation which admits nontrivial solutions.

2.2. Derivation of the SDD Equation from the Fokker–Planck Equation

In this section, we derive the SDD equation directly from the Fokker–Planck equation. We follow an approach similar to the one developed in Section 3 of [23] to derive the Lenard–Balescu equation.

2.2.1. Fokker–Planck Equation

Let us consider the evolution of a test particle of mass m moving in a spatially homogeneous medium and experiencing a stochastic perturbation $\Phi_e(\mathbf{r},t)$. The equations of motion of the test particle are

$$\frac{d\mathbf{r}}{dt}=\mathbf{v}\quad\text{and}\quad\frac{d\mathbf{v}}{dt}=-\nabla\delta\Phi_{\rm tot}(\mathbf{r},t), \tag{51}$$

where $\delta\Phi_{\rm tot}(\mathbf{r},t)$ is the total fluctuating potential acting on the particle. They can be written in Hamiltonian form as $md\mathbf{r}/dt = \partial(H+\delta H_{\rm tot})/\partial\mathbf{v}$ and $md\mathbf{v}/dt = -\partial(H+\delta H_{\rm tot})/\partial\mathbf{r}$, where $H=(1/2)mv^2$ is the mean Hamiltonian and $\delta H_{\rm tot}$ is the total fluctuating Hamiltonian. At leading order, the test particle follows a rectilinear trajectory at constant velocity $\mathbf{v}$ (the mean force vanishes) but it also experiences a small stochastic perturbation $\delta\Phi_{\rm tot}=\Phi_e+\delta\Phi$ which is equal to the external potential Φ_e plus the fluctuating potential $\delta\Phi$ produced by the system itself (collective effects). Equations (51) can be formally integrated into

$$\mathbf{r}(t)=\mathbf{r}+\int_0^t \mathbf{v}(t')\,dt',\qquad \mathbf{v}(t)=\mathbf{v}-\int_0^t \nabla\delta\Phi_{\rm tot}(\mathbf{r}(t'),t')\,dt',\tag{52}$$

where we have assumed that, initially, the test particle is at position $\mathbf{r}$ with velocity $\mathbf{v}$. Since the fluctuations $\delta\Phi_{\rm tot}$ of the potential are small, the changes in the velocity of the test particle are also small. On the other hand, the fluctuation time is short with respect to the evolution time of the distribution function. As a result, the dynamics of the test particle can be represented by a stochastic process governed by a Fokker–Planck equation [34,112]. The Fokker–Planck equation can be derived from the Master equation by using the Kramers–Moyal [110,111] expansion truncated at the level of the second moments of the increment in velocity.[20] If we denote by $f(\mathbf{v},t)$ the probability density that the test particle has a velocity $\mathbf{v}$ at time t, the general form of this equation is

$$\frac{\partial f}{\partial t}=\frac{\partial^2}{\partial v_i\partial v_j}(D_{ij}f)-\frac{\partial}{\partial v_i}(fF_i^{\rm tot}).\tag{53}$$

The diffusion tensor and the friction force are defined by

$$D_{ij}(\mathbf{v})=\lim_{t\to+\infty}\frac{1}{2t}\langle(v_i(t)-v_i)(v_j(t)-v_j)\rangle=\frac{\langle\Delta v_i\Delta v_j\rangle}{2\Delta t},\tag{54}$$

$$F_i^{\rm tot}(\mathbf{v})=\lim_{t\to+\infty}\frac{1}{t}\langle v_i(t)-v_i\rangle=\frac{\langle\Delta v_i\rangle}{\Delta t}.\tag{55}$$

In writing these limits, we have implicitly assumed that the time t is long compared to the fluctuation time but short compared to the evolution time of the distribution function.

As shown in our previous papers [8,23,59], it is relevant to rewrite the Fokker–Planck equation in the alternative form

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial v_i}\left(D_{ij}\frac{\partial f}{\partial v_j}-fF_i^{\rm pol}\right).\tag{56}$$

The total friction can be written as

$$F_i^{\rm tot}=F_i^{\rm pol}+\frac{\partial D_{ij}}{\partial v_j},\tag{57}$$

where $\mathbf{F}_{\rm pol}$ is the friction by polarization (see Section 3.3 of [23]) while the second term is due to the variation of the diffusion tensor with the velocity $\mathbf{v}$ (see Section 3.4 of [23]). As explained in [8,23,59,136] in the context of the Lenard–Balescu equation, the friction by polarization $\mathbf{F}_{\rm pol}$ arises from the retroaction (response) of the system to the perturbation caused by the test particle, just like in a polarization process. It represents, however, only one component of the dynamical friction $\mathbf{F}_{\rm tot}$ experienced by the test particle, the other component being $\partial_j D_{ij}$.

Remark 5. *The two expressions (53) and (56) of the Fokker–Planck equation have their own interest. The expression (53) where the diffusion tensor is placed after the second derivative $\partial^2(DP)$ involves the total friction $\mathbf{F}_{\rm tot}$ while the expression (56) where the diffusion tensor is placed between the* *derivatives $\partial D \partial P$ isolates the friction by polarization $\mathbf{F}_{\rm pol}$. It is shown in [23] that this second form is directly related to the Lenard–Balescu equation. This is also the form corresponding to the SDD Equation (36). It has therefore a clear physical meaning.*

2.2.2. Absence of Friction by Polarization

In the limit $N \to +\infty$ with $m \sim 1/N$ where the collisions between the particles are negligible, the friction by polarization (which is proportional to m [23]) vanishes

$$\mathbf{F}_{\rm pol} = \mathbf{0}. \tag{58}$$

Indeed, the perturbation on the system caused by the test particle is negligible. In that case, the Fokker–Planck Equation (56) reduces to

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i}\left(D_{ij}\frac{\partial f}{\partial v_j}\right). \tag{59}$$

We note that, in Equation (59), the diffusion tensor is "sandwiched" between the two derivatives $\partial/\partial v$ in agreement with Equation (36). As mentioned previously, this is not the usual form of the Fokker–Planck equation which is given by Equation (53). Therefore, the test particle experiences a friction (see Equation (57))

$$F_i^{\rm tot} = \frac{\partial D_{ij}}{\partial v_j}, \tag{60}$$

arising from the velocity dependence of the diffusion tensor.[21] Using Equations (54) and (55), this relation can be written as

$$\frac{\langle \Delta v_i \rangle}{\Delta t} = \frac{1}{2}\frac{\partial}{\partial v_j}\frac{\langle \Delta v_i \Delta v_j \rangle}{\Delta t}. \tag{61}$$

Remark 6. *This relation is sometimes referred to as a fluctuation-dissipation theorem because it links the friction to the diffusion. However, this is not the usual fluctuation-dissipation theorem which arises in connection to the Fokker–Planck form of the Lenard–Balescu Equation [136]. Equation (60) links D_{ij} and $\mathbf{F}_{\rm tot}$ when $\mathbf{F}_{\rm pol} = \mathbf{0}$ while the usual fluctuation–dissipation theorem $F_i^{\rm pol} = -\beta m D_{ij} v_j$ [23] links D_{ij} and $\mathbf{F}_{\rm pol}$ at statistical equilibrium (i.e., for the Maxwellian distribution function).*

2.2.3. First Calculation of D_{ij}

We now calculate the diffusion tensor from Equation (54) following the approach developed in Section 3.2 of [23]. According to Equation (51) the increment in velocity of the test particle is

$$\Delta \mathbf{v} = -\int_0^t \nabla \delta\Phi_{\rm tot}(\mathbf{r}(t'), t')\, dt'. \tag{62}$$

Substituting Equation (62) into Equation (54) and assuming that the correlations of the fluctuating force persist for a time less than the time for the trajectory of the test particle to be much altered, we can make a linear trajectory approximation[22]

$$\mathbf{r}(t') = \mathbf{r} + \mathbf{v}t', \qquad \mathbf{v}(t') = \mathbf{v}, \tag{63}$$

and write

$$D_{ij} = \lim_{t\to+\infty}\frac{1}{2t}\int_0^t dt' \int_0^t dt'' \left\langle \frac{\partial \delta\Phi_{\rm tot}}{\partial x_i}(\mathbf{r}+\mathbf{v}t', t')\frac{\partial \delta\Phi_{\rm tot}}{\partial x_j}(\mathbf{r}+\mathbf{v}t'', t'')\right\rangle. \tag{64}$$

Introducing the Fourier transform of the total fluctuating potential, we obtain

$$\left\langle \frac{\partial\delta\Phi_{\rm tot}}{\partial x_i}(\mathbf{r}+\mathbf{v}t',t')\frac{\partial\delta\Phi_{\rm tot}}{\partial x_j}(\mathbf{r}+\mathbf{v}t'',t'')\right\rangle = \int d\mathbf{k}\int\frac{d\omega}{2\pi}\int d\mathbf{k}'\int\frac{d\omega'}{2\pi}\,k_ik_j' \times e^{i\mathbf{k}\cdot(\mathbf{r}+\mathbf{v}t')}e^{-i\omega t'}e^{-i\mathbf{k}'\cdot(\mathbf{r}+\mathbf{v}t'')}e^{i\omega' t''}\langle\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\omega)\delta\hat{\Phi}_{\rm tot}(\mathbf{k}',\omega')^*\rangle. \quad (65)$$

Substituting Equation (25) into Equation (65), and carrying out the integrals over $\mathbf{k}'$ and ω', we end up with the result

$$\left\langle \frac{\partial\delta\Phi_{\rm tot}}{\partial x_i}(\mathbf{r}+\mathbf{v}t',t')\frac{\partial\delta\Phi_{\rm tot}}{\partial x_j}(\mathbf{r}+\mathbf{v}t'',t'')\right\rangle = \int d\mathbf{k}\int\frac{d\omega}{2\pi}\,k_ik_je^{i(\mathbf{k}\cdot\mathbf{v}-\omega)(t'-t'')}P(\mathbf{k},\omega). \quad (66)$$

This expression shows that the correlation function appearing in Equation (64) depends only on the difference of times $s=t'-t''$. Using the identity

$$\begin{aligned}\int_0^t dt'\int_0^t dt''\,f(t'-t'') &= 2\int_0^t dt'\int_{t'}^t dt''\,f(t'-t'') = 2\int_0^t dt'\int_0^{t-t'} ds\,f(s)\\ &= 2\int_0^t ds\int_0^{t-s} dt'\,f(s) = 2\int_0^t ds\,(t-s)f(s),\end{aligned} \quad (67)$$

and assuming that the force auto-correlation function $f(s)$ decreases more rapidly than s^{-1},[23] we find for $t\to+\infty$ that[24]

$$D_{ij} = \int_0^{+\infty}\left\langle \frac{\partial\delta\Phi_{\rm tot}}{\partial x_i}(\mathbf{r},0)\frac{\partial\delta\Phi_{\rm tot}}{\partial x_j}(\mathbf{r}+\mathbf{v}s,s)\right\rangle ds. \quad (68)$$

Therefore, as in the theory of Brownian motion [34,112,138–140], turbulent fluids [141], plasma physics [15,16,22,49,142], and stellar dynamics [8,31,137,143–147], the diffusion tensor of the test particle is equal to the integral of the temporal auto-correlation function $\langle F_i(0)F_j(t)\rangle$ of the fluctuating force acting on it along the unperturbed motions:

$$D_{ij} = \int_0^{+\infty}\langle F_i(0)F_j(t)\rangle\,dt. \quad (69)$$

Replacing the auto-correlation function by its expression from Equation (66), which can be written as

$$\langle F_i(0)F_j(t)\rangle = \int d\mathbf{k}\int\frac{d\omega}{2\pi}\,k_ik_je^{i(\mathbf{k}\cdot\mathbf{v}-\omega)t}P(\mathbf{k},\omega), \quad (70)$$

we obtain

$$D_{ij} = \int_0^{+\infty}dt\int d\mathbf{k}\int\frac{d\omega}{2\pi}\,k_ik_je^{i(\mathbf{k}\cdot\mathbf{v}-\omega)t}P(\mathbf{k},\omega). \quad (71)$$

Making the change of variables $t\to-t$, $\mathbf{k}\to-\mathbf{k}$ and $\omega\to-\omega$, and using the fact that $P(-\mathbf{k},-\omega)=P(\mathbf{k},\omega)$, we see that we can replace $\int_0^{+\infty}dt$ by $(1/2)\int_{-\infty}^{+\infty}dt$ in Equation (71). As a result,

$$D_{ij} = \frac{1}{2}\int_{-\infty}^{+\infty}dt\int d\mathbf{k}\int\frac{d\omega}{2\pi}\,k_ik_je^{i(\mathbf{k}\cdot\mathbf{v}-\omega)t}P(\mathbf{k},\omega). \quad (72)$$

Using the identity (16), we obtain

$$D_{ij} = \pi\int d\mathbf{k}\int\frac{d\omega}{2\pi}\,k_ik_j\delta(\mathbf{k}\cdot\mathbf{v}-\omega)P(\mathbf{k},\omega). \quad (73)$$

The time integration has given a δ-function which creates a resonance condition for interaction. Integrating over the δ-function (resonance), we find that

$$D_{ij} = \frac{1}{2}\int d\mathbf{k}\, k_i k_j P(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}). \tag{74}$$

This is the general expression of the diffusion tensor of a test particle submitted to a stochastic perturbation in a spatially homogeneous system with long-range interactions. Since $P(\mathbf{k}, \omega) > 0$ (see Section 2.1.3), the diffusion tensor is positive definite. Using the relation between the power spectrum of the total fluctuating potential and the spectral auto-correlation function of the external perturbation (see Equation (26)), we obtain

$$D_{ij} = \frac{1}{2}\int d\mathbf{k}\, k_i k_j \frac{\hat{C}(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2}. \tag{75}$$

We thus recover the expressions from Equations (37) and (38) obtained from the Klimontovich approach. Therefore, the Klimontovich approach and the Fokker–Planck approach coincide.

2.2.4. Second Calculation of D_{ij}

In order to compute the diffusion tensor, we can also proceed as follows. The change in velocity of the test particle due to the total fluctuating potential is given by (see Equation (51))

$$\frac{d\mathbf{v}}{dt} = -\nabla\delta\Phi_{\rm tot}(\mathbf{r}, t). \tag{76}$$

Integrating this equation between 0 and t, we obtain

$$\begin{aligned} \Delta\mathbf{v} &= -\int_0^t \nabla\delta\Phi_{\rm tot}(\mathbf{r}(t'), t')\, dt' \\ &= -\int_0^t \nabla\delta\Phi_{\rm tot}(\mathbf{r}+\mathbf{v}t', t')\, dt', \end{aligned} \tag{77}$$

where we have used the unperturbed equations of motion (63) in the second equality (this accounts for the fact that the test particle follows a rectilinear trajectory at leading order). Decomposing the total fluctuating potential $\delta\Phi_{\rm tot}$ in Fourier modes and integrating over time t', we obtain

$$\begin{aligned} \Delta\mathbf{v} &= -\int_0^t dt' \nabla \int d\mathbf{k} \int \frac{d\omega}{2\pi}\, e^{i\mathbf{k}\cdot(\mathbf{r}+\mathbf{v}t')} e^{-i\omega t'} \delta\hat{\Phi}_{\rm tot}(\mathbf{k}, \omega) \\ &= -\int d\mathbf{k} \int \frac{d\omega}{2\pi}\, i\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} \delta\hat{\Phi}_{\rm tot}(\mathbf{k}, \omega) \int_0^t dt'\, e^{i(\mathbf{k}\cdot\mathbf{v}-\omega)t'} \\ &= -\int d\mathbf{k} \int \frac{d\omega}{2\pi}\, i\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} \delta\hat{\Phi}_{\rm tot}(\mathbf{k}, \omega) \frac{e^{i(\mathbf{k}\cdot\mathbf{v}-\omega)t} - 1}{i(\mathbf{k}\cdot\mathbf{v}-\omega)}. \end{aligned} \tag{78}$$

Substituting Equation (78) into Equation (54), we find that

$$\begin{aligned} D_{ij} = \lim_{t\to+\infty} \frac{1}{2t} \int d\mathbf{k} \int d\mathbf{k}' \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} k_i k'_j e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} \langle \delta\hat{\Phi}_{\rm tot}(\mathbf{k}, \omega) \delta\hat{\Phi}_{\rm tot}(\mathbf{k}', \omega')^* \rangle \\ \times \frac{e^{i(\mathbf{k}\cdot\mathbf{v}-\omega)t} - 1}{i(\mathbf{k}\cdot\mathbf{v}-\omega)} \frac{e^{-i(\mathbf{k}'\cdot\mathbf{v}-\omega')t} - 1}{-i(\mathbf{k}'\cdot\mathbf{v}-\omega')}. \end{aligned} \tag{79}$$

Introducing the power spectrum from Equation (25) into Equation (79) and carrying out the integrals over $\mathbf{k}'$ and ω' we are left with

$$D_{ij} = \lim_{t\to+\infty} \frac{1}{2t} \int d\mathbf{k} \int \frac{d\omega}{2\pi}\, k_i k_j \frac{|e^{i(\mathbf{k}\cdot\mathbf{v}-\omega)t} - 1|^2}{(\mathbf{k}\cdot\mathbf{v}-\omega)^2} P(\mathbf{k}, \omega). \tag{80}$$

This equation can also be directly obtained from Equation (64) with Equation (66) by integrating over t' and t''. Taking the limit $t \to +\infty$ and using the identity (see, e.g., Appendix E1 of [60])

$$\lim_{t\to+\infty} \frac{|e^{ixt}-1|^2}{x^2 t} = 2\pi\delta(x), \tag{81}$$

we find that

$$D_{ij} = \pi \int d\mathbf{k} \int \frac{d\omega}{2\pi} k_i k_j \delta(\mathbf{k}\cdot\mathbf{v} - \omega) P(\mathbf{k}, \omega). \tag{82}$$

Integrating over the δ-function (resonance), we obtain

$$D_{ij} = \frac{1}{2} \int d\mathbf{k}\, k_i k_j P(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}), \tag{83}$$

which returns Equation (74). Then, using Equation (26), we obtain Equation (75).

Remark 7. *If we do not take the limit $t \to +\infty$ in Equation (80), we obtain a time-dependent diffusion tensor of the form*

$$D_{ij}(t) = \pi \int d\mathbf{k} \int \frac{d\omega}{2\pi} k_i k_j \Delta(\mathbf{k}\cdot\mathbf{v} - \omega, t) P(\mathbf{k}, \omega) \tag{84}$$

with the regularized function

$$\Delta(x, t) = \frac{1}{2\pi t} \frac{|e^{ixt}-1|^2}{x^2} = \frac{1-\cos(xt)}{\pi t x^2}. \tag{85}$$

When $t \to +\infty$, we can make the replacement $\Delta(x, t) \to \delta(x)$.

2.2.5. Third Calculation of D_{ij}

We can make the calculations of the previous section in a slightly different manner. In Equation (77) we decompose the total fluctuating potential in Fourier modes in position but not in time. In that case, we obtain

$$\begin{aligned} \Delta\mathbf{v} &= -\int_0^t dt' \, \nabla \int d\mathbf{k}\, e^{i\mathbf{k}\cdot(\mathbf{r}+\mathbf{v}t')} \delta\hat{\Phi}_{\rm tot}(\mathbf{k}, t') \\ &= -\int_0^t dt' \int d\mathbf{k}\, i\mathbf{k} e^{i\mathbf{k}\cdot(\mathbf{r}+\mathbf{v}t')} \delta\hat{\Phi}_{\rm tot}(\mathbf{k}, t'). \end{aligned} \tag{86}$$

Substituting Equation (86) into Equation (54), we obtain

$$D_{ij} = \lim_{t\to+\infty} \tfrac{1}{2t} \int_0^t dt' \int_0^t dt'' \int d\mathbf{k} d\mathbf{k}' k_i k_j' e^{i\mathbf{k}\cdot(\mathbf{r}+\mathbf{v}t')} e^{-i\mathbf{k}'\cdot(\mathbf{r}+\mathbf{v}t'')} \langle \delta\hat{\Phi}_{\rm tot}(\mathbf{k}, t') \delta\hat{\Phi}_{\rm tot}(\mathbf{k}', t'')^* \rangle. \tag{87}$$

Using the relation

$$\langle \delta\hat{\Phi}_{\rm tot}(\mathbf{k}, t) \delta\hat{\Phi}_{\rm tot}(\mathbf{k}', t')^* \rangle = \delta(\mathbf{k} - \mathbf{k}') \mathcal{P}(\mathbf{k}, t - t'), \tag{88}$$

where $\mathcal{P}(\mathbf{k}, t)$ is the temporal inverse Fourier transform of $P(\mathbf{k}, \omega)$, we can rewrite the foregoing equation as

$$D_{ij} = \lim_{t\to+\infty} \frac{1}{2t} \int_0^t dt' \int_0^t dt'' \int d\mathbf{k}\, k_i k_j e^{i\mathbf{k}\cdot\mathbf{v}(t'-t'')} \mathcal{P}(\mathbf{k}, t' - t''). \tag{89}$$

Using the identity (67), we obtain

$$D_{ij} = \lim_{t\to+\infty} \frac{1}{t} \int_0^t ds\, (t - s) \int d\mathbf{k}\, k_i k_j e^{i\mathbf{k}\cdot\mathbf{v}s} \mathcal{P}(\mathbf{k}, s). \tag{90}$$

Assuming that $\mathcal{P}(\mathbf{k}, s)$ decreases more rapidly than s^{-1}, we find that

$$D_{ij} = \int_0^{+\infty} ds \int d\mathbf{k}\, k_i k_j e^{i\mathbf{k}\cdot\mathbf{v}s} \mathcal{P}(\mathbf{k}, s). \tag{91}$$

Making the change of variables $s \to -s$ and $\mathbf{k} \to -\mathbf{k}$, and using the fact that $\mathcal{P}(-\mathbf{k}, -s) = \mathcal{P}(\mathbf{k}, s)$, we see that we can replace $\int_0^{+\infty} ds$ by $(1/2)\int_{-\infty}^{+\infty} ds$. Therefore,

$$D_{ij} = \frac{1}{2}\int_{-\infty}^{+\infty} ds \int d\mathbf{k}\, k_i k_j e^{i\mathbf{k}\cdot\mathbf{v}s} \mathcal{P}(\mathbf{k}, s). \tag{92}$$

Taking the inverse Fourier transform of $\mathcal{P}(\mathbf{k}, s)$ we obtain

$$D_{ij} = \frac{1}{2}\int d\mathbf{k}\, k_i k_j P(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}), \tag{93}$$

which returns Equation (74). Then, using Equation (26), we obtain Equation (75). We note that $\mathcal{P}(\mathbf{k}, s)$ is complex while $P(\mathbf{k}, \omega)$ is real. They satisfy the identities $\mathcal{P}(-\mathbf{k}, s) = \mathcal{P}(\mathbf{k}, s)^* = \mathcal{P}(\mathbf{k}, -s)$ and $P(\mathbf{k}, \omega) = P(\mathbf{k}, \omega)^* = P(-\mathbf{k}, -\omega)$. We also note that the static power spectrum is $P(\mathbf{k}) = \mathcal{P}(\mathbf{k}, 0) = \int \frac{d\omega}{2\pi} P(\mathbf{k}, \omega)$.

Remark 8. *If we introduce the temporal Fourier transform of $\mathcal{P}(\mathbf{k}, t)$ in Equation (89) we obtain*

$$D_{ij} = \lim_{t\to+\infty} \frac{1}{2t}\int_0^t dt' \int_0^t dt'' \int d\mathbf{k} \int \frac{d\omega}{2\pi}\, k_i k_j e^{i\mathbf{k}\cdot\mathbf{v}(t'-t'')} e^{-i\omega(t'-t'')} P(\mathbf{k}, \omega), \tag{94}$$

which is equivalent to Equation (64) with Equation (66). If we integrate over t' and t'', we recover Equation (80).

2.3. Energy of Fluctuations

The energy of fluctuations

$$\mathcal{E} = \frac{1}{2}\langle \delta\rho_{\rm tot}\delta\Phi_{\rm tot}\rangle, \tag{95}$$

where $\delta\rho_{\rm tot} = \delta\rho + \rho_e$ is the total perturbed density, can be calculated as follows. Decompositing the fluctuations of density and potential in Fourier modes, we obtain

$$\mathcal{E} = \frac{1}{2}\int d\mathbf{k}d\mathbf{k}' \int \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} e^{-i(\mathbf{k}'\cdot\mathbf{r}-\omega' t)} \langle \delta\hat{\rho}_{\rm tot}(\mathbf{k}, \omega)\delta\hat{\Phi}_{\rm tot}(\mathbf{k}', \omega')^*\rangle. \tag{96}$$

Using Equation (19) we can rewrite the foregoing equation as

$$\mathcal{E} = \frac{1}{2}\int d\mathbf{k}d\mathbf{k}' \int \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} e^{-i(\mathbf{k}'\cdot\mathbf{r}-\omega' t)} \frac{1}{(2\pi)^d \hat{u}(k)} \langle \delta\hat{\Phi}_{\rm tot}(\mathbf{k}, \omega)\delta\hat{\Phi}_{\rm tot}(\mathbf{k}', \omega')^*\rangle. \tag{97}$$

Introducing the power spectrum from Equation (25) and integrating over $\mathbf{k}'$ and ω', we obtain

$$\mathcal{E} = \frac{1}{2}\int d\mathbf{k} \int \frac{d\omega}{2\pi} \frac{P(\mathbf{k}, \omega)}{(2\pi)^d \hat{u}(k)}. \tag{98}$$

Finally, using Equation (26), we can rewrite the energy of fluctuations as

$$\mathcal{E} = \frac{1}{2}\int d\mathbf{k} \int \frac{d\omega}{2\pi} \frac{\hat{C}(\mathbf{k}, \omega)}{(2\pi)^d \hat{u}(k) |\epsilon(\mathbf{k}, \omega)|^2}. \tag{99}$$

Remark 9. *In the gravitational case, the energy of fluctuations is negative since $\hat{C}(\mathbf{k}, \omega) > 0$ and $\hat{u}(k) < 0$ in agreement with the relation $\mathcal{E} = -\frac{1}{8\pi G}\langle (\nabla\delta\Phi_{\rm tot})^2\rangle$ that can be deduced directly from Equation (95) by using the Poisson equation $\Delta\Phi = S_d G\rho$. More generally, the energy is positive* *for repulsive potentials with $\hat{u}(k) > 0$ (like plasmas) and negative for attractive potentials with $\hat{u}(k) < 0$ (like gravity).*

2.4. Connection between the SDD Equation and the Multi-Species Lenard–Balescu Equation

In this section, we discuss the connection between the SDD Equation (36) with Equation (38) and the multi-species Lenard–Balescu equation. The multi-species Lenard–Balescu equation reads (see, e.g., [8,148])

$$\frac{\partial f_a}{\partial t} = \sum_b \pi(2\pi)^d \frac{\partial}{\partial v_i} \int d\mathbf{k}d\mathbf{v}'\, k_i k_j \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} \delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')] \times \left(m_b \frac{\partial}{\partial v_j} - m_a \frac{\partial}{\partial v'_j} \right) f_a(\mathbf{v}, t) f_b(\mathbf{v}', t). \tag{100}$$

This equation governs the evolution of the distribution function $f_a(\mathbf{v}, t)$ of particles of species a under the effects of "collisions" with particles of all species "b" (including the particles of species a) with distribution function $f_b(\mathbf{v}', t)$. The dielectric function is given by Equation (21) where $f(\mathbf{v}, t)$ denotes the total distribution function $\sum_b f_b(\mathbf{v}, t)$. The set of Equations (100), in which all the distribution functions $f_a(\mathbf{v}, t)$ evolve in a self-consistent manner, is closed.

We now make the following approximations to simplify these equations. The particles of mass m_a with a distribution function $f_a(\mathbf{v}, t)$ form our system. They will be called the test particles. The particles of masses $\{m_b\}_{b\neq a}$ with distribution functions $\{f_b(\mathbf{v}, t)\}_{b\neq a}$ form the external—background—medium. They will be called the field particles. We take into account the collisions induced by the field particles on the test particles but we neglect the collisions induced by the test particles on the field particles and on themselves. This approximation is valid for very light test particles $m_a \ll m_b$ or, more precisely, in the limit $N_a \to +\infty$ with $m_a \sim 1/N_a$. The distribution functions $\{f_b(\mathbf{v}, t)\}_{b\neq a}$ of the field particles are either assumed to be fixed (bath) or evolve according to their own dynamics (i.e., following equations that we do not write explicitly). Under these conditions, the kinetic Equation (100) reduces to

$$\frac{\partial f_a}{\partial t} = \sum_{b\neq a} \pi(2\pi)^d \frac{\partial}{\partial v_i} \int d\mathbf{k}d\mathbf{v}'\, k_i k_j \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} \delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')] \left(m_b \frac{\partial}{\partial v_j} \right) f_a(\mathbf{v}, t) f_b(\mathbf{v}'). \tag{101}$$

This equation can be interpreted as a Fokker–Planck equation. The diffusion arises from the discrete distribution of the field particles which creates a fluctuating gravitational force (Poisson shot noise) acting on the test particles. For that reason, the diffusion tensor is proportional to the masses $\{m_b\}$ of the field particles. The diffusion tensor has no contribution from particles of species a. The condition $m_a \ll m_b$ justifies neglecting the fluctuations induced by the test particles on themselves.[25] On the other hand, the friction by polarization vanishes ($\mathbf{F}_{\rm pol} = \mathbf{0}$). Indeed, since the mass m_a of the test particles is small, the test particles do not significantly perturb the distribution function of the medium, so there is no friction by polarization (no retroaction).[26] As a result, the mass m_a of the particles of species a does not appear in the kinetic Equation (101).

Equation (101) can be written as an SDD equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left(D_{ij}[\mathbf{v}, f] \frac{\partial f}{\partial v_j} \right) \tag{102}$$

with a diffusion tensor

$$D_{ij}[\mathbf{v}, f] = \sum_b \pi(2\pi)^d \int d\mathbf{k}d\mathbf{v}'\, k_i k_j \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} \delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]\, m_b f_b(\mathbf{v}'), \tag{103}$$

where we have dropped the subscript a for clarity. One can show [136] that the expression (103) of the diffusion tensor is consistent with Equation (38) where

$$\hat{C}(\mathbf{k},\omega) = \sum_b (2\pi)^{d+1}\hat{u}(k)^2 \int d\mathbf{v}'\, \delta(\omega - \mathbf{k}\cdot\mathbf{v}') m_b f_b(\mathbf{v}') \tag{104}$$

is the bare correlation function of the potential created by a discrete collection of field particles of masses $\{m_b\}$. To exactly recover the SDD Equation (36) with Equation (38), we have to replace $f(\mathbf{v},t) + \sum_b f_b(\mathbf{v})$ by $f(\mathbf{v},t)$ in the dielectric function. This assumes that the external medium—field particles—is non polarizable (i.e., collective effects can be neglected) while our system—test particles—is polarizable (i.e., collective effects must be taken into account).[27] As we have already mentioned, the SDD Equation (102) is a nonlinear diffusion equation involving a diffusion tensor which depends on the distribution function of the system $f(\mathbf{v},t)$ itself. This is therefore a complicated integrodifferential equation. Combining Equations (26), (98) and (104) we obtain the dressed power spectrum

$$P(\mathbf{k},\omega) = \sum_b (2\pi)^{d+1} \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k},\omega)|^2} \int d\mathbf{v}'\, \delta(\omega - \mathbf{k}\cdot\mathbf{v}') m_b f_b(\mathbf{v}'), \tag{105}$$

the static power spectrum

$$P(\mathbf{k}) = (2\pi)^d \sum_b \int d\mathbf{v}\, \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2} m_b f_b(\mathbf{v}) = (2\pi)^d \sum_b \frac{\rho_b m_b \hat{u}(k)^2}{1 + (2\pi)^d \hat{u}(k)\beta\rho_b}, \tag{106}$$

and the energy of fluctuations

$$\mathcal{E} = \frac{1}{2}\sum_b \int d\mathbf{k} \int d\mathbf{v}\, \frac{\hat{u}(k)}{|\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2} m_b f_b(\mathbf{v}) = \frac{1}{2}\sum_b \int \frac{m_b \rho_b \hat{u}(k)}{1 + (2\pi)^d \hat{u}(k)\beta\rho_b}\, d\mathbf{k}. \tag{107}$$

The last equalities in Equations (106) and (107) are valid for the Maxwell–Boltzmann distribution (see Appendix C of [119]).

Remark 10. *This type of equations was first introduced and studied by Spitzer and Schwarzschild [95] in an astrophysical context.*[28] *These authors showed that the collisions of test particles with external massive perturbers such as molecular clouds can considerably increase their diffusion and heating. A recent discussion of this problem has been given in Appendix F of [135] where a self-similar solution of the Spitzer–Schwarzschild equation is obtained.*

3. Rotating Homogeneous Systems

We consider here a 3D homogeneous system of particles with long-range interactions rotating with a constant angular velocity $\mathbf{\Omega}$. We assume that the mean field force is balanced by the centrifugal force so that the system is in static equilibrium in the rotating frame. We also assume that the system is submitted to a weak external stochastic perturbation that slowly changes its velocity distribution function $f(\mathbf{v},t)$. We wish to derive the SDD equation in that context. Since the formalism is complicated and has been developed elsewhere, we just give the main steps of the derivation and refer to earlier works [53,55,149] for technical details. The fundamental equations of the quasilinear theory are

$$\frac{\partial f}{\partial t} - 2(\mathbf{\Omega}\times\mathbf{v})\cdot\frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\cdot\langle \delta f \nabla(\delta\Phi + \Phi_e)\rangle, \tag{108}$$

$$\frac{\partial \delta f}{\partial t} + \mathbf{v}\cdot\frac{\partial \delta f}{\partial \mathbf{r}} - 2(\mathbf{\Omega}\times\mathbf{v})\cdot\frac{\partial \delta f}{\partial \mathbf{v}} - \nabla(\delta\Phi + \Phi_e)\cdot\frac{\partial f}{\partial \mathbf{v}} = 0, \tag{109}$$

where $2\mathbf{\Omega} \times \mathbf{v}$ is the Coriolis force. We assume that f is isotropic in the plane perpendicular to the axis of rotation $\mathbf{\Omega}$. Taking the Fourier transform of Equation (109), integrating over the azimuthal angle ϕ, and averaging over this angle (see Ref. [53] for details), we obtain

$$\delta\hat{f}(\mathbf{k},\mathbf{v},\omega) = \frac{i\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{v}}}{i(\mathbf{k}\cdot\mathbf{v}-\omega)+2\Omega\partial_\phi}\left[\delta\hat{\Phi}(\mathbf{k},\omega)+\hat{\Phi}_e(\mathbf{k},\omega)\right]. \tag{110}$$

In this equation, the notation

$$\frac{1}{i(\mathbf{k}\cdot\mathbf{v}-\omega)+2\Omega\partial_\phi} \tag{111}$$

stands for the operator defined in Appendix A of [149]. Integrating Equation (110) over $\mathbf{v}$ and using Equation (19), we find that the Fourier transform of the fluctuations of the potential is related to the Fourier transform of the external stochastic potential by

$$\delta\hat{\Phi}(\mathbf{k},\omega) = \frac{1-\epsilon(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}\hat{\Phi}_e(\mathbf{k},\omega), \tag{112}$$

where the dielectric function is defined by (see Appendix A of [149])

$$\epsilon(\mathbf{k},\omega) = 1-(2\pi)^3\hat{u}(k)\int d\mathbf{v}\sum_{n=-\infty}^{+\infty}\frac{J_n^2(k_\perp v_\perp/2\Omega)}{(\mathbf{k}\cdot\mathbf{v})_n-\omega}\left(\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{v}}\right)_n. \tag{113}$$

Following Wu [53,149], we have introduced the notations

$$\left(\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{v}}\right)_n = k_z\frac{\partial}{\partial v_z}+\frac{2n\Omega}{v_\perp}\frac{\partial}{\partial v_\perp} \tag{114}$$

and

$$(\mathbf{k}\cdot\mathbf{v})_n = k_z v_z + 2n\Omega. \tag{115}$$

The Fourier transform of the total potential acting on a particle, $\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\omega) = \delta\hat{\Phi}(\mathbf{k},\omega) + \hat{\Phi}_e(\mathbf{k},\omega)$, is therefore related to the Fourier transform of the external stochastic potential by

$$\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\omega) = \frac{\hat{\Phi}_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}. \tag{116}$$

We can use the foregoing expressions to compute the collision term appearing on the right hand side of Equation (108). Introducing the notation

$$C[f] = \frac{\partial}{\partial \mathbf{v}}\cdot\langle\delta f\nabla\delta\Phi_{\rm tot}\rangle, \tag{117}$$

and decomposing the fluctuations of distribution function and potential in Fourier modes, we obtain

$$C[f] = -i\int d\mathbf{k}\int\frac{d\omega}{2\pi}\int d\mathbf{k}'\int\frac{d\omega'}{2\pi}\left(\mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{v}}\right)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}e^{-i(\mathbf{k}'\cdot\mathbf{r}-\omega' t)}\langle\delta\hat{f}(\mathbf{k},\mathbf{v},\omega)\delta\hat{\Phi}_{\rm tot}^*(\mathbf{k}',\omega')\rangle. \tag{118}$$

Using Equations (110) and (116), we obtain successively

$$C[f] = -i\int d\mathbf{k}\int\frac{d\omega}{2\pi}\int d\mathbf{k}'\int\frac{d\omega'}{2\pi}\left(\mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{v}}\right)e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}e^{-i(\omega-\omega')t}$$
$$\times\frac{i\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{v}}}{i(\mathbf{k}\cdot\mathbf{v}-\omega)+2\Omega\partial_\phi}\langle\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\omega)\delta\hat{\Phi}_{\rm tot}^*(\mathbf{k}',\omega')\rangle \tag{119}$$

and

$$\mathcal{C}[f] = -i\int d\mathbf{k}\int \frac{d\omega}{2\pi}\int d\mathbf{k}'\int \frac{d\omega'}{2\pi}\left(\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}}\right)e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}e^{-i(\omega-\omega')t}$$
$$\times\frac{i\mathbf{k}\cdot\frac{\partial f}{\partial\mathbf{v}}}{i(\mathbf{k}\cdot\mathbf{v}-\omega)+2\Omega\partial_\phi}\frac{1}{\epsilon(\mathbf{k},\omega)}\frac{1}{\epsilon(\mathbf{k}',\omega')^*}\langle\hat{\Phi}_e(\mathbf{k},\omega)\hat{\Phi}_e(\mathbf{k}',\omega')^*\rangle. \quad (120)$$

Substituting Equation (24) into Equation (120), we find that

$$\mathcal{C}[f] = -i\int d\mathbf{k}\int \frac{d\omega}{2\pi}\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\frac{i\mathbf{k}\cdot\frac{\partial f}{\partial\mathbf{v}}}{i(\mathbf{k}\cdot\mathbf{v}-\omega)+2\Omega\partial_\phi}\frac{1}{|\epsilon(\mathbf{k},\omega)|^2}\hat{C}(\mathbf{k},\omega). \quad (121)$$

Averaging this expression over the azimuthal angle ϕ and using Equation (42) of [53], we get

$$\mathcal{C}[f] = i\int d\mathbf{k}\int \frac{d\omega}{2\pi}\sum_{n=-\infty}^{+\infty}\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)_n\frac{J_n^2(k_\perp v_\perp/2\Omega)}{\omega-(\mathbf{k}\cdot\mathbf{v})_n}\left(\mathbf{k}\cdot\frac{\partial f}{\partial\mathbf{v}}\right)_n\frac{1}{|\epsilon(\mathbf{k},\omega)|^2}\hat{C}(\mathbf{k},\omega). \quad (122)$$

Using the Landau prescription $\omega\to\omega+i0^+$ and the Sokhotski–Plemelj Formula (33), we can replace $1/((\mathbf{k}\cdot\mathbf{v})_n-\omega-i0^+)$ by $+i\pi\delta((\mathbf{k}\cdot\mathbf{v})_n-\omega)$. Accordingly,

$$\mathcal{C}[f] = \pi\int d\mathbf{k}\int \frac{d\omega}{2\pi}\sum_{n=-\infty}^{+\infty}\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)_n J_n^2\left(\frac{k_\perp v_\perp}{2\Omega}\right)\delta(\omega-(\mathbf{k}\cdot\mathbf{v})_n)\left(\mathbf{k}\cdot\frac{\partial f}{\partial\mathbf{v}}\right)_n\frac{\hat{C}(\mathbf{k},\omega)}{|\epsilon(\mathbf{k},\omega)|^2}. \quad (123)$$

The integration over ω, which corresponds to a condition of resonance, is straightforward and leads to the final expression of the collision term

$$\mathcal{C}[f] = \frac{1}{2}\int d\mathbf{k}\sum_{n=-\infty}^{+\infty}\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)_n J_n^2\left(\frac{k_\perp v_\perp}{2\Omega}\right)\frac{\hat{C}(\mathbf{k},(\mathbf{k}\cdot\mathbf{v})_n)}{|\epsilon(\mathbf{k},(\mathbf{k}\cdot\mathbf{v})_n)|^2}\left(\mathbf{k}\cdot\frac{\partial f}{\partial\mathbf{v}}\right)_n. \quad (124)$$

In conclusion, the secular evolution of the mean distribution function of the particles sourced by the external stochastic force is governed by a nonlinear diffusion equation of the form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i}\left(D_{ij}[f,\mathbf{v}]\frac{\partial f}{\partial v_j}\right), \quad (125)$$

where $D_{ij}[f,\mathbf{v}]$ is a diffusion tensor whose components are

$$D_{zz} = \frac{1}{2}\int d\mathbf{k}\sum_{n=-\infty}^{+\infty}k_z^2 J_n^2\left(\frac{k_\perp v_\perp}{2\Omega}\right)\frac{\hat{C}(\mathbf{k},(\mathbf{k}\cdot\mathbf{v})_n)}{|\epsilon(\mathbf{k},(\mathbf{k}\cdot\mathbf{v})_n)|^2}, \quad (126)$$

$$D_{\perp\perp} = \frac{1}{2}\int d\mathbf{k}\sum_{n=-\infty}^{+\infty}\left(\frac{2n\Omega}{v_\perp}\right)^2 J_n^2\left(\frac{k_\perp v_\perp}{2\Omega}\right)\frac{\hat{C}(\mathbf{k},(\mathbf{k}\cdot\mathbf{v})_n)}{|\epsilon(\mathbf{k},(\mathbf{k}\cdot\mathbf{v})_n)|^2}, \quad (127)$$

$$D_{\perp z} = D_{z\perp} = \frac{1}{2}\int d\mathbf{k}\sum_{n=-\infty}^{+\infty}k_z\frac{2n\Omega}{v_\perp}J_n^2\left(\frac{k_\perp v_\perp}{2\Omega}\right)\frac{\hat{C}(\mathbf{k},(\mathbf{k}\cdot\mathbf{v})_n)}{|\epsilon(\mathbf{k},(\mathbf{k}\cdot\mathbf{v})_n)|^2}. \quad (128)$$

The SDD equation for a spatially homogeneous rotating system with long-range interactions can be written as

$$\frac{\partial f}{\partial t} = \frac{1}{2}\int d\mathbf{k}\sum_{n=-\infty}^{+\infty}\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)_n J_n^2\left(\frac{k_\perp v_\perp}{2\Omega}\right)\frac{\hat{C}(\mathbf{k},(\mathbf{k}\cdot\mathbf{v})_n)}{|\epsilon(\mathbf{k},(\mathbf{k}\cdot\mathbf{v})_n)|^2}\left(\mathbf{k}\cdot\frac{\partial f}{\partial\mathbf{v}}\right)_n. \quad (129)$$

It can be compared to the multi-species Lenard–Balescu equation [53]

$$\frac{\partial f_a}{\partial t} = \pi(2\pi)^3 \sum_b \int d\mathbf{k} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}}\right)_n \int d\mathbf{v}' \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k},(\mathbf{k}\cdot\mathbf{v})_n)|^2} \delta\left[(\mathbf{k}\cdot\mathbf{v})_n - (\mathbf{k}\cdot\mathbf{v}')_m\right]$$
$$\times J_n^2\left(\frac{k_\perp v_\perp}{2\Omega}\right) J_m^2\left(\frac{k_\perp v'_\perp}{2\Omega}\right)\left[m_b\left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}}\right)_n - m_a\left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}'}\right)_m\right] f_a(\mathbf{v}) f_b(\mathbf{v}'). \quad (130)$$

Taking the limit $m_a \to 0$ and proceeding as in Section 2.4, we obtain the SDD Equation (129) with a function $\hat{C}(\mathbf{k},\omega)$ corresponding to the bare correlation function of the potential created by a discrete collection of N field particles of masses $\{m_b\}$:

$$\hat{C}(\mathbf{k},\omega) = \pi(2\pi)^3 \sum_b \sum_{m=-\infty}^{+\infty} \int d\mathbf{v}' \hat{u}(k)^2 \delta\left[\omega - (\mathbf{k}\cdot\mathbf{v}')_m\right] J_m^2\left(\frac{k_\perp v'_\perp}{2\Omega}\right) m_b f_b(\mathbf{v}'). \quad (131)$$

4. Spatially Inhomogeneous Systems

We now consider a spatially inhomogeneous system of particles with long-range interactions submitted to a weak external stochastic perturbation. We take the limit $N \to +\infty$ with $m \sim 1/N$ in which the collisions between the particles can be neglected. For sufficiently short times, the external perturbation can also be neglected. Therefore, the initial stage of the system's evolution is governed by the Vlasov equation (see Equations (8) and (9)) involving only mean field effects. During this regime, the system generically reaches a quasistationary (virialized) state as a result of a process of violent relaxation [5]. This is a stable steady state of the Vlasov equation. Then, on a longer timescale, the system undergoes a slow evolution due to the effect of the external stochastic perturbation. This slow evolution is described by the SDD equation. In order to deal with spatial inhomogeneity, it is convenient to introduce angle-action variables $(\mathbf{w},\mathbf{J})$ [46]. In a steady state of the Vlasov equation, the particles move through a mean potential $\Phi(\mathbf{r})$ on orbits characterized by isolating integrals $\mathbf{J}$ called the actions. These are adiabatic invariants. By construction, the mean field Hamiltonian H of a particle in angle and action variables depends only on the actions $\mathbf{J}$ that are constants of the motion: $H = H(\mathbf{J})$. The conjugate coordinates $\mathbf{w}$ are called the angles. The Hamilton equations become $\dot{\mathbf{J}} = -\partial H/\partial \mathbf{w} = \mathbf{0}$ and $\dot{\mathbf{w}} = \partial H/\partial \mathbf{J} = \mathbf{\Omega}(\mathbf{J})$, where $\mathbf{\Omega}(\mathbf{J}) = \partial H/\partial \mathbf{J}$ is the angular frequency (pulsation) of the orbit with action $\mathbf{J}$. The unperturbed equations of motion of the particles are simply straight lines in angle-action space traveled at constant action: $\mathbf{J}(t) = \mathbf{J}$ and $\mathbf{w}(t) = \mathbf{w} + \mathbf{\Omega}(\mathbf{J})t$. One can show that the distribution function of a perfectly relaxed collisionless system is a function $f = f(\mathbf{J})$ of the actions only (Jeans theorem) [2]. Such a distribution is a steady state of the Vlasov equation. Because of the external stochastic perturbation[29] the actions slowly (secularly) drift from their initial values. This causes the particles to diffuse. As a result, the distribution function f slowly evolves in time. If the perturbation is small the particles keep their action $\mathbf{J}$ for at least several orbital times. On a long timescale they slowly drift from one orbit of constant $\mathbf{J}$ to another. This is because there is a timescale separation between the dynamical time t_D which is the timescale during which the system readjusts itself to reach a new steady state and the diffusion time which is the timescale over which the system evolves under the effect of the external perturbation. Making an adiabatic approximation, we can assume that f remains at any given time a function of the actions only, though a function that changes slowly in time. Thus, we assume that the distribution function is a function of the form $f = f(\mathbf{J},t)$ which, at fixed secular time, only depends on the actions according to the Jeans theorem. Therefore, the system evolves through a sequence of quasistationary states that are steady states of the Vlasov equation, depending only on the actions $\mathbf{J}$, slowly changing in time as a result of the external stochastic perturbation. The system is approximately in mechanical equilibrium at each stage of the dynamics but the external perturbation makes it slowly diffuse in action space. Below, we derive the SDD equation from the Klimontovich equation and from the Fokker–Planck equation.

4.1. Derivation of the SDD Equation from the Klimontovich Equation

4.1.1. Quasilinear Theory and Bogoliubov Ansatz

For spatially inhomogeneous systems, the fundamental equations of the quasilinear theory written in terms of angle-action variables[30] are [59,99,104]

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left\langle \delta f \frac{\partial}{\partial \mathbf{w}} (\delta \Phi + \Phi_e) \right\rangle, \tag{132}$$

$$\frac{\partial \delta f}{\partial t} + \mathbf{\Omega} \cdot \frac{\partial \delta f}{\partial \mathbf{w}} - \frac{\partial}{\partial \mathbf{w}} (\delta \Phi + \Phi_e) \cdot \frac{\partial f}{\partial \mathbf{J}} = 0. \tag{133}$$

In order to solve Equation (133) for the fluctuations, we resort to the Bogoliubov ansatz. We assume that there exists a timescale separation between a slow and a fast dynamics and we regard $\mathbf{\Omega}(\mathbf{J})$ and $f(\mathbf{J})$ in Equation (133) as "frozen" (independent of time) at the scale of the fast dynamics.[31] This amounts to neglecting the temporal variations of the mean field when we consider the evolution of the fluctuations. This is possible when the mean distribution function evolves on a secular timescale that is long compared to the dynamical time (the time over which the correlations of the fluctuations have their essential support). Since we ignore collisions, we can solve Equation (133) with Fourier transforms.[32] The Fourier transform of the fluctuations of the distribution function $\delta f(\mathbf{w}, \mathbf{J}, t)$ is defined by

$$\delta \hat{f}(\mathbf{k}, \mathbf{J}, \omega) = \int \frac{d\mathbf{w}}{(2\pi)^d} \int_{-\infty}^{+\infty} dt\, e^{-i(\mathbf{k}\cdot\mathbf{w} - \omega t)} \delta f(\mathbf{w}, \mathbf{J}, t) \tag{134}$$

and its inverse Fourier transform is given by

$$\delta f(\mathbf{w}, \mathbf{J}, t) = \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{w} - \omega t)} \delta \hat{f}(\mathbf{k}, \mathbf{J}, \omega). \tag{135}$$

Similar expressions hold for the fluctuations of the potential $\delta\Phi(\mathbf{w}, \mathbf{J}, t)$ and for the external potential $\Phi_e(\mathbf{w}, \mathbf{J}, t)$. Since the angles $\mathbf{w}$ are 2π-periodic, we have introduced discrete Fourier expansions with respect to these variables. For future reference, we recall the Fourier representation of the Dirac δ-function

$$\delta(t) = \int_{-\infty}^{+\infty} e^{-i\omega t} \frac{d\omega}{2\pi}, \qquad \delta_{\mathbf{k},\mathbf{0}} = \int e^{-i\mathbf{k}\cdot\mathbf{w}} \frac{d\mathbf{w}}{(2\pi)^d}. \tag{136}$$

4.1.2. Dielectric Tensor

We are now ready to solve the equation for the fluctuations for a given external perturbation. Taking the Fourier transform of Equation (133) in $\mathbf{w}$ and t, we obtain

$$\delta \hat{f}(\mathbf{k}, \mathbf{J}, \omega) = \frac{\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{J}}}{\mathbf{k} \cdot \mathbf{\Omega} - \omega} \left[\delta \hat{\Phi}(\mathbf{k}, \mathbf{J}, \omega) + \hat{\Phi}_e(\mathbf{k}, \mathbf{J}, \omega) \right]. \tag{137}$$

The fluctuations of the potential are related to the fluctuations of the density by Equation (18). The coupled Equations (18) and (133) are difficult to solve because their natural coordinates are different. The natural coordinates for solving the linearized Vlasov equation are angle-action variables which make the solution by Fourier transform easy. On the other hand, the natural coordinates for solving Equation (18) are the position coordinates $\mathbf{r}$. If we transform the fluctuating potential to angle-action variables, we obtain

$$\delta \hat{\Phi}(\mathbf{k}, \mathbf{J}, \omega) = (2\pi)^d \sum_{\mathbf{k}'} \int d\mathbf{J}' A_{\mathbf{k},\mathbf{k}'}(\mathbf{J}, \mathbf{J}') \delta \hat{f}(\mathbf{k}', \mathbf{J}', \omega), \tag{138}$$

where $A_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}')$ is the Fourier transform of the potential of interaction in angle-action variables [59]. If we substitute Equation (137) into Equation (138) we obtain a Fredholm integral equation

$$\delta\hat{\Phi}(\mathbf{k},\mathbf{J},\omega) = (2\pi)^d \sum_{\mathbf{k}'} \int d\mathbf{J}' A_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}') \frac{\mathbf{k}'\cdot\frac{\partial f'}{\partial \mathbf{J}'}}{\mathbf{k}'\cdot\mathbf{\Omega}'-\omega} \left[\delta\hat{\Phi}(\mathbf{k}',\mathbf{J}',\omega) + \hat{\Phi}_e(\mathbf{k}',\mathbf{J}',\omega)\right], \quad (139)$$

which determines $\delta\hat{\Phi}(\mathbf{k},\mathbf{J},\omega)$. However, this equation is complicated to solve explicitly because of the summation on $\mathbf{k}'$ and the integration over $\mathbf{J}'$. In order to solve Equation (137) with Equation (18), we follow Kalnajs' matrix method [150]. We assume that the fluctuations of density $\delta\rho(\mathbf{r},t)$ and potentials $\delta\Phi(\mathbf{r},t)$ and $\Phi_e(\mathbf{r},t)$ can be expanded on a complete biorthonormal basis such that

$$\delta\rho(\mathbf{r},t) = \textstyle\sum_\alpha A_\alpha(t)\rho_\alpha(\mathbf{r}), \qquad \delta\Phi(\mathbf{r},t) = \sum_\alpha A_\alpha(t)\Phi_\alpha(\mathbf{r}), \qquad \Phi_e(\mathbf{r},t) = \sum_\alpha A^e_\alpha(t)\Phi_\alpha(\mathbf{r}), \quad (140)$$

where the functions $\rho_\alpha(\mathbf{r})$ and $\Phi_\alpha(\mathbf{r})$ satisfy[33]

$$\Phi_\alpha(\mathbf{r}) = \int u(|\mathbf{r}-\mathbf{r}'|)\rho_\alpha(\mathbf{r}')\, d\mathbf{r}', \qquad \int \rho_\alpha(\mathbf{r})\Phi_{\alpha'}(\mathbf{r})^*\, d\mathbf{r} = -\delta_{\alpha\alpha'}. \quad (141)$$

Multiplying the first relation in Equation (140) by $\Phi^*_\alpha(\mathbf{r})$, integrating over $\mathbf{r}$, and using Equation (141), we obtain

$$A_\alpha(t) = -\int \delta\rho(\mathbf{r},t)\Phi^*_\alpha(\mathbf{r})\, d\mathbf{r}. \quad (142)$$

Substituting the relation

$$\delta\rho(\mathbf{r},t) = \int \delta f(\mathbf{r},\mathbf{v},t)\, d\mathbf{v} = \int d\mathbf{v} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{w}} \delta\hat{f}(\mathbf{k},\mathbf{J},t) \quad (143)$$

into Equation (142), we find that

$$\begin{aligned} A_\alpha(t) &= -\int d\mathbf{r} d\mathbf{v} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{w}} \delta\hat{f}(\mathbf{k},\mathbf{J},t)\Phi^*_\alpha(\mathbf{r}) \\ &= -\int d\mathbf{w} d\mathbf{J} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{w}} \delta\hat{f}(\mathbf{k},\mathbf{J},t)\Phi^*_\alpha(\mathbf{w},\mathbf{J}) \\ &= -(2\pi)^d \int d\mathbf{J} \sum_{\mathbf{k}} \delta\hat{f}(\mathbf{k},\mathbf{J},t)\hat{\Phi}^*_\alpha(\mathbf{k},\mathbf{J}), \end{aligned} \quad (144)$$

where we have use the fact that the transformation $(\mathbf{r},\mathbf{v}) \to (\mathbf{w},\mathbf{J})$ is canonical so that $d\mathbf{r}d\mathbf{v} = d\mathbf{w}d\mathbf{J}$. Taking the temporal Fourier transform of Equation (144), we obtain

$$\hat{A}_\alpha(\omega) = -(2\pi)^d \int d\mathbf{J} \sum_{\mathbf{k}} \delta\hat{f}(\mathbf{k},\mathbf{J},\omega)\hat{\Phi}^*_\alpha(\mathbf{k},\mathbf{J}). \quad (145)$$

Using Equations (137) and (140), we find that

$$\begin{aligned} \hat{A}_\alpha(\omega) &= -(2\pi)^d \int d\mathbf{J} \sum_{\mathbf{k}} \frac{\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{J}}}{\mathbf{k}\cdot\mathbf{\Omega}-\omega} \left[\delta\hat{\Phi}(\mathbf{k},\mathbf{J},\omega) + \hat{\Phi}_e(\mathbf{k},\mathbf{J},\omega)\right]\hat{\Phi}^*_\alpha(\mathbf{k},\mathbf{J}) \\ &= -(2\pi)^d \int d\mathbf{J} \sum_{\mathbf{k}} \frac{\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{J}}}{\mathbf{k}\cdot\mathbf{\Omega}-\omega} \sum_{\alpha'} \left[\hat{A}_{\alpha'}(\omega) + \hat{A}^e_{\alpha'}(\omega)\right]\hat{\Phi}_{\alpha'}(\mathbf{k},\mathbf{J})\hat{\Phi}^*_\alpha(\mathbf{k},\mathbf{J}). \end{aligned} \quad (146)$$

Introducing the dielectric tensor

$$\epsilon_{\alpha\alpha'}(\omega) = \delta_{\alpha\alpha'} + (2\pi)^d \int d\mathbf{J} \sum_{\mathbf{k}} \frac{\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{J}}}{\mathbf{k}\cdot\mathbf{\Omega}-\omega} \hat{\Phi}_{\alpha'}(\mathbf{k},\mathbf{J})\hat{\Phi}^*_\alpha(\mathbf{k},\mathbf{J}), \quad (147)$$

the foregoing equation can be rewritten as

$$\hat{A}_\alpha(\omega) = \sum_{\alpha'} \left[\hat{A}_{\alpha'}(\omega) + \hat{A}^e_{\alpha'}(\omega)\right](\delta_{\alpha\alpha'} - \epsilon_{\alpha\alpha'}(\omega)). \tag{148}$$

Solving this equation for $\hat{A}_\alpha(\omega)$, we obtain

$$\hat{A}_\alpha(\omega) = \sum_{\alpha'} \left[(\epsilon^{-1})_{\alpha\alpha'}(\omega) - \delta_{\alpha\alpha'}\right] \hat{A}^e_{\alpha'}(\omega). \tag{149}$$

Without external perturbation, the system would be described by the distribution function $f(\mathbf{J})$. The external perturbation $\Phi_e(\mathbf{w}, \mathbf{J}, t)$ polarizes the system and creates a small change in the distribution function $\delta f(\mathbf{w}, \mathbf{J}, t)$ (see Equation (133)) producing in turn a weak potential $\delta\Phi(\mathbf{w}, \mathbf{J}, t)$ (see Equation (18)). As a result, the total potential acting on a particle, which is sometimes called the dressed or effective potential, is $\delta\Phi_{\rm tot}(\mathbf{w}, \mathbf{J}, t) = \Phi_e(\mathbf{w}, \mathbf{J}, t) + \delta\Phi(\mathbf{w}, \mathbf{J}, t)$. This is the sum of the external potential plus the potential fluctuation induced by the system itself (i.e., the system's own response). Equations (18) and (133) are coupled together and written in terms of different variables. One manner to solve this loop is to use a biorthonormal basis and Fourier transforms as we have shown above. Using Equation (149), we find that the Fourier amplitudes $\hat{A}^{\rm tot}_\alpha(\omega) = \hat{A}_\alpha(\omega) + \hat{A}^e_\alpha(\omega)$ of the total fluctuating potential $\delta\Phi_{\rm tot} = \delta\Phi + \Phi_e$ acting on a particle are related to the Fourier amplitudes of the external stochastic potential Φ_e by[34]

$$\hat{A}^{\rm tot}_\alpha(\omega) = \sum_{\alpha'} (\epsilon^{-1})_{\alpha\alpha'}(\omega) \hat{A}^e_{\alpha'}(\omega). \tag{150}$$

Although not explicitly written, we must use the Landau prescription $\omega \to \omega + i0^+$ in Equation (147). As a result, $(\epsilon^{-1})_{\alpha\alpha'}(\omega)$ is a complex tensor which plays the role of the response function in plasma physics. It determines the response of the system $\hat{A}^{\rm tot}_\alpha(\omega)$ to an external perturbation $\hat{A}^e_{\alpha'}(\omega)$ through Equation (150). The dielectric tensor takes into account the polarization of the medium caused by the self-interaction of the particles. This corresponds to the so-called "collective effects". Depending on the form of the self-interaction and on the geometry of the system, the polarization cloud may amplify or shield the action of the imposed external perturbation. Therefore, the action of the external potential is modified by collective effects. This amounts to replacing the bare potential $\hat{\Phi}_e(\mathbf{k}, \mathbf{J}, \omega)$ by the dressed potential $\delta\hat{\Phi}_{\rm tot}(\mathbf{k}, \mathbf{J}, \omega)$ which is the potential $\hat{\Phi}_e(\mathbf{k}, \mathbf{J}, \omega)$ dressed by the polarization cloud. Without self-interaction, or if we neglect collective effects, we just have $\delta\hat{\Phi}_{\rm tot}(\mathbf{k}, \mathbf{J}, \omega) = \hat{\Phi}_e(\mathbf{k}, \mathbf{J}, \omega)$, corresponding to $\epsilon = 1$ and $\delta\Phi = 0$.

Remark 11. *The dielectric tensor (147) can be written as $\epsilon = 1 - M$ where M is the polarization tensor such that $\hat{A} = M\hat{A}_{\rm tot}$. On the other hand, $R = \epsilon^{-1} - 1$ is the response tensor such that $\hat{A} = R\hat{A}_e$. Equation (150) may be written in matrix form as $\epsilon\hat{A}_{\rm tot} = \hat{A}_e$. When $\hat{A}_e = 0$ this equation has nontrivial solutions only if $\det[\epsilon(\omega)] = 0$. This is the dispersion relation which determines the proper complex pulsations ω of the system associated with the distribution function $f(\mathbf{J})$. It can be used to study the (Vlasov) linear dynamical stability of the system. A zero of the dispersion relation $\det[\epsilon(\omega)] = 0$ for $\omega_i > 0$ signifies an unstable growing mode.*

4.1.3. Power Spectrum

We assume that the time evolution of the perturbing potential is a stationary stochastic process and write the auto-correlation function of its different components as

$$\langle A^e_\alpha(t) A^e_{\alpha'}(t')^* \rangle = C_{\alpha\alpha'}(t - t'). \tag{151}$$

The function $C_{\alpha\alpha'}(t-t')$ describes a possibly colored noise. The spectral auto-correlation function of the components of the external stochastic potential is

$$\langle \hat{A}^e_\alpha(\omega)\hat{A}^e_{\alpha'}(\omega')^*\rangle = 2\pi\delta(\omega-\omega')\hat{C}_{\alpha\alpha'}(\omega), \tag{152}$$

where $\hat{C}_{\alpha\alpha'}(\omega)$ is the temporal Fourier transform of $C_{\alpha\alpha'}(t-t')$ (Wiener–Khinchin theorem). The auto-correlation function of the external potential is

$$\langle \hat{\Phi}_e(\mathbf{k},\mathbf{J},t)\hat{\Phi}_e(\mathbf{k},\mathbf{J},t')^*\rangle = C(\mathbf{k},\mathbf{J},t-t'), \tag{153}$$

and its temporal Fourier transform is

$$\langle \hat{\Phi}_e(\mathbf{k},\mathbf{J},\omega)\hat{\Phi}_e(\mathbf{k},\mathbf{J},\omega')^*\rangle = 2\pi\delta(\omega-\omega')\hat{C}(\mathbf{k},\mathbf{J},\omega). \tag{154}$$

According to Equation (140) we have

$$\langle \hat{\Phi}_e(\mathbf{k},\mathbf{J},\omega)\hat{\Phi}_e(\mathbf{k},\mathbf{J},\omega')^*\rangle = \sum_{\alpha\alpha'}\langle \hat{A}^e_\alpha(\omega)\hat{A}^e_{\alpha'}(\omega')^*\rangle\hat{\Phi}_\alpha(\mathbf{k},\mathbf{J})\hat{\Phi}_{\alpha'}(\mathbf{k},\mathbf{J})^*. \tag{155}$$

Comparing Equations (152), (154) and (155) we find that

$$\hat{C}(\mathbf{k},\mathbf{J},\omega) = \sum_{\alpha\alpha'}\hat{\Phi}_\alpha(\mathbf{k},\mathbf{J})\hat{C}_{\alpha\alpha'}(\omega)\hat{\Phi}_{\alpha'}(\mathbf{k},\mathbf{J})^*. \tag{156}$$

Similarly, we write the spectral auto-correlation function of the components of the total fluctuating potential as

$$\langle \hat{A}^{\rm tot}_\alpha(\omega)\hat{A}^{\rm tot}_{\alpha'}(\omega')^*\rangle = 2\pi\delta(\omega-\omega')P_{\alpha\alpha'}(\omega), \tag{157}$$

where $P_{\alpha\alpha'}(\omega)$ is the power spectrum tensor. The auto-correlation function of the total fluctuating potential in Fourier space is given by

$$\langle \delta\hat{\Phi}_{\rm tot}(\mathbf{k},\mathbf{J},\omega)\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\mathbf{J},\omega')^*\rangle = 2\pi\delta(\omega-\omega')P(\mathbf{k},\mathbf{J},\omega), \tag{158}$$

where $P(\mathbf{k},\mathbf{J},\omega)$ is the power spectrum. According to Equation (140) we have

$$\langle \delta\hat{\Phi}_{\rm tot}(\mathbf{k},\mathbf{J},\omega)\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\mathbf{J},\omega')^*\rangle = \sum_{\alpha\alpha'}\langle \hat{A}^{\rm tot}_\alpha(\omega)\hat{A}^{\rm tot}_{\alpha'}(\omega')^*\rangle\hat{\Phi}_\alpha(\mathbf{k},\mathbf{J})\hat{\Phi}_{\alpha'}(\mathbf{k},\mathbf{J})^*. \tag{159}$$

Comparing Equations (157)–(159) we find that

$$P(\mathbf{k},\mathbf{J},\omega) = \sum_{\alpha\alpha'}\hat{\Phi}_\alpha(\mathbf{k},\mathbf{J})P_{\alpha\alpha'}(\omega)\hat{\Phi}_{\alpha'}(\mathbf{k},\mathbf{J})^*. \tag{160}$$

On the other hand, using Equation (150), we obtain

$$\langle \hat{A}^{\rm tot}_\alpha(\omega)\hat{A}^{\rm tot}_{\alpha'}(\omega')^*\rangle = \sum_{\alpha''\alpha'''}\epsilon^{-1}_{\alpha\alpha''}(\omega)\langle \hat{A}^e_{\alpha''}(\omega)\hat{A}^e_{\alpha'''}(\omega')^*\rangle\epsilon^{-1}_{\alpha'\alpha'''}(\omega')^*. \tag{161}$$

Equations (152), (157) and (161) then yield

$$P_{\alpha\alpha'}(\omega) = \left[\epsilon^{-1}\hat{C}(\epsilon^{-1})^\dagger\right]_{\alpha\alpha'}(\omega). \tag{162}$$

Substituting Equation (162) into Equation (160), we finally obtain

$$P(\mathbf{k},\mathbf{J},\omega) = \sum_{\alpha\alpha'}\hat{\Phi}_\alpha(\mathbf{k},\mathbf{J})\left[\epsilon^{-1}\hat{C}(\epsilon^{-1})^\dagger\right]_{\alpha\alpha'}(\omega)\,\hat{\Phi}_{\alpha'}(\mathbf{k},\mathbf{J})^*, \tag{163}$$

where we have used $(\epsilon^{-1})^\dagger_{\alpha\alpha'} = (\epsilon^{-1})^*_{\alpha'\alpha}$. This equation relates the power spectrum $P(\mathbf{k},\mathbf{J},\omega)$ of the total fluctuating potential acting on the particles to the auto-correlation function $\hat{C}_{\alpha\alpha'}(\omega)$ of the external stochastic potential. We note that $\hat{C}_{\alpha\alpha'}(\omega)$, $\hat{C}(\mathbf{k},\mathbf{J},\omega)$, $P_{\alpha\alpha'}(\omega)$ and $P(\mathbf{k},\mathbf{J},\omega)$ are real and positive. The power spectrum $P(\mathbf{k},\mathbf{J},\omega)$ takes into account collective effects through the dielectric tensor $\epsilon_{\alpha\alpha'}$. It can be seen as a dressed correlation function. If we neglect collective effects ($\epsilon = 1$) we obtain the bare power spectrum (or the bare correlation function)

$$P_{\mathrm{bare}}(\mathbf{k},\mathbf{J},\omega) = \sum_{\alpha\alpha'} \hat{\Phi}_{\alpha}(\mathbf{k},\mathbf{J})\hat{C}_{\alpha\alpha'}(\omega)\hat{\Phi}_{\alpha'}(\mathbf{k},\mathbf{J})^* = \hat{C}(\mathbf{k},\mathbf{J},\omega). \tag{164}$$

It can be directly obtained from Equations (154) and (158) with $\delta\hat{\Phi}_{\mathrm{tot}}(\mathbf{k},\mathbf{J},\omega) = \hat{\Phi}_e(\mathbf{k},\mathbf{J},\omega)$.

4.1.4. SDD Equation

The basic equations governing the evolution of the mean distribution function $f(\mathbf{J},t)$ of a spatially inhomogeneous system of particles with long-range interactions forced by an external perturbation are given by Equations (132) and (133). Introducing the total fluctuating potential $\delta\Phi_{\mathrm{tot}} = \delta\Phi + \Phi_e$, we can rewrite Equation (132) as

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left\langle \delta f \frac{\partial}{\partial \mathbf{w}} \delta\Phi_{\mathrm{tot}} \right\rangle. \tag{165}$$

Introducing the Fourier transforms of the fluctuations of distribution function and potential, we obtain

$$\frac{\partial f}{\partial t} = -i\sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} \sum_{\mathbf{k}'} \int \frac{d\omega'}{2\pi} \left(\mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{J}}\right) e^{i(\mathbf{k}\cdot\mathbf{w}-\omega t)} e^{-i(\mathbf{k}'\cdot\mathbf{w}-\omega' t)} \langle \delta\hat{f}(\mathbf{k},\mathbf{J},\omega)\delta\hat{\Phi}^*_{\mathrm{tot}}(\mathbf{k}',\mathbf{J},\omega')\rangle. \tag{166}$$

On the other hand, Equation (133) can be written in Fourier space as (see Equation (137))

$$\delta\hat{f}(\mathbf{k},\mathbf{J},\omega) = \frac{\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{J}}}{\mathbf{k} \cdot \mathbf{\Omega} - \omega} \delta\hat{\Phi}_{\mathrm{tot}}(\mathbf{k},\mathbf{J},\omega). \tag{167}$$

Substituting Equation (167) into Equation (166), we obtain

$$\begin{aligned} \frac{\partial f}{\partial t} = -i\sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} \sum_{\mathbf{k}'} \int \frac{d\omega'}{2\pi} \left(\mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{J}}\right) e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{w}} e^{-i(\omega-\omega')t} \\ \times \frac{\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{J}}}{\mathbf{k} \cdot \mathbf{\Omega} - \omega} \langle \delta\hat{\Phi}_{\mathrm{tot}}(\mathbf{k},\mathbf{J},\omega)\delta\hat{\Phi}^*_{\mathrm{tot}}(\mathbf{k}',\mathbf{J},\omega')\rangle. \end{aligned} \tag{168}$$

Since the mean distribution function f depends only on the action $\mathbf{J}$, we can average the collision term over the angle $\mathbf{w}$. This brings a Kronecker factor $\delta_{\mathbf{k},\mathbf{k}'}$ which amounts to taking $\mathbf{k}' = \mathbf{k}$. Therefore,

$$\frac{\partial f}{\partial t} = -i\sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{J}}\right) e^{-i(\omega-\omega')t} \frac{\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{J}}}{\mathbf{k} \cdot \mathbf{\Omega} - \omega} \langle \delta\hat{\Phi}_{\mathrm{tot}}(\mathbf{k},\mathbf{J},\omega)\delta\hat{\Phi}^*_{\mathrm{tot}}(\mathbf{k},\mathbf{J},\omega')\rangle. \tag{169}$$

Introducing the power spectrum from Equation (158) and performing the integration over ω', we find that

$$\frac{\partial f}{\partial t} = -i\sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} \left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{J}}\right) \frac{\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{J}}}{\mathbf{k} \cdot \mathbf{\Omega} - \omega} P(\mathbf{k},\mathbf{J},\omega). \tag{170}$$

Recalling the Landau prescription $\omega \to \omega + i0^+$ and using the Sokhotski–Plemelj Formula (33), we can replace $1/(\mathbf{k} \cdot \mathbf{\Omega} - \omega - i0^+)$ by $+i\pi\delta(\mathbf{k} \cdot \mathbf{\Omega} - \omega)$. Accordingly,

$$\frac{\partial f}{\partial t} = \pi\sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} \left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{J}}\right) \delta(\mathbf{k} \cdot \mathbf{\Omega} - \omega) P(\mathbf{k},\mathbf{J},\omega) \left(\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{J}}\right). \tag{171}$$

Integrating over the δ-function (resonance), we obtain the kinetic equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\sum_{\mathbf{k}}\left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\right)P(\mathbf{k},\mathbf{J},\mathbf{k}\cdot\mathbf{\Omega})\left(\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{J}}\right). \tag{172}$$

Therefore, for spatially inhomogeneous systems with long-range interactions, the secular evolution of the mean distribution function $f(\mathbf{J},t)$ of the particles sourced by an external stochastic force is governed by a nonlinear diffusion equation of the form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial J_i}\left(D_{ij}[f,\mathbf{J}]\frac{\partial f}{\partial J_j}\right), \tag{173}$$

with an anisotropic diffusion tensor

$$D_{ij}[f,\mathbf{J}] = \frac{1}{2}\sum_{\mathbf{k}} k_i k_j P(\mathbf{k},\mathbf{J},\mathbf{k}\cdot\mathbf{\Omega}). \tag{174}$$

This equation expresses the diffusion tensor in terms of the power spectrum of the fluctuations at the resonances $\omega = \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})$.

Using Equation (163), we can express the diffusion tensor in terms of the correlation tensor of the external perturbation as

$$D_{ij}[f,\mathbf{J}] = \frac{1}{2}\sum_{\mathbf{k}}\sum_{\alpha\alpha'} k_i k_j \hat{\Phi}_{\alpha}(\mathbf{k},\mathbf{J})\left[\epsilon^{-1}\hat{C}(\epsilon^{-1})^{\dagger}\right]_{\alpha\alpha'}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))\hat{\Phi}_{\alpha'}(\mathbf{k},\mathbf{J})^{*}. \tag{175}$$

The diffusion tensor depends on the spectral auto-correlation tensor of the external perturbation $\hat{C}_{\alpha\alpha'}(\omega)$ and on the inverse dielectric tensor $\epsilon^{-1}(\omega)$ both evaluated at the resonance frequencies $\omega = \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})$.[35] As a result, the diffusion tensor $D_{ij}[f,\mathbf{J}]$ depends not only on the action $\mathbf{J}$ but is also a functional of the distribution function $f(\mathbf{J},t)$ itself through the dielectric tensor $\epsilon_{\alpha\alpha'}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))$ defined by Equation (147). It also depends implicitly on f through the orbital frequencies $\mathbf{\Omega}(\mathbf{J})$. Equation (173) with the diffusion tensor from Equation (175) is therefore a complicated integrodifferential equation called the inhomogeneous SDD equation. In the spatially homogeneous case, we recover Equation (36) with Equation (38).

When collective effects are neglected, i.e., when we make $\epsilon = 1$ in Equation (175) and use Equation (156), or replace $P(\mathbf{k},\mathbf{J},\mathbf{k}\cdot\mathbf{\Omega})$ by $P_{\rm bare}(\mathbf{k},\mathbf{J},\mathbf{k}\cdot\mathbf{\Omega})$ (given by Equation (164)) in Equation (174), the diffusion tensor reduces to[36]

$$D_{ij}(\mathbf{J}) = \frac{1}{2}\sum_{\mathbf{k}} k_i k_j \hat{C}(\mathbf{k},\mathbf{J},\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})). \tag{176}$$

In that case, it depends only on $\mathbf{J}$ and $\mathbf{\Omega}(\mathbf{J})$. Equation (173) with Equation (176) is called the inhomogeneous SBD equation.

4.1.5. Properties of the SDD Equation

Some general properties of the inhomogeneous SDD Equation (173) can be given. First of all, the total mass $M = \int f\, d\mathbf{J}$ of the system is conserved since the right hand side of Equation (173) is the divergence of a current in action space. By contrast, the energy of the system is not conserved, contrary to the case of the inhomogeneous Lenard–Balescu equation [58,59], since the system is forced by an external medium. Taking the time derivative of the energy

$$E = \int f H(\mathbf{J})\, d\mathbf{v}, \tag{177}$$

using Equation (173), and integrating by parts, we obtain

$$\dot{E} = -\int D_{ij}[f,\mathbf{J}]\frac{\partial f}{\partial J_j}\Omega_i \, d\mathbf{J}. \tag{178}$$

In general, $\dot{E}$ has not a definite sign. However, when $f = f(H)$ with $f'(H) \leq 0$, we obtain $\dot{E} = -\int f'(H)D_{ij}\Omega_i\Omega_j \, d\mathbf{J} \geq 0$ (since $D_{ij}\Omega_i\Omega_j \geq 0$) so that energy is injected in the system. Finally, introducing the H-functions

$$S = -\int C(f) \, d\mathbf{J}, \tag{179}$$

where $C(f)$ is any convex function, we obtain

$$\dot{S} = \int C''(f)\frac{\partial f}{\partial J_i} D_{ij}[f,\mathbf{J}]\frac{\partial f}{\partial J_j} \, d\mathbf{J}. \tag{180}$$

Because of the convexity condition $C'' \geq 0$ and the fact that the quadratic form $x_i D_{ij} x_j$ is definite positive (see Section 4.1.4), we find that $\dot{S} \geq 0$. Therefore, all the H-functions increase monotonically with time. This is different from the case of the inhomogeneous Lenard–Balescu equation where only the Boltzmann entropy increases monotonically [58,59].

If we heuristically introduce a dissipative term in the inhomogeneous SDD equation by analogy with the inhomogeneous Kramers equation (see Section 3.6 of [59]), we obtain an equation of the form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial J_i}\left(D_{ij}[f,\mathbf{J}]\frac{\partial f}{\partial J_j} + \xi f \Omega_i(\mathbf{J})\right), \tag{181}$$

where $\xi > 0$ is the friction coefficient. Here, $\mathbf{\Omega}(\mathbf{J}) = \partial H/\partial \mathbf{J}$ is either a given external "force" or the pulsation generated by the distribution function $f(\mathbf{J}, t)$ itself in a self-consistent manner. The general behavior of this nonlinear equation is difficult to predict. It may relax towards a non-Boltzmannian steady state determined by the equation

$$D_{ij}[f,\mathbf{J}]\frac{\partial f}{\partial J_j} + \xi f \Omega_i(\mathbf{J}) = 0, \tag{182}$$

or exhibit a complicated (e.g., periodic) dynamics. Since the diffusion tensor is a functional of f, the nonlinear SDD equation presents a rich and complex behavior.

4.1.6. Stochastic SDD Equation

The SDD Equation (181), which is a deterministic partial differential equation, describes the evolution of the mean distribution function $f(\mathbf{J}, t)$. If we take fluctuations into account, by analogy with the results presented in [124], we expect that the mesoscopic distribution function $\overline{f}(\mathbf{J}, t)$ will satisfy a stochastic partial differential equation of the form

$$\frac{\partial \overline{f}}{\partial t} = \frac{\partial}{\partial J_i}\left(D_{ij}[\overline{f},\mathbf{J}]\frac{\partial \overline{f}}{\partial J_j} + \xi \overline{f} \Omega_i(\mathbf{J})\right) + \zeta(\mathbf{J}, t), \tag{183}$$

where $\zeta(\mathbf{J}, t)$ is a noise term with zero mean that generally depends on $\overline{f}(\mathbf{J}, t)$. When $D_{ij} = D\delta_{ij}$ is constant and isotropic and when the fluctuation–dissipation theorem is fulfilled so that $\xi = D\beta m$, as in the case of Brownian particles with long-range interactions, the noise term is given by [124]

$$\zeta(\mathbf{J}, t) = \frac{\partial}{\partial \mathbf{J}} \cdot \left(\sqrt{2Dm\overline{f}} \, \mathbf{Q}(\mathbf{J}, t)\right), \tag{184}$$

where $\mathbf{Q}(\mathbf{J},t)$ is a Gaussian white noise satisfying $\langle Q_i(\mathbf{J},t)\rangle = 0$ and $\langle Q_i(\mathbf{J},t)Q_j(\mathbf{J}',t')\rangle = \delta_{ij}\delta(\mathbf{J}-\mathbf{J}')\delta(t-t')$. This expression can be obtained from an adaptation of the theory of fluctuating hydrodynamics [124]. In that case, the stochastic partial differential equation reads

$$\frac{\partial\overline{f}}{\partial t} = \frac{\partial}{\partial\mathbf{J}}\cdot\left[D\left(\frac{\partial\overline{f}}{\partial\mathbf{J}} + \beta m\overline{f}\mathbf{\Omega}(\mathbf{J})\right)\right] + \frac{\partial}{\partial\mathbf{J}}\cdot\left(\sqrt{2Dm\overline{f}}\,\mathbf{Q}(\mathbf{J},t)\right), \tag{185}$$

and the deterministic equation for the mean distribution function (i.e., Equation (185) without the noise term) relaxes towards the Boltzmann distribution $f = Ae^{-\beta mH(\mathbf{J})}$. When $D[\overline{f}]$ is a functional of $\overline{f}$, the noise term may be more complicated (see note 19). When the deterministic Equation (181) admits several equilibrium states, the noise term in Equation (183) can trigger random transitions from one state to the other (see, e.g., [128,129]).

Remark 12. *We may alternatively consider a stochastic partial differential equation of the form*

$$\frac{\partial\overline{f}}{\partial t} = \frac{\partial}{\partial J_i}\left(D_{ij}[\overline{f},\mathbf{J}]\frac{\partial\overline{f}}{\partial J_j}\right) - \nu\overline{f} + \zeta(\mathbf{J},t), \tag{186}$$

where $-\nu\overline{f}$ is a linear damping term with $0<\nu\ll 1$. The stationary solutions of the corresponding deterministic equation (Equation (186) with $\zeta = 0$) are determined by

$$\frac{\partial}{\partial J_i}\left(D_{ij}[f,\mathbf{J}]\frac{\partial f}{\partial J_j}\right) - \nu f = 0. \tag{187}$$

Since $D_{ij}[f]$ is a functional of f, Equation (187) is a very nonlinear equation which admits nontrivial solutions.

4.2. Derivation of the SDD Equation from the Fokker–Planck Equation

In this section, we derive the SDD equation directly from the Fokker–Planck equation. We follow an approach similar to the one developed in Section 3 of [59] to derive the inhomogeneous Lenard–Balescu equation.

4.2.1. Fokker–Planck Equation

Let us consider the evolution of a test particle of mass m moving in a spatially inhomogeneous medium and experiencing a stochastic perturbation $\Phi_e(\mathbf{r},t)$. The equations of motion of the test particle, written with angle-action variables, are

$$\frac{d\mathbf{w}}{dt} = \mathbf{\Omega}(\mathbf{J}) + \frac{\partial\delta\Phi_{\rm tot}}{\partial\mathbf{J}}(\mathbf{w},\mathbf{J},t), \qquad \frac{d\mathbf{J}}{dt} = -\frac{\partial\delta\Phi_{\rm tot}}{\partial\mathbf{w}}(\mathbf{w},\mathbf{J},t), \tag{188}$$

where $\delta\Phi_{\rm tot}(\mathbf{w},\mathbf{J},t)$ is the total fluctuating potential acting on the particle. They can be written in Hamiltonian form as $\dot{\mathbf{w}} = \partial(H+\delta H_{\rm tot})/\partial\mathbf{J}$ and $\dot{\mathbf{J}} = -\partial(H+\delta H_{\rm tot})/\partial\mathbf{w}$, where H is the mean Hamiltonian and $\delta H_{\rm tot}$ is the total fluctuating Hamiltonian. At leading order, the test particle moves on an orbit characterized by a constant action $\mathbf{J}$ and a pulsation $\mathbf{\Omega}(\mathbf{J}) = \partial H/\partial\mathbf{J}$ but it also experiences a small stochastic perturbation $\delta\Phi_{\rm tot} = \Phi_e + \delta\Phi$ which is equal to the external potential Φ_e plus the fluctuating potential $\delta\Phi$ produced by the system itself (collective effects). Equations (188) can be formally integrated into

$$\mathbf{w}(t) = \mathbf{w} + \int_0^t \mathbf{\Omega}(\mathbf{J}(t'))\,dt' + \int_0^t \frac{\partial\delta\Phi_{\rm tot}}{\partial\mathbf{J}}(\mathbf{w}(t'),\mathbf{J}(t'),t')\,dt', \tag{189}$$

$$\mathbf{J}(t) = \mathbf{J} - \int_0^t \frac{\partial\delta\Phi_{\rm tot}}{\partial\mathbf{w}}(\mathbf{w}(t'),\mathbf{J}(t'),t')\,dt', \tag{190}$$

where we have assumed that, initially, the test particle has an angle $\mathbf{w}$ and an action $\mathbf{J}$. Since the fluctuations $\delta\Phi_{\rm tot}$ of the potential are small, the changes in the action of the test particle are also small. On the other hand, the fluctuation time is short with respect to the evolution time of the distribution function. As a result, the dynamics of the test particle can be represented by a stochastic process governed by a Fokker–Planck equation [34,112]. The Fokker–Planck equation can be derived from the Master equation by using the Kramers–Moyal [110,111] expansion truncated at the level of the second moments of the increment in action. If we denote by $f(\mathbf{J}, t)$ the probability density that the test particle has an action $\mathbf{J}$ at time t, the general form of this equation is

$$\frac{\partial f}{\partial t} = \frac{\partial^2}{\partial J_i \partial J_j}(D_{ij} f) - \frac{\partial}{\partial J_i}(f F_i^{\rm tot}). \tag{191}$$

The diffusion tensor and the friction force are defined by

$$D_{ij}(\mathbf{J}) = \lim_{t\to+\infty} \frac{1}{2t}\langle (J_i(t) - J_i)(J_j(t) - J_j)\rangle = \frac{\langle \Delta J_i \Delta J_j \rangle}{2\Delta t}, \tag{192}$$

$$F_i^{\rm tot}(\mathbf{J}) = \lim_{t\to+\infty} \frac{1}{t}\langle J_i(t) - J_i\rangle = \frac{\langle \Delta J_i \rangle}{\Delta t}. \tag{193}$$

In writing these limits, we have implicitly assumed that the time t is long compared to the fluctuation time but short compared to the evolution time of the distribution function.

As shown in our previous paper [59], it is relevant to rewrite the Fokker–Planck equation in the alternative form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial J_i}\left(D_{ij} \frac{\partial f}{\partial J_j} - f F_i^{\rm pol}\right). \tag{194}$$

The total friction can be written as

$$F_i^{\rm tot} = F_i^{\rm pol} + \frac{\partial D_{ij}}{\partial J_j}, \tag{195}$$

where $\mathbf{F}_{\rm pol}$ is the friction by polarization (see Section 3.3 of [59]) while the second term is due to the variation of the diffusion tensor with $\mathbf{J}$ (see Section 3.4 of [59]). As explained in [8,23,59,136] in the context of the Lenard–Balescu equation, the friction by polarization $\mathbf{F}_{\rm pol}$ arises from the retroaction (response) of the system to the perturbation caused by the test particle, just like in a polarization process. It represents, however, only one component of the dynamical friction $\mathbf{F}_{\rm tot}$ experienced by the test particle, the other component being $\partial_j D_{ij}$.

Remark 13. *The two expressions (191) and (194) of the Fokker–Planck equation have their own interest. The expression (191) where the diffusion tensor is placed after the second derivative $\partial^2(DP)$ involves the total friction $\mathbf{F}_{\rm tot}$ and the expression (194) where the diffusion tensor is placed between the derivatives $\partial D \partial P$ isolates the friction by polarization $\mathbf{F}_{\rm pol}$. It is shown in [59] that this second form is directly related to the inhomogeneous Lenard–Balescu equation. This is also the form corresponding to the inhomogeneous SDD Equation (173). It has therefore a clear physical meaning.*

4.2.2. Absence of friction by polarization

In the limit $N \to +\infty$ with $m \sim 1/N$ where the collisions between the particles are negligible, the friction by polarization (which is proportional to m [59]) vanishes

$$\mathbf{F}_{\rm pol} = \mathbf{0}. \tag{196}$$

Indeed, the perturbation on the system caused by the test particle is negligible. In that case, the Fokker–Planck Equation (194) reduces to

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial J_i}\left(D_{ij}\frac{\partial f}{\partial J_j}\right). \tag{197}$$

We note that, in Equation (197), the diffusion tensor is "sandwiched" between the two derivatives $\partial/\partial J$ in agreement with Equation (173). As mentioned previously, this is not the usual form of the Fokker–Planck equation which is given by Equation (191). Therefore, the test particle experiences a friction [see Equation (195)]

$$F_i^{\rm tot} = \frac{\partial D_{ij}}{\partial J_j}, \tag{198}$$

arising from the inhomogeneity of the diffusion tensor.[37] Using Equations (192) and (193), this relation can be written as

$$\frac{\langle \Delta J_i\rangle}{\Delta t} = \frac{1}{2}\frac{\partial}{\partial J_j}\frac{\langle \Delta J_i \Delta J_j\rangle}{\Delta t}. \tag{199}$$

Remark 14. *The relation (199) was first derived by Binney and Lacey [94] (see their Equation (3.10)) from general considerations (see their Appendix A). As a result, they obtained the diffusion Equation (197) where the diffusion tensor is placed between the two action derivatives. They computed the diffusion tensor D_{ij} but did not take into account collective effects. Weinberg [99] independently considered the same problem. He computed the diffusion tensor D_{ij} and the friction force $\mathbf{F}_{\rm tot}$ by taking collective effects into account but did not notice that they satisfy the relation from Equation (198). As a result, he wrote the Fokker–Planck equation under the general form of Equation (191) but did not obtain the simpler form of Equation (197) resulting from Equation (199).*

4.2.3. First Calculation of D_{ij}

We now calculate the diffusion tensor from Equation (192) following the approach developed in Section 3.2 of [59]. According to Equation (188) the increment in action of the test particle is

$$\Delta \mathbf{J} = -\int_0^t \frac{\partial \delta\Phi_{\rm tot}}{\partial \mathbf{w}}(\mathbf{w}(t'),\mathbf{J}(t'),t')\,dt'. \tag{200}$$

Substituting Equation (200) into Equation (192) and assuming that the correlations of the fluctuating force persist for a time less than the time for the trajectory of the test particle to be much altered, we can make a linear trajectory approximation in angle-action space (i.e., we assume that the test particle follows an unperturbed orbit when calculating the dynamics of the fluctuations)

$$\mathbf{w}(t') = \mathbf{w} + \mathbf{\Omega} t', \qquad \mathbf{J}(t') = \mathbf{J}, \tag{201}$$

and write

$$D_{ij} = \lim_{t\to+\infty}\frac{1}{2t}\int_0^t dt'\int_0^t dt''\left\langle \frac{\partial\delta\Phi_{\rm tot}}{\partial w_i}(\mathbf{w}+\mathbf{\Omega}t',\mathbf{J},t')\frac{\partial\delta\Phi_{\rm tot}}{\partial w_j}(\mathbf{w}+\mathbf{\Omega}t'',\mathbf{J},t'')\right\rangle. \tag{202}$$

Introducing the Fourier transform of the total fluctuating potential, we obtain

$$\left\langle \frac{\partial\delta\Phi_{\rm tot}}{\partial w_i}(\mathbf{w}+\mathbf{\Omega}t',\mathbf{J},t')\frac{\partial\delta\Phi_{\rm tot}}{\partial w_j}(\mathbf{w}+\mathbf{\Omega}t'',\mathbf{J},t'')\right\rangle = \sum_{\mathbf{k}}\int\frac{d\omega}{2\pi}\sum_{\mathbf{k}'}\int\frac{d\omega'}{2\pi}\,k_i k_j' \\ \times e^{i\mathbf{k}\cdot(\mathbf{w}+\mathbf{\Omega}t')}e^{-i\omega t'}e^{-i\mathbf{k}'\cdot(\mathbf{w}+\mathbf{\Omega}t'')}e^{i\omega' t''}\,\langle\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\mathbf{J},\omega)\delta\hat{\Phi}_{\rm tot}(\mathbf{k}',\mathbf{J},\omega')^*\rangle. \tag{203}$$

Since the distribution function depends only on the action, we can average the diffusion tensor over $\mathbf{w}$. This brings a Kronecker factor $\delta_{\mathbf{k},\mathbf{k}'}$ which amounts to taking $\mathbf{k}' = \mathbf{k}$. Then, substituting Equation (158) into Equation (203), and carrying out the integrals over ω', we end up with the result

$$\left\langle \frac{\partial\delta\Phi_{\rm tot}}{\partial w_i}(\mathbf{w}+\mathbf{\Omega}t',\mathbf{J},t')\frac{\partial\delta\Phi_{\rm tot}}{\partial w_j}(\mathbf{w}+\mathbf{\Omega}t'',\mathbf{J},t'')\right\rangle = \sum_{\mathbf{k}}\int \frac{d\omega}{2\pi}\, k_i k_j e^{i(\mathbf{k}\cdot\mathbf{\Omega}-\omega)(t'-t'')} P(\mathbf{k},\mathbf{J},\omega). \tag{204}$$

This expression shows that the correlation function appearing in Equation (202) depends only on the difference of times $s = t' - t''$. Using the identity (67) and assuming that the auto-correlation function of the total fluctuating force $f(s)$ decreases more rapidly than s^{-1}, we find for $t \to +\infty$ that[38]

$$D_{ij} = \int_0^{+\infty} \left\langle \frac{\partial\delta\Phi_{\rm tot}}{\partial w_i}(\mathbf{w},\mathbf{J},0)\frac{\partial\delta\Phi_{\rm tot}}{\partial w_j}(\mathbf{w}+\mathbf{\Omega}s,\mathbf{J},s)\right\rangle ds. \tag{205}$$

Therefore, as in the theory of Brownian motion [34,112,138–140], turbulent fluids [141], plasma physics [15,16,22,49,142], and stellar dynamics [8,31,137,143–147], the diffusion tensor of the test particle is equal to the integral of the temporal auto-correlation function $\langle \mathcal{F}_i(0)\mathcal{F}_j(t)\rangle$ of the fluctuating force acting on it:

$$D_{ij} = \int_0^{+\infty} \langle \mathcal{F}_i(0)\mathcal{F}_j(t)\rangle\, dt. \tag{206}$$

Replacing the auto-correlation function by its expression from Equation (204), which can be written as

$$\langle \mathcal{F}_i(0)\mathcal{F}_j(t)\rangle = \sum_{\mathbf{k}}\int \frac{d\omega}{2\pi} k_i k_j e^{-i(\mathbf{k}\cdot\mathbf{\Omega}-\omega)t} P(\mathbf{k},\mathbf{J},\omega), \tag{207}$$

we obtain

$$D_{ij} = \int_0^{+\infty} dt \sum_{\mathbf{k}}\int \frac{d\omega}{2\pi} k_i k_j e^{-i(\mathbf{k}\cdot\mathbf{\Omega}-\omega)t} P(\mathbf{k},\mathbf{J},\omega). \tag{208}$$

Making the change of variables $t \to -t$, $\mathbf{k} \to -\mathbf{k}$ and $\omega \to -\omega$, and using the fact that $P(-\mathbf{k},\mathbf{J},-\omega) = P(\mathbf{k},\mathbf{J},\omega)$, we see that we can replace $\int_0^{+\infty} dt$ by $(1/2)\int_{-\infty}^{+\infty} dt$ in Equation (208). As a result,

$$D_{ij} = \frac{1}{2}\int_{-\infty}^{+\infty} dt \sum_{\mathbf{k}}\int \frac{d\omega}{2\pi} k_i k_j e^{-i(\mathbf{k}\cdot\mathbf{\Omega}-\omega)t} P(\mathbf{k},\mathbf{J},\omega). \tag{209}$$

Using the identity (136), we obtain

$$D_{ij} = \pi \sum_{\mathbf{k}}\int \frac{d\omega}{2\pi} k_i k_j \delta(\mathbf{k}\cdot\mathbf{\Omega}-\omega) P(\mathbf{k},\mathbf{J},\omega). \tag{210}$$

The time integration has given a δ-function which creates a resonance condition for interaction. Integrating over the δ-function (resonance), we arrive at the following equation

$$D_{ij} = \frac{1}{2}\sum_{\mathbf{k}} k_i k_j P(\mathbf{k},\mathbf{J},\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})). \tag{211}$$

This is the general expression of the diffusion tensor of a test particle submitted to a stochastic perturbation in a spatially inhomogeneous system with long-range interactions.

Using the relation between the power spectrum and the correlation function of the external perturbation (see Equation (163)), we obtain

$$D_{ij}[f,\mathbf{J}] = \frac{1}{2}\sum_{\mathbf{k}}\sum_{\alpha\alpha'} k_i k_j \hat{\Phi}_\alpha(\mathbf{k},\mathbf{J})\left[\epsilon^{-1}\hat{C}(\epsilon^{-1})^\dagger\right]_{\alpha\alpha'}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))\hat{\Phi}_{\alpha'}(\mathbf{k},\mathbf{J})^*. \tag{212}$$

We thus recover the expressions from Equations (174) and (175) obtained from the Klimontovich approach. Therefore, the Klimontovich approach and the Fokker–Planck approach coincide.

4.2.4. Second Calculation of D_{ij}

In order to compute the diffusion tensor, we can also proceed as follows. The change in action of the test particle due to the total fluctuating potential is given by see (Equation (188))

$$\frac{d\mathbf{J}}{dt} = -\frac{\partial\delta\Phi_{\rm tot}}{\partial\mathbf{w}}(\mathbf{r}(\mathbf{w},\mathbf{J}),t). \tag{213}$$

Integrating this equation between 0 and t, we obtain

$$\begin{aligned}\Delta\mathbf{J} &= -\int_0^t \frac{\partial\delta\Phi_{\rm tot}}{\partial\mathbf{w}}(\mathbf{w}(t'),\mathbf{J}(t'),t')\,dt'\\ &= -\int_0^t \frac{\partial\delta\Phi_{\rm tot}}{\partial\mathbf{w}}(\mathbf{w}+\mathbf{\Omega}t',\mathbf{J},t')\,dt',\end{aligned} \tag{214}$$

where we have used the unperturbed equations of motion (201) in the second equation (this accounts for the fact that the test particle follows the mean field trajectory at leading order). Decomposing the potential in Fourier modes, we obtain

$$\begin{aligned}\Delta\mathbf{J} &= -\int_0^t dt'\,\frac{\partial}{\partial\mathbf{w}}\sum_{\mathbf{k}}\int\frac{d\omega}{2\pi}e^{i\mathbf{k}\cdot(\mathbf{w}+\mathbf{\Omega}t')}e^{-i\omega t'}\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\mathbf{J},\omega)\\ &= -\sum_{\mathbf{k}}\int\frac{d\omega}{2\pi}i\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{w}}\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\mathbf{J},\omega)\int_0^t e^{i(\mathbf{k}\cdot\mathbf{\Omega}-\omega)t'}\,dt'\\ &= -\sum_{\mathbf{k}}\int\frac{d\omega}{2\pi}i\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{w}}\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\mathbf{J},\omega)\frac{e^{i(\mathbf{k}\cdot\mathbf{\Omega}-\omega)t}-1}{i(\mathbf{k}\cdot\mathbf{\Omega}-\omega)}.\end{aligned} \tag{215}$$

Substituting Equation (215) into Equation (192) and averaging over $\mathbf{w}$, we find that

$$\begin{aligned}D_{ij} &= \lim_{t\to+\infty}\frac{1}{2t}\int\frac{d\mathbf{w}}{(2\pi)^d}\sum_{\mathbf{k}\mathbf{k}'}\int\frac{d\omega}{2\pi}\int\frac{d\omega'}{2\pi}k_ik'_je^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{w}}\\ &\times\ \langle\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\mathbf{J},\omega)\delta\hat{\Phi}_{\rm tot}(\mathbf{k}',\mathbf{J},\omega')^*\rangle\frac{e^{i(\mathbf{k}\cdot\mathbf{\Omega}-\omega)t}-1}{i(\mathbf{k}\cdot\mathbf{\Omega}-\omega)}\frac{e^{-i(\mathbf{k}'\cdot\mathbf{\Omega}-\omega')t}-1}{-i(\mathbf{k}'\cdot\mathbf{\Omega}-\omega')}.\end{aligned} \tag{216}$$

Performing the integral over $\mathbf{w}$ and the sum over $\mathbf{k}'$, we obtain

$$\begin{aligned}D_{ij} = \lim_{t\to+\infty}\frac{1}{2t}\sum_{\mathbf{k}}\int\frac{d\omega}{2\pi}\int\frac{d\omega'}{2\pi}k_ik_j\langle\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\mathbf{J},\omega)\delta\hat{\Phi}_{\rm tot}(\mathbf{k},\mathbf{J},\omega')^*\rangle\\ \times\frac{e^{i(\mathbf{k}\cdot\mathbf{\Omega}-\omega)t}-1}{i(\mathbf{k}\cdot\mathbf{\Omega}-\omega)}\frac{e^{-i(\mathbf{k}\cdot\mathbf{\Omega}-\omega')t}-1}{-i(\mathbf{k}\cdot\mathbf{\Omega}-\omega')}.\end{aligned} \tag{217}$$

Introducing the power spectrum from Equation (158), the foregoing equation can be rewritten as

$$D_{ij} = \lim_{t\to+\infty}\frac{1}{2t}\sum_{\mathbf{k}}\int\frac{d\omega}{2\pi}k_ik_jP(\mathbf{k},\mathbf{J},\omega)\frac{|e^{i(\mathbf{k}\cdot\mathbf{\Omega}-\omega)t}-1|^2}{(\mathbf{k}\cdot\mathbf{\Omega}-\omega)^2}. \tag{218}$$

This equation can also be directly obtained from Equation (202) with Equation (204) by integrating over t' and t''. Taking the limit $t \to +\infty$ and using the identity (81) we find that

$$D_{ij} = \pi \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} k_i k_j P(\mathbf{k}, \mathbf{J}, \omega) \delta(\mathbf{k} \cdot \mathbf{\Omega} - \omega). \tag{219}$$

Integrating over the δ-function (resonance), we obtain

$$D_{ij} = \frac{1}{2} \sum_{\mathbf{k}} k_i k_j P(\mathbf{k}, \mathbf{J}, \mathbf{k} \cdot \mathbf{\Omega}), \tag{220}$$

which returns Equation (211). Then, using Equation (163), we obtain Equation (212).

Remark 15. *If we do not take the limit $t \to +\infty$ in Equation (218), we obtain a time-dependent diffusion tensor of the form*

$$D_{ij}(t) = \pi \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} k_i k_j P(\mathbf{k}, \mathbf{J}, \omega) \Delta(\mathbf{k} \cdot \mathbf{\Omega} - \omega, t) \tag{221}$$

with the regularized function (85).

4.2.5. Third Calculation of D_{ij}

We can make the calculations of the previous section in a slightly different manner. In Equation (214) we decompose the total fluctuating potential in Fourier modes in angle but not in time. In that case, we obtain

$$\begin{aligned} \Delta \mathbf{J} &= -\int_0^t dt' \frac{\partial}{\partial \mathbf{w}} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{w} + \mathbf{\Omega} t')} \delta \hat{\Phi}_{\rm tot}(\mathbf{k}, \mathbf{J}, t') \\ &= -\int_0^t dt' \sum_{\mathbf{k}} i\mathbf{k} e^{i\mathbf{k} \cdot (\mathbf{w} + \mathbf{\Omega} t')} \delta \hat{\Phi}_{\rm tot}(\mathbf{k}, \mathbf{J}, t'). \end{aligned} \tag{222}$$

Substituting Equation (222) into Equation (192), we obtain

$$D_{ij} = \lim_{t \to +\infty} \frac{1}{2t} \int_0^t dt' \int_0^t dt'' \sum_{\mathbf{k}\mathbf{k}'} k_i k'_j e^{i\mathbf{k} \cdot (\mathbf{w} + \mathbf{\Omega} t')} e^{-i\mathbf{k}' \cdot (\mathbf{w} + \mathbf{\Omega} t'')} \langle \delta \hat{\Phi}_{\rm tot}(\mathbf{k}, \mathbf{J}, t') \delta \hat{\Phi}_{\rm tot}(\mathbf{k}', \mathbf{J}, t'')^* \rangle. \tag{223}$$

Since the mean distribution function f depends only on the action $\mathbf{J}$, we can average D_{ij} over the angle $\mathbf{w}$. This amounts to taking $\mathbf{k}' = \mathbf{k}$, yielding

$$D_{ij} = \lim_{t \to +\infty} \frac{1}{2t} \int_0^t dt' \int_0^t dt'' \sum_{\mathbf{k}} k_i k_j e^{i\mathbf{k} \cdot \mathbf{\Omega} (t' - t'')} \langle \delta \hat{\Phi}_{\rm tot}(\mathbf{k}, \mathbf{J}, t') \delta \hat{\Phi}_{\rm tot}(\mathbf{k}, \mathbf{J}, t'')^* \rangle. \tag{224}$$

Using the relation

$$\langle \delta \hat{\Phi}_{\rm tot}(\mathbf{k}, \mathbf{J}, t) \delta \hat{\Phi}_{\rm tot}(\mathbf{k}, \mathbf{J}, t')^* \rangle = \mathcal{P}(\mathbf{k}, \mathbf{J}, t - t'), \tag{225}$$

where $\mathcal{P}(\mathbf{k}, \mathbf{J}, t)$ is the temporal inverse Fourier transform of $P(\mathbf{k}, \mathbf{J}, \omega)$, we can rewrite the foregoing equation as

$$D_{ij} = \lim_{t \to +\infty} \frac{1}{2t} \int_0^t dt' \int_0^t dt'' \sum_{\mathbf{k}} k_i k_j e^{i\mathbf{k} \cdot \mathbf{\Omega} (t' - t'')} \mathcal{P}(\mathbf{k}, \mathbf{J}, t' - t''). \tag{226}$$

Using the identity (67), we obtain

$$D_{ij} = \lim_{t \to +\infty} \frac{1}{t} \int_0^t ds\, (t - s) \sum_{\mathbf{k}} k_i k_j e^{i\mathbf{k} \cdot \mathbf{\Omega} s} \mathcal{P}(\mathbf{k}, \mathbf{J}, s). \tag{227}$$

Assuming that $\mathcal{P}(\mathbf{k},\mathbf{J},s)$ decreases more rapidly than s^{-1}, we find that

$$D_{ij} = \int_0^{+\infty} ds \sum_{\mathbf{k}} k_i k_j e^{i\mathbf{k}\cdot\mathbf{\Omega}s} \mathcal{P}(\mathbf{k},\mathbf{J},s). \tag{228}$$

Making the change of variables $s \to -s$ and $\mathbf{k} \to -\mathbf{k}$, and using the fact that $\mathcal{P}(-\mathbf{k},\mathbf{J},-s) = \mathcal{P}(\mathbf{k},\mathbf{J},s)$, we see that we can replace $\int_0^{+\infty} ds$ by $(1/2)\int_{-\infty}^{+\infty} ds$. Therefore,

$$D_{ij} = \frac{1}{2}\int_{-\infty}^{+\infty} ds \sum_{\mathbf{k}} k_i k_j e^{i\mathbf{k}\cdot\mathbf{\Omega}s} \mathcal{P}(\mathbf{k},\mathbf{J},s). \tag{229}$$

Taking the inverse Fourier transform of $\mathcal{P}(\mathbf{k},\mathbf{J},s)$ we obtain

$$D_{ij} = \frac{1}{2}\sum_{\mathbf{k}} k_i k_j P(\mathbf{k},\mathbf{J},\mathbf{k}\cdot\mathbf{\Omega}), \tag{230}$$

which returns Equation (211). Then, using Equation (163), we obtain Equation (212). We note that $\mathcal{P}(\mathbf{k},\mathbf{J},s)$ is complex while $P(\mathbf{k},\mathbf{J},\omega)$ is real. They satisfy the identities $\mathcal{P}(-\mathbf{k},\mathbf{J},s) = \mathcal{P}(\mathbf{k},\mathbf{J},s)^* = \mathcal{P}(\mathbf{k},\mathbf{J},-s)$ and $P(\mathbf{k},\mathbf{J},\omega) = P(\mathbf{k},\mathbf{J},\omega)^* = P(-\mathbf{k},\mathbf{J},-\omega)$. We also note that the static power spectrum is $P(\mathbf{k},\mathbf{J}) = \mathcal{P}(\mathbf{k},\mathbf{J},0) = \int \frac{d\omega}{2\pi} P(\mathbf{k},\mathbf{J},\omega)$.

Remark 16. *If we introduce the temporal Fourier transform of $\mathcal{P}(\mathbf{k},\mathbf{J},t)$ in Equation (226) we obtain*

$$D_{ij} = \lim_{t\to+\infty}\frac{1}{2t}\int_0^t dt' \int_0^t dt'' \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} k_i k_j e^{i\mathbf{k}\cdot\mathbf{\Omega}(t'-t'')} e^{-i\omega(t'-t'')} P(\mathbf{k},\mathbf{J},\omega), \tag{231}$$

which is equivalent to Equation (202) with Equation (204). If we integrate over t' and t'', we recover Equation (218).

4.3. Energy of Fluctuations

The energy of fluctuations

$$\mathcal{E} = \frac{1}{2}\int \langle \delta\rho_{\rm tot}\delta\Phi_{\rm tot}\rangle \, d\mathbf{r}, \tag{232}$$

where $\delta\rho_{\rm tot} = \delta\rho + \rho_e$ is the total perturbed density, can be calculated as follows. Decompositing the fluctuations of density and potential on the biorthonormal basis, using Equation (140), we obtain

$$\mathcal{E} = \frac{1}{2}\sum_{\alpha\alpha'} \langle A_\alpha^{\rm tot}(t) A_{\alpha'}^{\rm tot}(t)^*\rangle \int \rho_\alpha(\mathbf{r})\Phi_{\alpha'}^*(\mathbf{r})\, d\mathbf{r}. \tag{233}$$

Using the orthonormalization condition from Equation (141), we find that

$$\mathcal{E} = -\frac{1}{2}\sum_\alpha \langle |A_\alpha^{\rm tot}(t)|^2\rangle. \tag{234}$$

Decomposing the amplitudes in temporal Fourier modes, we can rewrite the foregoing equation as

$$\mathcal{E} = -\frac{1}{2}\sum_\alpha \int \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi} e^{-i\omega t} e^{i\omega' t} \langle \hat{A}_\alpha^{\rm tot}(\omega) \hat{A}_\alpha^{\rm tot}(\omega')^*\rangle. \tag{235}$$

Introducing the power spectrum tensor from Equation (157) and integrating over ω', we obtain

$$\mathcal{E} = -\frac{1}{2}\sum_\alpha \int \frac{d\omega}{2\pi} P_{\alpha\alpha}(\omega). \tag{236}$$

Finally, using Equation (162), we can rewrite the energy of fluctuations as

$$\mathcal{E} = -\frac{1}{2}\sum_{\alpha}\int \frac{d\omega}{2\pi}\left[\epsilon^{-1}\hat{C}(\epsilon^{-1})^{\dagger}\right]_{\alpha\alpha}(\omega). \tag{237}$$

This is the proper generalization of Equation (99) to spatially inhomogeneous systems.

4.4. Connection between the SDD Equation and the Multi-Species Lenard–Balescu Equation

We can readily adapt to the case of spatially inhomogeneous systems the discussion given in Section 2.4 concerning the connection between the SDD equation and the multi-species Lenard–Balescu equation. The multi-species inhomogeneous Lenard–Balescu equation reads [60,136]

$$\frac{\partial f_a}{\partial t} = \pi(2\pi)^d \frac{\partial}{\partial \mathbf{J}}\cdot\sum_b\sum_{\mathbf{k},\mathbf{k}'}\int d\mathbf{J}'\,\mathbf{k}\,|A^d_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\mathbf{\Omega})|^2\delta(\mathbf{k}\cdot\mathbf{\Omega}-\mathbf{k}'\cdot\mathbf{\Omega}')$$
$$\times\left(m_b f'_b\mathbf{k}\cdot\frac{\partial f_a}{\partial \mathbf{J}} - m_a f_a \mathbf{k}'\cdot\frac{\partial f'_b}{\partial \mathbf{J}'}\right), \tag{238}$$

where f' stands for $f(\mathbf{J}')$, $\mathbf{\Omega}'$ stands for $\mathbf{\Omega}(\mathbf{J}')$ and $A^d_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega)$ is the Fourier transform of the dressed potential of interaction in angle-action variables defined by[39]

$$A^d_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega) = -\sum_{\alpha\alpha'}\hat{\Phi}_{\alpha}(\mathbf{k},\mathbf{J})(\epsilon^{-1})_{\alpha\alpha'}(\omega)\hat{\Phi}^*_{\alpha'}(\mathbf{k}',\mathbf{J}'). \tag{239}$$

Let us focus on the species of particles of mass m_a (test particles). We take into account the collisions induced by the field particles of masses $\{m_b\}_{b\neq a}$ on the test particles but we neglect the collisions induced by the test particles on the field particles and on themselves. This approximation is valid for very light test particles $m_a \ll m_b$ or, more precisely, in the limit $N_a \to +\infty$ with $m_a \sim 1/N_a$. In that case, the inhomogeneous Lenard–Balescu Equation (238) reduces to

$$\frac{\partial f_a}{\partial t} = \pi(2\pi)^d \frac{\partial}{\partial \mathbf{J}}\cdot\sum_{b\neq a}\sum_{\mathbf{k},\mathbf{k}'}\int d\mathbf{J}'\,\mathbf{k}\,|A^d_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\mathbf{\Omega})|^2\delta(\mathbf{k}\cdot\mathbf{\Omega}-\mathbf{k}'\cdot\mathbf{\Omega}')\left(m_b f'_b\mathbf{k}\cdot\frac{\partial f_a}{\partial \mathbf{J}}\right). \tag{240}$$

We note that the friction by polarization vanishes ($\mathbf{F}_{\rm pol} = \mathbf{0}$) and that the diffusion tensor has no contribution from particles of species a. As a result, the mass m_a of the particles of species a does not appear anymore in the kinetic equation.

Equation (240) can be written as an inhomogeneous SDD equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial J_i}\left(D_{ij}[f,\mathbf{J}]\frac{\partial f}{\partial J_j}\right) \tag{241}$$

with a diffusion tensor

$$D_{ij}[f,\mathbf{J}] = \pi(2\pi)^d\sum_b\sum_{\mathbf{k},\mathbf{k}'}\int d\mathbf{J}'\,k_i k_j\,|A^d_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\mathbf{\Omega})|^2\delta(\mathbf{k}\cdot\mathbf{\Omega}-\mathbf{k}'\cdot\mathbf{\Omega}')m_b f'_b, \tag{242}$$

where we have dropped the subscript a for clarity. One can show [136] that the expression (242) of the diffusion tensor is consistent with Equation (175) where

$$\hat{C}_{\alpha\alpha'}(\omega) = (2\pi)^{d+1}\sum_b m_b\sum_{\mathbf{k}}\int \delta(\omega-\mathbf{k}\cdot\mathbf{\Omega})\hat{\Phi}^*_{\alpha}(\mathbf{k},\mathbf{J})\hat{\Phi}_{\alpha'}(\mathbf{k},\mathbf{J})f_b(\mathbf{J})\,d\mathbf{J} \tag{243}$$

is the bare correlation function of the potential created by a discrete collection of field particles of masses $\{m_b\}$. Combining Equations (160), (162), (236) and (243) we obtain the dressed power spectrum

$$P_{\alpha\alpha'}(\omega) = (2\pi)^{d+1}\sum_b m_b \sum_{\mathbf{k}} \int \delta(\omega - \mathbf{k}\cdot\mathbf{\Omega})\left[\epsilon^{-1}(\omega)\hat{\Phi}^*(\mathbf{k},\mathbf{J})\right]_\alpha \left[\epsilon^{-1}(\omega)^*\hat{\Phi}(\mathbf{k},\mathbf{J})\right]_{\alpha'} f_b(\mathbf{J})\, d\mathbf{J}, \quad (244)$$

$$P(\mathbf{k},\mathbf{J},\omega) = (2\pi)^{d+1}\sum_b m_b \sum_{\mathbf{k}'} \int \delta(\omega - \mathbf{k}'\cdot\mathbf{\Omega}')|A^d_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega)|^2 f_b(\mathbf{J}')\, d\mathbf{J}', \quad (245)$$

the static power spectrum

$$P_{\alpha\alpha'} = (2\pi)^{d}\sum_b m_b \sum_{\mathbf{k}} \int \left[\epsilon^{-1}(\mathbf{k}\cdot\mathbf{\Omega})\hat{\Phi}^*(\mathbf{k},\mathbf{J})\right]_\alpha \left[\epsilon^{-1}(\mathbf{k}\cdot\mathbf{\Omega})^*\hat{\Phi}(\mathbf{k},\mathbf{J})\right]_{\alpha'} f_b(\mathbf{J})\, d\mathbf{J}, \quad (246)$$

$$P(\mathbf{k},\mathbf{J}) = (2\pi)^{d}\sum_b m_b \sum_{\mathbf{k}'} \int d\mathbf{J}'\, |A^d_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}'\cdot\mathbf{\Omega}')|^2 f_b(\mathbf{J}'), \quad (247)$$

and the energy of fluctuations

$$\mathcal{E} = -\frac{1}{2}(2\pi)^{d}\sum_b m_b \sum_{\mathbf{k}} \int |\epsilon^{-1}(\mathbf{k}\cdot\mathbf{\Omega})\hat{\Phi}^*(\mathbf{k},\mathbf{J})|^2 f_b(\mathbf{J})\, d\mathbf{J}, \quad (248)$$

where $|\epsilon^{-1}(\mathbf{k}\cdot\mathbf{\Omega})\hat{\Phi}^*(\mathbf{k},\mathbf{J})|^2 = \sum_\alpha [\epsilon^{-1}(\mathbf{k}\cdot\mathbf{\Omega})\hat{\Phi}^*(\mathbf{k},\mathbf{J})]_\alpha [\epsilon^{-1}(\mathbf{k}\cdot\mathbf{\Omega})^*\hat{\Phi}(\mathbf{k},\mathbf{J})]_\alpha$. If we neglect collective effects, the energy of fluctuations can be written as

$$\mathcal{E}_{\rm bare} = \frac{1}{2}(2\pi)^{d}\sum_b m_b \sum_{\mathbf{k}} \int A_{\mathbf{k},\mathbf{k}}(\mathbf{J},\mathbf{J}) f_b(\mathbf{J})\, d\mathbf{J}. \quad (249)$$

5. Derivation of the SDD Equation in Physical Space

5.1. SDD Equation

In this section, we derive the SDD equation in physical space without using Fourier transforms. This formal derivation provides some insight on the physical structure of this equation. We consider a possibly spatially inhomogeneous system of particles with long-range interactions submitted to an external stochastic perturbation $\Phi_e(\mathbf{r},t)$. Denoting by $\mathcal{F}_{\rm tot} = -\nabla(\delta\Phi + \Phi_e)$ the total fluctuating force by unit of mass experienced by a particle, the fundamental Equations (10) and (11) of the quasilinear theory can be rewritten as

$$\frac{\partial f}{\partial t} + Lf = -\frac{\partial}{\partial \mathbf{v}}\cdot\langle \delta f \mathcal{F}_{\rm tot}\rangle, \quad (250)$$

$$\frac{\partial \delta f}{\partial t} + L\delta f = -\mathcal{F}_{\rm tot}\cdot\frac{\partial f}{\partial \mathbf{v}}, \quad (251)$$

where $L = \mathbf{v}\cdot\partial/\partial\mathbf{r} - \nabla\Phi\cdot\partial/\partial\mathbf{v}$ is the mean field Liouvillian (or Vlasov) operator, i.e., the advection term in phase space. Equation (251) for the fluctuations can be formally integrated into

$$\delta f(\mathbf{r},\mathbf{v},t) = -\int_0^{+\infty} G(t,t-\tau)\mathcal{F}_{\rm tot}(\mathbf{r},t-\tau)\cdot\frac{\partial f}{\partial \mathbf{v}}(\mathbf{r},\mathbf{v},t-\tau)\, d\tau, \quad (252)$$

where

$$G(t_2,t_1) = \exp\left\{-\int_{t_1}^{t_2} L(t)\, dt\right\} \quad (253)$$

is the Greenian operator constructed with the mean field Liouvillian operator. We have taken $\delta f(\mathbf{r},\mathbf{v},-\infty) = 0$ because we neglect the fluctuations due to finite N effects (we do not consider the collisions between the particles of the system but only the influence of the external perturbation).[40] This is valid for $N \to +\infty$ with $m \sim 1/N$. Substituting Equation (252) into Equation (250), we obtain the non-Markovian kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial v_i} \int_0^{+\infty} d\tau \, \langle \mathcal{F}_i^{\rm tot}(\mathbf{r}, t) \mathcal{F}_j^{\rm tot}(\mathbf{r}(t-\tau), t-\tau) \rangle \times \frac{\partial f}{\partial v_j}(\mathbf{r}(t-\tau), \mathbf{v}(t-\tau), t-\tau), \tag{254}$$

where the notations $\mathbf{r}(t-\tau)$ and $\mathbf{v}(t-\tau)$ recall that the particles move along the mean field trajectories determined by the time-dependent potential $\Phi(\mathbf{r}(t-\tau), t-\tau)$ according to

$$\frac{d\mathbf{r}}{dt}(t-\tau) = \mathbf{v}(t-\tau), \qquad \frac{d\mathbf{v}}{dt}(t-\tau) = -\nabla\Phi(\mathbf{r}(t-\tau), t-\tau). \tag{255}$$

Making a Markovian approximation, we can replace $f(\mathbf{r}(t-\tau), \mathbf{v}(t-\tau), t-\tau)$ by $f(\mathbf{r}(t-\tau), \mathbf{v}(t-\tau), t)$ and $\Phi(\mathbf{r}(t-\tau), t-\tau)$ by $\Phi(\mathbf{r}(t-\tau), t)$. In this manner, we obtain the inhomogeneous SDD equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial v_i} \int_0^{+\infty} d\tau \, \langle \mathcal{F}_i^{\rm tot}(\mathbf{r}, t) \mathcal{F}_j^{\rm tot}(\mathbf{r}(t-\tau), t-\tau) \rangle \times \frac{\partial f}{\partial v_j}(\mathbf{r}(t-\tau), \mathbf{v}(t-\tau), t). \tag{256}$$

When integrating over τ, we must consider that the mean field is "frozen" at its value at time t (Bogoliubov ansatz). In that case, the particles move along the mean field trajectories determined by the static potential $\Phi(\mathbf{r}(t-\tau), t)$ according to

$$\frac{d\mathbf{r}}{dt}(t-\tau) = \mathbf{v}(t-\tau), \qquad \frac{d\mathbf{v}}{dt}(t-\tau) = -\nabla\Phi(\mathbf{r}(t-\tau), t). \tag{257}$$

Although appealing, Equation (256) is formal because $\delta\mathbf{F}(\mathbf{r}, t) = -\nabla\delta\Phi(\mathbf{r}, t)$ in $\mathcal{F}^{\rm tot}(\mathbf{r}, t) = \delta\mathbf{F}(\mathbf{r}, t) + \mathbf{F}_e(\mathbf{r}, t)$ depends on $\delta f(\mathbf{r}, \mathbf{v}, t)$ through Equation (18) which itself depends on $\delta\mathbf{F}(\mathbf{r}, t)$ through Equation (251). We can solve this loop by introducing a formal generalization of the dielectric function in physical space (instead of Fourier space). In that case, $\mathcal{F}^{\rm tot}(\mathbf{r}, t)$ can be interpreted as the dressed force acting on the particles and $\langle \mathcal{F}_i^{\rm tot}(\mathbf{r}, t) \mathcal{F}_j^{\rm tot}(\mathbf{r}(t-\tau), t-\tau) \rangle$ must be viewed as a functional of f which accounts for collective effects. If we neglect collective effects and take $\mathcal{F}^{\rm tot}(\mathbf{r}, t) \simeq \mathbf{F}_e(\mathbf{r}, t)$, Equation (256) reduces to

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial v_i} \int_0^{+\infty} d\tau \, \langle F_i^e(\mathbf{r}, t) F_j^e(\mathbf{r}(t-\tau), t-\tau) \rangle \frac{\partial f}{\partial v_j}(\mathbf{r}(t-\tau), \mathbf{v}(t-\tau), t). \tag{258}$$

In this manner, we obtain the inhomogeneous SBD equation.[41]

For spatially homogeneous systems, we have $f = f(\mathbf{v}, t)$ and $\langle \mathbf{F} \rangle = \mathbf{0}$. Since the mean field force vanishes, the mean field trajectories of the particles are just straight lines traveled at constant velocity:

$$\mathbf{r}(t-\tau) = \mathbf{r} - \mathbf{v}\tau, \qquad \mathbf{v}(t-\tau) = \mathbf{v}. \tag{259}$$

In that case, Equation (254) reduces to

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \int_0^{+\infty} d\tau \, \langle F_i^{\rm tot}(\mathbf{r}, t) F_j^{\rm tot}(\mathbf{r} - \mathbf{v}\tau, t-\tau) \rangle \frac{\partial f}{\partial v_j}(\mathbf{v}, t-\tau). \tag{260}$$

If we make a Markovian approximation, we obtain the homogeneous SDD equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \int_0^{+\infty} d\tau \, \langle F_i^{\rm tot}(\mathbf{r}, t) F_j^{\rm tot}(\mathbf{r} - \mathbf{v}\tau, t-\tau) \rangle \frac{\partial f}{\partial v_j}(\mathbf{v}, t). \tag{261}$$

If we neglect collective effects, we obtain the homogeneous SBD equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \int_0^{+\infty} d\tau \, \langle F_i^e(\mathbf{r},t) F_j^e(\mathbf{r}-\mathbf{v}\tau, t-\tau) \rangle \frac{\partial f}{\partial v_j}(\mathbf{v},t). \tag{262}$$

It can be written as a diffusion equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left(D_{ij} \frac{\partial f}{\partial v_j} \right) \tag{263}$$

with the anisotropic diffusion tensor

$$D_{ij} = \int_0^{+\infty} d\tau \, \langle F_i^e(\mathbf{r},t) F_j^e(\mathbf{r}-\mathbf{v}\tau, t-\tau) \rangle. \tag{264}$$

For a colored noise, the diffusion tensor depends on the velocity $\mathbf{v}$ of the test particle. When the external force is δ-correlated in time (white noise), i.e.,

$$\langle F_i^e(\mathbf{r},t) F_j^e(\mathbf{r}',t') \rangle = C_{ij}(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{265}$$

the Markovian approximation becomes exact and we obtain

$$D_{ij} = \frac{1}{2} C_{ij}(\mathbf{0}). \tag{266}$$

In that case, the spatial correlation of the external perturbation is unimportant (only the value of $C_{ij}(\mathbf{0})$ matters) and the diffusion tensor is constant. We recover the results of the discussion following Equation (39).

Remark 17. *Equations (263) and (264) (or more generally Equation (261)) are equivalent to Equations (59) and (68) obtained in Section 2.2 from the Fokker–Planck approach. We stress, however, that the derivation based on the Klimontovich equation exposed in this section is very different from the one relying on the Fokker–Planck approach in Section 2.2. In particular, we do not have to use Equations (53), (56), (58) and (64). Therefore, the Klimontovich approach is another manner to derive the Fokker–Planck Equations (59) and (68) without using the Master equation and the Kramers–Moyal expansion. We note that the Klimontovich approach directly leads to a diffusion equation of the form of Equation (56) where the diffusion tensor is sandwiched between the velocity derivatives whereas, in the usual Fokker–Planck approach, the diffusion tensor is placed after the two velocity derivatives (see Equation (53)). Since* $\mathbf{F}_{\rm pol} = \mathbf{0}$ *the two equations are equivalent as discussed in Section 2.2.2. The same remarks apply to the results of Section 2.1.*

5.2. Application to the Theory of Brownian Motion

Let us consider a system of self-interacting Brownian particles immersed in a fluid. We write the equations of motion of the Brownian particles as

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = -\nabla\Phi(\mathbf{r}_i) - \xi\mathbf{v}_i - \nabla\Phi_e(\mathbf{r}_i,t). \tag{267}$$

The friction force $-\xi\mathbf{v}$ and the external stochastic force $\mathbf{F}_{\rm e} = -\nabla\Phi_e$ model the "collisions" between the Brownian particles and the molecules of the fluid in which they move.[42] In stellar dynamics, these equations can describe the motion of test stars a experiencing a stochastic force which may be due to collisions with other stars or have a more general origin. For simplicity, we neglect discreteness effects due to the Brownian particles.[43] Therefore, $\Phi(\mathbf{r}) = \int u(|\mathbf{r}-\mathbf{r}'|)\rho(\mathbf{r}')\, d\mathbf{r}'$ represents the self-consistent mean field potential produced by the Brownian particles. Repeating the calculations of Section 5.1 with the additional friction force (which is a deterministic term), we obtain the Fokker–Planck equation

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}-\nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}}=\frac{\partial}{\partial \mathbf{v}}\cdot\left(D\frac{\partial f}{\partial \mathbf{v}}+\xi f\mathbf{v}\right), \tag{268}$$

like in the theory of Brownian motion [34,43]. This is the so-called Kramers or Klein–Kramers–Chandrasekhar equation [34,35,37–39,43,110,132].[44] However, it is derived here in a completely different manner without using the Langevin equations, the Master equation, and the Kramers–Moyal expansion. The diffusion coefficient is given by (see Equation (264))

$$D=\frac{1}{3}\int_0^{+\infty}\langle\mathbf{F}(0)\cdot\mathbf{F}(t)\rangle\,dt, \tag{269}$$

where $\mathbf{F}$ represents the stochastic force acting on the Brownian particle (the factor $1/3$ accounts for the dimensionality of space $d=3$). For simplicity, we have assumed that the diffusion tensor is isotropic ($D_{ij}=D\delta_{ij}$) and constant (independent of f and $\mathbf{v}$). This is the case when the external potential $\Phi_e(\mathbf{r},t)$ is δ-correlated in time (white noise) and when collective effects can be neglected (e.g., when the Brownian particles are noninteracting). The diffusion coefficient can be approximated by

$$D\sim\frac{1}{3}\tau\langle F^2\rangle, \tag{270}$$

where τ is a typical decorrelation time and $\langle F^2\rangle$ is the variance of the inter-molecular forces. In the case of stellar systems, the distribution of the gravitational force produced by a random distribution of stars is a particular Lévy law called the Holtsmark [162] distribution (see, e.g., [163]) and the diffusion coefficient can be calculated from a stochastic approach [8,31,32,36,40,41,49,137,143,144].

If we apply the general argument of Chandrasekhar [34,43] and require that the kinetic Equation (268) relaxes towards the Maxwell–Boltzmann distribution function of statistical equilibrium[45]

$$f=Ae^{-\beta m[v^2/2+\Phi(\mathbf{r})]}, \tag{271}$$

where $\beta=1/(k_BT)$ in the inverse temperature, we find that the friction coefficient is related to the diffusion coefficient by the Einstein relation

$$\xi=D\beta m. \tag{272}$$

We can then write the Fokker–Planck equation as

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}-\nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}}=\frac{\partial}{\partial \mathbf{v}}\cdot\left[D\left(\frac{\partial f}{\partial \mathbf{v}}+\beta m f\mathbf{v}\right)\right]. \tag{273}$$

Combining Equations (269) and (272), we obtain the formula

$$\xi=\frac{1}{3}\beta m\int_0^{+\infty}\langle\mathbf{F}(0)\cdot\mathbf{F}(t)\rangle\,dt. \tag{274}$$

This is the so-called Green–Kubo [139,140,153] relation expressing the fluctuation–dissipation theorem. Actually, this formula was first derived by Kirkwood [152] by a direct calculation of the friction force (see also [8,136,139,140,144–147,153–159]).

If D is constant, the Langevin equations associated with the Fokker–Planck Equation (268) are [112][46]

$$\frac{d\mathbf{r}}{dt}=\mathbf{v},\qquad \frac{d\mathbf{v}}{dt}=-\nabla\Phi(\mathbf{r})-\xi\mathbf{v}+\sqrt{2D}\,\mathbf{R}(t), \tag{275}$$

where $\mathbf{R}(t)$ is a Gaussian white noise satisfying $\langle R_i(t)\rangle = 0$ and $\langle R_i(t)R_j(t')\rangle = \delta_{ij}\delta(t-t')$.[47]

If we take fluctuations into account at a mesoscopic level of description, we obtain a stochastic partial differential equation of the form (see [124] and Section 2.1.6)

$$\frac{\partial \overline{f}}{\partial t} + \mathbf{v}\cdot\frac{\partial \overline{f}}{\partial \mathbf{r}} - \nabla\overline{\Phi}\cdot\frac{\partial \overline{f}}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(D\frac{\partial \overline{f}}{\partial \mathbf{v}} + \xi\overline{f}\mathbf{v}\right) + \frac{\partial}{\partial \mathbf{v}}\cdot\left[\sqrt{2Dm\overline{f}}\,\mathbf{Q}(\mathbf{r},\mathbf{v},t)\right], \tag{276}$$

where $\mathbf{Q}(\mathbf{r},\mathbf{v},t)$ is a Gaussian white noise satisfying $\langle Q_i(\mathbf{r},\mathbf{v},t)\rangle = 0$ and $\langle Q_i(\mathbf{r},\mathbf{v},t)Q_j(\mathbf{r}',\mathbf{v}',t')\rangle = \delta_{ij}\delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{v}-\mathbf{v}')\delta(t-t')$. When the deterministic Equation (268) admits several equilibrium states, the noise term in Equation (276) can trigger random transitions from one state to the other (see, e.g., [128,129]).

Remark 18. *In the original approach of Einstein [130], the value of the friction coefficient was assumed to be known (e.g., being given by the Stokes [165] formula) and Equation (272) was used to determine the diffusion coefficient D from the friction ξ, the temperature T, and the particle mass m.*[48] *Alternatively, in the approach of Kirkwood [152], the friction coefficient is determined via Equation (274) by the auto-correlation function of the inter-molecular forces. This is a very different point of view. In this sense, Equation (274) may be viewed as a generalization of the Stokes formula.*

5.3. Generalization for a Velocity-Dependent Diffusion Coefficient

If we account for the possibility that the diffusion coefficient depends on the velocity, we must write the equations of motion of the Brownian particles in our approach as

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = \mathbf{F}_i^{\rm pol} - \nabla\Phi(\mathbf{r}_i) - \nabla\Phi_e(\mathbf{r}_i,t). \tag{277}$$

In that case, the Fokker–Planck equation becomes (see Section 5.1)

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\cdot\left[D(\mathbf{v})\frac{\partial f}{\partial \mathbf{v}} - f\mathbf{F}_{\rm pol}\right]. \tag{278}$$

If we require that the kinetic Equation (278) relaxes towards the Maxwell–Boltzmann distribution function (271), we find that the friction force (more precisely the friction by polarization) is related to the diffusion coefficient by the generalized Einstein relation

$$\mathbf{F}_{\rm pol} = -D(\mathbf{v})\beta m\mathbf{v}. \tag{279}$$

The Fokker–Planck equation can then be rewritten as

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\cdot\left[D(\mathbf{v})\left(\frac{\partial f}{\partial \mathbf{v}} + \beta m f\mathbf{v}\right)\right]. \tag{280}$$

The friction by polarization is of the form

$$\mathbf{F}_{\rm pol} = -\xi(\mathbf{v})\mathbf{v} \quad \text{with} \quad \xi(\mathbf{v}) = D(\mathbf{v})\beta m. \tag{281}$$

The friction coefficient depends on the velocity.

If $D = D(\mathbf{v})$, the Langevin equations associated with the Fokker–Planck Equation (280) are [8,112]

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \qquad \frac{d\mathbf{v}}{dt} = -\nabla\Phi(\mathbf{r}) - \xi(\mathbf{v})\mathbf{v} + \frac{1}{2}\frac{\partial D}{\partial \mathbf{v}} + \sqrt{2D(\mathbf{v})}\mathbf{R}(t), \tag{282}$$

where $\mathbf{R}(t)$ is a Gaussian white noise satisfying $\langle R_i(t)\rangle = 0$ and $\langle R_i(t)R_j(t')\rangle = \delta_{ij}\delta(t-t')$. Since the diffusion coefficient depends on the velocity, we have a multiplicative noise. In that case, the Langevin equations involve a term $\frac{1}{2}\frac{\partial D}{\partial \mathbf{v}}$ in addition to the naive friction force $-\xi(\mathbf{v})\mathbf{v}$. This spurious drift is not very intuitive. In contrast, the equations of motion (277) with Equation (281) do not involve a spurious drift; they are therefore more intuitive. This shows that the approach based on Equation (277) with Equation (279) is fundamentally different from the (Langevin) approach based on the Equation (282).

5.4. Generalization for an Anisotropic Diffusion Tensor

If we account for the possibility that the diffusion tensor is anisotropic, we must write the equations of motion of the Brownian particles in our approach as

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = \mathbf{F}_i^{\rm pol} - \nabla\Phi(\mathbf{r}_i) - \nabla\Phi_e(\mathbf{r}_i, t). \tag{283}$$

In that case, the Fokker–Planck equation becomes (see Section 5.1)

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial v_i}\cdot\left(D_{ij}\frac{\partial f}{\partial v_j} + fF_i^{\rm pol}\right). \tag{284}$$

The diffusion tensor is given by

$$D_{ij} = \int_0^{+\infty}\langle \mathcal{F}_i(0)\mathcal{F}_j(t)\rangle\, dt. \tag{285}$$

In the general case, $D_{ij}[f, \mathbf{v}]$ depends on $\mathbf{v}$ and is a functional of f. If we require that the kinetic Equation (284) relaxes towards the Maxwell–Boltzmann distribution function (271), we find that the friction force (more precisely the friction by polarization) is related to the diffusion tensor by the generalized Einstein relation

$$F_i^{\rm pol} = -D_{ij}\beta m v_j, \tag{286}$$

so that

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial v_i}\cdot\left[D_{ij}\left(\frac{\partial f}{\partial v_j} + \beta m f v_j\right)\right]. \tag{287}$$

Combining Equations (285) and (286), we obtain the generalized Green–Kubo formula

$$F_i^{\rm pol} = -\beta m v_j \int_0^{+\infty}\langle \mathcal{F}_i(0)\mathcal{F}_j(t)\rangle\, dt \tag{288}$$

expressing the fluctuation-dissipation theorem.

If $D_{ij} = D_{ij}(\mathbf{v})$, the Langevin equations associated with the Fokker–Planck Equation (287) are [112]

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \qquad \frac{d\mathbf{v}}{dt} = -\nabla\Phi(\mathbf{r}) + \mathbf{F}_{\rm pol} + g_{ij}\frac{\partial g_{kj}}{\partial v_k} + g_{ij}\Gamma_j(t), \tag{289}$$

where $D_{ij} = g_{ik}g_{jk}$ and $\mathbf{\Gamma}(t)$ is a Gaussian white noise satisfying $\langle \Gamma_i(t)\rangle = 0$ and $\langle \Gamma_i(t)\Gamma_j(t')\rangle = 2\delta_{ij}\delta(t-t')$. Using Equation (57) the Langevin equations can also be written as

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \qquad \frac{d\mathbf{v}}{dt} = -\nabla\Phi(\mathbf{r}) + \mathbf{F}_{\rm tot} - g_{kj}\frac{\partial g_{ij}}{\partial v_k} + g_{ij}\Gamma_j(t). \tag{290}$$

As mentioned above, these equations are less intuitive than Equation (283). This shows the physical interest of our formulation.

6. Conclusions

We have reviewed the SDD equation for spatially homogeneous and inhomogeneous systems with long-range interactions submitted to an externally imposed stochastic force. The SDD equation describes the evolution of a near-equilibrium system caused by scattering of orbits by fluctuations in the potential due to the external perturbation. The evolution timescale is intermediate between the violent collisionless relaxation timescale and the collisional relaxation timescale. We have derived the SDD equation either from the Klimontovich approach or from the Fokker–Planck approach. We have considered the case of an external stochastic force with an arbitrary correlation function. When the external noise is produced by a random distribution of N field particles, we have made the connection between the SDD equation and the Lenard–Balescu equation (see also [136]).

The SDD equation may have several applications in astrophysics and cosmology. Some examples have been given in the Introduction and in Section 4 in relation to the secular evolution of stellar discs. In this context, Fouvry et al. [64,104–106] have computed the initial flux $(\partial f/\partial t)_0$ due to the external forcing in thin and thick discs and compared the theoretical prediction with direct numerical simulations (similar calculations have been performed with the Lenard–Balescu equation taking into account internal Poisson shot noise [62–64]). Here, we consider the case of dark matter halos. Classical N-body cosmological simulations suggest that dark matter halos have a universal density profile, the so-called Navarro–Frenk–White (NFW) profile [166]. This profile results from a process of collisionless violent relaxation.[49] It decreases as r^{-3} at large distances and presents a r^{-1} cusp for $r \to 0$. This cuspy density profile is in conflict with observations [167] which reveal that dark matter halos have a flat core, not a cusp. This core-cusp problem could be solved by baryonic feedback that can transform cusps into cores [168–170] or by quantum mechanics which prevents infinite densities if dark matter is made of fermions or bosons with a very small mass (see, e.g., [135] and references therein). In addition to baryonic feedback and quantum effects, the SDD equation could also be used in the context of galactic halos to predict their density profile resulting from a (slow) secular evolution due to external perturbations (e.g., fly-by encounters, satellite mergers, black holes, baryons, etc.). The evolution of haloes in noisy environments was first considered by Weinberg [171] who predicted a universal density profile with a flattened core. He mentioned that results of large-scale structure simulations and observations of substructures in the Milky Way halo provide evidence that galaxies are not in equilibrium and that dynamical evolution does not stop after the galaxy formation. A wide variety of instabilities and interactions may drive a galaxy away from equilibrium in the 10 Gyrs subsequent to formation. Weinberg [171] emphasized that galaxies often have very weakly damped modes and these result in large amplification when excited. If the noise excites the weakly damped mode, he showed that the repetitive stochastic response of the galaxy drives the equilibrium toward some universal profile independent of its initial condition and from the detailed characteristics of the noise. The halo profile obtained by Weinberg [171] differs from the NFW profile because it is due to a repetitive excitation from equilibrium and subsequent (secular) relaxation to a new equilibrium while the cosmological mechanism leading to the NFW profile is rapid and violent. For technical reasons Weinberg [171] did not consider cuspy density profiles. However, he showed that the noise results in flattening the profile of the dark matter distribution. This suggests that external perturbations can transform cusps into cores. The SDD equation reviewed in the present paper could be used to study this possibility. It may also be interesting to derive hydrodynamic equations from the kinetic theory and investigate their possible applications in cosmology on a long timescale (see, e.g., [172]).

The SDD equation also occurs in other systems with long-range interactions such as the HMF model [7,23,102,103]. When the dynamics includes a form of dissipation, the system may relax towards a non-Boltzmannian distribution function arising from the competition between the nonlinear diffusion and the friction, or display a complicated (e.g., oscillatory) dynamics. On a mesoscopic scale, these systems may present out-of-equilibrium phase transitions. These interesting results may be worth investigating further. This could be a topic for future research. Similar results are expected to hold for 2D point vortices and for geophysical flows submitted to an external stochastic perturbation (see [88] and references therein).

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Notes

1 Prigogine and Balescu [14] previously derived a kinetic equation without collective effects being unaware of the earlier work of Landau [9]. Their equation (which corresponds to the Lenard–Balescu equation with the dielectric function $\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})$ replaced by 1) involves the Fourier transform of the potential of interaction $\hat{u}(k)$ and exhibits a condition of resonance $\delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')]$. If we perform the integral over the wavenumber $\mathbf{k}$ (see, e.g., Appendix C of [8]) we recover the original form of the Landau equation [9].

2 Note that the kinetic approach developed by Chandrasekhar [31–43] and Rosenbluth et al. [44] allowed them to take strong collisions into account (unlike Landau [9]) so that no divergence occurs at small scales in their Fokker–Planck equation. In particular, the gravitational Landau length appears naturally in their formalism.

3 Self-gravitating systems are never spatially homogeneous even in theory. Indeed, an infinite homogeneous distribution of stars is not a steady state of the Vlasov–Poisson equations (except in an expanding background). Even if we advocate the "Jeans swindle" [46], or take into account the expansion of the Universe, we find that an infinite homogeneous distribution of stars is linearly unstable to perturbations whose wavelengths are above the Jeans length λ_J. Therefore, Chandrasekhar [31–43] and Rosenbluth et al. [44] actually considered spatially inhomogeneous stellar systems with a size of the order of the Jeans length λ_J and made a "local approximation" so as to treat them as if they were spatially homogeneous (this could be referred to as "Chandrasekhar's swindle"). This local approximation gives relatively good results in the case of globular clusters when an orbit average approximation is implemented, leading to the orbit-averaged Fokker–Planck equation [47,48]. In his works, Chandrasekhar [31–43] argued that the Coulombian factor $\ln \Lambda$ should be cut-off at the interparticle distance but it was later recognized [49] that this claim is incorrect and that the Jeans length (i.e., the system's size) provides a better prescription for the large scale cut-off.

4 This rough argument may sometimes be misleading (see below).

5 It is interesting to point out the analogies between the effect of a solid rotation on a stellar system and the effect of a constant magnetic field in plasma physics. The small oscillations of a fully ionized plasma in a constant magnetic field were first studied by Bernstein [51] and the kinetic equation of a plasma in a constant magnetic field was first derived by Rostoker [52]. Of course, there are also differences between stellar systems and plasmas since gravity is attractive whereas electricity is repulsive.

6 Heyvaerts [58] derived the inhomogeneous Lenard–Balescu equation from the BBGKY hierarchy and Chavanis [59] derived it from the Klimontovich formalism. The inhomogeneous Lenard–Balescu equation can also be directly derived from the Fokker–Planck equation as shown in [59–61]. The inhomogeneous Landau and Lenard–Balescu equations are structurally very different from the orbit-averaged-Fokker–Planck equation [47,48] used to describe globular clusters.

7 The η-formalism which treats the fluctuating force acting on a particle as a stochastic force is closely related to the Fokker–Planck approach developed in [59,60]. However, it relies on the rather technical Novikov theorem [109] while the Fokker–Planck approach is based on the more physical Kramers–Moyal [110,111] expansion (see, e.g., Ref. [112]).

8 We generically call it the "collision" term although it may have a more general meaning due to the contribution of the external perturbation. Actually, in this paper, we shall be exclusively interested by the contribution of the external perturbation. A more proper name could be the "correlational" term.

9 For the simplicity of the presentation, we have assumed that the external potential Φ_e is of zero mean. If there is an external mean component $\langle \Phi_e \rangle$, it can be included in Φ by making the substitution $\Phi \to \Phi + \langle \Phi_e \rangle$. In other words, Φ represents the total mean potential including the mean potential produced by the system of particles and by the external perturbation. In Section 4, this amounts to replacing the pulsation $\mathbf{\Omega}$ of a particle in an inhomogeneous system by the total pulsation $\mathbf{\Omega} + \mathbf{\Omega}_e$.

10 In the following, we assume that the system remains spatially homogeneous during its whole evolution. This may not always be the case. Even if we start from a spatially homogeneous distribution $f(\mathbf{v})$, the collision term will change it. The system may become dynamically (Vlasov) unstable and undergo a dynamical phase transition from an homogeneous phase to an inhomogeneous phase (see, e.g., Ref. [113]).

11 In our approach, the initial condition is rejected to the infinite past but we have to add a small imaginary term $i0^+$ in the pulsation ω of the Fourier transform to ensure the vanishing of the fluctuations and the convergence of the integrals for $t \to -\infty$. In a sense, our procedure amounts to using a Laplace transform in time but neglecting the initial condition.

12 Equation (21) is the generalization, for an arbitrary binary potential of interaction, of the dielectric function introduced in plasma physics [114]. We generically call it the "dielectric" function although it has a more general meaning.

13 It is implicit throughout the calculations that a small imaginary term $i0^+$ should be added to the pulsation when necessary. For convenience, we do not write this term at each step of the calculations and refer to the "Landau prescription" when necessary. The Landau prescription $\omega \to \omega + i0^+$ ensures the vanishing of the Fourier modes $\delta f_\omega(t) \propto e^{-i\omega t}$ for $t \to -\infty$ and regularizes the resonant denominators (see note 11).

14 Since we have assumed that the system is linearly Vlasov stable, we have $\epsilon(\mathbf{k}, \omega) \neq 0$ for any real pulsation ω (the solution of the dispersion relation $\epsilon(\mathbf{k}, \omega) = 0$ is of the form $\omega = \omega_r + i\omega_i$ with $\omega_i < 0$) so that Equation (26) is well-defined.

15 We call it "effective" because it is not due to collisions (finite N effects) between particles but to the external perturbation (see note 8).

16 Here, and in the following, we use the Einstein summation convention on repeated indices.

17 We have assumed that the system is stable so $\epsilon(\mathbf{k}, \omega)$ does not vanish for any real $\omega = \mathbf{k} \cdot \mathbf{v}$ (see note 14).

18 This mesoscopic level of description characterized by $\overline{f}$ (coarse-grained dynamics) is intermediate between the discrete level of description characterized by f_d (exact dynamics) and the smooth—locally averaged—level of description characterized by f (mean dynamics). It takes into account fluctuations with respect to the mean dynamics f but replaces the sum of δ-functions in f_d by a continuous (yet stochastic) distribution function $\overline{f}$. If we average $\overline{f}$ over the noise, we recover f.

19 A general approach to obtain the noise term and the corresponding action is to use the theory of large deviations [127].

20 The idea of this expansion was introduced by Einstein [130] who used it to derive the diffusion equation, thereby improving the heuristic approach of Fick [131]. It was then generalized by Fokker [17–19] and Planck [20] to derive the Fokker–Planck equation (the same results were obtained independently by Klein [132] a little later). Landau [9] used a similar expansion to derive the Landau equation from the Boltzmann equation. Actually, this type of Taylor expansion was introduced by Lord Rayleigh [133] long before all the classic papers on Brownian theory. He considered the dynamics of massive particles bombarded by numerous small projectiles. This paper can be seen as a precursor of the theory of Brownian motion that is usually considered to start with the seminal work of Einstein [130] (see Refs. [129,134,135] for some additional comments about the paper of Lord Rayleigh and the history of Brownian theory).

21 This formula is established by a direct calculation in Section 3.4 of [23].

22 This protocol is usually called the quasilinear approximation, i.e., we assume that the test particle follows an unperturbed trajectory when calculating the dynamics of the fluctuations.

23 The auto-correlation of the gravitational force in $d = 3$ in an infinite homogeneous medium decreases precisely as s^{-1} [41,49]. This leads to logarithmic corrections as discussed by Lee [137].

24 This formula can also be obtained by using the identity

$$D_{ij} = \frac{1}{2}\frac{d}{dt}\langle \Delta v_i \Delta v_j \rangle = \frac{1}{2}\langle \dot{v}_i \Delta v_j + \dot{v}_j \Delta v_i \rangle = \frac{1}{2}\left\langle \int_0^t dt' \left[F_i(\mathbf{r}, 0) F_j(\mathbf{r}(t'), t') + F_j(\mathbf{r}, 0) F_i(\mathbf{r}(t'), t') \right] \right\rangle, \tag{291}$$

where $\mathbf{F} = -\nabla \delta \Phi_{\rm tot}$ is the total fluctuating force (by unit of mass) experienced by the test particle. Making a linear trajectory approximation and taking the limit $t \to +\infty$, we obtain

$$D_{ij} = \int_0^{+\infty} \left\langle F_i(\mathbf{r}, 0) F_j(\mathbf{r} + \mathbf{v}s, s) \right\rangle ds, \tag{292}$$

in agreement with Equation (68).

25 A more precise criterion for neglecting the contribution of the test particles on the diffusion is $m_a f_a \ll \sum_b m_b f_b$.

26 A more precise criterion for neglecting the friction by polarization is $m_a f_a \partial f / \partial \mathbf{v} \ll \sum_b m_b f_b \partial f_a / \partial \mathbf{v}$.

27 In this sense Equation (102) with Equation (103) is more general than the SDD Equation (36) with Equation (38) because it takes into account the polarizability of the external medium.

28 In these studies, collective effects were neglected for the reasons given in the introduction.

29 In the context of stellar discs, the stochastic perturbation may be due to weak mergers, transient spiral structure, orbiting blobs, giant molecular clouds, massive sub-halos around the disc, spiral arms, the presence of a bar, dwarf or large satellites, a halo of super-massive black holes, gas accretion, orbiting dwarf galaxies, debris streams, dark clusters, dwarf galaxy mergers, disrupting dwarfs, fly-by encounters, etc. (see [94,99,101] and references therein).

30 We assume that the background potential of the system is stationary and integrable so that we can always remap the usual phase space coordinates $(\mathbf{r}, \mathbf{v})$ to the angle-action coordinates $(\mathbf{w}, \mathbf{J})$. This is consistent with the Bogoliubov ansatz discussed below. We also assume that the system remains Vlasov stable during the whole evolution. This may not always be the case. Even if we start from a Vlasov stable distribution function $f_0(\mathbf{J})$, the "collision" term (r.h.s. in Equation (132)) will change it and induce a temporal evolution of $f(\mathbf{J}, t)$. The system may become dynamically (Vlasov) unstable and undergo a dynamical phase transition

from one state to the other. We assume here that this transition does not take place or we consider a period of time preceding this transition.

31 For convenience, we shall often write f for $f(\mathbf{J})$ and $\mathbf{\Omega}$ for $\mathbf{\Omega}(\mathbf{J})$.

32 In order to derive the inhomogeneous Lenard–Balescu equation describing the collisional evolution of the system due to finite N effects, we have to solve an initial value problem and use Laplace transforms in time as explained in Section 2 of [59]. This involves the initial perturbed distribution function $\delta\hat{f}(\mathbf{k},\mathbf{J},0)$ which accounts for the granularities of the system.

33 We consider an attractive self-interaction, such as gravity, hence the sign $-$ in the second term of Equation (141).

34 In matrix form, we have $\hat{A} = (1-\epsilon)(\hat{A}+\hat{A}_e)$ yielding $\epsilon\hat{A} = (1-\epsilon)\hat{A}_e$, then $\hat{A} = \epsilon^{-1}(1-\epsilon)\hat{A}_e = (\epsilon^{-1}-1)\hat{A}_e$, and finally $\hat{A}_{\rm tot} = \hat{A}+\hat{A}_e = \epsilon^{-1}\hat{A}_e$. If we neglect collective effects, we have $\hat{A}_{\rm tot} = \hat{A}_e$ corresponding to $\epsilon = 1$ and $\hat{A} \simeq (1-\epsilon)\hat{A}_e \simeq 0$.

35 We have assumed that the system is stable so the dielectric tensor is invertible on the real axis.

36 When collective effects are neglected, it is not necessary to introduce a biorthonormal basis.

37 This formula is established by a direct calculation in Section 3.4 of [59].

38 This formula can also be obtained by using the identity

$$D_{ij} = \frac{1}{2}\frac{d}{dt}\langle \Delta J_i \Delta J_j\rangle = \frac{1}{2}\langle \dot{J}_i\Delta J_j + \dot{J}_j \Delta J_i\rangle = \frac{1}{2}\left\langle \int_0^t dt' \left[\mathcal{F}_i(\mathbf{w},\mathbf{J},0)\mathcal{F}_j(\mathbf{w}(t'),\mathbf{J}(t'),t') + \mathcal{F}_j(\mathbf{w},\mathbf{J},0)\mathcal{F}_i(\mathbf{w}(t'),\mathbf{J}(t'),t')\right]\right\rangle, \quad (293)$$

where $\mathcal{F} = -\partial\delta\Phi_{\rm tot}/\partial\mathbf{w}$ is the total fluctuating force (by unit of mass) experienced by the test particle. Making a linear trajectory approximation in angle-action space and taking the limit $t \to +\infty$, we obtain

$$D_{ij} = \int_0^{+\infty} \left\langle \mathcal{F}_i(\mathbf{w},\mathbf{J},0)\mathcal{F}_j(\mathbf{w}+\mathbf{\Omega}(\mathbf{J})s,s)\right\rangle ds, \quad (294)$$

in agreement with Equation (205).

39 The Fourier transform of the dressed potential of interaction in angle-action variables is written as $-1/D_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega)$ in Refs. [58,59]. The Fourier transform of the bare potential of interaction in angle-action variables is recovered by taking $\epsilon = 1$ yielding $A_{\mathbf{k},\mathbf{k}'}(\mathbf{J},\mathbf{J}') = -\sum_\alpha \hat{\Phi}_\alpha(\mathbf{k},\mathbf{J})\hat{\Phi}^*_\alpha(\mathbf{k}',\mathbf{J}')$.

40 See the difference with Section 2.1 of [147] where we derive the collision term due to finite N effects by considering an initial value problem.

41 This equation is exact for noninteracting systems forced by an external perturbation.

42 Note that we depart from the standard Langevin [151] approach where the equations of motion are written as

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = -\xi\mathbf{v}_i - \nabla\Phi(\mathbf{r}_i) + \mathbf{F}_i(t), \quad (295)$$

where $\mathbf{F}_i(t)$ is a random force (per unit of mass) acting on particle i [34,43,112]. The difference of formalism will become more apparent in the following sections.

43 Actually, the friction force arises precisely from discreteness effects. The friction force is introduced here in a purely heuristic manner as in the original works on Brownian motion [34]. Chandrasekhar [37–39,43] also introduced the friction force in an ad hoc manner in his early works on stellar dynamics based on the analogy with Brownian motion. We refer to [8,136,139,140,144–147,152–161], in addition to the references given in the introduction, for a self-consistent treatment of discreteness effects and a direct calculation of the friction force.

44 This equation (without the advection term) was introduced by Lord Rayleigh [133] long before all the classic papers on Brownian theory (see additional comments in Refs. [129,134,135]). This equation was further studied by Uhlenbeck and Ornstein [138].

45 This argument was also advocated by Klein [132], Uhlenbeck and Ornstein [138], and Kramers [110]. Actually, it was first given by Lord Rayleigh [133] long before all the classic papers on Brownian theory, including the seminal paper of Einstein [130] (see Refs. [129,134,135] for additional comments on this point of history).

46 For simplicity, we omit the index labeling the particles.

47 Following Uhlenbeck and Ornstein [138], the diffusion coefficient defined by Equation (54) can be obtained from the Langevin equations by writing the autocorrelation function of the noise term as $\langle \mathbf{F}(t)\cdot\mathbf{F}(t')\rangle = K\delta(t-t')$ where $\delta(x)$ is a function with a very sharp maximum at $x=0$ (like the δ-function). Using the relation (see Equation (67))

$$D = \frac{\langle(\Delta\mathbf{v})^2\rangle}{6\Delta t} = \frac{1}{6\Delta t}\int_0^{\Delta t} dt' \int_0^{\Delta t} dt'' \langle \mathbf{F}(t')\cdot\mathbf{F}(t'')\rangle = \frac{1}{3}\int_0^{+\infty}\langle\mathbf{F}(0)\cdot\mathbf{F}(t)\rangle\, dt, \quad (296)$$

one finds that the constant K is given by $K = 6D$ so that $\langle \mathbf{F}(t)\cdot\mathbf{F}(t')\rangle = 6D\delta(t-t')$. This justifies the expression of the stochastic force in Equation (275). Following Chandrasekhar [34,43], the diffusion coefficient can also be obtained from the Langevin equations by writing $\Delta\mathbf{v} = \int_0^{\Delta t} \mathbf{F}(t')\, dt' = \sum_{i=1}^{\mathcal{N}} \delta\mathbf{v}_i$ with $\mathcal{N} \gg 1$. Therefore, the net increment in velocity $\Delta\mathbf{v}$ is a sum of $\mathcal{N}$ random variables $\delta\mathbf{v}_i$. We then have $\langle(\Delta\mathbf{v})^2\rangle = \mathcal{N}\langle(\delta\mathbf{v})^2\rangle$, yielding

$$D = \frac{\langle(\Delta\mathbf{v})^2\rangle}{6\Delta t} = \frac{1}{6} n\langle(\delta\mathbf{v})^2\rangle, \tag{297}$$

where n is the number of collisions per unit time between a Brownian particle and the molecules of the surrounding fluid (about 10^{21} collisions per second) and $\langle(\delta\mathbf{v})^2\rangle$ is the mean square increment in velocity of a particle per collision. The Brownian particle experiences a great number of weak deflections. One can show that the probability of occurrence of different net increments of velocity $\Delta\mathbf{v}$ during an interval Δt is given by the Gaussian distribution [34,43]

$$W(\Delta\mathbf{v}) = \frac{1}{(4\pi D\Delta t)^{3/2}} e^{-\frac{(\Delta\mathbf{v}+\xi\mathbf{v}\Delta t+\nabla\Phi\Delta t)^2}{4D\Delta t}}. \tag{298}$$

This is a consequence of the central limit theorem [164] according to which the distribution of the sum of $\mathcal{N} \gg 1$ random variables of finite variance is a Gaussian.

48 Note that Einstein [130] was working in physical space (instead of phase space) by considering a strong friction limit $\xi \to +\infty$. Therefore, the original Einstein relation reads $D = \mu k_B T$ where $\mu = 1/(\xi m)$ is the mobility.

49 The NFW profile is, however, different from the prediction of Lynden-Bell [5] based on statistical mechanics. This discrepancy may possibly be due to incomplete relaxation.

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