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Yukawa–Casimir Wormholes in $f(Q)$ Gravity

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Abstract: Casimir energy is always suggested as a possible source to create a traversable wormhole. It is also used to demonstrate the existence of negative energy, which can be created in a lab. To generalize this idea, Yukawa modification of a Casimir source has been considered in Remo Garattini (Eur. Phys. J. C 81 no.9, 824, 2021). In this work, we explore the Yukawa–Casimir wormholes in symmetric teleparallel gravity. We have taken four different forms of $f(Q)$ to obtain wormhole solutions powered by the original Casimir energy source and Yukawa modification of the Casimir energy source. In power law form $f(Q) = \alpha Q^2 + \beta$ and quadratic form $f(Q) = \alpha Q^2 + \beta Q + \gamma$, where α, β, γ are constants and Q is non-metricity scalar, we analyze that wormhole throat is filled with non-exotic matter. We find self-sustained traversable wormholes in the Casimir source where null energy conditions are violated in all specific forms of $f(Q)$, while after Yukawa modification, it is observed that violation of null energy conditions is restricted to some regions in the vicinity of the throat.

Keywords: Yukawa–Casimir; wormholes; $f(Q)$ gravity; ECs

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1. Introduction

One of the greatest desires of mankind is to explore and to travel to the distant regions in the universe, which is only possible by faster-than-light travel, and this is where the concept of a wormhole comes into play, which can be considered as a tool, hypothetically, to provide faster than light inter-universe, as well intra-universe travel. Additionally, this the idea of a wormhole is associated with the probability of time-travel. The classical solution to the Einstein field equations define the wormhole, which is free from horizon and singularity. Wormholes are hypothetical structures that are supposed to be tunnel-like in shape and have two openings at each end connected with a throat. Although the wormholes are completely hypothetical up until now and there is no experimental proof of their existence, still the curiosity regarding them is exponentially increasing, as they are assumed to be a potential instrument for time travel and interstellar travel in an affordable time span, where affordable signifies being within the human life time. The lensing and light deflection caused by wormholes are investigated in this regard [1–3].

The idea of static wormholes in the form of Schwarschild solutions, joining two asymptotically flat regions in space-time, was first given by Flamm [4]. The solution of Einstein field equations coincides with the solution of traversable wormholes [5]. The idea of traversable wormholes, also called Einstein Rosen bridges, was first conceived by Einstein and Nathan Rosen [6]. Eventually, the solutions close to the Einstein Rosen bridge connecting two Schwarschild solutions were also achieved [7,8]. Morrice and Thorne worked on the physical requirements to construct such an apparatus and presented the idea of spherically symmetric traversable wormholes, which subsequently proved to be in synchronization with tachyonic massless scalar fields. Later on, Yurtsever, along with them,

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introduced the concept of a time machine, thus giving a possible solution of a traversable wormhole [9–14].

It is evident that the most astounding theories in science are associated with Einstein, and the most revolutionary is the general theory of relativity. With the emergence of the techniques such as EHT, VIRGO, and gravitational lenses, it is experimentally proven to be an eighty percent correct theory of gravity [15–18]. From September 2015 to January 2016, advanced LIGO, for the first time, detected the gravitational waves from the merger of black-holes [19], justifying another conjecture of general relativity that gravitational waves are the result of the distortion in the curvature of space time. The general relativity is inconsistent with the present acceleration of the universe [20]. Within the framework of general relativity, only with the help of extremely fine-tuned cosmological or a typical dark energy source, this phenomenon of the universe can be described [21]. This inconsistency leads to the evolution of extensions of general relativity in the form of modified theories of gravity. The most simple theories among these extensions are the scalar tensor theories that can justify the accelerated expansion of the universe [22,23]. Secondly, nowadays, researchers are more interested in the solution of traversable wormholes free from any kind of singularity and any event horizon, as the Schwarzschild metric shatters close to the event horizon. Hence, some adjustments have to be made in the metric, along with some restrictions on the wormhole throat by implementing Birkho theorem [24], in which case, mass energy becomes less than the radial tension [25–28], leading to the violation of null energy conditions (NEC). This violation, in turn, leads to the presence of exotic matter. To avoid this presence of exotic matter in space-time or wormholes, various modified theories of gravity were evolved, where higher order curvature terms account for the violation of null energy conditions. A thin shell traversable wormhole solution was produced in the framework of $f(R)$ gravity. The coupling of geometry with the matter terms gave rise to the $f(R,T)$ theory of gravity, and several wormhole solutions were studied for several forms of $f(R,T)$ [29–47].

Teleparallel and other modified theories of gravity also played a major role in exploring the solutions of traversable wormholes [48–53]. Teleparallel theory [54,55] has been proven to be one of the potential alternatives of general relativity. Here, the geometry is free from torsion and curvature, while the gravitational interactions of the space-time are portrayed by the non-metricity term Q. The Lagrangian of $f(Q)$ gravity is reformed in explicit function of red-shift, $f(z)$ to justify these teleparallel gravity models by means of various observational tools, such as gamma ray bursts, cosmic microwave background, Baryon acoustic oscillations data, quasars etc [56,57]. Various researchers established different gravitational modification classes implementing different function forms of $f(Q)$ and discussed the wormhole solutions and other cosmographical results [58–67].

As stated earlier, an unfortunate consequence in general relativity is the unavoidable association of wormhole traversability of wormhole with the violation of NEC, which implies the exotic matter treading toward the wormhole throat which has negative energy. As the classical matter justifies the null energy condition, the wormhole solution can be obtained in the framework of a semi-classical or quantum field. In that case Casimir energy density of the Casimir device could be an appropriate source of exotic energy, as until now, Casimir energy was the only artificial source of negative energy [68]. The Casimir energy arises in a vacuum between two closely placed, uncharged, plane parallel metallic plates. An attractive force appears in this event, which was predicted in 1948 and later on confirmed experimentally in the Philips Laboratories [69,70] and by other investigators in recent years [71–73].

A modification to the original Casimir shape function [74] is made by implementing the Yukawa term. The zero tidal force condition (ZTF) was imposed by modifying the original profile with Yukawa-type modifications to procure different solutions to obtain the signals of traversable wormholes and the possibility of obtaining negative energy density more concentrated in the proximity of throat, as they stated in their article [75]. Following [62,76], in the current work, we are interested in the probable solutions of traversable wormholes with ordinary matter devising the Yukawa–Casimir shape function in the backdrop of symmetric teleparallel gravity. In [62,76,77], the authors explored the possible existence of wormhole solutions in a recently proposed symmetric teleparallel gravity, or $f(Q)$ gravity, as the WHs are supported by the exotic matter, and that is an entirely unsolved problem. The key difference between symmetric teleparallel gravity and general relativity (GR) is the role played by the affine connection, $\Gamma^{\alpha}_{\mu\nu}$, rather than the physical manifold. Most remarkably, $f(Q)$ gravity is equivalent to GR in flat space [55]. It is important to keep in mind that, similar to the $f(T)$ gravity, $f(Q)$ gravity also features in second order field equations, while gravitational field equations of $f(R)$ gravity are of the fourth-order [78]. Therefore, $f(Q)$ gravity provides a different geometric description of gravity, which is nevertheless equivalent to GR [76]. However, this work is different from the work [62,76,77] in different ways. Here, we investigate traversable wormhole solutions with the Yukawa–Casimir shape function in the backdrop of $f(Q)$ gravity, which has not been performed previously. We focused on exploring the solution of traversable wormhole space-time threaded by the ordinary matter. We tried to find such a solution where the negative energy required holding the throat open for requisite time can be sourced from some other source of negative energy. We studied four different functional forms of $f(Q)$ gravity. The first case showcases the linear form. The second and third cases analyze Yukawa–Casimir wormhole solutions for power law forms. The fourth case is the inverse power law form of teleparallel gravity.

2. Symmetric Teleparallel Gravity i.e., $f(Q)$-Gravity

The action corresponding to the symmetric teleparallel gravity, taken in this article, is [79]

$$S = \frac{1}{2}\int \left[f(Q) + 2\mathcal{L}_{\mu}\right]\sqrt{-g}d^4x. \tag{1}$$

Here, the function $f(Q)$ regards to the non-metricity term Q, whereas the matter Lagrangian density can be described as $\mathcal{L}_{\mu}$, and the determinant of the metric g_{xy} is given as g. The equation for non-metricity tensor is written as

$$Q_{\lambda xy} = \nabla_{\lambda} g_{xy}. \tag{2}$$

The two independent traces corresponding to non-metricity tensor come out to be

$$Q_{\phi} = Q_{\phi}{}^{x}{}_{x}, \qquad \tilde{Q}_{\phi} = Q^{x}{}_{\phi x}. \tag{3}$$

Jiminez et al. [55] addressed this new geometry as new general relativity. Additionally, they defined the non-metricity conjugate related to the super-potential of this so-called new general relativity as

$$P^{\phi}_{xy} = \frac{1}{4}\left[-Q^{\phi}{}_{xy} + 2Q^{\phi}_{xy} + Q^{\phi}g_{xy} - \tilde{Q}^{\phi}g_{xy} - \delta^{\phi}_{(xQ_y)}\right], \tag{4}$$

which can be snatched by contemplating the following form of non-metricity tensor

$$Q = -Q_{\phi xy}P^{\phi xy}. \tag{5}$$

The features of matter threading the space-time are defined by the energy momentum tensor, which is stated as

$$T_{xy} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_{\mu}\right)}{\delta g^{xy}}. \tag{6}$$

We obtain the motion equations by varying the action (1), with respect to the metric tensor g_{xy}, as

$$\frac{2\nabla_{\eta}}{\sqrt{-g}}\left(\sqrt{-g}f_Q P^{\eta}{}_{xy}\right) + \frac{1}{2}g_{xy}f + f_Q\left(P_{x\eta\zeta}Q_y{}^{\eta\zeta} - 2Q_{\eta\zeta x}P^{\eta}{}_{\zeta y}\right) = -T_{xy}. \tag{7}$$

Here, f_Q represents the total derivative of f, with respect to Q. The variation of Equation (1), regarding the connections, gives the following relation

$$\nabla_x \nabla_y \left(\sqrt{-g} f_Q P^{\eta}{}_{xy}\right) = 0. \tag{8}$$

The field equations assert conserving the energy momentum tensor with the conformity of specified $f(Q)$ gravity. In the present study, we focus on specifying the gravitational field equations commanding the static and spherically symmetric solutions [80] to the wormhole geometry.

3. Wormhole Geometry and Solution of Field Equations in $f(Q)$ Gravity

The usual spherically symmetric and static line element of Morris–Thorne class to specify the wormhole geometry is written as

$$ds^2 = -\exp(2\phi(r))dt^2 + \left(\frac{r - b(r)}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{9}$$

Here, the redshift function of the intervening object with regards to the radial coordinate r is given by $\phi(r)$. By the definition of radial coordinate r as $0 < r_0 \leq r \leq \infty$, its non-monotonic behavior is indicated. One can easily collect that r falls from ∞ and approaches $b(r_0)$, which is the minimum value r_0 and after attaining this minimum value, it again proceeds to infinity. This minimum r_0 is the throat radius. To comply with the traversability of a wormhole, it has to be prevented from any event horizon or the presence of singularity. The redshift function, therefore, is restricted to attain only a non-zero finite value to avoid such occurrence. In our present article, we are examining the Yukawa–Casimir wormhole [75] solutions for which the shape function and redshift functions are fixed and are obtained by introducing Yukawa terms in the original Casimir wormhole [81]. Therefore, the shape function also satisfies all the required constraints, such as the flaring out condition, throat condition, asymptotically flatness condition, etc.

The matter fluid that threads the wormhole throat is considered anisotropic, which has the stress–energy–momentum tensor

$$T_x^y = (\rho + p_t) u_x u^y - p_t \delta_x^y + (p_r - p_t) v_x v^y. \tag{10}$$

Here, v_x and u_x are unitary space-like vectors in radial direction, and four velocities, respectively, ρ, p_r, p_t, represent the energy density and principal pressures, respectively. The trace Q of non-metricity tensor for the line element Equation (9), within $F(Q)$ gravity is given by

$$Q = -\frac{2}{r^3}[r - b(r)][2.r.\phi'(r) + 1]. \tag{11}$$

We solve Equations (7), (9), and (10) to obtain the expressions for energy density (ρ), radial pressure (p_r), and tangential pressure (p_t), in terms of radial coordinate r, as

$$\rho = \left[\frac{1}{r^3}\left(2r\left(r - b(r)\phi'(r) - rb'(r) - b(r) + r\right)\right)\right] f_Q - \frac{2}{r^2}(b(r) - r) f_Q' + \frac{f}{2}, \tag{12}$$

$$p_r = -\frac{1}{r^3}\left[2(r - b(r))(2r\phi'(r) + 1) - 1\right] f_Q - \frac{f}{2}, \tag{13}$$

$$\begin{aligned} p_t &= -\left[\frac{1}{r^3}\left(\left[r^2\phi''(r) + r\phi'(r)(r\phi'(r) + 3) + 1\right][r - b(r)]\right.\right. \\ &- \left.\left.\frac{1}{2}[r\phi'(r) + 1][rb'(r) - b(r)]\right)\right] f_Q - \frac{1}{r^2}[r\phi'(r) + 1][r - b(r)] f_Q' - \frac{f}{2}. \end{aligned} \tag{14}$$

With appropriate choices of the shape function $b(r)$ and the redshift function $\phi(r)$, one can analyze the plausible solutions for wormhole geometry in the backdrop of modified gravity. In this paper, we take the specified Yukawa–Casimir shape function and redshift function to acknowledge the negative energy in the wormhole geometry.

The Energy Conditions

The exotic properties of the matter field are the necessary requirements of wormhole space-time. The energy conditions are the tools that play the prime role in defining these exotic properties. These energy conditions are particular constraints made on the matter from the stress–energy–momentum tensor, which demonstrates the basic features of various forms of matter. The idea of energy conditions and their various implications are summarized in [82]. These constraints are extracted from the Raychaudhuri equations that narrate the temporal evolution of expansion scalar θ for the congruences of the time-like vectors u^l and k^l, which, in turn, describes the null geodesics as

$$\frac{d\theta}{d\tau} - \omega_{lm}\omega^{lm} + \sigma_{lm}\sigma^{lm} + \frac{1}{3}\theta^2 + R_{lm}u^l u^m = 0, \tag{15}$$

$$\frac{d\theta}{d\tau} - \omega_{lm}\omega^{lm} + \sigma_{lm}\sigma^{lm} + \frac{1}{2}\theta^2 + R_{lm}k^l k^m = 0. \tag{16}$$

These conditions are used to obtain an idea about the basic features of fluid threading the wormhole throat. The energy conditions are normally examined with regard to principle pressures and energy density. The justification or infringement of energy conditions are responsible for the occurrence, existence, and stability of the traversable wormholes. These conditions coincide with the Rai Chaudhary conditions for $\theta < 0$, i.e., for attractive geometry or, in other words, for positive energy.

$$R_{lm}u^l u^m \geq 0, \tag{17}$$

$$R_{lm}k^l k^m \geq 0. \tag{18}$$

Here, the energy conditions are discussed for the anisotropic matter fluid. The energy conditions can be expressed in terms of energy density ρ, radial pressure p_r, and tangential pressure p_t as $\forall i$, $\rho(r) + p_i \geq 0$. The tensor form of NEC (null energy condition) is given as $T_{lm}k^l k^m \geq 0$. The NEC implies the non-negativity of the principle pressures. The traversability of wormholes is unfortunately consequent to the violation of the NEC in GR. The weak energy condition or WEC assures that the energy density of a time-like vector can not be negative. With regard to principle pressures, WEC is given as $\rho(r) \geq 0$ and $\forall i,\ \rho(r) + p_i \geq 0$, while the tensor form of WEC is $T_{lm}k^l k^m \geq 0$. $(T_{lm} - \frac{T}{2}g_{lm})k^l k^m \geq 0$ is the tensor form of strong energy condition, whereas in principle pressures SEC can be written as $T = -\rho(r) + \sum_j p_j$ and $\forall j,\ \rho(r) + p_j \geq 0,\ \rho(r) + \sum_j p_j \geq 0$, whose violation is mandatory for the inflation of the universe. The dominant energy condition (DEC) is the one that puts a restriction on the transmission of energy and limits its rate to the speed of light. Both the tensor and principle pressure forms of DEC are expressed as $T_{lm}k^l k^m \geq 0$ and $\rho(r) \geq 0$ and $\forall i,\ \rho \pm p_i \geq 0$, respectively.

4. The Yukawa–Casimir Wormhole Model

In this article, we examine the Yukawa–Casimir wormhole solutions in the backdrop of four different form functions of $f(Q)$ representing the symmetric teleparallel gravity. For the Yukawa–Casimir wormhole model, it is assumed that the exotic matter is substituted by the Casimir energy density. Thus far, Casimir energy is the only known artificial negative energy source, which has the energy density $\rho_0 = -\frac{hc\pi^2}{720d^4}$.

The stress–energy–tensor (SET) for Yukawa–Casimir energy is

$$T_{\mu\nu} = \frac{1}{r_0 \kappa}\left[diag\left(-\frac{1}{3r_0} - \mu, -\frac{1}{r_0}, \frac{1}{2}\left(\frac{4}{3r_0} + \mu\right), \frac{1}{2}\left(\frac{4}{3r_0} + \mu\right)\right)\right], \tag{19}$$

and stress–energy–tensor (SET) in the limit $\mu \to 0$, i.e., for Casimir energy is

$$T_{\mu\nu} = \frac{1}{3r_0^2 \kappa}[dia(-1, -3, 2, 2)] = \frac{hc\pi^2}{720d^4}[dia(-1, -3, 2, 2)], \tag{20}$$

here, d represents the plate separation, which is traceless and whose divergence is also null. By establishing the connection between above SET and the space-time metric, the Casimir shape function $b(r)$ and redshift function $\phi(r)$ are obtained as [74]

$$b(r) = \frac{2r_0}{3} + \frac{{r_0}^2}{3r}, \quad \phi(r) = \ln\left(\frac{3r}{3r + r_0}\right). \tag{21}$$

The Casimir wormhole described by the above shape function does not satisfy the zero tidal force (ZTF) condition. In [75], the authors deformed the Casimir wormhole shape function by inducing the Yukawa profile of the form $\exp(-\mu(r - r_0))$, which also satisfy the ZTF condition. They found a possibility of a new family of solutions having vanishing redshift function. The Yukawa–Casimire shape function is, thus, obtained as

$$b(r) = \frac{2r_0}{3} + \frac{{r_0}^2}{3r}\exp(-\mu(r - r_0)), \tag{22}$$

here, μ is a positive mass scale and the original Casimir profile is achieved again by putting $\mu = 0$.

4.1. Linear Form: $f(Q) = \alpha Q$

The linear functional form $f(Q) = \alpha Q$ (where α is constant) is considered here to analyze the Yukava-Casimir wormhole solution. This particular form helps in comparing the wormhole solutions with the usual wormholes and identifies with the symmetric teleparallel equivalent of general relativity. The energy density ρ and energy conditions regarding the Casimir energy source, i.e., for $\mu = 0$, and Yukawa–Casimir energy source, i.e., for $\mu = 1, 2, 3$, are plotted against the radial coordinate r and throat radius r_0 in Figure 1–3. To investigate the Yukawa–Casimir wormhole geometry, we analyze the implications of these figures. From Figure 1a, it is evident that energy density ρ is non-negative for all values $\mu = 0, 1, 2, 3$ in the region $r \geq 1.6$. The radial and tangential NECs are mapped in Figures 1b and 2a, which shows that $\rho + p_r \geq 0$ for $r \in (1.6, \infty)$, while $\rho + p_l < 0$ for $r \geq r_0$ for $\mu = 0, 1, 2$, but only for $\mu = 3$; the tangential NEC is also validated for radial parameter $r \in (3.5, \infty)$. Therefore it is gathered that both the null energy conditions are satisfied only for the particular value $\mu = 3$ for the region $r > 3.5$, which indicates that Yukawa modification in the Casimir source is useful to minimize the exotic matter in wormhole formation. For this particular case, a solution with ordinary matter threading the throat is obtained, and it can be the Casimir energy that is providing the required exotic or negative energy to sustain the wormhole. For other cases, even in the case of $\mu = 0$, when the system reduces to the Casimir wormhole, the NECs are violated, and this violation indicates the existence of exotic matter near the throat, which supports the wormhole geometry. The SEC and DECs are also plotted in Figures 2b and 3, which are also violated.

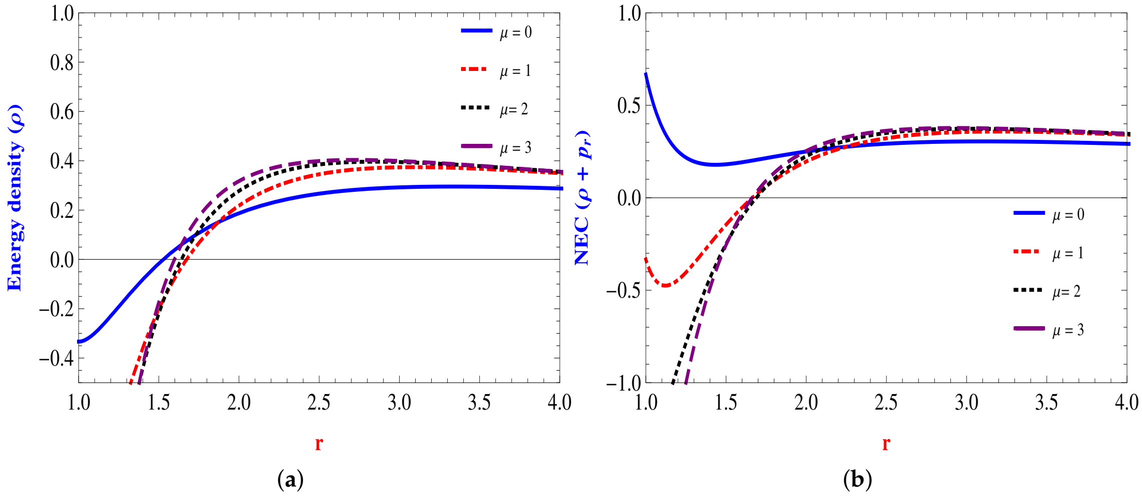


Figure 1. Plots: (**a**) Energy density (ρ) and (**b**) Radial NEC ($\rho + p_r$) with throat radius $r_0 = 1$ and $\alpha = 1$ in $f(Q) = \alpha Q$ gravity.

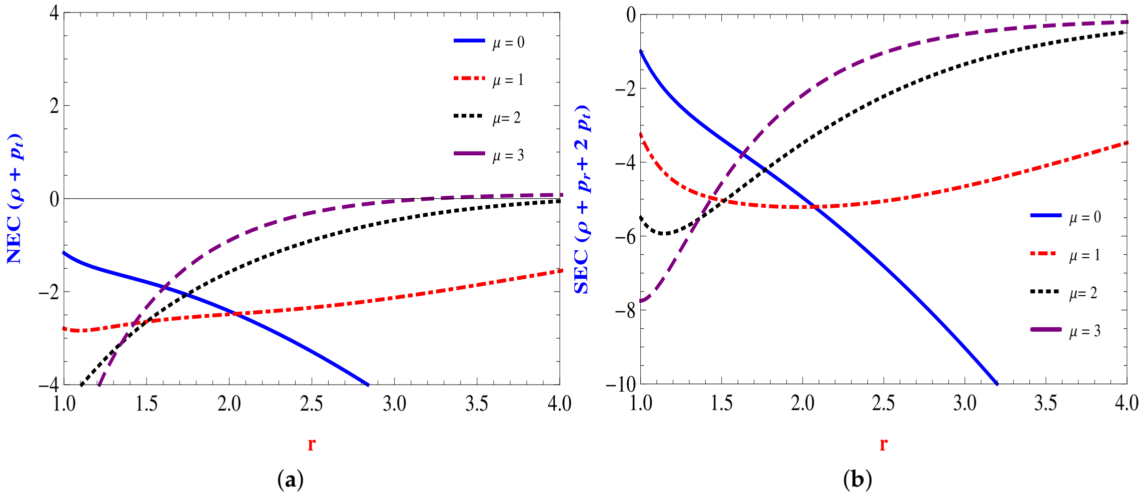


Figure 2. Plots: (**a**) Tangential NEC ($\rho + p_t$) and (**b**) SEC ($\rho + p_r + 2p_t$) with throat radius $r_0 = 1$ and $\alpha = 1$ in $f(Q) = \alpha Q$ gravity.

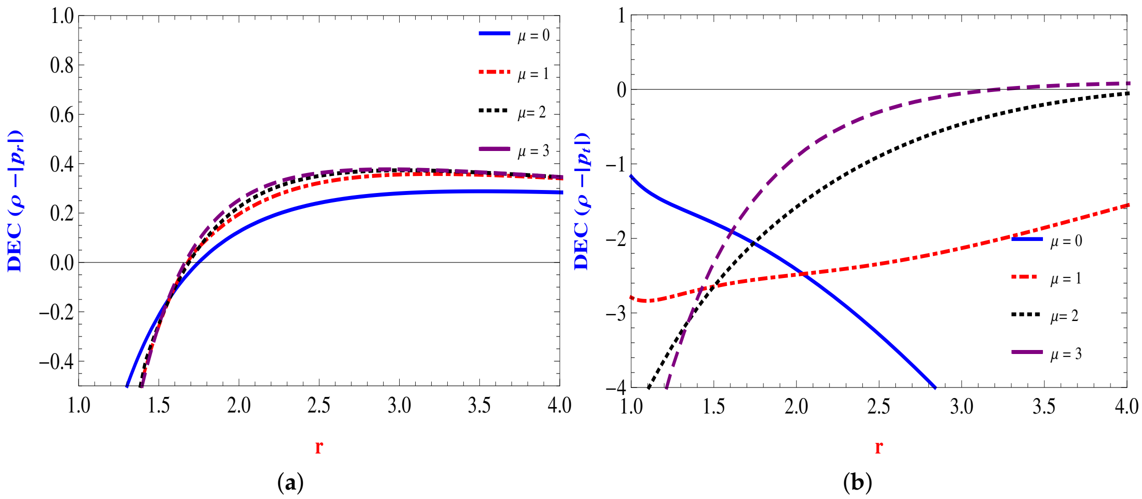
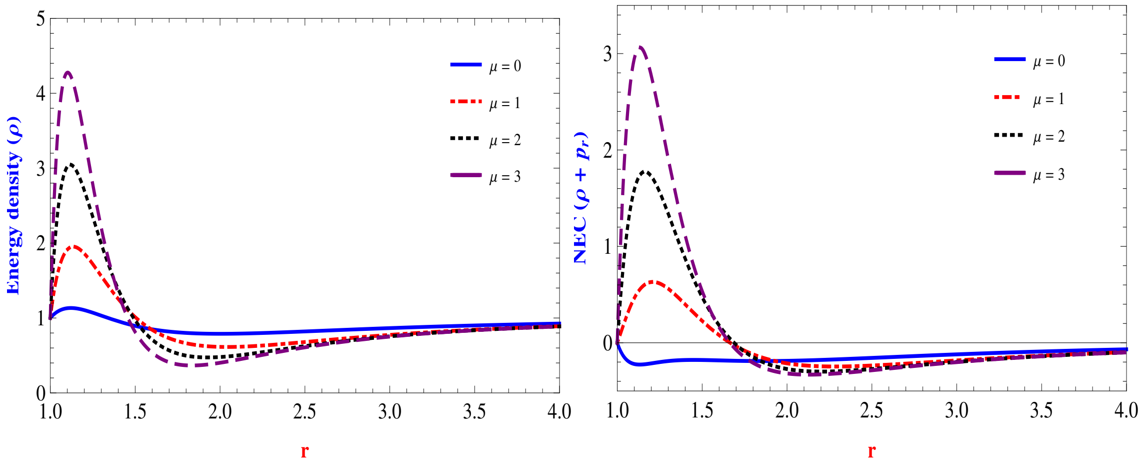


Figure 3. Plots: (**a**) Radial DEC ($\rho - |p_r|$) and (**b**) Tangential DEC ($\rho - |p_t|$) with throat radius $r_0 = 1$ and $\alpha = 1$ in $f(Q) = \alpha Q$ gravity.

4.2. *Power Law Form:* $f(Q) = \alpha Q^2 + \beta$

As we saw in the above linear case that, only for a particular case, the traversable wormhole geometry filled with ordinary matter is found, we further proceed to investigate the Yukawa–Casimir wormhole solution with ordinary matter in the framework of power law form $f(Q) = \alpha Q^2 + \beta$, where α and β are constants. The radiation and CDM dominated background has already been acknowledged by this power law form. The energy density ρ and the energy conditions for this case are mapped again for $\mu = 0, 1, 2, 3$. The energy density for this case, as depicted in Figure 4a, is positive for all four values of μ and $\forall r \geq r_0$. Here, we are concerned about the solutions that are supported by the ordinary matter, which is achieved by the non-violation of NECs. From Figure 4b, we found the radial NEC is satisfied for the Yukawa–Casimir energy source at the throat and in $r \in (1, 1.65)$. It is interesting to see that radial NEC is violated in the case of Casimir energy for $\forall r \geq r_0$. From Figure 5a, one can clearly see that the tangential NEC is satisfied throughout the region for every value of μ, including the case of the Casimir wormhole, where $\mu = 0$, which signifies that, except for the case of $\mu = 0$, the wormhole solutions with ordinary matter near the throat are obtained. The SEC is plotted in Figure 5b, which is violated at the throat in each case. The DECs are also shown in Figure 6, which are also satisfied at the throat for $\mu = 1, 2, 3$ and radial DEC is violated for $\mu = 0$.

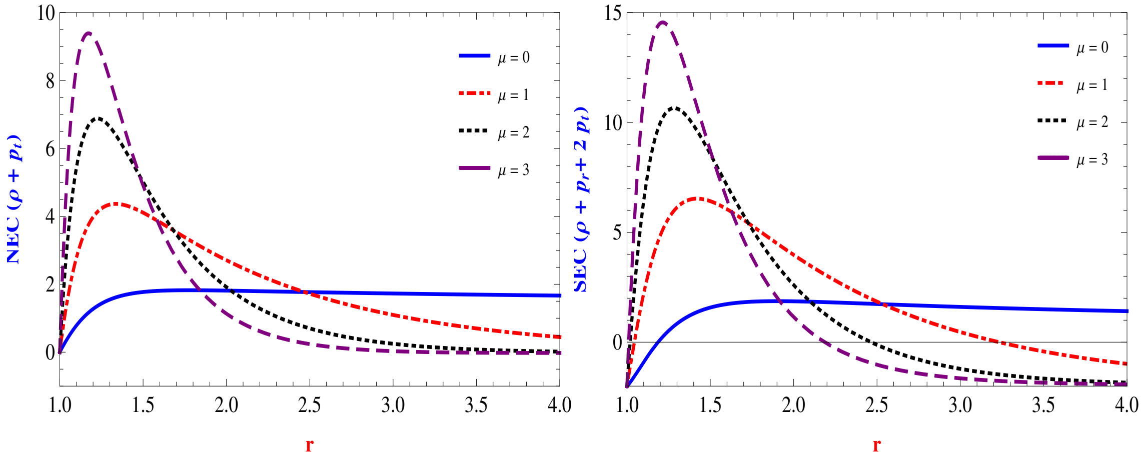


(a) **(b)**

Figure 4. Plots: (**a**) Energy density (ρ) and (**b**) radial NEC ($\rho + p_r$) with throat radius $r_0 = 1$, $\alpha = 1$, and $\beta = 2$ in $f(Q) = \alpha Q^2 + \beta$ gravity.

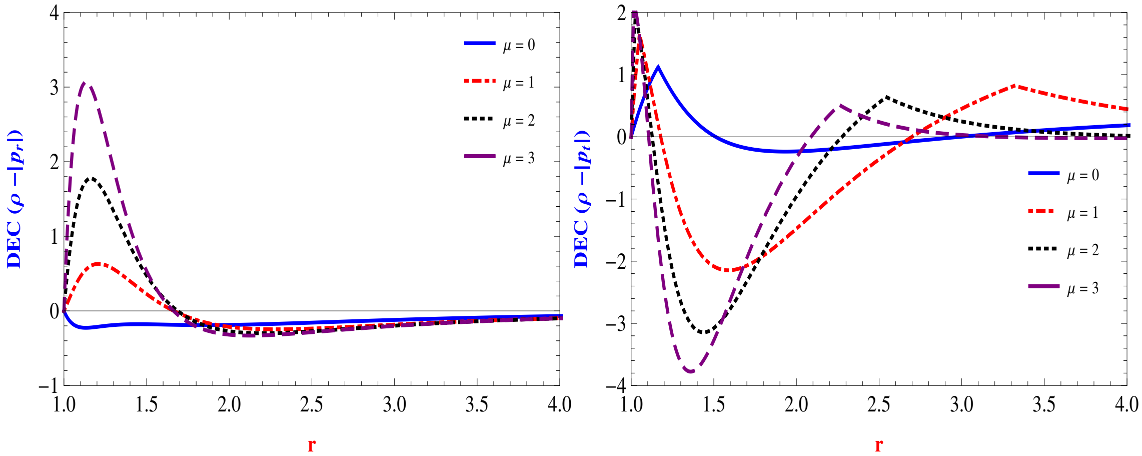
4.3. *Quadratic Form:* $f(Q) = \alpha Q^2 + \beta Q + \gamma$

In this section, we take a more general quadratic form of function $f(Q)$, i.e., $f(Q) = \alpha Q^2 + \beta Q + \gamma$, where α, β, and γ are constants. We examine this case for the possible TWH solution with non-exotic matter. Figure 7a depicts the energy density ρ against all four values $0, 1, 2, 3$ of scalar μ. As we interpret from the figure, energy density is positive near the throat and throughout the region $r \geq r_0$. The NECs can be analyzed from the plots, namely Figures 7b and 8a. It is very interesting to see that similar to the previous case, both the NECs are satisfied near the throat implying the presence of non-exotic matter near the wormhole throat, except for the case of the original Casimir wormhole, i.e., at $\mu = 0$. For $\mu = 0$ only the radial null energy condition is satisfied, but the tangential NEC is violated. Here, we also found the range of radial coordinates where the NEC terms are satisfied. The radial NEC is satisfied for $r \in (1.04, 1.65)$, and tangential NEC is also satisfied in $r \in (1.04, 2.1)$ for Yukawa–Casimir energy, but the tangential NEC is violated everywhere in the region for Casimir source. The nature of SEC and both DECs are shown in Figures 8b and 9.



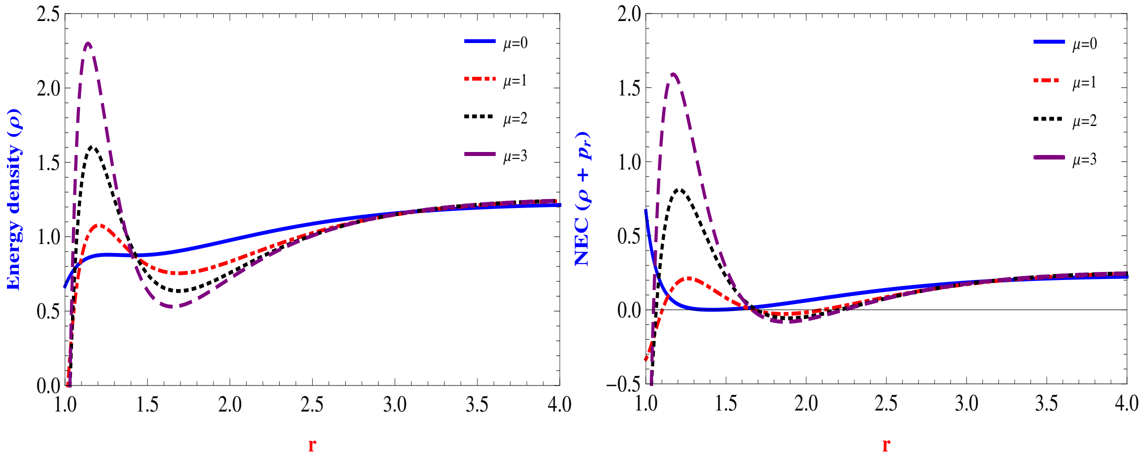
(**a**) (**b**)

Figure 5. Plots: (**a**) Tangential NEC ($\rho + p_t$) and (**b**) SEC ($\rho + p_r + 2p_t$) with throat radius $r_0 = 1$, $\alpha = 1$, and $\beta = 2$ in $f(Q) = \alpha Q^2 + \beta$ gravity.



(**a**) (**b**)

Figure 6. Plots: (**a**) Radial DEC ($\rho - |p_r|$), and (**b**) Tangential DEC ($\rho - |p_t|$) with throat radius $r_0 = 1$, $\alpha = 1$, and $\beta = 2$ in $f(Q) = \alpha Q^2 + \beta$ gravity.



(**a**) (**b**)

Figure 7. Plots: (**a**) Energy density (ρ) and (**b**) Radial NEC ($\rho + p_r$) with throat radius $r_0 = 1$, $\alpha = 1$, $\beta = 1$, and $\gamma = 2$ in $f(Q) = \alpha Q^2 + \beta Q + \gamma$ gravity.

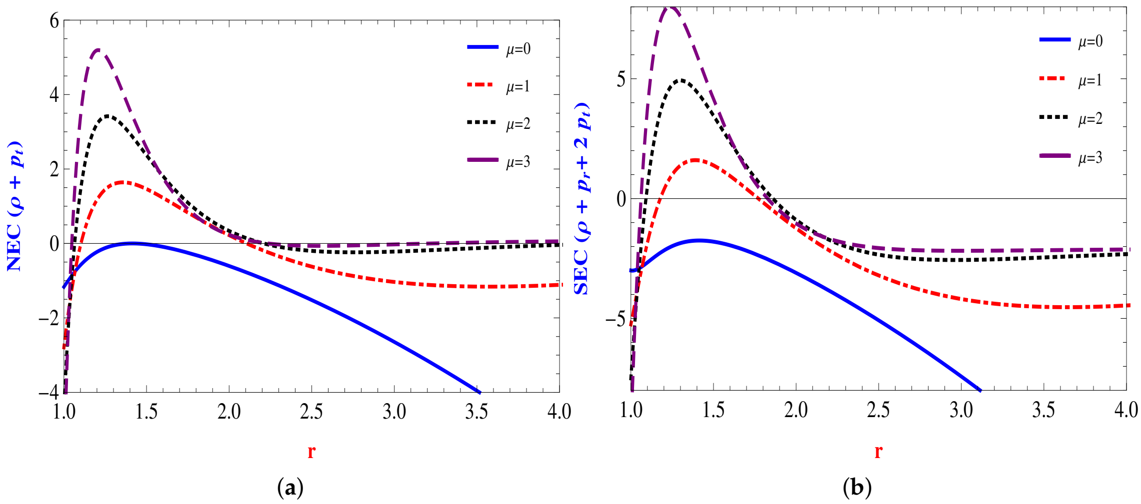


Figure 8. Plots: (**a**) Tangential NEC $(\rho + p_t)$ and (**b**) SEC $(\rho + p_r + 2p_t)$ with throat radius $r_0 = 1$, $\alpha = 1$, $\beta = 1$, and $\gamma = 2$ in $f(Q) = \alpha Q^2 + \beta Q + \gamma$ gravity.

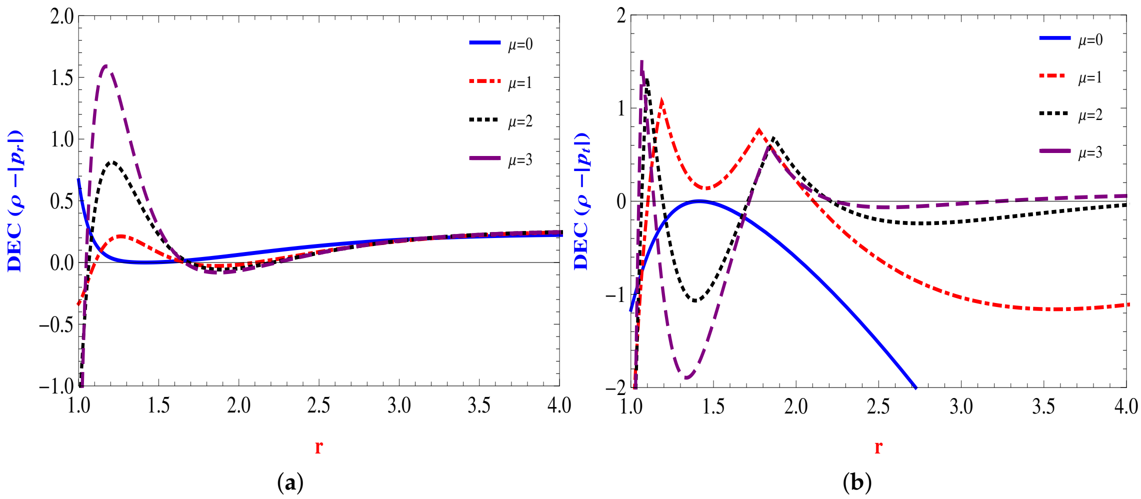


Figure 9. Plots: (**a**) Radial DEC $(\rho - |p_r|)$, and (**b**) Tangential DEC $(\rho - |p_t|)$ with throat radius $r_0 = 1$, $\alpha = 1$, $\beta = 1$, and $\gamma = 2$ in $f(Q) = \alpha Q^2 + \beta Q + \gamma$ gravity.

4.4. Inverse Power Law Form: $f(Q) = Q + \frac{\alpha}{Q}$

This section is devoted to the results obtained on the stage of the inverse power law form of teleparallel gravity [76]. In power law form, various energy conditions and other profiles are coined in Figures 10–12. The graphs are plotted against $\mu = 0, 1, 2, 3$. The original Casimir energy density is found to be positive in the region $r \in (1.3, \infty)$, and Yukawa–Casimir energy density is also observed to be positive for every non-zero value of μ in the region $r \in (1.55, \infty)$, which is plotted in Figure 10a. The radial NEC is validated everywhere in the region for Casimir energy, and it is also validated in the region $r \in (1.55, \infty)$ for Yukawa–Casimir energy. In the case of Casimir energy, the tangential NEC is violated for $\forall r \geq r_0$ in the region, but it is satisfied for $\mu = 3, 4$ when radial coordinates $r \in (4.3, \infty)$ and $r \in (3.2, \infty)$, respectively. The SEC is violated in both the cases in Figure 11b for all values of radial coordinates in the region. The radial DEC in Figure 12a is validated in $r \in (1.45, \infty)$ for $\mu = 0$ and $r \in (1.65, \infty)$ for $\mu = 1, 2, 3$. In Figure 12b) the tangential DEC is violated in both cases. One can see the expressions for energy density (ρ), radial pressure (p_r) and tangential pressure (p_t) for all four forms of $f(Q)$ gravity in the Appendix A.

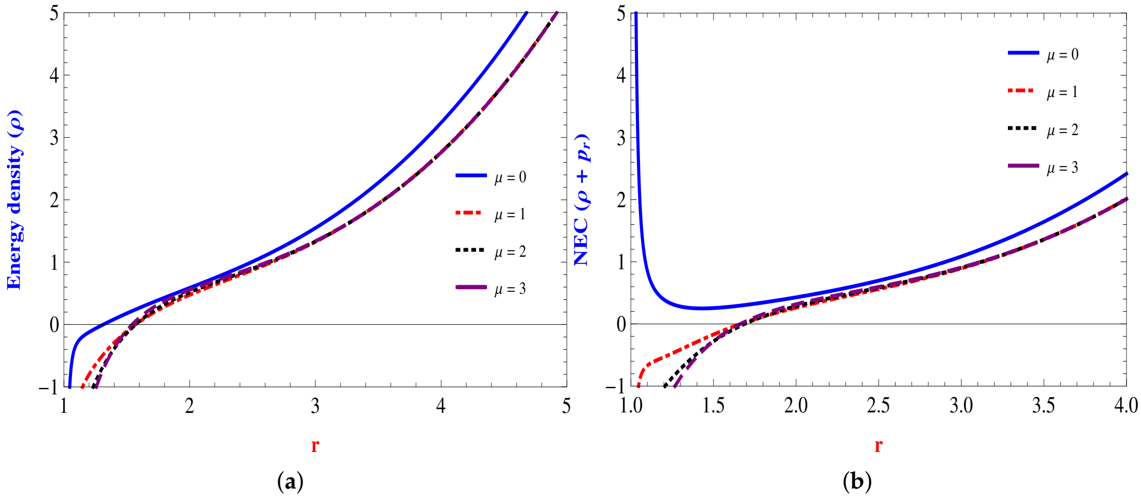


Figure 10. Plots: (**a**) Energy density (ρ) and (**b**) Radial NEC ($\rho + p_r$) with throat radius $r_0 = 1$ and $\alpha = -0.1$ in $f(Q) = Q + \frac{\alpha}{Q}$ gravity.

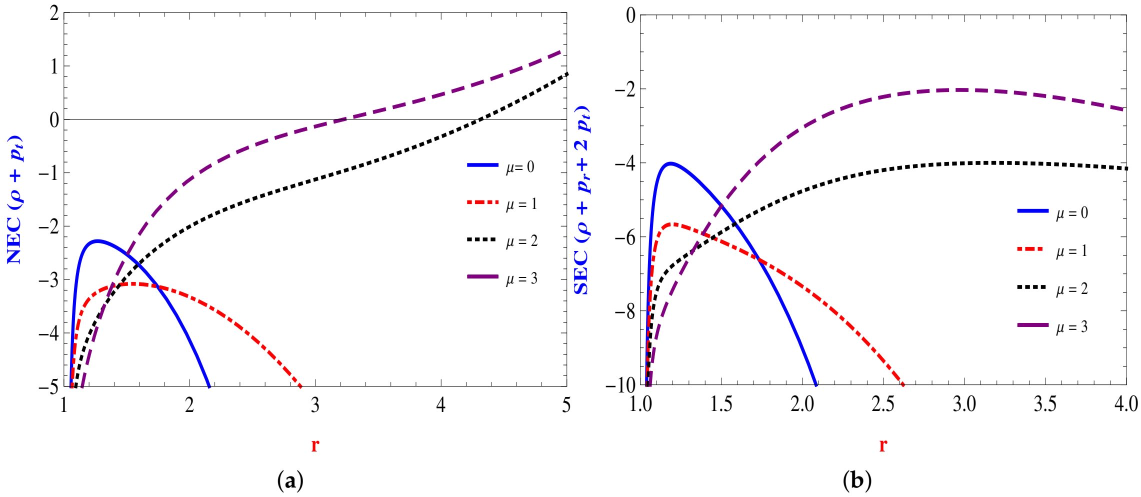


Figure 11. Plots: (**a**) Tangential NEC ($\rho + p_t$) and (**b**) SEC ($\rho + p_r + 2p_t$) with throat radius $r_0 = 1$ and $\alpha = -0.1$ in $f(Q) = Q + \frac{\alpha}{Q}$ gravity.

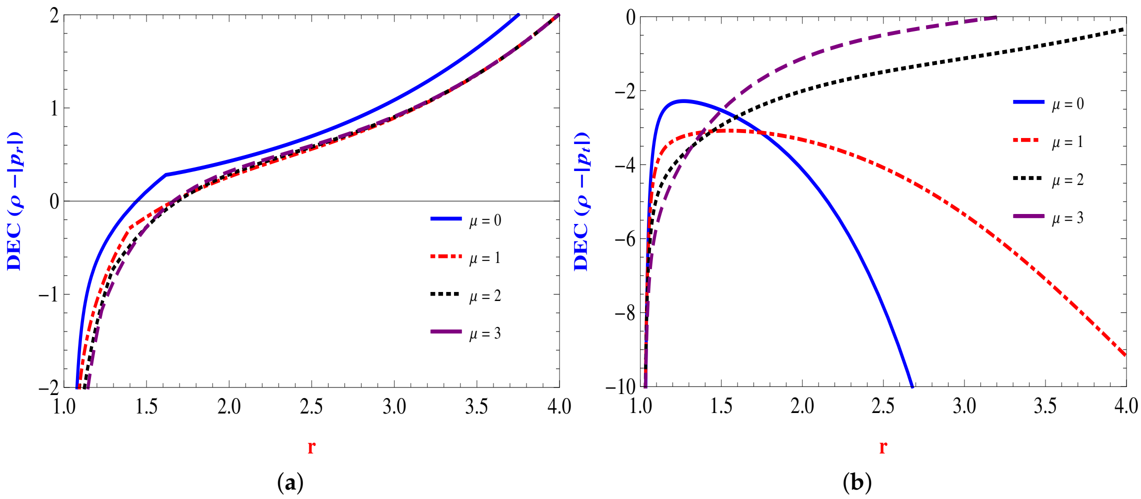


Figure 12. Plots: (**a**) Radial DEC ($\rho - |p_r|$), and (**b**) Tangential DEC ($\rho - |p_t|$) with throat radius $r_0 = 1$ and $\alpha = -0.1$ in $f(Q) = Q + \frac{\alpha}{Q}$ gravity.

5. Discussion and Conclusions

An unfortunate repercussion of general relativity, regarding the traversability of the wormhole, is the unavoidable breach of null energy conditions, which leads to the presence of exotic or non-ordinary matter in the WH throat. The focus nowadays is on finding the solutions of traversable wormholes threaded by ordinary matter or with a minimal amount of exotic matter by using the modified gravity framework. The usual energy conditions are satisfied by the classical matter. Hence, it is possible that wormholes must come from the domain of semi-classical or, more likely, a feasible quantum theory of the gravitational field. It is a well-known fact that Casimir energy is the only source of negative or exotic energy for various physical systems. In [83], the author conjectured that the traversability could be supported by quantum fluctuations as an effective source of the semi-classical Einstein equation. In this present work, we aim to find the solution for the traversable wormhole with ordinary matter threading the wormhole throat. In this regard, we examine the static and spherically symmetric wormhole in the backdrop of symmetric teleparallel gravity, where the non-metricity term Q defines the gravitational interactions of space-time.

The motivation of this current study comes from [76], where various forms of $f(Q)$ gravity were discussed for the plausible solutions of traversable wormholes. The outcomes in [76] indicate the wormhole solutions supported by exotic matter, which gives the requisite negative energy to sustain the wormhole throat. As stated earlier in this section, the violation of NEC is accountable for the traversability, which indicates the existence of exotic matter in the wormhole throat. To circumvent this situation and to approach a more realistic solution supported by ordinary matter, we induce the Yukawa- Casimir wormhole apparatus in the backdrop of symmetric teleparallel gravity taking four function forms of $f(Q)$. The Yukawa modification of the Casimir wormhole is obtained where zero tidal force is imposed with the help of the equation of state [75]. The Casimir energy is the source of exotic energy, which is negative energy and can support the traversability of the wormhole geometry. Introducing the Yukawa term in the original Casimir wormhole the Yukawa–Casimir wormhole is constructed. We try to set up the wormhole geometry threaded by the non-exotic matter, where the requisite negative energy is sourced from the Casimir energy instead of exotic matter.

In the case of linear form $f(Q) = \alpha Q$, we gather from the plots that, in all the cases of original Casimir and Yukawa–Casimir, the energy density ρ is positive for $r \geq 1.6$. We also found that wormhole throat is filled with exotic matter, where NECs are violated in Casimir energy source and Yukawa–Casimir energy source for $\mu = 1, 2$, but for $\mu = 3$, NECs are satisfied in region $r \in (3.5, \infty)$. In the second attempt of power law form $f(Q) = \alpha Q^2 + \beta$, we observed that energy density ρ is positive for all values of radial coordinate in both cases. We also found the wormhole throat is filled with the non-exotic matter, as both NECs are satisfied at the throat, and in region $r \in (1, \infty)$ after Yukawa modification in Casimir energy. It is interesting to see that radial NEC is violated here in the original Casimir energy source. In the next case, we have taken a more general quadratic form $f(Q) = \alpha Q^2 + \beta Q + \gamma$—we again found energy density positive, and both NECs are satisfied in a specific region after Yukawa modification. Radial NEC is satisfied in $r \in (1.04, 1.65)$, and tangential NEC is also satisfied in $r \in (1.04, 2.1)$. We again analyzed the clear difference between Casimir energy and Yukawa–Casimir energy as tangential NEC is violated in Casimir energy for $\forall r \geq r_0$. In the last case we investigated inverse power law form $f(Q) = Q + \frac{\alpha}{Q}$, we observed that tangential NEC is violated in Casimir energy and Yukawa–Casimir for $\mu = 1$ but it is also satisfied for $\mu = 2, 3$ in regions $r \in (4.3, \infty)$ and $r \in (3.2, \infty)$, respectively. Here, we again found a specific region in which both NECs are satisfied after Yukawa modification in Casimir energy.

In this investigation, we found, for all specific forms of $f(Q)$ in Casimir-source, that the null energy conditions are violated and wormholes are self-sustained and traversable, but after the Yukawa modification in the Casimir energy source violation of the null energy conditions is restricted to some regions in the vicinity of the throat. These results are completely different from the previous wormhole solutions mentioned in [76], where the authors investigated traversable wormholes violating null energy conditions in $f(Q)$ gravity. Hence, we may conclude that, with semi-classical gravity theory, with appropriate choices, the solution of traversable wormholes can be obtained, where the traversability is achieved with the non-exotic matter.

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Appendix A

Appendix A.1. Linear Form: $f(Q) = \alpha Q$

We devise the Yukawa–Casimir system in the Equations (12)–(14) to obtain the Yukawa–Casimir energy density and principal pressures as

$$\begin{aligned} \rho &= \frac{1}{3r^4(3r+r_0)}\Big\{\alpha e^{\mu(-r)}\Big(6r^2e^{\mu r}\Big(3r^2+r(r_0-3)-3r_0\Big)-r_0e^{\mu r_0}\Big(6(\mu+2)r^3 \\ &+ r^2(5(\mu+2)r_0-12)+rr_0((\mu+2)r_0-15)-5{r_0}^2\Big)\Big)\Big\}, \end{aligned} \tag{A1}$$

$$p_r = \frac{1}{r^4(3r+r_0)}\Big\{\alpha r_0 e^{\mu(r_0-r)}\Big(-2r^2\Big(e^{\mu(r-r_0)}-1\Big)+3rr_0+{r_0}^2\Big)\Big\}, \tag{A2}$$

$$\begin{aligned} p_t &= -\frac{1}{6r^4(3r+r_0)^2}\Big\{\alpha e^{\mu(-r)}\Big(6r^2e^{\mu r}\Big(9r^3+9r^2r_0+rr_0(2r_0-3)+{r_0}^2\Big) \\ &+ r_0e^{\mu r_0}\Big(18\mu r^9+9r^8(3\mu r_0+2)+r^7r_0(13\mu r_0+36)+2r^6{r_0}^2(\mu r_0+11) \\ &+ 4r^5{r_0}^3-36r^4-54r^3r_0+2r^2r_0(6-13r_0)+2r{r_0}^2(1-2r_0)-2{r_0}^3\Big)\Big)\Big\}. \end{aligned} \tag{A3}$$

Appendix A.2. Power Law Form: $f(Q) = \alpha Q^2 + \beta$

For this case, the energy components are obtained by solving the field equations, as given below

$$\begin{aligned} \rho &= \frac{1}{6r^8(3r+r_0)^2}\Big\{e^{2\mu(r_0-r)}\Big(24\alpha r^2r_0(r+r_0)e^{\mu(r-r_0)}\Big(6(\mu+4)r^3+r^2(5(\mu+4)r_0-18) \\ &+ rr_0((\mu+4)r_0-24)-8{r_0}^2\Big)-4\alpha{r_0}^2\Big(2r^2+3rr_0+{r_0}^2\Big)\Big(12(\mu+2)r^3 \\ &+ 2r^2(5(\mu+2)r_0-9)+rr_0(2(\mu+2)r_0-21)-7{r_0}^2\Big)+3r^4e^{2\mu(r-r_0)}\Big(9\beta r^6 \\ &+ 6\beta r^5r_0+\beta r^4{r_0}^2-144\alpha r^3-12\alpha r^2(16r_0-9)-24\alpha rr_0(2r_0-9)+108\alpha{r_0}^2\Big)\Big)\Big\}, \end{aligned} \tag{A4}$$

$$
\begin{aligned}
p_r &= \frac{1}{2r^8(3r+r_0)^2}\Big\{e^{2\mu(r_0-r)}\Big(12\alpha r_0{}^2\Big(2r^2+3rr_0+r_0{}^2\Big)^2-16\alpha r^2 r_0\Big(6r^3+17r^2 r_0 \\
&+ 15rr_0{}^2+4r_0{}^3\Big)e^{\mu(r-r_0)}+r^4\Big(-e^{2\mu(r-r_0)}\Big)\Big(9\beta r^6+6\beta r^5 r_0+\beta r^4 r_0{}^2-36\alpha r^2 \\
&- 120\alpha rr_0-84\alpha r_0{}^2\Big)\Big)\Big\},
\end{aligned} \tag{A5}
$$

$$
\begin{aligned}
p_t &= \frac{1}{6r^8(3r+r_0)^3}\Big\{e^{2\mu(r_0-r)}\Big(-3r^4 e^{2\mu(r-r_0)}\Big(27\beta r^7+27\beta r^6 r_0+9\beta r^5 r_0{}^2 \\
&+ r^4\Big(\beta r_0{}^3-216\alpha\Big)-108r^3(\alpha+4\alpha r_0)-12\alpha r^2 r_0(22r_0+15)-12\alpha rr_0{}^2(4r_0+11) \\
&- 60\alpha r_0{}^3\Big)+12\alpha r^2 r_0(r+r_0)e^{\mu(r-r_0)}\Big(18\mu r^9+9r^8(3\mu r_0+2)+r^7 r_0(13\mu r_0+36) \\
&+ 2r^6 r_0{}^2(\mu r_0+11)+4r^5 r_0{}^3-72r^4-36r^3(3r_0+1)-2r^2 r_0(26r_0+21) \\
&- 8rr_0{}^2(r_0+4)-10r_0{}^3\Big)-4\alpha r_0{}^2\Big(2r^2+3rr_0+r_0{}^2\Big)\Big(18\mu r^9+9r^8(3\mu r_0+2) \\
&+ r^7 r_0(13\mu r_0+36)+2r^6 r_0{}^2(\mu r_0+11)+4r^5 r_0{}^3-36r^4-18r^3(3r_0+1) \\
&- r^2 r_0(26r_0+21)-4rr_0{}^2(r_0+4)-5r_0{}^3\Big)\Big)\Big\}.
\end{aligned} \tag{A6}
$$

Appendix A.3. Quadratic Form: $f(Q)=\alpha Q^2+\beta Q+\gamma$

Using this quadratic model with Yukawa–Casimir shape function, the field Equations (12)–(14) can be developed to give the stress energy components as

$$
\begin{aligned}
\rho &= \frac{1}{6r^8(3r+r_0)^2}\Big\{e^{-2\mu r}\Big(-4\alpha r_0{}^2\Big(2r^2+3rr_0+r_0{}^2\Big)e^{2\mu r_0}\Big(12(\mu+2)r^3+2r^2(5(\mu+2)r_0-9) \\
&+ rr_0(2(\mu+2)r_0-21)-7r_0{}^2\Big)+3r^4 e^{2\mu r}\Big(9\gamma r^6+6r^5(6\beta+\gamma r_0)+r^4\Big(\gamma r_0{}^2+12\beta(2r_0-3)\Big) \\
&- 4r^3(36\alpha-\beta(r_0-12)r_0)-12r^2\Big(\beta r_0{}^2+\alpha(16r_0-9)\Big)-24\alpha rr_0(2r_0-9)+108\alpha r_0{}^2\Big) \\
&- 2r^2 r_0 e^{\mu(r+r_0)}\Big(18\beta(\mu+2)r^6+3\beta r^5(7(\mu+2)r_0-12)+r^4(\beta r_0(8(\mu+2)r_0-57) \\
&- 72\alpha(\mu+4))+r^3\Big(\beta r_0{}^2((\mu+2)r_0-30)-12\alpha(11(\mu+4)r_0-18)\Big)-r^2 r_0\Big(5\beta r_0{}^2 \\
&+ 72\alpha((\mu+4)r_0-7)\big)-12\alpha rr_0{}^2((\mu+4)r_0-32)+96\alpha r_0{}^3\Big)\Big)\Big\},
\end{aligned} \tag{A7}
$$

$$
\begin{aligned}
p_r &= \frac{1}{2r^8(3r+r_0)^2}\Big\{e^{-2\mu r}\Big(12\alpha r_0{}^2\Big(2r^2+3rr_0+r_0{}^2\Big)^2 e^{2\mu r_0}+2r^2 r_0\Big(2r^2+3rr_0+r_0{}^2\Big) \\
&\times e^{\mu(r+r_0)}\Big(3\beta r^3+\beta r^2 r_0-24\alpha r-32\alpha r_0\Big)+r^4\Big(-e^{2\mu r}\Big)\Big(9\gamma r^6+6\gamma r^5 r_0+\gamma r^4 r_0{}^2 6r^8(3r+r_0)^3 \\
&+ 12\beta r^3 r_0+4r^2\Big(\beta r_0{}^2-9\alpha\Big)-120\alpha rr_0-84\alpha r_0{}^2\Big)\Big)\Big\},
\end{aligned} \tag{A8}
$$

$$
\begin{aligned}
p_t &= \frac{1}{6r^8(3r+r_0)^3}\Big\{e^{2\mu(r_0-r)}\Big(-3r^4e^{2\mu(r-r_0)}\Big(27\gamma r^7+27r^6(2\beta+\gamma r_0)+9r^5r_0(8\beta+\gamma r_0)\\
&+ r^4\Big(r_0\Big(\gamma r_0{}^2+6\beta(5r_0-3)\Big)-216\alpha\Big)+4r^3\Big(\beta r_0{}^3-27\alpha(4r_0+1)\Big)+2r^2r_0\Big(\beta r_0{}^2\\
&- 6\alpha(22r_0+15))-12\alpha rr_0{}^2(4r_0+11)-60\alpha r_0{}^3\Big)-4\alpha r_0{}^2\Big(2r^2+3rr_0+r_0{}^2\Big)\Big(18\mu r^9\\
&+ 9r^8(3\mu r_0+2)+r^7r_0(13\mu r_0+36)+2r^6r_0{}^2(\mu r_0+11)+4r^5r_0{}^3-36r^4-18r^3(3r_0+1)\\
&- r^2r_0(26r_0+21)-4rr_0{}^2(r_0+4)-5r_0{}^3\Big)-r^2r_0e^{\mu(r-r_0)}\Big(54\beta\mu r^{12}+9\beta r^{11}(11\mu r_0+6)\\
&+ 6r^{10}\Big(-36\alpha\mu+11\beta\mu r_0{}^2+21\beta r_0\Big)+r^9\Big(\beta r_0{}^2(19\mu r_0+102)-108\alpha(5\mu r_0+2)\Big)\\
&+ 2r^8r_0\Big(\beta r_0{}^2(\mu r_0+17)-12\alpha(20\mu r_0+27)\Big)+4r^7\Big(-27\beta+\beta r_0{}^4-45\alpha\mu r_0{}^3-174\alpha r_0{}^2\Big)\\
&- 6r^6\Big(4\alpha\mu r_0{}^4+52\alpha r_0{}^3+33\beta r_0\Big)-12r^5\Big(4\alpha\Big(r_0{}^4-18\Big)+\beta r_0(11r_0-3)\Big)\\
&+ 2r^4\Big(\beta(9-19r_0)r_0{}^2+216\alpha(5r_0+1)\Big)-4r^3r_0\Big(\beta r_0{}^2(r_0+1)-6\alpha(80r_0+39)\Big)\\
&+ r^2\Big(-2\beta r_0{}^4+720\alpha r_0{}^3+888\alpha r_0{}^2\Big)+24\alpha rr_0{}^3(4r_0+21)+120\alpha r_0{}^4\Big)\Big)\Big\}. \qquad \text{(A9)}
\end{aligned}
$$

Appendix A.4. Inverse Power Law Form: $f(Q) = Q + \frac{\alpha}{Q}$

The energy profile for stress-energy tensor, in terms of radial pressure p_r, tangential pressure p_t, and the energy density ρ, is obtained from (12)–(14) as

$$
\begin{aligned}
\rho &= \frac{1}{12r^4(r+r_0)^2(3r+r_0)\big(r_0(2r+r_0)-3r^2e^{\mu(r-r_0)}\big)^2}\Big\{e^{\mu(r_0-r)}\Big(24r^2r_0{}^2(r+r_0)^2(2r\\
&+ r_0)e^{\mu(r-r_0)}\Big(6(\mu+3)r^3+r^2(5(\mu+3)r_0-18)+rr_0((\mu+3)r_0-24)-8r_0{}^2\Big)\\
&- 4r_0{}^3\Big(2r^2+3rr_0+r_0{}^2\Big)^2\Big(6(\mu+2)r^3+r^2(5(\mu+2)r_0-12)+rr_0((\mu+2)r_0-15)-5r_0{}^2\Big)\\
&- 6r^6\Big(27\alpha r^8+27\alpha r^7r_0+9\alpha r^6r_0{}^2+\alpha r^5r_0{}^3-108r^4-36r^3(7r_0-3)-36r^2r_0(5r_0-9)\\
&- 36r(r_0-9)r_0{}^2+108r_0{}^3\Big)e^{3\mu(r-r_0)}+r^4r_0e^{2\mu(r-r_0)}\Big(54\alpha(\mu+2)r^9+81\alpha(\mu+2)r^8r_0\\
&+ 9\alpha r^7r_0(5(\mu+2)r_0+3)+\alpha r^6r_0{}^2(11(\mu+2)r_0+27)+r^5\Big(-216(\mu+6)+\alpha(\mu+2)r_0{}^4\\
&+ 9\alpha r_0{}^3\Big)+r^4\Big(\alpha r_0{}^4-612(\mu+6)r_0+1296\Big)-36r^3r_0(17(\mu+6)r_0-123)\\
&- 36r^2r_0{}^2(7(\mu+6)r_0-155)-36rr_0{}^3((\mu+6)r_0-85)+612r_0{}^4\Big)\Big)\Big\}, \qquad \text{(A10)}
\end{aligned}
$$

$$
\begin{aligned}
p_r &= \frac{1}{4r^4(r+r_0)^2(3r+r_0)\big(r_0(2r+r_0)-3r^2e^{\mu(r-r_0)}\big)^2}\Big\{e^{\mu(r_0-r)}\Big(-8r^2r_0{}^2(3r+4r_0)\Big(2r^2\\
&+ 3rr_0+r_0{}^2\Big)^2e^{\mu(r-r_0)}+4r_0{}^3\Big(2r^2+3rr_0+r_0{}^2\Big)^3-3r^4r_0\Big(2r^2+3rr_0+r_0{}^2\Big)\Big(9\alpha r^6\\
&+ 6\alpha r^5r_0+\alpha r^4r_0{}^2-12r^2-40rr_0-28r_0{}^2\Big)e^{2\mu(r-r_0)}+2r^6\Big(27\alpha r^7+54\alpha r^6r_0+27\alpha r^5r_0{}^2\\
&+ 4\alpha r^4r_0{}^3-36r^2r_0-72rr_0{}^2-36r_0{}^3\Big)e^{3\mu(r-r_0)}\Big)\Big\}, \qquad \text{(A11)}
\end{aligned}
$$

$$
\begin{aligned}
p_t &= \frac{1}{24r^4(r+r_0)^2(3r+r_0)^2\left(r_0(2r+r_0)-3r^2e^{\mu(r-r_0)}\right)^2}\Big\{e^{\mu(r_0-r)}\Big(24r^2{r_0}^2(r+r_0)^2(2r+r_0) \\
&\times\ e^{\mu(r-r_0)}\Big(18\mu r^9+9r^8(3\mu r_0+2)+r^7r_0(13\mu r_0+36)+2r^6{r_0}^2(\mu r_0+11)+4r^5{r_0}^3 \\
&-\ 54r^4-81r^3r_0+3r^2r_0(6-13r_0)+3r{r_0}^2(1-2r_0)-3{r_0}^3\Big)-4{r_0}^3\Big(2r^2+3rr_0+{r_0}^2\Big)^2 \\
&\times\ \Big(18\mu r^9+9r^8(3\mu r_0+2)+r^7r_0(13\mu r_0+36)+2r^6{r_0}^2(\mu r_0+11)+4r^5{r_0}^3-36r^4 \\
&-\ 54r^3r_0+2r^2r_0(6-13r_0)+2r{r_0}^2(1-2r_0)-2{r_0}^3\Big)+6r^6\Big(81\alpha r^9+27\alpha r^8(5r_0+6) \\
&+\ 27\alpha r^7r_0(3r_0+11)+3\alpha r^6{r_0}^2(7r_0+69)+r^5\Big(2\alpha{r_0}^4+63\alpha{r_0}^3-324\Big)+r^4r_0\Big(7\alpha{r_0}^3 \\
&-\ 972)-36r^3r_0(29r_0-3)-36r^2{r_0}^2(13r_0-5)+36r{r_0}^3(1-2r_0)-36{r_0}^4\Big)e^{3\mu(r-r_0)} \\
&+\ r^4r_0e^{2\mu(r-r_0)}\Big(162\alpha\mu r^{15}+27\alpha r^{14}(13\mu r_0+6)+27\alpha r^{13}r_0(11\mu r_0+16)+3\alpha r^{12}{r_0}^2(41\mu r_0 \\
&+\ 144)+r^{11}\Big(-648\mu+25\alpha\mu{r_0}^4+204\alpha{r_0}^3\Big)+2r^{10}\Big(\alpha\Big(\mu{r_0}^5+23{r_0}^4-162\Big)-162(7\mu r_0 \\
&+\ 2))+r^9\Big(-648\alpha+4\alpha{r_0}^5-3060\mu{r_0}^2-54(13\alpha+48)r_0\Big)-18r^8r_0\Big(84\alpha+110\mu{r_0}^2 \\
&+\ (33\alpha+224)r_0)-6r^7{r_0}^2\Big(237\alpha+102\mu{r_0}^2+(41\alpha+504)r_0\Big)-2r^6\Big(36\mu{r_0}^5+5(5\alpha \\
&+\ 108){r_0}^4+333\alpha{r_0}^3-1944\Big)-2r^5r_0\Big(2(\alpha+36){r_0}^4+77\alpha{r_0}^3-6804\Big)-2r^4r_0\Big(7\alpha{r_0}^4 \\
&-\ 9180r_0+648)+216r^3{r_0}^2(55r_0-13)+216r^2{r_0}^3(17r_0-7)+216r{r_0}^4(2r_0+1) \\
&+\ 216{r_0}^5\Big)\Big)\Big\}. \qquad \text{(A12)}
\end{aligned}
$$

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