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Dynamically Implementing the $\bar{\mu}$ -Scheme in Cosmological and Spherically Symmetric Models in an Extended Phase Space Model

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Abstract: We consider an extended phase space formulation for cosmological and spherically symmetric models in which the choice of a given $\bar{\mu}$ -scheme can be implemented dynamically. These models are constructed in the context of the relational formalism by using a canonical transformation on the extended phase space, which provides a Kuchař decomposition of the extended phase space. The resulting model can be understood as a gauge-unfixed model of a given $\bar{\mu}$ -scheme. We use this formalism to investigate the restrictions to the allowed $\bar{\mu}$ -scheme from this perspective and discuss the differences in the cosmological and spherically symmetric case. This method can be useful, for example, to obtain a $\bar{\mu}$ -scheme in a top-down derivation from full LQG to symmetry-reduced effective models, where, for some models, only the μ_0 -scheme has been obtained thus far.

Keywords: loop quantum cosmology; spherically symmetric models; effective models



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1. Introduction

The investigation of symmetry-reduced models for loop quantum gravity (LQG) in the context of cosmological or spherically symmetric spacetimes is a topic of growing interest in recent years; for spherically symmetric models, see, for instance, [1–32] or [33] for a recent review and [34] for a review on loop quantum cosmology (LQC). In this context, effective models for LQC or quantum black holes are a useful framework to obtain insights on the underlying symmetry-reduced quantum model and its characteristic properties.

A common feature that such effective models share is that holonomy operators involved in full LQG are replaced by so-called polymerization functions that mimic the loop-inspired quantization of these symmetry-reduced models. Such a polymerization function is always accompanied by a so-called polymerization parameter, usually denoted by μ , which is the relevant scale for, e.g., holonomy corrections in these effective models. The first formulation of such models used a fixed polymerization parameter that is independent of the phase space variables—the so-called μ_0 -scheme.

The usage of a fixed parameter is inspired by the regularization of holonomies of connections used in full loop quantum gravity [35]. One problem of the μ_0 -scheme in effective LQC dynamics is that the critical density at the bounce depends on the initial values and is possibly non-Planckian. In the context of the so-called improved dynamics, a $\bar{\mu}$ -scheme for LQC was then proposed to overcome this problem [36], in which the polymerization parameter is related to the minimal area gap and, thus, depends on the triad operators.

Going beyond cosmological models, improved dynamics were then proposed for spherically symmetric models [4,9,10,14,28–31,37]. We note that there are different implementations of the improved dynamics in these models, unlike in LQC where the choice is unique if we require scale invariance. To establish a link between full LQG and the LQC effective dynamics, a model for a coherent state-path integral for LQG was introduced in [38].

However, obtaining the improved $\bar{\mu}$ -scheme from such top-down derivations of full LQG is still an open problem [38,39]. If one generalizes the models in [38,39] to the extended phase space introduced here, we expect that these models will yield corresponding effective models with a $\bar{\mu}$ -scheme in cosmology or spherically symmetric models. Thus, in these top-down derivations, the difference is whether one works with or without the extended phase space to obtain a model in either a μ_0 - or $\bar{\mu}$ -scheme.

There have been some attempts to generalize models in different directions [40–42] to obtain effective models with an $\bar{\mu}$ -scheme, which involves either a different regularization of the dynamics or an ensemble of microstates labeled by different graphs. The approach to the implementation of the $\bar{\mu}$ -scheme in this work is more along the lines introduced in the effective cosmological models—namely, as a canonical transformation on the phase space of gravitational degrees of freedom.

In this paper, we aim at constructing a method based on an extended phase space that allows the implementation of a general $\bar{\mu}$ -scheme dynamically, and we investigate the restrictions for the allowed $\bar{\mu}$ -scheme. We discuss the model in the framework of the relational formalism [43–47] where we can formulate the model in terms of Dirac observables with respect to the constraints related to the choice of a given $\bar{\mu}$ -scheme. The construction of such Dirac observables allows us to define a canonical transformation on the extended phase space, which maps the original kinematical variables into a new set of variables where the constraints related to the $\bar{\mu}$ -scheme become new elementary variables in phase space.

By this, we achieve a so-called Kuchař-decomposition [48,49] of the phase space in which the constraints as well as the Dirac observables with respect to these constraints become the new elementary-phase space variables. Working in extended phase space allows the choice of a particular $\bar{\mu}$ -scheme to be implemented as a canonical transformation. Although this is also possible directly in the reduced phase space of cosmological models, it is no longer possible in reduced phase space for general $\bar{\mu}$ -schemes in spherically symmetric models.

With the detailed construction of the cosmological model in Section 2 and spherically symmetric model in Section 3, we show that the model in terms of these Dirac observables can be understood as the gauge-unfixed version of a given $\bar{\mu}$ model. The model includes constraints that fix $\bar{\mu}$ to some chosen phase space functions that is implementing a specific $\bar{\mu}$ -scheme, and we call such functions gauge-fixing functions in the following.

By requiring that the elementary canonical variables encoding the constraints vanish, we obtain the Hamiltonian constraint in the $\bar{\mu}$ -scheme characterized by the choice of the gauge-fixing functions, which can be taken as the starting point to further study the quantum theory and effective dynamics. In addition, we also investigate the corresponding gauge-fixed model for both the cosmological and spherically symmetric case. The method that we propose here can, for instance, be applied to the path integral formulation of full LQG [38], thereby, allowing us to obtain $\bar{\mu}$ -like effective dynamics from the top-down derivations of full LQG.

As our analysis shows, in the cosmological model, the gauge-fixing function is encoded in the model, and thus the choice for $\bar{\mu}$ has no restrictions to the form of the $\bar{\mu}$ function. Any gauge-fixing function that one chooses, which depends on the scale factor and extrinsic curvature, can lead to a well-defined Hamiltonian in the partially reduced phase space equipped with the standard canonical commutation relations (CCR). The form of the $\bar{\mu}$ -scheme is fixed only when we further require scale invariance of the polymerization function, which then reduces to the standard $\bar{\mu}$ -scheme used in LQC.

In the spherically symmetric model, which is a 1 + 1-dimensional field theory, this is no longer the case. Due to the requirement for a consistent density weight of the effective constraints, the dynamical implementation of a chosen $\bar{\mu}$ -scheme is only possible if the polymerization depends on a combination of the extrinsic curvature and the triad such that they have density weight zero. This gives a restriction to the possible gauge-fixing functions for spherical symmetric models.

We note that, if one only works with the interior of a black hole, which can be described as a homogeneous Kantowski–Sachs spacetime, the gauge-fixing functions are again free from any restriction similar to the case of cosmological models.

The paper is structured as follows: In Section 2, we present the dynamical implementation of the $\bar{\mu}$ -scheme for cosmological models. In Section 2.1, we use the framework of the relational formalism and discuss, in detail, how a canonical transformation can be constructed. Using the transformed variables, we show on the one hand that we obtain a gauge-unfixed model for each given choice of a $\bar{\mu}$ -scheme, including also the special choice of μ_0 and, on the other hand, how the reduced phase space can be obtained.

In Section 2.2, we present the corresponding gauge-fixed model for the cosmological case. In Section 3, we generalize our strategy to spherically symmetric models where in general more than one polymerization parameter is present and we consider field theory models. The sections on the cosmological and the spherically symmetric models should be self-contained, so that the reader interested in only one of the two cases can go directly to the appropriate section.

As shown in Section 3.1 a similar canonical transformation on the extended phase space can be constructed using the relational formalism. To show that the transformation is indeed canonical requires a slightly more effort compared to the cosmological case but works analogously. Again, the resulting model can be understood as a gauge-unfixed model of a given $\bar{\mu}$ -scheme. In addition, we present, in Section 3.2, the corresponding gauge-fixed model. We summarize and conclude in Section 3.2 and provide further details for some calculations in the appendix.

2. Implementing the $\bar{\mu}$ -Scheme in Cosmology

We aimed to construct a model on an extended phase space that allows us to implement the $\bar{\mu}$ -scheme in cosmology dynamically. Note that we will not consider a further gauge fixing or the construction of Dirac observables with respect to the Hamiltonian constraint but see the model presented here rather as a starting point for a cosmological model with a given Hamiltonian constraints for which on the usual kinematical phase space a $\bar{\mu}$ -scheme needs to be chosen. Afterwards, one can still consider a further gauge fixing of the Hamiltonian constraint or construct the fully reduced phase space that also involves the Hamiltonian constraint because such a step is usually performed within an already given $\bar{\mu}$ -scheme.

2.1. Extended Phase Space for Cosmology: Canonical Transformation and Dirac Observables

We consider the usual kinematical phase space of cosmology with elementary-phase space variables denoted by $\theta, p, N, p_N, \phi^A, \pi_A$, where N is the lapse function, θ, p the gravitational variables and ϕ^A, π_A some generic matter degrees of freedom where $A \in 1, \dots, L_\#$ is a generic label to allow more than one matter component in general. We extend this phase space by two additional canonical pairs (λ, p_λ) and (μ, p_μ) . The non-vanishing CCR in the $(8 + 2L_\#)$ dimensional phase space are given by

$$\{\theta, p\} = \frac{\gamma\kappa}{6} \quad \{\lambda, p_\lambda\} = \{\mu, p_\mu\} = 1, \quad \{\phi^A, \pi_B\} = \delta_B^A, \quad \{N, p_N\} = 1, \quad (1)$$

where γ denotes the Barbero–Immirzi parameter and $\kappa = 8\pi G$ with G being Newton’s constant. The model that consider has a primary Hamiltonian of the form

$$H^P = NH_0(\theta, p, \mu, \phi^A, \pi_A) + \lambda(\mu - f(p, \theta)) + \Lambda_\lambda p_\lambda + \Lambda_\mu p_\mu + \Lambda_N p_N, \quad (2)$$

where $\Lambda_\lambda, \Lambda_\mu, \Lambda_N$ are Lagrange multipliers, and $H_0(\theta, p, \mu, \phi^A, \pi_A)$ takes the following form

$$H_0(\theta, p, \mu, \phi^A, \pi_A) = -\frac{6\sqrt{p}h(\mu\theta)}{\gamma^2\kappa\mu^2} + H_0^{\text{matter}}(\phi^A, \pi_A, p). \quad (3)$$

We consider a polymerization of the connection variable θ in the gravitational sector with a generic function $h(\mu\theta)$ satisfying

$$\lim_{\mu \rightarrow 0} \frac{h(\mu\theta)}{\mu^2} = \theta^2. \quad (4)$$

This provides the possibility to either consider this model as a classical starting point for a later loop quantization or to stay at the classical level and consider it as an effective model. Choosing this function to be $h(\mu\theta) = \sin(\mu\theta)^2$ gives the usual LQC effective Hamiltonian. The function $f(\theta, p)$ is an up to now arbitrary function and can be later chosen to have a specific form in order to reproduce the wanted $\bar{\mu}$ -scheme. The system has three primary constraints

$$C_{p_\lambda} = p_\lambda \approx 0 \quad C_{p_\mu} = p_\mu \approx 0 \quad C_{p_N} = p_N \approx 0. \tag{5}$$

The stability of C_{p_λ} leads to the following gauge fixing condition for μ ,

$$C_\mu = \dot{C}_{p_\lambda} = \mu - f_\mu(\theta, p) \approx 0 \tag{6}$$

The stability of C_{p_μ} gives

$$C_\lambda = \dot{C}_{p_\mu} = \lambda + \frac{12\sqrt{p}h(\mu\theta)}{\gamma^2\kappa\mu^3} - \frac{6\theta\sqrt{p}h'(\mu\theta)}{\gamma^2\kappa\mu^2} =: \lambda + g(\theta, p, \mu) \approx 0 \tag{7}$$

with $h'(X) = \partial_X h(X)|_{X=\mu\theta}$. Note that the matter contribution H_0^{matter} does not contribute to C_λ because it does not depend on μ as long as we do not consider a polymerization of the matter part. Furthermore, the requirement of the stability for C_{p_N} yields the Hamiltonian constraint

$$C_0 = H_0(\theta, p, \mu, \varphi^A, \pi_A) \approx 0. \tag{8}$$

The stability of C_μ and C_λ fix Lagrangian multiplier Λ_μ and Λ_λ , respectively, whereas C_0 is already stable.

The canonical transformation that we want to construct is a so-called partial Kuchař decomposition of the original phase space. We call it partial here because we perform such a decomposition only with respect to part of the constraints, namely those related to the choice of the $\bar{\mu}$ -scheme. Strictly speaking, we should also call the Dirac observable partial Dirac observables because they span the partially reduced phase space but not the fully reduced one, which needs to involve observables with respect to the Hamiltonian constraints.

In the following, we still call the partial Dirac observables simply Dirac observables but have in mind that the Hamiltonian constraint has not been considered yet but could easily be included following, for instance, the strategy used in [50] by using an additional observable map based on matter clocks and where details on the construction of such observable maps are presented.

For this purpose, we use the observable map as well as its dual version introduced in [51] where a Kuchař decomposition [48,49] of the phase space was constructed in the context of linearized gravity. This is necessary because originally we want to choose some of the involved constraints as new canonical coordinates, and this requires that each constraint commutes with all but one. As of this, we cannot use the original set of constraints for this purpose. Applying different observable maps successively will, however, yield new phase space coordinates that can be used for this purpose. We then show that the application of the different observables maps is indeed a canonical transformation on the extended phase space.

As a first step, we consider the two first class constraints C_{p_μ} and C_{p_λ} . As there are elementary momentum variables in phase space, they mutually commute with all remaining phase space variables except μ, λ and as a consequence C_μ, C_λ . In particular θ and p are obviously Dirac observables with respect to C_{p_μ}, C_{p_λ} . This means that the corresponding observable maps are defined as

$$\mathcal{O}_f^{p_\mu} := \sum_{n=0}^{\infty} \frac{(-1)^n (C_\mu)^n}{n!} \{f, C_{p_\mu}\}_{(n)}, \quad \mathcal{O}_f^{p_\lambda} := \sum_{n=0}^{\infty} \frac{(-1)^n (C_\lambda)^n}{n!} \{f, C_{p_\lambda}\}_{(n)}, \tag{9}$$

where $\{.,.\}_{(n)}$ denotes the nested Poisson bracket with $\{f, g\}_{(0)} = f$ and $\{f, g\}_{(n+1)} = \{\{f, g\}_{(n)}, g\}$ and f is a function on phase space that becomes the identity map for functions f that depend on (θ, p) only. In the language of the relational formalism the constraints C_μ, C_λ play the role of clocks. Now, given the observable map in (9), we can define its dual map along the lines of the work in [51] where the role of clocks and first class constraints is interchanged.

However, if we want to consider dual observable maps for both pairs (C_μ, C_{p_μ}) and $(C_\lambda, C_{p_\lambda})$ we need that the ‘clocks’ C_μ, C_λ commute at least weakly. Considering the Dirac matrix in (19), we see that this is not the case. Since we aim at constructing a canonical transformation we even need the stronger requirement that the clocks strongly commute. This can be achieved by successively applying the dual observable map as follows: in the first step, we consider the following set of variables

$$Q^1 := C_\lambda, \quad P_1 := C_{p_\lambda}, \quad Q^2 := \mathcal{O}_{C_\mu}^{C_\lambda}, \quad P_2 := \mathcal{O}_{C_{p_\mu}}^{C_\lambda}, \tag{10}$$

with the dual observable map

$$\mathcal{O}_f^{C_\lambda} := \sum_{n=0}^{\infty} \frac{(C_{p_\lambda})^n}{n!} \{f, C_\lambda\}_{(n)}. \tag{11}$$

By construction, we have $\{\mathcal{O}_f^{C_\lambda}, C_\lambda\} = 0$. Using the result in [47] on the Poisson algebra of the so-constructed Dirac observables and carrying it over to the dual observable map, we obtain

$$\{\mathcal{O}_{C_\mu}^{C_\lambda}, \mathcal{O}_{C_{p_\mu}}^{C_\lambda}\} = \mathcal{O}_{\{C_\mu, p_\mu\}_D^{C_\lambda}} = \mathcal{O}_{\{C_\mu, p_\mu\}} = 1, \tag{12}$$

where $\{.,.\}_D^{C_\lambda}$ denotes the submatrix of the Dirac bracket in (19) that includes the Poisson brackets of the pair $(C_\lambda, C_{p_\lambda})$ only. Given this, we already have $\{Q^j, P_k\} = \delta_k^j$ with $j, k = 1, 2$. To extend that set by variables corresponding to the gravitational and matter degrees of freedom that have the property to mutually commute with Q^j, P_j for $j = 1, 2$ and satisfy standard CCR, we define

$$Q^3 := \mathcal{O}_\theta^{C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}}, \quad P_3 := \mathcal{O}_p^{C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}}, \quad Q^4 := N, \quad P_4 := p_N, \quad Q^I := \mathcal{O}_{\phi^A}^{C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}}, \quad P_I := \mathcal{O}_{\pi_A}^{C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}}, \tag{13}$$

with $I \in 5, \dots, L_\# + 4$ being an index set covering the matter degrees of freedom for all A at the level of the observables. The notation $\mathcal{O}_f^{C_\lambda, \mathcal{O}_{C_{p_\mu}}^{C_\lambda}}$ means that, in the first step, we apply the dual observable map from (11) on f and afterwards the following dual observable map

$$\mathcal{O}_f^{\mathcal{O}_{C_\mu}^{C_\lambda}} := \sum_{n=0}^{\infty} \frac{(\mathcal{O}_{C_{p_\mu}}^{C_\lambda})^n}{n!} \{f, \mathcal{O}_{C_\mu}^{C_\lambda}\}_{(n)}. \tag{14}$$

For the canonical pair (N, p_N) , we used that both variables trivially commute with C_λ, C_μ , and thus these maps act similar to the identity map on N, p_N .

Now, it remains to show that $Q^3, P_3, Q^4, P_4, Q^I, P_I$ satisfy standard CCR and have vanishing Poisson brackets with Q^1, Q^2, P_1, P_2 . For the pair (Q^4, P_4) , this is the case because we have $\{Q^4, P_4\} = 1$, and none of the constructed observables depends on N, p_N so that they mutually commute with them. To show that it also for the remaining set Q^3, P_3, Q^I, P_I , somewhat more work is needed. By construction, C_λ and $\mathcal{O}_{C_\mu}^{C_\lambda}$ commute and hence the order of the application of the two (dual) observable maps is irrelevant and will yield a quantity that commutes with Q^1 as well as Q^2 by construction. To compute the Poisson bracket with the momenta, we use

$$\mathcal{O}_f^{C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}} = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathcal{O}_{C_{p_\mu}}^{C_\lambda})^n \{\mathcal{O}_f^{C_\lambda}, \mathcal{O}_{C_\mu}^{C_\lambda}\}_{(n)} = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathcal{O}_{C_{p_\mu}}^{C_\lambda})^n \mathcal{O}_{\{f, C_\mu\}_D^{C_\lambda}}^{C_\lambda}, \tag{15}$$

where $\{f, g\}_{D(n)}^{C_\lambda}$ denotes the iterated Dirac bracket with respect to the pair $(C_\lambda, C_{p_\lambda})$ with $\{f, g\}_{D(0)}^{C_\lambda} = f$ and $\{f, g\}_{D(n+1)}^{C_\lambda} = \{\{f, g\}_{D(n)}^{C_\lambda}, g\}_D^{C_\lambda}$.

The last step in (15) can be easily shown using again the result on the observable algebra from [47] iteratively as well as a proof by induction, which we present in the Lemma 1 below. The Lemma 1 is not discussed in [47] but can be easily proven using the theorem presented in [47], and we present the proof in Appendix C.

Lemma 1. *For the iterated Poisson bracket of observables, we have*

$$\{\mathcal{O}_f^{C_\lambda}, \mathcal{O}_g^{C_\lambda}\}_{(n)} = \mathcal{O}_{\{f, g\}_{D(n)}^{C_\lambda}}^{C_\lambda}. \tag{16}$$

Using these results, the Poisson brackets with the momenta yield

$$\begin{aligned} \{Q^3, P_2\} &= \{\mathcal{O}_\theta^{C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}}, \mathcal{O}_{C_{p_\mu}}^{C_\lambda}\} = \sum_{n=0}^{\infty} \frac{1}{n!} \{(\mathcal{O}_{C_{p_\mu}}^{C_\lambda})^n \mathcal{O}_{\{\theta, C_\mu\}_{D(n)}^{C_\lambda}}^{C_\lambda}, \mathcal{O}_{C_{p_\mu}}^{C_\lambda}\} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (\mathcal{O}_{C_{p_\mu}}^{C_\lambda})^n \{\mathcal{O}_{\{\theta, C_\mu\}_{D(n)}^{C_\lambda}}^{C_\lambda}, \mathcal{O}_{C_{p_\mu}}^{C_\lambda}\} = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathcal{O}_{C_{p_\mu}}^{C_\lambda})^n \mathcal{O}_{\{\{\theta, C_\mu\}_{D(n)}^{C_\lambda}, C_{p_\mu}\}_D^{C_\lambda}}, \end{aligned}$$

where we used the result of Lemma 1 in the second and one before the last step.

Now, we consider the Dirac bracket $\{\theta, C_\mu\}_{D(n)}^{C_\lambda}$. For $n = 0$, we have $\{\theta, C_\mu\}_{D(0)}^{C_\lambda} = \theta$. For $n \geq 1$, we find

$$\{\theta, C_\mu\}_{D(n)}^{C_\lambda} = \{\theta, C_\mu\}_{(n)} = \{\theta, f_\mu\}_{(n)},$$

where, by assumption, the function f_μ depends on the gravitational degrees of freedom only, and thus $\{\theta, f_\mu\}_{(n)}$ will do so as well. Here, we used, in the first step, that θ, C_μ both commute with p_λ . Since any partial derivative of f as well as θ commutes with p_λ we obtain for $n \geq 1$

$$\{\{\theta, C_\mu\}_{D(n)}^{C_\lambda}, C_{p_\mu}\}_D^{C_\lambda} = \{\{\theta, C_\mu\}_{D(n)}^{C_\lambda}, C_{p_\mu}\} = \frac{\partial}{\partial \mu} \{\theta, f_\mu\}_{(n)} = 0.$$

For $n = 0$, the nested Dirac bracket also vanishes because $\{\theta, C_{p_\mu}\} = 0$. From these results, we immediately end up with

$$\{Q^3, P_2\} = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathcal{O}_{C_{p_\mu}}^{C_\lambda})^n \mathcal{O}_{\{\{\theta, C_\mu\}_{D(n)}^{C_\lambda}, C_{p_\mu}\}_D^{C_\lambda}} = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathcal{O}_{C_{p_\mu}}^{C_\lambda})^n \mathcal{O}_0^{C_\lambda} = 0. \tag{17}$$

For the momentum P_1 , the calculation of the Poisson bracket leads to

$$\{Q^3, P_1\} = \{\mathcal{O}_\theta^{C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}}, C_{p_\lambda}\} = \{\mathcal{O}_\theta^{C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}}, C_{p_\lambda}\} = \frac{\partial}{\partial \lambda} \sum_{n=0}^{\infty} \frac{(\mathcal{O}_{C_{p_\mu}}^{C_\lambda})^n}{n!} \{\mathcal{O}_\theta^{C_\lambda}, \mathcal{O}_{C_\mu}^{C_\lambda}\}_{(n)} = 0.$$

The last step follows from the fact that, if we consider (15), we see that, for functions f that do not depend on p_λ , the term linear in λ in C_λ does not contribute to the nested Poisson bracket. Hence, neither $\mathcal{O}_{C_{p_\mu}}^{C_\lambda}$ nor $\mathcal{O}_{C_\mu}^{C_\lambda}$ depend on λ . Since also $\partial_\lambda \mathcal{O}_\theta^{C_\lambda} = 0$, the nested Poisson bracket in the above equation is independent of λ too.

The same steps can be performed if we replace Q^3 by P_3 as well as the matter degrees of freedom Q^I, P_I , and thus we have shown that, indeed, Q^3, P_3, Q^I, P_I mutually commutes with Q^1, Q^2, P_1, P_2 .

Finally, we still need to show that Q^3, P_3 as well as Q^I, P_I satisfy standard CCR and mutually commute. For showing that they satisfy standard CCR, we restrict our discussion to the canonical pair Q^3, P_3 and discuss the matter variables afterwards. For this purpose it

is more convenient to directly work with the second class constraints $(C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}, C_{p_\lambda}, \mathcal{O}_{p_\mu}^{C_\lambda})$ and the variables $\mathcal{O}_\theta^{C_\lambda}, \mathcal{O}_p^{C_\lambda}$ and their corresponding algebra. Then, we obtain

$$\{\mathcal{O}_\theta^{C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}}, \mathcal{O}_p^{C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}}\} = \mathcal{O}_{\{\mathcal{O}_\theta^{C_\lambda}, \mathcal{O}_p^{C_\lambda}\}_D^{C_\mu}} = \mathcal{O}_{\{\mathcal{O}_\theta^{C_\lambda}, \mathcal{O}_p^{C_\lambda}\}} = \frac{\gamma\kappa}{6} \tag{18}$$

with

$$\begin{aligned} \{\mathcal{O}_\theta^{C_\lambda}, \mathcal{O}_p^{C_\lambda}\}_D^{C_\mu} &:= \{\mathcal{O}_\theta^{C_\lambda}, \mathcal{O}_p^{C_\lambda}\} + \{\mathcal{O}_\theta^{C_\lambda}, \mathcal{O}_{C_\mu}^{C_\lambda}\}\{\mathcal{O}_{C_{p_\mu}}^{C_\lambda}, \mathcal{O}_p^{C_\lambda}\} - \{\mathcal{O}_\theta^{C_\lambda}, \mathcal{O}_{C_{p_\mu}}^{C_\lambda}\}\{\mathcal{O}_{C_\mu}^{C_\lambda}, \mathcal{O}_p^{C_\lambda}\} \\ &= \{\mathcal{O}_\theta^{C_\lambda}, \mathcal{O}_p^{C_\lambda}\} = \frac{\gamma\kappa}{6}. \end{aligned}$$

Here, we used that

$$\begin{aligned} \{\mathcal{O}_\theta^{C_\lambda}, \mathcal{O}_p^{C_\lambda}\} &= \mathcal{O}_{\{\theta, p\}_D^{C_\lambda}} = \mathcal{O}_{\{\theta, p\}} = \frac{\gamma\kappa}{6}, \\ \{\mathcal{O}_{C_{p_\mu}}^{C_\lambda}, \mathcal{O}_p^{C_\lambda}\} &= \mathcal{O}_{\{C_{p_\mu}, p\}_D^{C_\lambda}} = \mathcal{O}_{\{C_{p_\mu}, p\}} = 0 \\ \{\mathcal{O}_\theta^{C_\lambda}, \mathcal{O}_{C_{p_\mu}}^{C_\lambda}\} &= \mathcal{O}_{\{\theta, C_{p_\mu}\}_D^{C_\lambda}} = \mathcal{O}_{\{\theta, C_{p_\mu}\}} = 0, \end{aligned}$$

where, in all three cases, the Dirac bracket agrees with the Poisson bracket because θ, p commute with C_{p_λ} , and we further use that θ, p also Poisson commute with C_{p_μ} . The result in (18) can be directly carried over to Q^I, P_I the observables corresponding to the elementary matter phase space variables ϕ^A, π_A because the crucial property needed for this result is that both θ, p Poisson commute with C_{p_λ} and C_{p_μ} , which is also given for all ϕ^A, π_A . Moreover, this property is also sufficient to show that the sets Q^3, P_3, Q^I, P_I mutually commute because, also in this computation, the relevant Dirac brackets reduce to their corresponding Poisson brackets, and then we can use that the original phase space variables p, θ and ϕ^A, π_A are sets of mutually commuting elementary variables in phase space.

Summarizing the canonical transformation that allows to perform a (partial) Kuchař decomposition of the $(8 + 2L_\#)$ dimensional phase space separating the physical degrees of freedom from the gauge degrees of freedom is given by

$$\begin{aligned} Q^1 &:= C_\lambda, \quad P_1 := C_{p_\lambda}, \quad Q^2 := \mathcal{O}_{C_\mu}^{C_\lambda}, \quad P_2 := \mathcal{O}_{C_{p_\mu}}^{C_\lambda}, \quad Q^3 := \mathcal{O}_\theta^{C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}}, \quad P_3 := \mathcal{O}_p^{C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}}, \\ Q^4 &:= N, \quad P_4 := p_N, \quad Q^I := \mathcal{O}_{\phi^A}^{C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}}, \quad P_I := \mathcal{O}_{\pi_A}^{C_\lambda, \mathcal{O}_{C_\mu}^{C_\lambda}}, \end{aligned}$$

with $I = 5, \dots, L_\# + 4$. In terms of the original variables $\theta, p, \mu, p_\mu, \lambda, p_\lambda, \phi^A, \pi_A$, these are rather complicated functions consisting of partly nested power series with phase space depended coefficients. However, as shown above, the new variables satisfy standard CCR

$$\{Q^J, P_K\} = \tilde{\delta}_K^J, \quad \{Q^J, Q^K\} = 0, \quad \{P_J, P_K\} = 0, \quad J, K = 1, \dots, L_\# + 4$$

with

$$\tilde{\delta}_K^J = \delta_K^J \quad \text{for } J, K = 1, \dots, L_\# + 4 \quad \text{if } J = K \neq 3 \quad \text{and} \quad \tilde{\delta}_3^3 = \frac{\gamma\kappa}{6}.$$

The primary Hamiltonian in the new variables expressed as a function in terms of the old canonical variables looks similarly complicated but has, in the new variables, the following form:

$$\begin{aligned} H^P &= H_0(Q^3, P_3, Q^1 + f_\mu(Q^3, P_3), Q^I, P_I) + (Q^2 - g(Q^3, P_3, Q^1 + f_\mu(Q^3, P_3)))Q^1 \\ &\quad + \Lambda_\lambda P_2 + \Lambda_\mu P_1. \end{aligned}$$

If we consider the Hamiltonian constraint C_0 in (8) on the partially reduced phase space with respect to the constraints encoded in Q^1, P_1, Q^2, P_2 that we obtain by setting $Q^1 = Q^2 = P_1 = P_2 = 0$, we end up with

$$C_0 = H_0(Q^3, P_3, f_\mu(Q^3, P^3), Q^I, P_I)$$

showing that we obtain the Hamiltonian constraint in the $\bar{\mu}$ -scheme characterized by the choice of the function f_μ in (6) and, in this sense, dynamically implemented the $\bar{\mu}$ by a canonical transformation on an extended phase space in terms of a Kuchař-decomposition and the corresponding partial reduction afterwards. This can now be taken as the starting point to either implement a Dirac quantization or reduced quantization with respect to the Hamiltonian constraint and the remaining primary constraint p_N , which are both first class or work with corresponding effective models.

2.2. Gauge-Fixed Model for Cosmology

For the corresponding gauge fixed version of the model introduced in Section 2.1, we use that the set of constraints $C = \{C_\lambda, C_\mu, C_{p_\lambda}, C_{p_\mu}\}$ form a second class system. The Dirac matrix has the form

$$M = \begin{pmatrix} 0 & A_{\lambda\mu} & 1 & A_{\lambda p_\mu} \\ -A_{\lambda\mu} & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ -A_{\lambda p_\mu} & -1 & 0 & 0 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & A_{\lambda p_\mu} & -1 \\ 1 & -A_{\lambda p_\mu} & 0 & A_{\lambda\mu} \\ 0 & 1 & -A_{\lambda\mu} & 0 \end{pmatrix} \quad (19)$$

with

$$A_{\lambda p_\mu} = -\frac{6\sqrt{p}(6h(\theta\mu) + \theta\mu(-4h'(\theta\mu) + \theta\mu h''(\theta\mu)))}{\gamma^2\kappa\mu^4} \quad (20)$$

$$A_{\lambda\mu} = \frac{\partial_\theta f_\mu(2h(\theta\mu) - \theta\mu h'(\theta\mu)) + 2\mu p \partial_p f_\mu(-h'(\theta\mu) + \theta\mu h''(\theta\mu))}{2\gamma\mu^3\sqrt{p}}. \quad (21)$$

The Poisson brackets between θ, p and constraints C are non-trivial for $C_{\lambda,\mu}$ only. As a result, the Dirac brackets between θ and p recovers their Poisson brackets for generic gauge-fixing function f , namely

$$\{\theta, p\}_D = \frac{\gamma^\kappa}{6} \quad \{\theta, \theta\}_D = \{p, p\}_D = 0 \quad (22)$$

Likewise, we obtain for the matter degrees of freedom

$$\{\phi^A, \pi_B\}_D = \delta_B^A \quad \{\phi_A, \phi_B\}_D = \{\pi_A, \pi_B\}_D = 0, \quad A, B \in 1, \dots, L_\#. \quad (23)$$

The remaining Hamiltonian constraint has, in the partially reduced phase space, the following form

$$C = H_0(\theta, p, f_\mu(\theta, p), \phi^A, \pi_A). \quad (24)$$

The canonical transformation in Section 2.1 and the gauge fixing here does not further restrict the function $f_\mu(\theta, p)$ and particularly allows to choose the special case of a constant function, which corresponds exactly to the μ_0 -scheme in LQC.

Now, if we require in addition, invariance under the following transformation

$$\theta \rightarrow \alpha\theta, \quad p \rightarrow \alpha^2 p \quad (25)$$

then this further restricts the form of $f_\mu(\theta, p)$ to $f_\mu(p) = \frac{c_1}{\sqrt{|p|}}$ where c_1 is some constant. In the case of the cosmological models, this is the only possible scale invariant choice. Choosing this constant to be $c_1 := \sqrt{\Delta}$ with $\Delta := 2\sqrt{3}\pi\gamma\ell_p^2$, where γ is the Barbero–

Immirzi parameter and ℓ_p denotes the Planck length, we end up with the usual $\bar{\mu}$ scheme used in LQC [52].

In this section, we demonstrate that we can implement the choice of a generic phase-space dependent function in the argument of the polymerization function dynamically in the following sense: in Section 2.1, this was achieved in terms of a canonical transformation on the extended phase space that allows a Kuchař decomposition [48,49] of the extended phase space. We extended the usual kinematical phase space used in LQC by two additional canonical pairs, which come along with two first class constraints so that the number of physical degrees of freedom does not change.

A later reduction to the physical sector associated with these additional first class constraints then corresponds to a specific choice of a $\bar{\mu}$ -scheme. Furthermore, we showed that, in this extended phase space, we can also understand the choice of a given $\bar{\mu}$ -scheme as a gauge fixing. We extend our analysis to the spherically symmetric case in the next section.

3. Implementing the $\bar{\mu}$ -Scheme in Spherically Symmetric Models

In this section, we want to generalize the formalism used for cosmology in Section 2.1 to spherically symmetric models. The main difference is that on the one hand, we deal with a 1 + 1-dimensional field theory, and thus the diffeomorphism constraint is no longer trivial. Moreover, since the number of gravitational (field) variables is larger here, we need to extend the phase space by more variables in order to be able to allow an independent polymerization for the different connection variables involved. In case we describe the Schwarzschild black hole using isometry with Kantowski–Sachs vacuum cosmology, where the diffeomorphism is trivially satisfied, the system is similar to cosmological models with a larger phase space.

3.1. Extended Phase Space for Spherically Symmetric Models: Canonical Transformation and Dirac Observables

In spherically symmetric symmetry-reduced case after implementing the Gauß constraint, the Ashtekar–Barbero variables (A_a^j, E_j^a) have the following form

$$\begin{aligned}
 A_a^j \tau_j dX^a &= 2\gamma K_x(x) \tau_1 dx + \left(\gamma K_\varphi(x) \tau_2 + \frac{\partial_x E^x(x)}{2E^\varphi(x)} \tau_3 \right) d\theta \\
 &\quad + \left(\gamma K_\varphi(x) \tau_3 - \frac{\partial_x E^x(x)}{2E^\varphi(x)} \tau_2 \right) \sin(\theta) d\varphi + \cos(\theta) \tau_1 d\varphi \\
 E_j^a \tau_j \frac{\partial}{\partial X^a} &= E^x(x) \sin(\theta) \tau_1 \partial_x + (E^\varphi(x) \tau_2) \sin(\theta) \partial_\theta + (E^\varphi(x) \tau_3) \partial_\varphi,
 \end{aligned}$$

where $X^a = (x, \theta, \varphi)$, $a = 1, 2, 3$ denote spherical coordinates, as before, γ the Barbero–Immirzi parameter and $\tau_j = -\frac{1}{2}\sigma_j$ with σ_j being the Pauli matrices. The with respect to the Gauß constraint partially gauge-fixed phase space includes two canonical pairs denoted by $(K_x(x), E^x(x))$ and $(K_\varphi(x), E^\varphi(x))$. In addition, as in the cosmological case, we consider a set of matter degrees of freedom that we do not further specify whose phase space variables are $(\phi^A(x), \pi_A(x))$ with $A \in 1, \dots, L_\#$. Their non-vanishing Poisson brackets read

$$\{K_x(x), E^x(y)\} = G\delta(x, y) \quad \{K_\varphi(x), E^\varphi(y)\} = G\delta(x, y) \quad \{\phi^A(x), \pi_B(y)\} = \delta_B^A \delta(x, y),$$

with G being Newton’s constant. We consider this phase space as the starting point for our extension. We introduce the following additional set of canonical pairs

$$(e_1(x), p_{e_1}(x)), \quad (e_2(x), p_{e_2}(x)), \quad (\mu_x(x), p_{\mu_x}(x)), \quad (\mu_\varphi(x), p_{\mu_\varphi}(x)).$$

with standard CCR

$$\{e_I(x), p_{e_J}(y)\} = \delta_I^J \delta(x, y), \quad I, J = 1, 2 \quad \{\mu_K(x), p_{\mu_L}(y)\} = \delta_L^K \delta(x, y), \quad K, L = x, \varphi.$$

Note that, here, we considered already the partially reduced phase space with respect to the constraint $p_N(x)$ associated with the momentum of the lapse function $N(x)$ such that the latter becomes a Lagrange multiplier here and is not part of the elementary-phase space variables. We start with the following primary Hamiltonian

$$H = \int dx \left[\mathcal{C}_{\text{tot}}^\Delta + N^x \mathcal{C}_x^{\text{tot}} + e_1(\mu_x - f_{\mu_x}(E^x, E^\varphi, K_x, K_\varphi)) + e_2(\mu_\varphi - f_{\mu_\varphi}(E^x, E^\varphi, K_x, K_\varphi)) + \Lambda_{N^x} p_{N^x} + \Lambda_{e_1} p_{e_1} + \Lambda_{e_2} p_{e_2} + \Lambda_{\mu_x} p_{\mu_x} + \Lambda_{\mu_\varphi} p_{\mu_\varphi} \right](x), \tag{26}$$

where we considered a partial gauge fixing that yields for the lapse function $N = 1$. This can, for instance, be achieved by choosing a suitable external matter field that is chosen as a temporal reference field such that $N = 1$ is implemented. As a consequence, in contrast to the cosmological model in Section 2.1 here the polymerized contribution $\mathcal{C}_{\text{tot}}^\Delta$ is not a secondary constraint but plays the role of a physical Hamiltonian instead, once the reduction with respect to the spatial diffeomorphism constraint has also been performed. We consider here a partial gauge fixing because, in general, we also expect further restrictions on the possible polymerizations from the closure of the algebra of constraints. Such a general analysis is beyond the scope of this article and is planned for future work, although the analysis performed here will be useful as a first step.

Here, $\mathcal{C}_{\text{tot}}^\Delta$ and $\mathcal{C}_x^{\text{tot}}$ are given by

$$\mathcal{C}_{\text{tot}}^\Delta = \mathcal{C}^\Delta + \mathcal{C}^{\text{matter}}(E^x, E^\varphi, \phi^A, \pi_A) \quad \mathcal{C}_x^{\text{tot}} = \mathcal{C}_x + \mathcal{C}_x^{\text{matter}}(\phi^A, \pi_A) \tag{27}$$

with

$$\mathcal{C}^\Delta(x) = \frac{1}{2G} \left[\frac{E^\varphi}{\sqrt{E^x}} \left(\frac{f_1(\mu_x K_x \mu_\varphi K_\varphi)}{\mu_\varphi^2} + \Gamma^2 - 1 - 4E^x \frac{f_2(\mu_x K_x \mu_\varphi K_\varphi)}{\mu_x \mu_\varphi E^\varphi} \right) + 2\sqrt{E^x} \Gamma' \right](x) \tag{28}$$

$$\mathcal{C}_x = \frac{1}{G} \left(E^\varphi K_\varphi' - K_x E^{x'} \right)(x) \tag{29}$$

with $\Gamma(x) = \left(\frac{E^{x'}}{2E^\varphi} \right)(x)$ the spin connection, and f_1, f_2 represent general polymerization functions satisfying

$$\lim_{\mu_\varphi \rightarrow 0} \frac{f_1(\mu_x K_x, \mu_\varphi K_\varphi)}{\mu_\varphi^2} = K_\varphi^2 \quad \lim_{\mu_x, \mu_\varphi \rightarrow 0} \frac{f_2(\mu_x K_x, \mu_\varphi K_\varphi)}{\mu_x \mu_\varphi} = K_x K_\varphi \tag{30}$$

In order that \mathcal{C}^Δ is a scalar density of weight 1, we are forced to choose μ_x being as scalar with density weight -1 since K_x has density weight 1. Furthermore, μ_φ needs to have density weight 0. As a result, the only allowed functions f_{μ_x} and f_{μ_φ} will be of the form

$$f_{\mu_x}(E^x, E^\varphi, K_x, K_\varphi) = (E^\varphi)^{-1} \tilde{f}_{\mu_x}(E^x, K_\varphi, (E^\varphi)^{-1} K_x), \tag{31}$$

$$f_{\mu_\varphi}(E^x, E^\varphi, K_x, K_\varphi) = \tilde{f}_{\mu_\varphi}(E^x, K_\varphi, (E^\varphi)^{-1} K_x). \tag{32}$$

Here, the appearance of E^φ , and the combination of $(E^\varphi)^{-1} K_x$ are essential in order to have density weight of one and zero for f_{μ_x} and f_{μ_φ} , respectively. This completely removes the possibility of the usual μ_0 scheme in the effective dynamics if we have polymerization of K_x . If we require, in addition, the scaling invariance of f_1 and f_2 and remove the K dependence in the gauge-fixing function, the only possible ansatz for f_{μ_x} and f_{μ_φ} is the $\bar{\mu}$ scheme used in [4,29,31,41], which is given by

$$f_{\mu_x}(E^x, E^\varphi) = \alpha_1 \sqrt{E^x} (E^\varphi)^{-1}, \quad f_{\mu_\varphi}(E^x, E^\varphi) = \alpha_2 \sqrt{E^x} \tag{33}$$

for some constants α_1, α_2 proportional to the minimal area gaps introduced in LQG.

Note that the conditions (31) and (32) come from the fact that K_x has a non-trivial density weight, in contrast to K_φ , which has density weight 0. If we exclude the polymerization of K_x and consider the polymerization of K_φ only:

$$f_1(\mu_x K_x, \mu_\varphi K_\varphi) = \tilde{f}_1(\mu_\varphi K_\varphi), \quad f_2(\mu_x K_x, \mu_\varphi K_\varphi) = \mu_x K_x \tilde{f}_2(\mu_\varphi K_\varphi) \tag{34}$$

e.g., as in the models presented in [6,9,10,16,17]. In this case, we can remove the condition on f_{μ_x} , and a gauge fixing leading to a μ_0 -scheme is still allowed.

The model has five primary constraints

$$C_{\text{Pri}} = [p_{N^x}, p_{e_1}, p_{e_2}, p_{\mu_x}, p_{\mu_\varphi}] \approx 0. \tag{35}$$

Starting from (26), we obtain the following five secondary constraints:

$$C_{\text{Sec}} = \{H, C_{\text{Pri}}\} = [C_x^{\text{tot}}, C_{\mu_x}, C_{\mu_\varphi}, C_{e_1}, C_{e_2}] \approx 0. \tag{36}$$

Here, C_{μ_x, μ_φ} are the constraints related to implementing the $\bar{\mu}$ -scheme, they read

$$C_{\mu_x} = \mu_x - f_{\mu_x}, \quad C_{\mu_\varphi} = \mu_\varphi - f_{\mu_\varphi} \tag{37}$$

and C_{e_1, e_2} fix e_1, e_2 to be:

$$C_{e_1} = e_1 + \frac{4E^x f_2 \mu_\varphi - K_x \mu_x (E^\varphi \mu_x \partial_X f_1 + 4E^x \mu_\varphi \partial_X f_2)}{2\sqrt{E^x} \mu_x^2 \mu_\varphi^2}, \tag{38}$$

$$C_{e_2} = e_2 + \frac{E^\varphi \mu_x (2f_1 - K_\varphi \mu_\varphi \partial_Y f_1) + 4E^x \mu_\varphi (f_2 - K_\varphi \mu_\varphi \partial_Y f_2)}{2\sqrt{E^x} \mu_x \mu_\varphi^3} \tag{39}$$

where we defined $X := \mu_x K_x$, $Y := \mu_\varphi K_\varphi$. The stability of $C_{e_1}, C_{e_2}, C_{\mu_x}, C_{\mu_\varphi}$ can be ensured by choosing the Lagrangian multipliers $\Lambda_{e_1}, \Lambda_{e_2}, \Lambda_{\mu_x}, \Lambda_{\mu_\varphi}$ appropriately, while the stability of C_x^{tot} is automatically satisfied,

$$\{H, C_x^{\text{tot}}\} \approx 0. \tag{40}$$

The sets C_{Pri} and C_{Sec} from (35) and (36) include all constraints of the model, so we end up with ten constraints

$$C_{\text{all}} = [p_{N^x}, p_{e_1}, p_{e_2}, p_{\mu_x}, p_{\mu_\varphi}, C_x^{\text{tot}}, C_{\mu_x}, C_{\mu_\varphi}, C_{e_1}, C_{e_2}] \approx 0.$$

It remains to classify the constraints (or suitable combinations thereof) as first or second class. We immediately see that p_{N^x} is first class. Since C_x^{tot} may not be first class as it will have non-trivial brackets with the constraints $C_{\mu_x}, C_{\mu_\varphi}$ as well as the polymerization functions in $C_{e_1}, C_{e_2}, C_{\mu_x}, C_{\mu_\varphi}$. However, to circumvent this, we introduce an extended diffeomorphism \tilde{C}_x^{tot} on the entire phase space by taking into account the density weight of the individual phase space variables:

$$\begin{aligned} \tilde{C}_x^{\text{tot}}(x) = & C_x^{\text{tot}}(x) - 2p_{\mu_x}(x)\mu'_x(x) - p'_{\mu_x}(x)\mu_x(x) - p_{\mu_\varphi}(x)\mu'_\varphi(x) \\ & + p_{e_1}(x)e'_1(x) + 2p'_{e_1}(x)e_1(x) + p'_{e_2}(x)e_2(x) \end{aligned} \tag{41}$$

One can verify that \tilde{C}_x^{tot} is first class, the explicit derivation of this result is presented in Appendix A. Note that this extension is always possible and does not change the constraint structure because the linear momenta involved in the extra terms are all constraints themselves. The remaining constraints are second class. Hence, from now on, we work with the following set of constraints

$$\tilde{C}_{\text{all}} = [p_{N^x}, p_{e_1}, p_{e_2}, p_{\mu_x}, p_{\mu_\varphi}, \tilde{C}_x^{\text{tot}}, C_{\mu_x}, C_{\mu_\varphi}, C_{e_1}, C_{e_2}] \approx 0,$$

including two first class and eight second class constraints.

In an analogous manner to the cosmological case in Section 2.1, we construct a canonical transformation on the extended phase space that allows to perform a Kuchař decomposition of the extended phase space introduced above. For this purpose, we consider the

subset of second class constraints $C = [p_{e_1}, p_{e_2}, p_{\mu_x}, p_{\mu_\phi}, C_{e_1}, C_{e_2}, C_{\mu_x}, C_{\mu_\phi}]$. Similar to the cosmological case, $p_{\mu_x}, p_{\mu_\phi}, C_{\mu_x}, C_{\mu_\phi}$ do not commute with C_{e_1}, C_{e_2} , in addition, C_{e_1}, C_{e_2} do not mutually commute. Therefore, as a first step, we introduce the following set of variables

$$\begin{aligned} Q^1 &:= C_{e_2}, \quad P_1 := p_{e_2}, \quad Q^2 := \mathcal{O}_{C_{e_1}}^{C_{e_2}} = \mathcal{O}_{C_{e_1}}^{Q^1}, \quad P_2 := \mathcal{O}_{C_{p_{e_1}}}^{C_{e_2}} = \mathcal{O}_{C_{p_{e_1}}}^{Q^1}, \\ Q^3 &:= \mathcal{O}_{C_{\mu_x}}^{C_{e_2}, \mathcal{O}_{C_{e_1}}^{C_{e_2}}} = \mathcal{O}_{C_{\mu_x}}^{Q^1, Q^2}, \quad P_3 := \mathcal{O}_{p_{\mu_x}}^{C_{e_2}, \mathcal{O}_{C_{e_1}}^{C_{e_2}}} = \mathcal{O}_{p_{\mu_x}}^{Q^1, Q^2}, \\ Q^4 &:= \mathcal{O}_{C_{\mu_\phi}}^{C_{e_2}, \mathcal{O}_{C_{e_1}}^{C_{e_2}}, \mathcal{O}_{C_{\mu_x}}^{C_{e_2}, \mathcal{O}_{C_{e_1}}^{C_{e_2}}}} = \mathcal{O}_{C_{\mu_\phi}}^{Q^1, Q^2, Q^3}, \quad P_4 := \mathcal{O}_{p_{\mu_\phi}}^{C_{e_2}, \mathcal{O}_{C_{e_1}}^{C_{e_2}}, \mathcal{O}_{C_{\mu_x}}^{C_{e_2}, \mathcal{O}_{C_{e_1}}^{C_{e_2}}}} = \mathcal{O}_{p_{\mu_\phi}}^{Q^1, Q^2, Q^3}. \end{aligned}$$

In the spherically symmetric models, the individual dual observable maps can be defined as

$$\mathcal{O}_f^{Q^I}(x) := \sum_{n=0}^{\infty} \int dy_1 \cdots \int dy_n \int \frac{(P_I(y_1) \cdots P_I(y_n))}{n!} \{ \dots \{ \{ f(x), Q_I(y_1) \}, Q_I(y_2) \} \dots \}, Q_I(y_n) \}. \tag{42}$$

For these subsets of canonical variables, we have

$$\{Q^I(x), P_J(y)\} = \delta_J^I \delta(x, y), \quad \{Q^I(x), Q^J(y)\} = 0, \quad \{P_I(x), P_J(y)\} = 0, \quad I = 1 \cdots 4.$$

This is explicitly shown in Appendix D for the canonical pair (Q^4, P_4) . If we consider any of the other pairs (Q^I, P_I) the computations work similarly with fewer steps since for the construction of these observables fewer individual observable maps have been applied. As the computation in Appendix D show in these cases the number of observable maps fit again well such that the involved Poisson brackets can be mapped to their corresponding Dirac brackets successively. By construction the observables Q^I mutually commute. It only remains to show that all observables related to momenta mutually commute and that these commute with all but one Q^I . In the two latter cases, the number of applied observable maps for a given Q^I and P_I will not be the same. Therefore, to show that these Poisson brackets vanish works slightly differently in these cases. As an example, we consider

$$\begin{aligned} \{P_4(x), P_2(y)\} &= \{ \mathcal{O}_{p_{\mu_\phi}}^{Q^1, Q^2, Q^3}, \mathcal{O}_{p_{e_1}}^{Q^1}(y) \} = \{ \mathcal{O}_{p_{\mu_\phi}}^{Q^2, Q^3, Q^1}(x), \mathcal{O}_{p_{e_1}}^{Q^1}(y) \} = \mathcal{O}_{\{ \mathcal{O}_{p_{\mu_\phi}}^{Q^2, Q^3}(x), p_{e_1}(y) \}_D}^{Q^1} \\ &= \mathcal{O}_{\{ \mathcal{O}_{p_{\mu_\phi}}^{Q^2, Q^3}(x), p_{e_1}(y) \}}^{Q^1} \end{aligned}$$

In the second step, we used that the Q^I s mutually commute and in the last step that p_{e_1} commutes with $Q^1 = C_{e_2}, P_1 = p_{e_2}$. Finally, from

$$\{ \mathcal{O}_{p_{\mu_\phi}}^{Q^2, Q^3}(x), p_{e_1}(y) \} = \frac{\delta}{\delta e_1(y)} \mathcal{O}_{p_{\mu_\phi}}^{Q^2, Q^3}(x) = 0 \quad \text{we find} \quad \{P_4(x), P_2(y)\} = 0.$$

That the Poisson bracket above vanishes follows from the fact that $e_1(y)$ is involved in C_{e_1} and when it is used in the observable map for other phase space variables than p_{e_1} the part linearly in $e_1(y)$ does not contribute to the nested Poisson bracket. Now, we extend the set of canonical pairs in (42) by the corresponding observables with respect to Q^1, \dots, Q^4 of the geometrical and matter degrees of freedom. For this purpose let us introduce the notation $\mathcal{O}_f^{Q^{1\dots 4}}$ for the observable of the function f with respect to the constraints $Q^{1\dots 4}$. Then, we consider the following canonical transformed variables

$$\begin{aligned} Q^5 &:= \mathcal{O}_{K_x}^{Q^{1\dots 4}}, \quad P_5 := \mathcal{O}_{E^x}^{Q^{1\dots 4}}, \quad Q^6 := \mathcal{O}_{K_\phi}^{Q^{1\dots 4}}, \quad P_6 := \mathcal{O}_{E^\phi}^{Q^{1\dots 4}}, \\ Q^J &:= \mathcal{O}_{\phi^A}^{Q^{1\dots 4}}, \quad P_J := \mathcal{O}_{\pi_A}^{Q^{1\dots 4}}, \quad J = 7, \dots, L_\# + 6 \end{aligned}$$

As shown in the Appendix D with these additional variables, we obtain a set of canonical variables satisfying standard CCR given by

$$\{Q^I(x), P_J(y)\} = \tilde{\delta}_J^I \delta(x, y), \quad \{Q^I(x), Q^J(y)\} = 0, \quad \{P_I(x), P_J(y)\} = 0, \quad I, J = 1, \dots, L_{\#} + 6.$$

with

$$\tilde{\delta}_J^I = \delta_J^I \quad \text{for } I, J = 1, \dots, L_{\#} + 6 \quad \text{if } I = J \neq 5, 6 \quad \text{and} \quad \tilde{\delta}_5^5 = \tilde{\delta}_6^6 = G.$$

In the new coordinates (Q^I, P_I) , a Kuchař decomposition [48,49] of the kinematical phase space can be achieved. In the explicit computations of the individual Poisson brackets one sees that the reason why they satisfy CCR is very close to the case of the cosmological model but generalized to the field theory case and a larger number of phase space variables. In both cases it turns out to be crucial that the constraints involving the gauge-fixing functions f in the cosmological case and $f_{\mu_x}, f_{\mu_\varphi}$ are given in a form in which they split into a part depending on the variables μ and μ_x, μ_φ , respectively, and a further part that depends on the gravitational degrees of freedom only, where the latter, in principle, can be generalized to a dependence on the matter degrees of freedom as well. Furthermore, also the fact that λ and e_1, e_2 , respectively, only enter linearly in the respective constraints is important.

3.2. Gauge-Fixed Model

Similar to the cosmological model, we also discuss the corresponding gauge-fixed model in the spherically symmetric case. As discussed above the set of constraints $C = \{p_{e_1}, p_{e_2}, p_{\mu_x}, p_{\mu_\varphi}, C_{e_1}, C_{e_2}, C_{\mu_x}, C_{\mu_\varphi}\}$ forms a second class system. The Dirac matrix has a similar form as (19), which reads

$$D_{BH} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{35} & D_{36} & -1 & 0 \\ 0 & 0 & 0 & 0 & D_{45} & D_{46} & 0 & -1 \\ 1 & 0 & -D_{35} & -D_{45} & 0 & D_{56} & D_{57} & D_{58} \\ 0 & 1 & -D_{36} & -D_{46} & -D_{56} & 0 & D_{67} & D_{68} \\ 0 & 0 & 1 & 0 & -D_{57} & -D_{67} & 0 & D_{78} \\ 0 & 0 & 0 & 1 & -D_{58} & -D_{68} & -D_{78} & 0 \end{pmatrix}, \tag{43}$$

where the non-trivial components D_{ij} are given explicitly in Appendix B. Note that $D_{78} = 0$ if the functions f_{μ_x} and f_{μ_φ} in (31) do not contain K_x, K_φ . The inverse of the D_{BH} reads

$$D_{BH}^{-1} = \begin{pmatrix} 0 & \tilde{D} & D_{57} - D_{45}D_{78} & D_{58} + D_{35}D_{78} & 1 & 0 & D_{35} & D_{45} \\ -\tilde{D} & 0 & D_{67} - D_{46}D_{78} & D_{68} + D_{36}D_{78} & 0 & 1 & D_{36} & D_{46} \\ D_{45}D_{78} - D_{57} & D_{46}D_{78} - D_{67} & 0 & D_{78} & 0 & 0 & 1 & 0 \\ -D_{58} - D_{35}D_{78} & -D_{68} - D_{36}D_{78} & -D_{78} & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -D_{35} & -D_{36} & -1 & 0 & 0 & 0 & 0 & 0 \\ -D_{45} & -D_{46} & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{44}$$

with

$$\tilde{D} = D_{56} + D_{36}D_{57} + D_{46}D_{58} - D_{35}D_{67} - D_{45}D_{68} - D_{36}D_{45}D_{78} + D_{35}D_{46}D_{78}. \tag{45}$$

Since $K_x, K_\varphi, E^x, E^\varphi$ clearly commute with $p_{e_1}, p_{e_2}, p_{\mu_x}, p_{\mu_\varphi}$, and we have a vanishing D_{BH}^{-1} for its sub-matrix corresponding to $C_{e_1}, C_{e_2}, C_{\mu_x}, C_{\mu_\varphi}$, we recover the standard CCR between the gravitational degrees of freedom $K_x, K_\varphi, E^x, E^\varphi$

$$\{K_x(x), E^x(y)\}_D = \{K_\varphi(x), E^\varphi(y)\}_D = G\delta(x - y) \tag{46}$$

$$\{K_x(x), K_\varphi(y)\}_D = \{E^x(x), E^\varphi(y)\}_D = 0 \tag{47}$$

and, for matter contributions, we have

$$\{\phi^A(x), \pi_B(y)\}_D = \delta_B^A \delta(x, y), \quad \{\phi^A(x), \phi^B(y)\}_D = \{\pi_A(x), \pi_B(y)\}_D = 0, \tag{48}$$

where $A, B \in 1, \dots, L_\#$. With the allowed choice of the functions f_{μ_x} and f_{μ_φ} in (31), that can be understood as a gauge fixing for μ_x and μ_φ , respectively, the gravitational contribution to the Hamiltonian in the gauge-fixed model reads

$$\mathcal{C}^\Delta(x) = \frac{1}{2G} \left[\frac{E^\varphi}{\sqrt{E^x}} \left(\frac{f_1(\tilde{f}_{\mu_x} \frac{K_x}{E^\varphi}, \tilde{f}_{\mu_\varphi} K_\varphi)}{\tilde{f}_{\mu_\varphi}^2} - 4E^x \frac{f_2(\tilde{f}_{\mu_x} \frac{K_x}{E^\varphi}, \tilde{f}_{\mu_\varphi} K_\varphi)}{\tilde{f}_{\mu_x} \tilde{f}_{\mu_\varphi}} + \Gamma^2 - 1 \right) + 2\sqrt{E^x} \Gamma' \right] (x)$$

We can rewrite it as

$$\mathcal{C}^\Delta(x) = \frac{1}{2G} \left[\frac{E^\varphi}{\sqrt{E^x}} \left(f(E^x, K_\varphi, (E^\varphi)^{-1} K_x) + \Gamma^2 - 1 \right) + 2\sqrt{E^x} \Gamma' \right] (x), \tag{49}$$

where we combine f_1, f_2 and the gauge fixing conditions into a single polymerization function $f(E^x, K_\varphi, (E^\varphi)^{-1} K_x) := f_1 / \tilde{f}_{\mu_\varphi}^2 - 4E^x f_2 / (\tilde{f}_{\mu_\varphi} \tilde{f}_{\mu_x})$. The polymerization function depends on the combination of $(E^\varphi)^{-1} K_x$ rather than depending on either of them only. We remark that, in the gauge-fixed model with (49), we have

$$\{\mathcal{C}^\Delta(x), \mathcal{C}_x(y)\}_D = -\partial_x \delta(x - y) \mathcal{C}^\Delta(y) \tag{50}$$

which confirms that \mathcal{C}^Δ has density weight one. Further requiring scale invariance under the transformation

$$E^x \rightarrow \mu^2 E^x, \quad E^\varphi \rightarrow \mu^2 E^\varphi, \quad K_x \rightarrow \mu K_x, \quad K_\varphi \rightarrow \mu K_\varphi, \tag{51}$$

leads to the following \mathcal{C}^Δ ,

$$\mathcal{C}^\Delta(x) = \frac{1}{2G} \left[\frac{E^\varphi}{\sqrt{E^x}} \left(f \left(\frac{K_\varphi}{\sqrt{E^x}}, \frac{\sqrt{E^x} K_x}{E^\varphi} \right) + \Gamma^2 - 1 \right) + 2\sqrt{E^x} \Gamma' \right] (x), \tag{52}$$

which is compatible with the choice of $\bar{\mu}$ -scheme used in [4,28,29,31,37]. We note that there are special cases where one or both f_{μ_x} and f_{μ_φ} are Dirac observables and constants of motion under the effective dynamics, where f_{μ_x} and f_{μ_φ} can still appear as constants, e.g., similar to the cases in [15].

4. Conclusions

In this article, we demonstrated that we can implement the choice of a given $\bar{\mu}$ -scheme dynamically. Such an implementation corresponds to choosing a phase-space dependent function in the argument of a generic polymerization function, which replaces holonomies in effective models. We perform the analysis in homogeneous cosmological models as well as spherical symmetric models, which are 1 + 1-dimensional field theories. To construct the models, we extended the usual kinematical phase space by additional canonical pairs, which come along with a set of primary constraints so that the number of physical degrees of freedom does not change.

These additional constraints build a set of second class constraints for which the corresponding gauge-unfixed model can be constructed in the relational formalism. The dynamical implementation of a $\bar{\mu}$ -scheme can be achieved in terms of a canonical transformation on the extended phase space that allows a Kuchař decomposition [48,49] of the extended phase space where all constraints related to the dynamical implementation as

well as the Dirac observables with respect to these constraints are the elementary-phase space variables.

Furthermore, we showed that, in this extended phase space, we can also understand the choice of a given $\bar{\mu}$ -scheme as a gauge fixing, which gives the polymerized Hamiltonian with the given $\bar{\mu}$ -scheme in reduced phase space. This gives a possible way from a loop holonomy on a fixed lattice to a polymerization with a $\bar{\mu}$ -scheme. Such extension of the phase space can be included in the path integral formulation of full LQG [38] to obtain $\bar{\mu}$ -effective dynamics.

In cosmological models, the canonical transformation on the extended phase space and the gauge fixing does not further restrict the choice of gauge-fixing function. In particular, the special case of a constant function that corresponds exactly to the μ_0 -scheme in LQC is allowed. The $\bar{\mu}$ -scheme used in LQC is determined only when we require in addition, scale invariance. In contrast, in black hole models, the requirement of keeping the density weight of the scalar Hamiltonian restricts the choice of the gauge-fixing functions and, hence, the possible $\bar{\mu}$ -schemes.

In such case, a constant μ_0 -scheme is generally not allowed, unless one does not polymerize the densitized connection K_x carrying a density weight of one, e.g., as in the models presented in [6,9,10,16,17]. A special case is if the gauge-fixing functions themselves corresponding to some constant of motion under the effective dynamics, e.g., similar to the model presented in [14].

In this work, we only consider the polymerization of the gravitational variables. The analysis can be generalized to the models with polymerized matter degrees of freedom by introducing additional functions $f_{\mu_{\text{matter}}}$. With this, we can go beyond the standard μ_0 and $\bar{\mu}$ scheme and allow more general gauge-fixing functions, which depend on all variables containing both matter and geometry. Our results here, in particular, the construction of the canonical transformation on the extended phase space, will work similarly in this more general situation.

This becomes relevant for models, such as those in [21,53] in which the matter sector is also polymerized. In these cases, because the momentum of the matter variables is a quantity of density weight, one could also use it to construct quantities that yield a Hamiltonian with density weight one. Another interesting situation is to extend the analysis to cases beyond spherically symmetric spacetimes. We expect the canonical transformation on the extended phase space and the gauge-fixing method will still work, similar to the spherically symmetric case if the involved gauge-fixing functions depend on the matter and gravitational degrees of freedom only.

In the general case, the requirement to have a Hamiltonian with density weight one may give us more restrictions on the gauge-fixing functions, thus, constraining the allowed $\bar{\mu}$ -schemes. In the models studied in this work, the Gauß constraints were already gauge fixed. The gauge fixing of the Gauß constraint was also required in some early attempts to obtain a $\bar{\mu}$ -scheme effective dynamics from full LQG [41,42]. It is interesting to generalize the current approach to models that emerge from the LQG theory where the Gauß constraints have not been gauge fixed.

The presence of Gauß constraints may give strong constraints to the possible form of such gauge fixing conditions. Moreover, in this work, we did not consider a polymerization of the diffeomorphism constraints. Such an assumption must be dropped if, for example, we work with models based on a lattice theory on a fixed graph. We expect that work with polymerized diffeomorphism constraints will provide further constraints on admissible $\bar{\mu}$ -schemes.

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Appendix A. \tilde{C}_x^{tot} and Its Poisson Bracket

Here, we keep the gauge-fixing functions general without imposing (31), thus the weight of μ_x, μ_φ and e_1, e_2 are general. We derive the correct weight—and thus the restriction to gauge-fixing functions—such that we have a first class \tilde{C}_x^{tot} . We start with the following ansatz,

$$\tilde{C}_x^{tot}(x) = C_x^{tot}(x) + (a - 1)p_{\mu_x}(x)\mu'_x(x) + ap'_{\mu_x}(x)\mu_x(x) + (b - 1)p_{\mu_x}(x)\mu'_x(x) + bp'_{\mu_x}(x)\mu_x(x) + cp_{e_1}(x)e'_1(x) + (1 + c)p'_{e_1}(x)e_1(x) + dp_{e_2}(x)e'_2(x) + (1 + d)p'_{e_2}(x)e_2(x) \tag{A1}$$

The non-trivial Poisson bracket between \tilde{C}_x and $C_{e_1}, C_{e_2}, C_{\mu_x}, C_{\mu_\varphi}$ is given by

$$\{\tilde{C}_x(x), C_{\mu_x}(y)\} = -\partial_x\delta(x - y)\left(af_{\mu_x} - K_x\partial_{E^x}f_{\mu_x} - E^\varphi\partial_{K_\varphi}f_{\mu_x}\right)(y) \tag{A2}$$

$$\{\tilde{C}_x(x), C_{\mu_\varphi}(y)\} = -\partial_x\delta(x - y)\left(bf_{\mu_\varphi} - K_x\partial_{E^x}f_{\mu_\varphi} - E^\varphi\partial_{K_\varphi}f_{\mu_\varphi}\right)(y) \tag{A3}$$

$$\begin{aligned} \{\tilde{C}_x(x), C_{e_1}(y)\} &= \frac{\partial_x\delta(x-y)}{2\sqrt{E^x}f_{\mu_x}^2f_{\mu_\varphi}^2}\left(4(1 + 2a + b + c)E^xf_2f_{\mu_\varphi} - 4bE^xf_{\mu_\varphi}^2K_\varphi\partial_Yf_2 \right. \\ &\quad - (-1 + 2b + c)E^\varphi f_{\mu_x}^2K_x\partial_Xf_1 + f_{\mu_x}K_x(-4(1 + 2a + b + c)E^xf_{\mu_\varphi}\partial_Xf_2 \\ &\quad + bE^\varphi f_{\mu_x}f_{\mu_\varphi}K_\varphi\partial_X\partial_Yf_1 + (1 + a)E^\varphi f_{\mu_x}^2K_x\partial_X^2f_1 \\ &\quad \left. + 4E^xf_{\mu_\varphi}\left(bf_{\mu_\varphi}K_\varphi\partial_X\partial_Yf_2 + (1 + a)f_{\mu_x}K_x\partial_X^2f_2\right)\right)(y) \end{aligned} \tag{A4}$$

$$\begin{aligned} \{\tilde{C}_x(x), C_{e_2}(y)\} &= \frac{\partial_x\delta(x-y)}{2\sqrt{E^x}f_{\mu_x}f_{\mu_\varphi}^3}\left(E^\varphi f_{\mu_x}(2(3b + d)f_1 - 2(1 + a)f_{\mu_x}K_x\partial_Xf_1 \right. \\ &\quad + f_{\mu_\varphi}K_\varphi\left(-((4b + d)\partial_Yf_1) + bf_{\mu_\varphi}K_\varphi\partial_Y^2f_1 + (1 + a)f_{\mu_x}K_x\partial_X\partial_Yf_1\right)) \\ &\quad + 4E^xf_{\mu_\varphi}\left((1 + a + 2b + d)f_2 - (1 + a + 2b + d)f_{\mu_\varphi}K_\varphi\partial_Yf_2 \right. \\ &\quad \left. + bf_{\mu_\varphi}^2K_\varphi^2\partial_Y^2f_2 + (1 + a)f_{\mu_x}K_x\left(-\partial_Xf_2 + f_{\mu_\varphi}K_\varphi\partial_X\partial_Yf_2\right)\right)(y) \end{aligned} \tag{A5}$$

We notice that, in order to have $\{\tilde{C}_x, C_{e_1, e_2}\} \approx 0$ for generic f_1, f_2 , we need

$$a = -1, \quad b = 0, \quad c = 1, \quad d = 0 \tag{A6}$$

With this set of values, μ_x has density weight -1 and μ_φ has density weight 0 . $\{\tilde{C}_x, C_{\mu_x, \mu_\varphi}\}$ now becomes:

$$\{\tilde{C}_x(x), C_{\mu_x, \mu_\varphi}(y)\} = \partial_x\delta(x - y)\left(f_{\mu_x} + K_x\partial_{E^x}f_{\mu_x} - E^\varphi\partial_{K_\varphi}f_{\mu_x}\right)(y), \tag{A7}$$

$$\{\tilde{C}_x(x), C_{\mu_x, \mu_\varphi}(y)\} = \partial_x\delta(x - y)\left(K_x\partial_{E^x}f_{\mu_\varphi} - E^\varphi\partial_{K_\varphi}f_{\mu_\varphi}\right)(y) \tag{A8}$$

Requiring the right hand side to vanish gives the condition (31), which comes from requiring C^Δ to have the correct density weight. With this condition, one can easily check that

$$\{\tilde{C}_x(x), C_{tot}^\Delta(y)\} = \partial_x\delta(x - y)C^\Delta(y) \tag{A9}$$

where we use $\{\tilde{C}_x^{matter}(x), C^{matter}(y)\} = \partial_x\delta(x - y)C^{matter}(y)$. As a result, we also have $\{H, \tilde{C}_x^{tot}\} = 0$. With this fact, we finish the constraint analysis with $p_{N^x}, \tilde{C}_x^{tot}$ are first class, and $C = \{p_{e_1}, p_{e_2}, p_{\mu_x}, p_{\mu_\varphi}, C_{\mu_x}, C_{\mu_\varphi}, C_{e_1}, C_{e_2}\}$ form a second class system.

Appendix B. Explicit Form of the Dirac Matrix in BH

Here, we give, in detail, the elements $D_{ij} = D_{ij}\delta(x - y)$ of the Dirac matrix presented in (43). Since there are no partial derivatives appearing in the second class constraint, the

following elements are the same as these of the Dirac matrix for homogeneous Kontowski–Sachs spacetime.

$$\begin{aligned}
 D_{35} &= \frac{8E^x f_2 \mu_\varphi + K_x \mu_x (-8E^x \mu_\varphi \partial_X f_2 + K_x \mu_x (E^\varphi \mu_x \partial_X^2 f_1 + 4E^x \mu_\varphi \partial_X^2 f_2))}{2\sqrt{E^x} \mu_x^3 \mu_\varphi^2} \\
 D_{36} &= \frac{E^\varphi K_x (-2\partial_X f_1 + K_\varphi \mu_\varphi \partial_X \partial_Y f_1)}{2\sqrt{E^x} \mu_\varphi^3} \\
 &\quad + \frac{2\sqrt{E^x} (f_2 - K_\varphi \mu_\varphi \partial_Y f_2 - K_x \mu_x \partial_X f_2 + K_x K_\varphi \mu_x \mu_\varphi \partial_X \partial_Y f_2)}{\mu_x^2 \mu_\varphi^2} \\
 D_{45} &= \frac{E^\varphi K_x (-2\partial_X f_1 + K_\varphi \mu_\varphi \partial_X \partial_Y f_1)}{2\sqrt{E^x} \mu_\varphi^3} \\
 &\quad + \frac{2\sqrt{E^x} \mu_\varphi (f_2 - K_\varphi \mu_\varphi \partial_Y f_2 - K_x \mu_x \partial_X f_2 + K_x K_\varphi \mu_x \mu_\varphi \partial_X \partial_Y f_2)}{2\sqrt{E^x} \mu_x^2 \mu_\varphi^2} \\
 D_{46} &= \frac{E^\varphi (6f_1 + K_\varphi \mu_\varphi (-4\partial_Y f_1 + K_\varphi \mu_\varphi \partial_Y^2 f_1))}{2\sqrt{E^x} \mu_\varphi^4} \\
 &\quad + \frac{2\sqrt{E^x} (2f_2 + K_\varphi \mu_\varphi (-2\partial_Y f_2 + K_\varphi \mu_\varphi \partial_Y^2 f_2))}{\mu_x \mu_\varphi^3} \\
 D_{56} &= \frac{-2E^x K_x \mu_x \mu_\varphi (-E^\varphi \mu_x \partial_Y f_1 + K_\varphi \mu_\varphi (E^\varphi \mu_x \partial_Y^2 f_1 + 4E^x \mu_\varphi \partial_Y^2 f_2)) \partial_X f_1}{8E^{x^2} \mu_x^2 \mu_\varphi^5} \\
 &\quad + \frac{2E^x \mu_\varphi (-2f_1 + K_\varphi \mu_\varphi \partial_Y f_1) (-4E^x \mu_\varphi \partial_Y f_2 + K_x \mu_x (E^\varphi \mu_x \partial_X \partial_Y f_1 + 4E^x \mu_\varphi \partial_X \partial_Y f_2))}{8E^{x^2} \mu_x^2 \mu_\varphi^5} \\
 &\quad + \frac{(E^\varphi K_x \mu_x^2 \partial_X f_1 + 4E^x \mu_\varphi (f_2 - K_x \mu_x \partial_X f_2))}{8E^{x^2} \mu_x^2 \mu_\varphi^5} \times \\
 &\quad (-2E^\varphi \mu_x \partial_X f_1 + \mu_\varphi (-4E^x \partial_X f_2 + E^\varphi K_\varphi \mu_x \partial_X \partial_Y f_1 + 4E^x K_\varphi \mu_\varphi \partial_X \partial_Y f_2)) \\
 &\quad + \frac{(E^\varphi \partial_X f_1 + E^\varphi K_x \mu_x \partial_X^2 f_1 + 4E^x K_x \mu_\varphi \partial_X^2 f_2)}{8E^{x^2} \mu_x \mu_\varphi^5} \times \\
 &\quad (E^\varphi \mu_x (2f_1 - K_\varphi \mu_\varphi \partial_Y f_1) + 4E^x \mu_\varphi (-f_2 + K_\varphi \mu_\varphi \partial_Y f_2)) \\
 D_{57} &= \frac{-4E^x \mu_\varphi \partial_Y f_2 + K_x \mu_x (E^\varphi \mu_x \partial_X \partial_Y f_1 + 4E^x \mu_\varphi \partial_X \partial_Y f_2)}{2\sqrt{E^x} \mu_x^2 \mu_\varphi} \partial_{E^\varphi} \tilde{f}_{\mu_x} \\
 &\quad + \frac{E^\varphi K_x \mu_x^2 \partial_X f_1 + 4E^x \mu_\varphi (f_2 - K_x \mu_x \partial_X f_2)}{4E^{x^3/2} \mu_x^2 \mu_\varphi^2} \partial_{K_x} \tilde{f}_{\mu_x} \\
 &\quad + \frac{E^\varphi \partial_X f_1 + E^\varphi K_x \mu_x \partial_X^2 f_1 + 4E^x K_x \mu_\varphi \partial_X^2 f_2}{2\sqrt{E^x} \mu_\varphi^2} \partial_{E^x} \tilde{f}_{\mu_x} - \frac{K_x \partial_X f_1 \partial_{K_\varphi} \tilde{f}_{\mu_x}}{2\sqrt{E^x} \mu_\varphi^2} \\
 D_{58} &= \frac{-4E^x \mu_\varphi \partial_Y f_2 + K_x \mu_x (E^\varphi \mu_x \partial_X \partial_Y f_1 + 4E^x \mu_\varphi \partial_X \partial_Y f_2)}{2\sqrt{E^x} \mu_x^2 \mu_\varphi} \partial_{E^\varphi} \tilde{f}_{\mu_\varphi} \\
 &\quad + \frac{E^\varphi K_x \mu_x^2 \partial_X f_1 + 4E^x \mu_\varphi (f_2 - K_x \mu_x \partial_X f_2)}{4E^{x^3/2} \mu_x^2 \mu_\varphi^2} \partial_{K_x} \tilde{f}_{\mu_\varphi} \\
 &\quad + \frac{E^\varphi \partial_X f_1 + E^\varphi K_x \mu_x \partial_X^2 f_1 + 4E^x K_x \mu_\varphi \partial_X^2 f_2}{2\sqrt{E^x} \mu_\varphi^2} \partial_{E^x} \tilde{f}_{\mu_\varphi} - \frac{K_x \partial_X f_1 \partial_{K_\varphi} \tilde{f}_{\mu_\varphi}}{2\sqrt{E^x} \mu_\varphi^2}
 \end{aligned}$$

$$\begin{aligned}
 D_{67} &= \frac{-E^\varphi \mu_x \partial_Y f_1 + K_\varphi \mu_\varphi (E^\varphi \mu_x \partial_Y^2 f_1 + 4E^x \mu_\varphi \partial_Y^2 f_2)}{2\sqrt{E^x} \mu_x \mu_\varphi^2} \partial_{E^\varphi} \tilde{f}_{\mu_x} + \frac{2f_1 - K_\varphi \mu_\varphi \partial_Y f_1}{2\sqrt{E^x} \mu_\varphi^3} \partial_{K_\varphi} \tilde{f}_{\mu_x} \\
 &\quad - \frac{2E^\varphi f_1 \mu_x + 4E^x f_2 \mu_\varphi + K_\varphi \mu_\varphi (E^\varphi \mu_x \partial_Y f_1 - 4E^x \mu_\varphi \partial_Y f_2)}{4E^{x^3/2} \mu_x \mu_\varphi^3} \partial_{K_x} \tilde{f}_{\mu_x} \\
 &\quad - \frac{2E^\varphi \mu_x \partial_X f_1 + \mu_\varphi (-4E^x \partial_X f_2 + E^\varphi K_\varphi \mu_x \partial_X \partial_Y f_1 + 4E^x K_\varphi \mu_\varphi \partial_X \partial_Y f_2)}{2\sqrt{E^x} \mu_\varphi^3} \partial_{E^x} \tilde{f}_{\mu_x} \\
 D_{68} &= \frac{-E^\varphi \mu_x \partial_Y f_1 + K_\varphi \mu_\varphi (E^\varphi \mu_x \partial_Y^2 f_1 + 4E^x \mu_\varphi \partial_Y^2 f_2)}{2\sqrt{E^x} \mu_x \mu_\varphi^2} \partial_{E^\varphi} \tilde{f}_{\mu_\varphi} + \frac{2f_1 - K_\varphi \mu_\varphi \partial_Y f_1}{2\sqrt{E^x} \mu_\varphi^3} \partial_{K_\varphi} \tilde{f}_{\mu_\varphi} \\
 &\quad - \frac{2E^\varphi f_1 \mu_x + 4E^x f_2 \mu_\varphi + K_\varphi \mu_\varphi (E^\varphi \mu_x \partial_Y f_1 - 4E^x \mu_\varphi \partial_Y f_2)}{4E^{x^3/2} \mu_x \mu_\varphi^3} \partial_{K_x} \tilde{f}_{\mu_\varphi} \\
 &\quad - \frac{2E^\varphi \mu_x \partial_X f_1 + \mu_\varphi (-4E^x \partial_X f_2 + E^\varphi K_\varphi \mu_x \partial_X \partial_Y f_1 + 4E^x K_\varphi \mu_\varphi \partial_X \partial_Y f_2)}{2\sqrt{E^x} \mu_\varphi^3} \partial_{E^x} \tilde{f}_{\mu_\varphi} \\
 D_{78} &= -\partial_{E^\varphi} \tilde{f}_{\mu_\varphi} \partial_{K_\varphi} \tilde{f}_{\mu_x} + \partial_{E^\varphi} \tilde{f}_{\mu_x} \partial_{K_\varphi} \tilde{f}_{\mu_\varphi} - \partial_{E^x} \tilde{f}_{\mu_\varphi} \partial_{K_x} \tilde{f}_{\mu_x} + \partial_{E^x} \tilde{f}_{\mu_x} \partial_{K_x} \tilde{f}_{\mu_\varphi}
 \end{aligned}$$

Appendix C. Proof of Lemma 1

In this appendix, we present the proof of Lemma 1 used in the main text. For the benefit of the reader, we also show lemma one here again

Lemma A1. *For the iterated Poisson bracket of observables, we have*

$$\{\mathcal{O}_f^{C_\lambda}, \mathcal{O}_g^{C_\lambda}\}_{(n)} = \mathcal{O}_{\{\{f,g\}_{D(n)}\}^{C_\lambda}}^{C_\lambda} \tag{A10}$$

Proof. We start with the case $n = 2$, for which we have

$$\{\mathcal{O}_f^{C_\lambda}, \mathcal{O}_g^{C_\lambda}\}_{(2)} = \{\{\mathcal{O}_f^{C_\lambda}, \mathcal{O}_g^{C_\lambda}\}, \mathcal{O}_g^{C_\lambda}\} = \{\mathcal{O}_{\{f,g\}_D}^{C_\lambda}, \mathcal{O}_g^{C_\lambda}\} = \mathcal{O}_{\{\{f,g\}_D, g\}_D}^{C_\lambda} = \mathcal{O}_{\{f,g\}_{D(2)}}^{C_\lambda}$$

In the second step, we used

$$\{\mathcal{O}_f^{C_\lambda}, \mathcal{O}_g^{C_\lambda}\}_{(1)} = \{\mathcal{O}_f^{C_\lambda}, \mathcal{O}_g^{C_\lambda}\} = \mathcal{O}_{\{f,g\}_D}^{C_\lambda}$$

which was proven in [47] involving a weak equality sign there. In our case, due to $\{C_\lambda, p_\lambda\} = 1$, this becomes a strong equality here. Now, we assume that the statement is true for generic n and perform the induction step. We obtain

$$\begin{aligned}
 \{\mathcal{O}_f^{C_\lambda}, \mathcal{O}_g^{C_\lambda}\}_{(n+1)} &= \{\{\mathcal{O}_f^{C_\lambda}, \mathcal{O}_g^{C_\lambda}\}_{(n)}, \mathcal{O}_g^{C_\lambda}\} = \{\mathcal{O}_{\{f,g\}_{D(n)}}^{C_\lambda}, \mathcal{O}_g^{C_\lambda}\} \\
 &= \mathcal{O}_{\{\{f,g\}_{D(n)}, g\}_{D(n)}}^{C_\lambda} = \mathcal{O}_{\{f,g\}_{D(n+1)}}^{C_\lambda}
 \end{aligned}$$

□

Appendix D. Kuchař Decomposition for the Spherically Symmetric Model

In this appendix, we present more details regarding how the Kuchar decomposition can be explicitly constructed. For this purpose, first, we show that $\{Q^I(x), P_J(y)\} = \delta(x, y)$ for $I, J = 1, \dots, 4$, and afterwards we show that $\{Q^I(x), P_J(y)\}$ for $I, J = 5 \dots N + 6$ satisfies standard CCR and that Q^I, P_I for $I = 5 \dots N + 6$ commutes with all canonical pairs (Q^I, P_I) for $I = 1 \dots 4$. In both cases, we show this for one case only the remaining cases work exactly similar. For the first case, we consider Q^4, P_4 , and we have

$$\{Q^4(x), P_4(y)\} = \{\mathcal{O}_{C_{\mu\varphi}}^{Q^1, Q^2, Q^3}(x), \mathcal{O}_{p_{\mu\varphi}}^{Q^1, Q^2, Q^3}(y)\} = \mathcal{O}_{\{\mathcal{O}_{C_{\mu\varphi}}^{Q^1, Q^2}(x), \mathcal{O}_{p_{\mu\varphi}}^{Q^1, Q^2}(y)\}_D}^{Q^3}.$$

The involved Dirac bracket is given by

$$\begin{aligned} \{\mathcal{O}_{C_{\mu\varphi}}^{Q^1, Q^2}(x), \mathcal{O}_{p_{\mu\varphi}}^{Q^1, Q^2}(y)\}_D^{Q^3} &= \{\mathcal{O}_{C_{\mu\varphi}}^{Q^1, Q^2}(x), \mathcal{O}_{p_{\mu\varphi}}^{Q^1, Q^2}(y)\} - \int dz \{\mathcal{O}_{C_{\mu\varphi}}^{Q^1, Q^2}(x), Q^3(z)\} \{P_3(z), \mathcal{O}_{p_{\mu\varphi}}^{Q^1, Q^2}(y)\} \\ &+ \int dz \{\mathcal{O}_{C_{\mu\varphi}}^{Q^1, Q^2}(x), P_3(z)\} \{Q^3(z), \mathcal{O}_{p_{\mu\varphi}}^{Q^1, Q^2}(y)\}. \end{aligned} \tag{A11}$$

We further have

$$\{\mathcal{O}_{C_{\mu\varphi}}^{Q^1, Q^2}(x), P_3(z)\} = \mathcal{O}_{\{\mathcal{O}_{C_{\mu\varphi}}^{Q^1}(x), \mathcal{O}_{p_{\mu x}}^{Q^1}(z)\}_D}^{Q^2}$$

with

$$\begin{aligned} \{\mathcal{O}_{C_{\mu\varphi}}^{Q^1}(x), \mathcal{O}_{p_{\mu x}}^{Q^1}(z)\}_D^{Q^2} &= \{\mathcal{O}_{C_{\mu\varphi}}^{Q^1}(x), \mathcal{O}_{p_{\mu x}}^{Q^1}(y)\} - \int dz \{\mathcal{O}_{C_{\mu\varphi}}^{Q^1}(x), Q^2(z)\} \{P_2(z), \mathcal{O}_{p_{\mu\varphi}}^{Q^1}(y)\} \\ &+ \int dz \{\mathcal{O}_{C_{\mu\varphi}}^{Q^1}(x), P_2(z)\} \{Q^2(z), \mathcal{O}_{p_{\mu\varphi}}^{Q^1}(y)\}. \end{aligned}$$

Now, we can use

$$\{\mathcal{O}_{C_{\mu\varphi}}^{Q^1}(x), P_2(z)\} = \mathcal{O}_{\{C_{\mu\varphi}(x), p_{e_1}(z)\}_D}^{Q^1}$$

with

$$\begin{aligned} \{C_{\mu\varphi}(x), p_{e_1}(z)\}_D^{Q^1} &= \{C_{\mu\varphi}(x), p_{e_1}(z)\} - \int dz' \{C_{\mu\varphi}(x), C_{e_2}(z')\} \{p_{e_2}(z'), p_{e_1}(z)\} \\ &+ \int dz' \{C_{\mu\varphi}(x), p_{e_2}(z')\} \{C_{e_2}(z'), p_{e_1}(z)\} \\ &= 0. \end{aligned}$$

Likewise we obtain $\{P_2(z), \mathcal{O}_{p_{\mu\varphi}}^{Q^1}(y)\} = \{\mathcal{O}_{C_{\mu\varphi}}^{Q^1}(x), \mathcal{O}_{p_{\mu x}}^{Q^1}(y)\} = 0$. Hence, we obtain

$$\{\mathcal{O}_{C_{\mu\varphi}}^{Q^1, Q^2}(x), P_3(z)\} = 0, \quad \{P_3(z), \mathcal{O}_{p_{\mu\varphi}}^{Q^1, Q^2}(y)\} = 0,$$

Using this in (A11), we have

$$\{\mathcal{O}_{C_{\mu\varphi}}^{Q^1, Q^2}(x), \mathcal{O}_{p_{\mu\varphi}}^{Q^1, Q^2}(y)\}_D^{Q^3} = \{\mathcal{O}_{C_{\mu\varphi}}^{Q^1, Q^2}(x), \mathcal{O}_{p_{\mu\varphi}}^{Q^1, Q^2}(y)\}$$

Finally, we consider the remaining Poisson bracket

$$\{\mathcal{O}_{C_{\mu\varphi}}^{Q^1, Q^2}(x), \mathcal{O}_{p_{\mu\varphi}}^{Q^1, Q^2}(y)\} = \mathcal{O}_{\{\mathcal{O}_{C_{\mu\varphi}}^{Q^1}(x), \mathcal{O}_{p_{\mu\varphi}}^{Q^1}(y)\}_D}^{Q^2}$$

with

$$\begin{aligned} \{\mathcal{O}_{C_{\mu\varphi}}^{Q^1}(x), \mathcal{O}_{p_{\mu\varphi}}^{Q^1}(y)\}_D^{Q^2} &= \{C_{\mu\varphi}(x), p_{\mu\varphi}(y)\}_D^{Q^1} - \int dz \{C_{\mu\varphi}(x), C_{e_2}(z)\}_D^{Q^1} \{p_{e_2}(z), p_{\mu\varphi}(y)\}_D^{Q^1} \\ &+ \int dz \{C_{\mu\varphi}(x), p_{e_2}(z)\}_D^{Q^1} \{e_2(z), p_{\mu\varphi}(y)\}_D^{Q^1} \\ &= \{C_{\mu\varphi}(x), p_{\mu\varphi}(y)\} = \delta(x, y). \end{aligned}$$

Given this and reinserting it back into (A11), we finally have

$$\{Q^4(x), P_4(y)\} = \mathcal{O}_{\delta(x, y)}^{Q^3} = \delta(x, y).$$

Next we want to show that $\{Q^I(x), P_J(y)\}$ for $I, J = 5 \dots N + 6$ satisfy standard CCR and that Q^I, P_I for $I = 5 \dots N + 6$ commute with all canonical pairs (Q^I, P_I) for $I = 1 \dots 4$. Here, we consider the canonical pair (Q^5, P_5) as an example that we discuss in detail. The remaining combinations of variables work similarly. We start with

$$\{Q^5(x), P_5(y)\} = \{\mathcal{O}_{K_x}^{Q^{1\dots 4}}(x), \mathcal{O}_{E^x}^{Q^{1\dots 4}}(y)\} = \mathcal{O}_{\{\mathcal{O}_{K_x}^{Q^{1\dots 3}}(x), \mathcal{O}_{E^x}^{Q^{1\dots 3}}(y)\}_D^4}^{Q^4}$$

We have, using $Q^4 = \mathcal{O}_{C_{\mu\phi}}^{Q^{1\dots 3}}$ and $P_4 = \mathcal{O}_{p_{\mu\phi}}^{Q^{1\dots 3}}$,

$$\begin{aligned} \{\mathcal{O}_{K_x}^{Q^{1\dots 3}}(x), \mathcal{O}_{E^x}^{Q^{1\dots 3}}(y)\}_D^4 &= \{\mathcal{O}_{K_x}^{Q^{1\dots 3}}(x), \mathcal{O}_{E^x}^{Q^{1\dots 3}}(y)\} - \int dz \{\mathcal{O}_{K_x}^{Q^{1\dots 3}}(x), \mathcal{O}_{C_{\mu\phi}}^{Q^{1\dots 3}}(z)\} \{\mathcal{O}_{p_{\mu\phi}}^{Q^{1\dots 3}}(z), \mathcal{O}_{E^x}^{Q^{1\dots 3}}(y)\} \\ &+ \int dz \{\mathcal{O}_{K_x}^{Q^{1\dots 3}}(x), \mathcal{O}_{p_{\mu\phi}}^{Q^{1\dots 3}}(z)\} \{\mathcal{O}_{C_{\mu\phi}}^{Q^{1\dots 3}}(z), \mathcal{O}_{E^x}^{Q^{1\dots 3}}(y)\}. \end{aligned}$$

Next we want to show that the Poisson brackets involving $\mathcal{O}_{p_{\mu\phi}}^{Q^{1\dots 3}}$ all vanish. For this purpose we consider the following as an example

$$\{\mathcal{O}_{p_{\mu\phi}}^{Q^{1\dots 3}}(z), \mathcal{O}_{E^x}^{Q^{1\dots 3}}(y)\} = \mathcal{O}_{\{\mathcal{O}_{p_{\mu\phi}}^{Q^{1\dots 2}}(z), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\}_D^3}^{Q^3}$$

with

$$\begin{aligned} \{\mathcal{O}_{p_{\mu\phi}}^{Q^{1\dots 2}}(z), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\}_D^3 &= \{\mathcal{O}_{p_{\mu\phi}}^{Q^{1\dots 2}}(z), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\} - \int dz' \{\mathcal{O}_{p_{\mu\phi}}^{Q^{1\dots 2}}(z), \mathcal{O}_{C_{\mu x}}^{Q^{1\dots 2}}(z')\} \{\mathcal{O}_{p_{\mu x}}^{Q^{1\dots 2}}(z'), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\} \\ &+ \int dz' \{\mathcal{O}_{p_{\mu\phi}}^{Q^{1\dots 2}}(z), \mathcal{O}_{p_{\mu x}}^{Q^{1\dots 2}}(z')\} \{\mathcal{O}_{C_{\mu x}}^{Q^{1\dots 2}}(z'), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\}, \end{aligned}$$

where we used $Q^3 = \mathcal{O}_{C_{\mu x}}^{Q^{1\dots 2}}$ and $P_3 = \mathcal{O}_{p_{\mu x}}^{Q^{1\dots 2}}$. Here, we consider the Poisson brackets involving $\mathcal{O}_{p_{\mu x}}^{Q^{1\dots 2}}(z')$ further and find

$$\{\mathcal{O}_{p_{\mu x}}^{Q^{1\dots 2}}(z'), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\} = \mathcal{O}_{\{\mathcal{O}_{p_{\mu x}}^{Q^1}(z'), \mathcal{O}_{E^x}^{Q^1}(y)\}_D^2}^{Q^2}$$

with

$$\begin{aligned} \{\mathcal{O}_{p_{\mu x}}^{Q^1}(z'), \mathcal{O}_{E^x}^{Q^1}(y)\}_D^2 &= \{\mathcal{O}_{p_{\mu x}}^{Q^1}(z'), \mathcal{O}_{E^x}^{Q^1}(y)\} - \int dz'' \{\mathcal{O}_{p_{\mu x}}^{Q^1}(z'), \mathcal{O}_{C_{e_1}}^{Q^1}(z'')\} \{\mathcal{O}_{p_{e_1}}^{Q^1}(z''), \mathcal{O}_{E^x}^{Q^1}(y)\} \\ &+ \int dz'' \{\mathcal{O}_{p_{\mu x}}^{Q^1}(z'), \mathcal{O}_{p_{e_1}}^{Q^1}(z'')\} \{\mathcal{O}_{C_{e_1}}^{Q^1}(z''), \mathcal{O}_{E^x}^{Q^1}(y)\}. \end{aligned}$$

Given the last equation it is easy to show that $\{\mathcal{O}_{p_{\mu x}}^{Q^1}(z'), \mathcal{O}_{E^x}^{Q^1}(y)\}_D^2 = 0$ because we have

$$\begin{aligned} \{\mathcal{O}_{p_{\mu x}}^{Q^1}(z'), \mathcal{O}_{E^x}^{Q^1}(y)\} &= \mathcal{O}_{\{p_{\mu x}(z'), E^x(y)\}_D^1}^{Q^1} = 0 \\ \{\mathcal{O}_{p_{e_1}}^{Q^1}(z''), \mathcal{O}_{E^x}^{Q^1}(y)\} &= \mathcal{O}_{\{p_{e_1}(z''), E^x(y)\}_D^1}^{Q^1} = 0 \\ \{\mathcal{O}_{p_{\mu x}}^{Q^1}(z'), \mathcal{O}_{C_{e_1}}^{Q^1}(z'')\} &= \mathcal{O}_{\{p_{\mu x}(z'), C_{e_1}(z'')\}_D^1}^{Q^1} = 0. \end{aligned}$$

Using that $\{\mathcal{O}_{p_{\mu x}}^{Q^1}(z'), \mathcal{O}_{E^x}^{Q^1}(y)\}_D^2 = 0$ we can conclude $\{\mathcal{O}_{p_{\mu\phi}}^{Q^{1\dots 2}}(z), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\} = 0$, and likewise we obtain $\{\mathcal{O}_{p_{\mu\phi}}^{Q^{1\dots 2}}(z), \mathcal{O}_{p_{\mu x}}^{Q^{1\dots 2}}(z')\} = 0$. This yields

$$\{\mathcal{O}_{p_{\mu\phi}}^{Q^{1\dots 2}}(z), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\}_D^3 = \{\mathcal{O}_{p_{\mu\phi}}^{Q^{1\dots 2}}(z), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\}.$$

Now, with similar iterative steps we can show that

$$\begin{aligned} \{\mathcal{O}_{p_{\mu\varphi}}^{Q^{1\dots 2}}(z), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\}_D^{Q^3} &= \{\mathcal{O}_{p_{\mu\varphi}}^{Q^{1\dots 2}}(z), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\} \\ &= \mathcal{O}_{\{p_{\mu\varphi}(z), E^x(y)\}}^{Q^2} = 0. \end{aligned}$$

These results yield

$$\{\mathcal{O}_{p_{\mu\varphi}}^{Q^{1\dots 3}}(z), \mathcal{O}_{E^x}^{Q^{1\dots 3}}(y)\} = \mathcal{O}_{\{\mathcal{O}_{p_{\mu\varphi}}^{Q^{1\dots 2}}(z), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\}_D^{Q^3}}^{Q^3} = 0.$$

Then, for the second Poisson bracket involving $\mathcal{O}_{p_{\mu\varphi}}^{Q^{1\dots 3}}(z)$ we can perform exactly the same steps to end up with

$$\{\mathcal{O}_{K_x}^{Q^{1\dots 3}}(x), \mathcal{O}_{p_{\mu\varphi}}^{Q^{1\dots 3}}(z)\} = \mathcal{O}_{\{\mathcal{O}_{K_x}^{Q^{1\dots 2}}(x), \mathcal{O}_{p_{\mu\varphi}}^{Q^{1\dots 2}}(z)\}_D^{Q^3}}^{Q^3} = \mathcal{O}_{\{\mathcal{O}_{K_x}^{Q^{1\dots 2}}(x), \mathcal{O}_{p_{\mu\varphi}}^{Q^{1\dots 2}}(z)\}}^{Q^3} = \mathcal{O}_{\{K_x(x), p_{\mu\varphi}(z)\}}^{Q^{1\dots 3}} = 0.$$

This then leads to

$$\{\mathcal{O}_{K_x}^{Q^{1\dots 3}}(x), \mathcal{O}_{E^x}^{Q^{1\dots 3}}(y)\}_D^{Q^4} = \{\mathcal{O}_{K_x}^{Q^{1\dots 3}}(x), \mathcal{O}_{E^x}^{Q^{1\dots 3}}(y)\}.$$

The final step is to show that

$$\{\mathcal{O}_{K_x}^{Q^{1\dots 3}}(x), \mathcal{O}_{E^x}^{Q^{1\dots 3}}(y)\} = G\delta(x, y) \tag{A12}$$

This can be completed with similar iterative steps. First, we have

$$\{\mathcal{O}_{K_x}^{Q^{1\dots 3}}(x), \mathcal{O}_{E^x}^{Q^{1\dots 3}}(y)\} = \mathcal{O}_{\{\mathcal{O}_{K_x}^{Q^{1\dots 2}}(x), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\}_D^{Q^3}}^{Q^3} = \mathcal{O}_{\{\mathcal{O}_{K_x}^{Q^{1\dots 2}}(x), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\}}^{Q^3}'$$

where the reduction to the Poisson bracket in the last step can be performed because, as in the case of the Dirac brackets above, there will be always Poisson brackets involved in the additional terms in the Dirac bracket that include the momenta P_3 , which can be shown to vanish identically as above. Next, we have

$$\{\mathcal{O}_{K_x}^{Q^{1\dots 2}}(x), \mathcal{O}_{E^x}^{Q^{1\dots 2}}(y)\} = \mathcal{O}_{\{\mathcal{O}_{K_x}^{Q^1}(x), \mathcal{O}_{E^x}^{Q^1}(y)\}_D^{Q^2}}^{Q^2} = \mathcal{O}_{\{\mathcal{O}_{K_x}^{Q^1}(x), \mathcal{O}_{E^x}^{Q^1}(y)\}}^{Q^2}$$

and using that

$$\{\mathcal{O}_{K_x}^{Q^1}(x), \mathcal{O}_{E^x}^{Q^1}(y)\} = \mathcal{O}_{\{K_x(x), E^x(y)\}_D^{Q^1}}^{Q^1} = \mathcal{O}_{\{K_x(x), E^x(y)\}}^{Q^1} = G\delta(x, y)$$

we obtain directly

$$\{Q^5(x), P_5(y)\} = \{\mathcal{O}_{K_x}^{Q^{1\dots 4}}(x), \mathcal{O}_{E^x}^{Q^{1\dots 4}}(y)\} = \mathcal{O}_{\{\mathcal{O}_{K_x}^{Q^{1\dots 3}}(x), \mathcal{O}_{E^x}^{Q^{1\dots 3}}(y)\}_D^{Q^4}}^{Q^4} = G\delta(x, y).$$

Now, it remains to show that the set of variables (Q^I, P_I) for $I = 5 \dots N + 6$ mutually commute with all Q^I, P_I for $I = 1 \dots 4$. Here, in general, we have Poisson brackets of quantities to which a different number of observable maps have been applied. We demonstrate how the computation works in these cases for one specific example because the remaining ones can be computed similarly. We start with

$$\begin{aligned}
 \{Q^5(x), P_3(y)\} &= \{\mathcal{O}_{K_x}^{Q^{1...4}}(x), \mathcal{O}_{P_{\mu_x}}^{Q^{1...2}}(y)\} \\
 &= \sum_{n=0}^{\infty} \int dz_1 \cdots \int dz_n \left\{ \frac{(P_4(z_1) \cdots P_4(z_n))}{n!} \right. \\
 &\quad \left. \{ \dots \{ \{ \mathcal{O}_{K_x}^{Q^{1...3}}(x), Q_4(z_1) \}, Q_4(z_2) \cdots \}, \dots \}, Q_4(z_n) \}, \mathcal{O}_{P_{\mu_x}}^{Q^{1...2}}(y) \right\} \\
 &= \sum_{n=0}^{\infty} \int dz_1 \cdots \int dz_n \frac{(P_4(z_1) \cdots P_4(z_n))}{n!} \\
 &\quad \{ \{ \dots \{ \{ \mathcal{O}_{K_x}^{Q^{1...3}}(x), \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...3}}(z_1) \}, \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...3}}(z_2) \cdots \}, \dots \}, \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...3}}(z_n) \}, \mathcal{O}_{P_{\mu_x}}^{Q^{1...2}}(y) \}.
 \end{aligned}$$

Next, we consider the nested Poisson bracket more in detail. We have

$$\begin{aligned}
 &\{ \{ \dots \{ \{ \mathcal{O}_{K_x}^{Q^{1...3}}(x), \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...3}}(z_1) \}, \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...3}}(z_2) \cdots \}, \dots \}, \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...3}}(z_n) \} \\
 &= \mathcal{O}^{Q^3} \\
 &\quad \{ \{ \dots \{ \{ \mathcal{O}_{K_x}^{Q^{1...2}}(x), \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...2}}(z_1) \}, \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...2}}(z_2) \cdots \}, \dots \}, \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...2}}(z_n) \}_D^3 \\
 &= \mathcal{O}^{Q^3} \\
 &\quad \{ \{ \dots \{ \{ \mathcal{O}_{K_x}^{Q^{1...2}}(x), \mathcal{O}_{f_{\mu\varphi}}^{Q^{1...2}}(z_1) \}, \mathcal{O}_{f_{\mu\varphi}}^{Q^{1...2}}(z_2) \cdots \}, \dots \}, \mathcal{O}_{f_{\mu\varphi}}^{Q^{1...2}}(z_n) \}_D^3 \\
 &= \mathcal{O}^{Q^3} \\
 &\quad \{ \{ \dots \{ \{ \mathcal{O}_{K_x}^{Q^{1...2}}(x), \mathcal{O}_{f_{\mu\varphi}}^{Q^{1...2}}(z_1) \}, \mathcal{O}_{f_{\mu\varphi}}^{Q^{1...2}}(z_2) \cdots \}, \dots \}, \mathcal{O}_{f_{\mu\varphi}}^{Q^{1...2}}(z_n) \}'
 \end{aligned} \tag{A13}$$

where we used the gravitational degrees of freedom commute with all momenta whose corresponding observables are used as clocks in the individual observable maps to reduce the Dirac bracket to the corresponding Poisson bracket in the last step. Now, further, we obtain

$$\begin{aligned}
 &\{ \{ \dots \{ \{ \mathcal{O}_{K_x}^{Q^{1...2}}(x), \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...2}}(z_1) \}, \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...2}}(z_2) \cdots \}, \dots \}, \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...2}}(z_n) \} \\
 &= \mathcal{O}^{Q^{1...2}} \\
 &\quad \{ \{ \dots \{ \{ K_x(x), f_{\mu\varphi}(z_1) \}, f_{\mu\varphi}(z_2) \}, \dots \}, f_{\mu\varphi}(z_n) \}'
 \end{aligned}$$

We realize that, for all values of $n \geq 0$, the nested Poisson bracket involves gravitational degrees of freedom only. For a more compact notation, let us introduce the following abbreviation

$$F^{\text{grav}}(z_1, \dots, z_n) := \{ \{ \dots \{ \{ K_x(x), f_{\mu\varphi}(z_1) \}, f_{\mu\varphi}(z_2) \}, \dots \}, f_{\mu\varphi}(z_n) \},$$

the label ‘grav’ should indicate that the function depends on the gravitational degrees of freedom only. Note that the following steps will also hold if we generalize to the case where this function depends on gravitational as well as matter degrees of freedom. Now, we reinsert this function into (A13) and obtain

$$\begin{aligned}
 &\{ \{ \dots \{ \{ \mathcal{O}_{K_x}^{Q^{1...3}}(x), \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...3}}(z_1) \}, \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...3}}(z_2) \cdots \}, \dots \}, \mathcal{O}_{C_{\mu\varphi}}^{Q^{1...3}}(z_n) \} \\
 &= \mathcal{O}^{Q^3} \\
 &\quad \mathcal{O}_{F^{\text{grav}}(z_1, \dots, z_n)}^{Q^1 Q^2} \\
 &= \sum_{m=0}^{\infty} \frac{1}{m!} \int ds_1 \cdots \int ds_m (P_3(s_1) \cdots P_3(s_m)) \{ \{ \dots \{ \mathcal{O}_{F^{\text{grav}}(z_1, \dots, z_n)}^{Q^1 Q^2}, Q_3(s_1) \} \cdots \}, Q_3(s_m) \} \\
 &= \sum_{m=0}^{\infty} \frac{1}{m!} \int ds_1 \cdots \int ds_m (P_3(s_1) \cdots P_3(s_m)) \{ \{ \dots \{ \mathcal{O}_{F^{\text{grav}}(z_1, \dots, z_n)}^{Q^1 Q^2}, \mathcal{O}_{f_{\mu_x}}^{Q^1 Q^2}(s_1) \} \cdots \}, \mathcal{O}_{f_{\mu_x}}^{Q^1 Q^2}(s_m) \} \\
 &= \sum_{m=0}^{\infty} \frac{1}{m!} \int ds_1 \cdots \int ds_m (P_3(s_1) \cdots P_3(s_m)) \mathcal{O}_{\{ \dots \{ F^{\text{grav}}(z_1, \dots, z_n), f_{\mu_x}(s_1) \} \cdots \}, f_{\mu_x}(s_m)}^{Q^1 Q^2}.
 \end{aligned}$$

If we reinsert this result into the (A13) and use that $\{P_3(x), P_3(y)\} = 0$, finally, we end with

$$\begin{aligned} \{Q^5(x), P_3(y)\} &= \sum_{n=0}^{\infty} \int dz_1 \cdots \int dz_n \frac{(P_4(z_1) \cdots P_4(z_n))}{n!} \sum_{m=0}^{\infty} \int ds_1 \dots \int ds_m \frac{(P_3(s_1) \cdots P_3(s_m))}{m!} \\ &\quad \{ \mathcal{O}^{Q^1 Q^2} \{ \dots \{ F^{\text{grav}}(z_1, \dots, z_n), \mathcal{O}_{f_{\mu x}}^{Q^1 Q^2}(s_1) \} \dots \}, \mathcal{O}_{f_{\mu x}}^{Q^1 Q^2}(s_m) \}, \mathcal{O}_{p_{\mu x}}^{Q^1 Q^2}(y) \} \\ &= \sum_{n=0}^{\infty} \int dz_1 \cdots \int dz_n \frac{(P_4(z_1) \cdots P_4(z_n))}{n!} \sum_{m=0}^{\infty} \int ds_1 \cdots \int ds_m \frac{(P_3(s_1) \cdots P_3(s_m))}{m!} \\ &\quad \mathcal{O}^{Q^1 Q^2} \{ \dots \{ F^{\text{grav}}(z_1, \dots, z_n), \mathcal{O}_{f_{\mu x}}^{Q^1 Q^2}(s_1) \} \dots \}, \mathcal{O}_{f_{\mu x}}^{Q^1 Q^2}(s_m) \}, \mathcal{O}_{p_{\mu x}}^{Q^1 Q^2}(y) \} \\ &= 0, \end{aligned}$$

where we used, in the last step,

$$\{ \dots \{ F^{\text{grav}}(z_1, \dots, z_n), f_{\mu x}(s_1) \} \dots \}, f_{\mu x}(s_m) \}, p_{\mu x}(y) \} = 0$$

due to the fact that the involved iterated Poisson bracket depends on the gravitational degrees of freedom only, and thus a further Poisson bracket with $p_{\mu x}$ trivially vanishes along with, as a consequence, the application of the observables map $\mathcal{O}^{Q^1 \dots Q^2}$. The remaining combinations of elementary-phase space variables can be computed in a similar manner: some of them with the same and others with fewer iterations involved, which is why we considered the canonical pair (Q^5, P_3) as a representative example.

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