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Quantum Fractionary Cosmology: K-Essence Theory

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Abstract: Using a particular form of the quantum K-essence scalar field, we show that in the quantum formalism, a fractional differential equation in the scalar field variable, for some epochs in the Friedmann–Lemaβtre–Robertson–Walker (FLRW) model (radiation and inflation-like epochs, for example), appears naturally. In the classical analysis, the kinetic energy of scalar fields can falsify the standard matter in the sense that we obtain the time behavior for the scale factor in all scenarios of our Universe by using the Hamiltonian formalism, where the results are analogous to those obtained by an algebraic procedure in the Einstein field equations with standard matter. In the case of the quantum Wheeler–DeWitt (WDW) equation for the scalar field ϕ, a fractional differential equation of order $\beta = \frac{2\alpha}{2\alpha-1}$ is obtained. This fractional equation belongs to different intervals, depending on the value of the barotropic parameter; that is to say, when $\omega_X \in [0,1]$, the order belongs to the interval $1 \leq \beta \leq 2$, and when $\omega_X \in [-1,0)$, the order belongs to the interval $0 < \beta \leq 1$. The corresponding quantum solutions are also given.

Keywords: fractional derivative; fractional quantum cosmology; K-essence formalism

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1. Introduction

In general, fractional calculus (FC) is the natural generalization of ordinary calculus [1]. That is, FC considers integrals and derivatives of non-integer order. Despite the fact that there is no fully accepted physical and geometric interpretation of fractional derivatives, during the last three decades, FC has been the subject of intense theoretical and applied studies, because different types of fractional derivatives have emerged in the scientific literature, each with its advantages and disadvantages [2]. There are many works that investigate fractional calculus and its applications [3], being a powerful mathematical tool for describing complex processes, such as the tautochrone problem [4], models based on memory mechanism [5], anomalous diffusion [6], linear capacitor theory [7], non-local description of quantum mechanics [8], processing of medical images [9,10], and so on. FC has been quite successful in many areas of science and engineering [11–13].

With the exception of some fractional models that arise naturally, for example, in ref. [7], the vast majority of models starts from an ordinary differential equation corresponding to a certain physical model. Then, it is fractionated; that is, the derivatives of the system are taken as fractional by applying any of the definitions: Riemann–Liouville, Caputo, Caputo–Fabrizio, and Atangana–Baleanu.

Recently, the FC has been applied to the general theory of relativity, as in ref. [14], and in particular, to the Friedmann–Lemaβtre–Robertson–Walker model, with interesting results in cosmology [15–29], and quantum mechanics to quantum cosmology [30,31]. These have been featured recently in the chapter "Fractional Quantum Cosmology in Challenging Routes" in a Quantum Cosmology book [32], and in the publication revising fractional cosmology [33]. In a previous paper [34], we mentioned that in the quantum formalism, applied to different epochs for the K-essence theory, we would get a fractional Wheeler–DeWitt equation in the scalar field component. Now, we report our results in this direction. Additionally, one point worth mentioning is that by employing the classical information on the barotropic parameter of the scalar field, we present a relation for using a real parameter, which can reproduce the different epochs of our Universe (inflation-like phenomenon, radiation era, stiff and dust matter) for particular values. With this parameter, we found that the scale factor is analogous to those found in a previous work for the standard matter in the same scenarios [35,36]. In this sense, we can introduce the idea that the kinetic energy of the scalar field should falsify the standard matter by employing the K-essence formalism.

Usually, K-essence models are restricted to the Lagrangian density of the form [37–42]

$$S = \int d^4x \sqrt{-g}\,[f(\phi)\,\mathcal{G}(X)], \tag{1}$$

where the canonical kinetic energy is given by $\mathcal{G}(X) = X = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$, $f(\phi)$ is an arbitrary function of the scalar field ϕ, and g is the determinant of the metric. K-essence was originally proposed as a model for inflation, and then as a model for dark energy, along with explorations of unifying dark energy and dark matter [38,43,44]. Another motivation to consider this type of Lagrangian originates from string theory [45,46]. For more details on K-essence applied to dark energy, one can see ref. [47] and references therein.

On another front, the quantized version of this theory has not been constructed, perhaps due to the difficulties in building up the ADM formalism for it. Thus, we transform this theory to a conventional one, where the dimensionless scalar field is obtained from an energy-momentum tensor as an exotic matter component; and in this sense, we can use this structure for the quantization program, where the ADM formalism is well-known for different classes of matter [48].

This work is arranged as follows: in Section 1, we give some definitions of the fractional calculus that we employ in this work and the main ideas over the K-essence formalism, applied to obtain the classical solution to the scalar field, including the fractional parameter defined in a general way. In Section 2, we construct the Lagrangian and Hamiltonian densities for the plane FLRW cosmological model, considering a barotropic perfect fluid for the scalar field in the variable X, and present the general case. Next, the radiation particular scenario, where fractional momenta appear in the scalar field, will be used in Section 3, where we present the quantum regime for several cases of interest. Finally, Section 4 is devoted to discussions.

1.1. Basic Definitions from Fractional Calculus

Fractional derivatives are defined by means of analytical continuation of the Cauchy's formula for the multiple integral of an integer order as a single integral with a power-law kernel into the field of real order $\gamma > 0$, [1,11]. The fractional integral of order γ is written as:

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\gamma}}d\tau, \tag{2}$$

recovering an ordinary integral when $\gamma \to 1$. The Caputo fractional derivative of order $\gamma \geq 0$ of a function $f(t)$ is defined as the fractional order integral (2) of the integer order derivative (in the following, in the conventional notation, the sub-index 0 corresponds to the definition domain (0,x) by example, where other notations appear as (a,b), a+ and b-. In other words, they are the derivation limits).

$$\frac{d^\gamma}{dt^\gamma}f(t) = {}^C{}_0D_t^\gamma f(t) = I^{n-\gamma}\,{}_0D_t^n f(t) = \frac{1}{\Gamma(n-\gamma)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\gamma-n+1}}d\tau, \tag{3}$$

with $n-1<\gamma\leq n\in\mathbb{N}=1,2,\ldots$, and $\gamma\in\mathbb{R}$ is the order of the fractional derivative and $f^{(n)}$ are the ordinary integer derivatives, and $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$ is the gamma function. The Caputo derivative satisfies the following relations:

$$\begin{aligned} {}^{C}{}_{0}D_t^{\gamma}[f(t)+g(t)] &= {}^{C}{}_{0}D_t^{\gamma}f(t)+{}^{C}{}_{0}D_t^{\gamma}g(t), \quad (4)\\ {}^{C}{}_{0}D_t^{\gamma}c &= 0, \qquad \textit{where c is a constant}. \quad (5)\end{aligned}$$

The Laplace transform of the function $f(t)$ is defined as

$$\mathbb{L}[f(t)]=\int_0^\infty f(t)e^{-st}dt=F(s). \tag{6}$$

Then, the Laplace transform of the Caputo fractional derivative (3) has the form [1],

$$\mathbb{L}[{}^{C}{}_{0}D_t^{\gamma}f(t)]=s^{\gamma}F(s)-\sum_{k=0}^{n-1}s^{\gamma-k-1}f^{(k)}(0), \tag{7}$$

where $f^{(k)}$ is the ordinary derivative.

Another definition which will be used is the Mittag–Leffler generalized function [49,50] (and references therein), defined as the series expansion; in a general way, under a Maclaurin series, with $z\in\mathbb{C}$

$$E_{\chi,\sigma}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\chi n+\sigma)}, \qquad (\chi>0,\ \sigma>0), \tag{8}$$

and for $\sigma=1$, we have one parametric Mittag–Leffler function

$$E_{\chi}(z)=E_{\chi,1}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\chi n+1)}, \qquad \chi>0. \tag{9}$$

Some special cases are [49,50]:

$$\begin{aligned} &E_1(\pm z)=e^{\pm z}, \qquad E_2(z)=Cosh(\sqrt{z}), \qquad E_2(-z^2)=Cos(z),\\ &E_{2,2}(z^2)=\frac{Sinh(z)}{z}, \qquad E_{2,2}(-z^2)=\frac{Sin(z)}{z}. \end{aligned} \tag{10}$$

Laplace transform of the Mittag–Leffler function is given by Equation (1)

$$\int_0^\infty e^{-st}t^{\chi m+\sigma-1}E^{(m)}_{\chi,\sigma}(\pm at^{\chi})dt=\frac{m!\,s^{\chi-\sigma}}{(s^{\chi}\mp a)^{m+1}}. \tag{11}$$

Consequently, the inverse Laplace transform is

$$\mathbb{L}^{-1}\left[\frac{m!\,s^{\chi-\sigma}}{(s^{\chi}\mp a)^{m+1}}\right]=t^{\chi m+\sigma-1}E^{(m)}_{\chi,\sigma}(\pm at^{\chi}). \tag{12}$$

This expression is very useful for obtaining analytical solutions.

1.2. *K-Essence Theory*

One of the simplest K-essence Lagrangian densities is

$$\mathcal{L}_{geo}=(R+f(\phi)\mathcal{G}(X)), \tag{13}$$

where R is the scalar of curvature, $f(\phi)$ and $\mathcal{G}(X)$ have been defined before. Then, the field equations are given by

$$G_{\mu\nu} + f(\phi)\left[\mathcal{G}_X\phi_{,\mu}\phi_{,\nu} + \mathcal{G}g_{\mu\nu}\right] = T_{\mu\nu}, \tag{14}$$

$$f(\phi)\left[\mathcal{G}_X\phi^{;\nu}_{;\nu} + \mathcal{G}_{XX}X_{;\nu}\phi^{\nu}\right] + \frac{df}{d\phi}[\mathcal{G} - 2X\mathcal{G}_X] = 0, \tag{15}$$

where we have assumed the units with $8\pi G = 1$ and, as usual, the semicolon means a covariant derivative, and the subscript X denotes differentiation with respect to X. (Equations (14) and (15) are deduced in Appendix A).

The same set of Equations (14) and (15) is obtained if we consider the scalar field $X(\phi)$ as part of the matter content, to say $\mathcal{L}_{X,\phi} = f(\phi)\mathcal{G}(X)$, with the corresponding energy-momentum tensor

$$\mathcal{T}_{\mu\nu} = f(\phi)\left[\mathcal{G}_X\phi_{,\mu}\phi_{,\nu} + \mathcal{G}(X)g_{\mu\nu}\right]. \tag{16}$$

Additionally, considering the energy-momentum tensor of a barotropic perfect fluid,

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}, \tag{17}$$

with u_μ being the four-velocity satisfying the relation $u_\mu u^\mu = -1$, ρ the energy density, and P the pressure of the fluid. For simplicity, we consider a co-moving perfect fluid. The pressure and energy density, corresponding to the energy momentum tensor of the field X, are

$$P(X) = f(\phi)\mathcal{G}, \qquad \rho(X) = f(\phi)[2X\mathcal{G}_X - \mathcal{G}]; \tag{18}$$

thus, the barotropic parameter $\omega_X = \frac{P(X)}{\rho(X)}$ for the equivalent fluid is

$$\omega_X = \frac{\mathcal{G}}{2X\mathcal{G}_X - \mathcal{G}}. \tag{19}$$

Notice that the case of a constant barotropic index ω_X, (with the exception when $\omega_X = 0$) can be obtained by the $\mathcal{G}$ function

$$\mathcal{G} = X^{\frac{1+\omega_X}{2\omega_X}}. \tag{20}$$

Choosing the barotropic parameter as

$$\omega_X = \frac{2\kappa - 1}{2\kappa + 1}, \qquad \rightarrow \qquad \mathcal{G} = X^\alpha, \tag{21}$$

where the α parameter

$$\alpha = \frac{2\kappa}{2\kappa - 1}, \tag{22}$$

is relevant in our approach. Thus, we can write the barotropic parameter in terms of $\omega_X = \frac{1}{2\alpha - 1}$, when $\kappa = \frac{\alpha}{2(\alpha - 1)}$. With this, we can write the states in the evolution of our Universe as:

$$\begin{cases} \text{stiff matter :} & \kappa \rightarrow \infty, & \omega_X = 1, \rightarrow \mathcal{G}(X) = X. \\ \text{Radiation:} & \kappa = 1, & \omega_X = \frac{1}{3}, \rightarrow \mathcal{G}(X) = X^2. \\ \text{Radiation like:} & \kappa = \frac{5}{4}, & \omega_X = \frac{2}{3}, \rightarrow \mathcal{G}(X) = X^{5/2}. \\ \text{such as dust like:} & \kappa \rightarrow \frac{1}{2}, & \omega_X \rightarrow 0, \rightarrow \mathcal{G}(X) = X^m, \quad m \rightarrow \infty. \\ \text{inflation :} & \kappa = 0, & \omega_X = -1, \rightarrow \mathcal{G}(X) = 1,\ f(\phi) = \Lambda = constant. \\ \text{inflation such as} & \kappa = \frac{1}{4}, & \omega_X = -\frac{1}{3}, \rightarrow \mathcal{G}(X) = \frac{1}{X} \\ & \kappa = \frac{1}{10}, & \omega_X = -\frac{2}{3}, \rightarrow \mathcal{G}(X) = \frac{1}{\sqrt[4]{X}}. \end{cases}$$

The classical and quantum solutions for the stiff matter case $\omega_X = 1$ with the function $\mathcal{G}(X) = X$ were treated in the refs. [34,37], considering anisotropic cosmologies, which are the standard quintessence, such as $f(\phi) = constant$. For the inflation phenomenon, we chose the particular value for the cosmological constant function $f(\phi) = \Lambda$. The original Einstein field Equations (14) were reduced to the traditional problem with the cosmological constant with exponential time behavior for the scale factor [35].

It is clear that the stiff-matter case falls into the traditional treatment of quintessence cosmology, and in the other cases, a Hamiltonian density with a fractional momentum in the scalar field; then, the quantum Wheeler–DeWitt equation appears as a fractional differential equation. In the ref. [51], the authors present the classical analysis of the radiation era by using dynamic systems and obtaining rebound solutions.

In the following, by choosing the generic formula of the barotropic parameter $\omega_X = \frac{1}{2\alpha-1}$, we obtain the classical and quantum solutions.

1.3. Classical Cosmological FLRW Model, $f(\phi) = Constant$

The space-time background to be considered is the spatially flat FLRW with element

$$ds^2 = -N(t)^2 dt^2 + A^2(t)\left[dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\right], \tag{23}$$

where $N(t)$ represents the lapse function, $A(t) = e^{\Omega(t)}$ is the scale factor in the Misner parametrization, and Ω is a scalar function, whose interval is $(-\infty, \infty)$. If we consider the cosmological FLRW model, then the Equation (15) is written as (we use $\prime = \frac{d}{d\tau} = \frac{d}{Ndt}$, so $g_{\tau\tau} = -1$),

$$[\mathcal{G}_X + 2X\mathcal{G}_{XX}]X' + 6\frac{A'}{A}X\mathcal{G}_X = 0, \tag{24}$$

with the exact solution

$$X\mathcal{G}_X^2 = \eta A^{-6}, \tag{25}$$

where A is the scale factor of the cosmological FLRW model, and η is an integration constant, which is linked to the parameters of matter in the Universe epoch in study. This solution has been known for some time and was found by different authors [41,52,53]. (Equations (24) and (25) have been deduced in Appendix B).

In the following, we present the generic case, $\omega_X = \frac{1}{2\alpha-1}$, given by $\mathcal{G} = X^\alpha$ and substituting into (25), we have $\alpha^2 X^{2\alpha-1} = \eta A^{-6}$ with $X = \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2$, obtaining for the scalar field ϕ the equation

$$\frac{d\phi}{d\tau} = \sqrt{2}\left[\left(\frac{\eta}{\alpha^2}\right)^{\frac{1}{2(2\alpha-1)}} A^{-\frac{3}{2\alpha-1}}\right] = \sqrt{2}\left[\left(\frac{\eta}{\alpha^2}\right)^{\frac{1}{2(2\alpha-1)}} e^{-\frac{3}{2\alpha-1}\Omega}\right], \tag{26}$$

whose solution is

$$\Delta\phi = \sqrt{2}\left(\frac{\eta}{\alpha^2}\right)^{\frac{1}{2(2\alpha-1)}} \int A^{-\frac{3}{2\alpha-1}} d\tau = \sqrt{2}\left(\frac{\eta}{\alpha^2}\right)^{\frac{1}{2(2\alpha-1)}} \int e^{-\frac{3}{2\alpha-1}\Omega} d\tau, \tag{27}$$

which is dependent on the scale factor; it was obtained by using the relationship between the momenta and the Hamiltonian density constraint from the Lagrangian density, in the usual way.

2. Hamiltonian Cosmological Models

Introducing the line element (23) in Misner's parametrization, the Ricci scalar becomes $R = -6\frac{\ddot{\Omega}}{N^2} - 12\left(\frac{\dot{\Omega}}{N}\right)^2 + 6\frac{\dot{\Omega}\dot{N}}{N^3}$ and $\sqrt{-g} = Ne^{3\Omega}$; then, the total Lagrangian density (13) for the generic-like Universe is

$$\mathcal{L} = e^{3\Omega}\left[-6\frac{\ddot{\Omega}}{N} - 12\frac{(\dot{\Omega})^2}{N} + 6\frac{\dot{\Omega}\dot{N}}{N^2} - \left(\frac{1}{2}\right)^{\alpha}(\dot{\phi})^{2\alpha}N^{-2\alpha+1}\right]. \tag{28}$$

Thus, by using the total time derivative $\left(-6e^{3\Omega}\frac{\dot{\Omega}}{N}\right)^{\bullet} = -6e^{3\Omega}\frac{\ddot{\Omega}}{N} - 18\,e^{3\Omega}\frac{(\dot{\Omega})^2}{N} + 6e^{3\Omega}\frac{\dot{\Omega}\dot{N}}{N^2}$ in (28), we obtain

$$\mathcal{L} = e^{3\Omega}\left[6\frac{\dot{\Omega}^2}{N} - \left(\frac{1}{2}\right)^{\alpha}(\dot{\phi})^{2\alpha}N^{-2\alpha+1}\right]. \tag{29}$$

Using the standard definition of the momenta $\Pi_{q^\mu} = \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu}$, where $q^\mu = (\Omega, \phi)$, we obtain

$$\Pi_\Omega = \frac{12}{N}e^{3\Omega}\dot{\Omega}, \quad \rightarrow \quad \dot{\Omega} = \frac{N}{12}e^{-3\Omega}\Pi_\Omega, \tag{30}$$

$$\Pi_\phi = -\left(\frac{1}{2}\right)^{\alpha}\frac{2\alpha}{N^{2\alpha-1}}e^{3\Omega}\dot{\phi}^{2\alpha-1}, \quad \rightarrow \quad \dot{\phi} = -N\left[\frac{2^{\alpha-1}}{\alpha}e^{-3\Omega}\Pi_\phi\right]^{\frac{1}{2\alpha-1}}, \tag{31}$$

and introducing them into the Lagrangian density, we obtain the canonical Lagrangian $\mathcal{L}_{canonical} = \Pi_{q^\mu}\dot{q}^\mu - N\mathcal{H}$ as

$$\mathcal{L}_{canonical} = \Pi_{q^\mu}\dot{q}^\mu - \frac{N}{24}e^{-\frac{3}{2\alpha-1}\Omega}\left\{e^{-\frac{6(\alpha-1)}{2\alpha-1}\Omega}\Pi_\Omega^2 - \frac{12(2\alpha-1)}{\alpha}\Pi_\phi^{\frac{2\alpha}{2\alpha-1}}\right\}. \tag{32}$$

Performing the variation with respect to the lapse function N, $\delta\mathcal{L}_{canonical}/\delta N = 0$, the Hamiltonian constraint $\mathcal{H} = 0$ is obtained, where the classical density is written as

$$\mathcal{H} = \frac{1}{24}e^{-\frac{3}{2\alpha-1}\Omega}\left\{e^{-\frac{6(\alpha-1)}{2\alpha-1}\Omega}\Pi_\Omega^2 - \frac{12(2\alpha-1)}{\alpha}\left(\frac{2^{\alpha-1}}{\alpha}\right)^{\frac{1}{2\alpha-1}}\Pi_\phi^{\frac{2\alpha}{2\alpha-1}}\right\}. \tag{33}$$

In this point, we return to the equation of the scalar field (27) writing $d\tau = Ndt$

$$\Delta\phi = \sqrt{2}\left(\frac{\eta}{\alpha^2}\right)^{\frac{1}{2(2\alpha-1)}}\int e^{-\frac{3}{2\alpha-1}\Omega}Ndt, \tag{34}$$

and considering the gauge $N = 24e^{\frac{3}{2\alpha-1}\Omega}$; the classical scalar field goes like

$$\phi(t) = \phi_i(t_i) + 24\sqrt{2}\left(\frac{\eta}{\alpha^2}\right)^{\frac{1}{2(2\alpha-1)}}(t - t_i), \tag{35}$$

where t_i is the initial time for generic epoch and $\phi(t_i)$ is the scalar field evaluated in this time. In this way, the scalar field is present in the following epochs in our Universe. However, when we use the equation for momentum (31) in time τ, we have $\frac{d\phi}{d\tau} = -\left(\frac{2^{\alpha-1}}{\alpha}e^{-3\Omega}\Pi_\phi\right)^{\frac{1}{2\alpha-1}}$, and using the first time derivative of the scalar field (26), we obtain $\Pi_\phi^{\frac{1}{2\alpha-1}} = -\sqrt{2}\left(\frac{\eta}{2^{2(\alpha-1)}}\right)^{\frac{1}{2(2\alpha-1)}}$. Plugging this back into the Hamiltonian constraint, we fiind that the momenta in the variable Ω become

$$\Pi_\Omega = 2\sqrt{\frac{6(2\alpha-1)}{\alpha}}\left(\frac{2^{\alpha-1}}{\alpha}\right)^{\frac{1}{2(2\alpha-1)}}\left(\frac{\eta}{2^{2(\alpha-1)}}\right)^{\frac{\alpha}{2(2\alpha-1)}} e^{\frac{3(\alpha-1)}{2\alpha-1}\Omega}.$$

Now, using Equation (30) at time τ, we find that the scale factor becomes

$$A(\tau) = \left[\frac{\alpha}{2(2\alpha-1)}\sqrt{\frac{6(2\alpha-1)}{\alpha}}\left(\frac{2^{\alpha-1}}{\alpha}\right)^{\frac{1}{2(2\alpha-1)}}\left(\frac{\eta}{2^{2(\alpha-1)}}\right)^{\frac{\alpha}{2(2\alpha-1)}}(\tau-\tau_i)\right]^{\frac{2\alpha-1}{3\alpha}}, \tag{36}$$

which is consistent with the result obtained in the ref. [35], Equations (6) and (34), in the time τ, for ordinary matter $p = \omega\rho$. When we substitute the barotropic parameter $\omega = \frac{1}{2\alpha-1}$ in Equation (34) of the paper [35], we obtain the power law in the time τ, resulting in the same behavior as in (36). In this sense, we mention that the kinetic energy of the scalar field in the k-essence formalism falsifies the standard matter.

In the following, we will place all our effort in solving the quantum fractionary Wheeler–DeWitt equation.

3. Quantum Regime

The WDW equation for these models is obtained by making the usual substitution $\Pi_{q^\mu} = -i\hbar\partial_{q^\mu}$ into (33) and promoting the classical Hamiltonian density in the differential operator, applied to the wave function $\Psi(\Omega,\phi)$, $\hat{\mathcal{H}}\Psi = 0$. Then, we have

$$-\hbar^2 e^{-\frac{6(\alpha-1)}{2\alpha-1}\Omega}\frac{\partial^2\Psi}{\partial\Omega^2} - \frac{12(2\alpha-1)}{\alpha}\hbar^{\frac{2\alpha}{2\alpha-1}}\left(\frac{2^{\alpha-1}}{\alpha}\right)^{\frac{1}{2\alpha-1}}\frac{\partial^{\frac{2\alpha}{2\alpha-1}}}{\partial\phi^{\frac{2\alpha}{2\alpha-1}}}\Psi = 0. \tag{37}$$

We noted that the fractional differential equation with degree $\beta = \frac{2\alpha}{2\alpha-1}$ belongs to different intervals, depending on the value of the barotropic parameter; that is, when $\omega_X \in [0,1]$, the degree belongs to the interval $[1,2]$, and when $\omega_X \in [-1,0)$, the degree belongs to the interval $[0,1)$, for the scalar field ϕ (for this calculation, we remember that $\alpha = \frac{1}{2}\left(1+\frac{1}{\omega_X}\right)$). It is well-known in standard quantum cosmology that the best candidates for quantum solutions are those that have a damping behavior with respect to the scale factor; then, we use this conjecture in this formalism.

For simplicity, the factor $e^{-\frac{6(\alpha-1)}{2\alpha-1}\Omega}$ may be the factor ordered with $\hat{\Pi}_\Omega$ in many ways. Hartle and Hawking [54] suggested what might be called semi-general factor ordering, which, in this case, would order the terms $e^{-\frac{6(\alpha-1)}{2\alpha-1}\Omega}\hat{\Pi}^2_\Omega$ as $-e^{-(\frac{6(\alpha-1)}{2\alpha-1}-Q)\Omega}\partial_\Omega e^{-Q\Omega}\partial_\Omega = -e^{-\frac{6(\alpha-1)}{2\alpha-1}\Omega}\partial^2_\Omega + Q\,e^{-\frac{6(\alpha-1)}{2\alpha-1}\Omega}\partial_\Omega$, where Q is any real constant that measures the ambiguity in the factor ordering in the variables Ω and its corresponding momenta. We will assume in the following that this factor ordering for the Wheeler–DeWitt equation becomes

$$-\hbar^2 e^{-\frac{6(\alpha-1)}{2\alpha-1}\Omega}\frac{\partial^2\Psi}{\partial\Omega^2} + Q\hbar^2 e^{-\frac{6(\alpha-1)}{2\alpha-1}\Omega}\frac{\partial\Psi}{\partial\Omega} - \frac{12(2\alpha-1)}{\alpha}\hbar^{\frac{2\alpha}{2\alpha-1}}\left(\frac{2^{\alpha-1}}{\alpha}\right)^{\frac{1}{2\alpha-1}}\frac{\partial^{\frac{2\alpha}{2\alpha-1}}}{\partial\phi^{\frac{2\alpha}{2\alpha-1}}}\Psi = 0, \tag{38}$$

which when written in terms of the β parameter, becomes

$$-\hbar^2 e^{-3(2-\beta)\Omega}\frac{\partial^2\Psi}{\partial\Omega^2} + Q\hbar^2 e^{-3(2-\beta)\Omega}\frac{\partial\Psi}{\partial\Omega} - \frac{24}{\beta}\left(\frac{2^{\alpha-1}}{\alpha}\right)^{\frac{1}{2\alpha-1}}\hbar^\beta\frac{\partial^\beta}{\partial\phi^\beta}\Psi = 0. \tag{39}$$

By employing the separation variables method for the wave function $\Psi = \mathcal{A}(\Omega)\,\mathcal{B}(\phi)$, we have the following two differential equations for (Ω, ϕ)

$$\frac{d^2\mathcal{A}}{d\Omega^2} - Q\frac{d\mathcal{A}}{d\Omega} \mp \frac{\mu^2}{\hbar^2} e^{3(2-\beta)\Omega}\mathcal{A} = 0, \tag{40}$$

$$\frac{d^\beta \mathcal{B}_\pm}{d\phi^\beta} \pm \left(\frac{\alpha}{2^{\alpha-1}}\right)^{\frac{1}{2\alpha-1}} \frac{\mu^2\,\beta}{24\hbar^\beta}\mathcal{B}_\pm = 0, \tag{41}$$

where $\mathcal{B}_\pm$ considers the sign in the differential equation. The fractional differential Equation (41) can be given in the fractional frameworks, following [55,56] and identifying $\gamma = \frac{\beta}{2} = \frac{\alpha}{2\alpha-1}$, where now, γ is the order of the fractional derivative taking values in $0 < \gamma \leq 1$; then, we can write

$$\frac{d^{2\gamma}\mathcal{B}_\pm}{d\phi^{2\gamma}} \pm \left(\frac{\alpha}{2^{\alpha-1}}\right)^{\frac{1}{2\alpha-1}} \frac{\gamma\mu^2}{12\hbar^{2\gamma}}\mathcal{B}_\pm = 0, \qquad 0 < \gamma \leq 1, \tag{42}$$

the solution of the Equation (42) with a positive sign may be obtained by applying direct and inverse Laplace transforms [56], providing

$$\mathcal{B}_+(\phi,\gamma) = \mathbb{E}_{2\gamma}\left(-z^2\right), \qquad z = \left(\frac{\alpha}{2^{\alpha-1}}\right)^{\frac{1}{2(2\alpha-1)}} \frac{\sqrt{\gamma}\mu}{2\sqrt{3}\hbar^\gamma}\phi^\gamma, \qquad 0 < \gamma \leq 1. \tag{43}$$

In the ordinary case, $\gamma = 1$; then, the solution is [56],

$$\mathcal{B}_+(\phi,1) = \mathbb{E}_2\left[-\left(\frac{\mu}{2\sqrt{3}\hbar}(\phi-\phi_0)\right)^2\right] = cos\left(\frac{\mu}{2\sqrt{3}\hbar}(\phi-\phi_0)\right), \tag{44}$$

which is in agreement with the Equation (10), employing the Taylor series.

Following the book of Polyanin [57] (page 179.10), we discovered the solution for the first equation, considering different values in the factor ordering parameter (we take the corresponding sign minus in the constant μ^2)

$$\mathcal{A} = A_0\, e^{\frac{Q\Omega}{2}}\, Z_\nu\left[\frac{2\mu}{3\hbar(2-\beta)}\sqrt{-1}e^{\frac{3(2-\beta)}{2}\Omega}\right] = A_0\, e^{\frac{Q\Omega}{2}}\, K_\nu\left[\frac{\mu}{3\hbar(1-\gamma)}e^{3(1-\gamma)\Omega}\right], \tag{45}$$

with order $\nu = \pm\frac{Q}{6(1-\gamma)}$, where we had written the second expression in terms of the fractional order $\gamma = \frac{\beta}{2}$, and the solutions which become dependent on the sign of its argument; when $\sqrt{1}$ (for $\mathcal{B}_-$), the Bessel function Z_ν becomes the ordinary Bessel function J_ν. When $\sqrt{-1}$ (for $\mathcal{B}_+$), this becomes the modified Bessel function K_ν. For the particular values $\beta = 2$ ($\gamma = 1$), it will be necessary to solve the original differential equation for this variable.

Then, we have the probability density $|\Psi|^2$ by considering only $\mathcal{B}_+$, $\gamma \neq 1$,

$$|\Psi|^2 = \psi_0^2\, e^{Q\Omega}\, \mathbb{E}^2_{2\gamma}\left(-z^2\right)\, K_\nu\left[\frac{\mu}{3\hbar(1-\gamma)}e^{3(1-\gamma)\Omega}\right], \quad z = \left(\frac{\alpha}{2^{\alpha-1}}\right)^{\frac{1}{2(2\alpha-1)}} \frac{\sqrt{\gamma}\mu}{2\sqrt{3}\hbar^\gamma}\phi^\gamma. \tag{46}$$

On the other hand, it is well-known that in standard quantum cosmology, the wave function is unnormalized. There is no systematic method to do this, as the Hamiltonian density is not Hermitian. In particular cases, wave packets can be constructed, and from these wave packets we can construct a normalized wave function. In this work, we could not construct these wave packets. We hope to be able to do it in future studies.

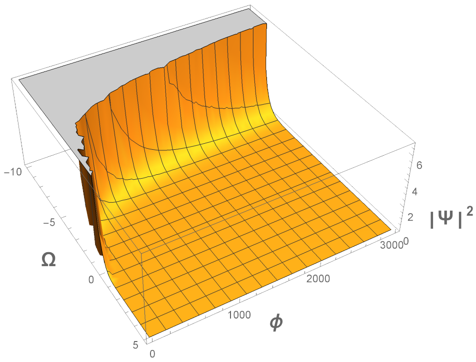
In the following, we present particular cases in the evolution of the Universe and some plots by employing the Equation (46), and for better viewing in the plots, we introduce by hand particular values to the constant ψ_0.

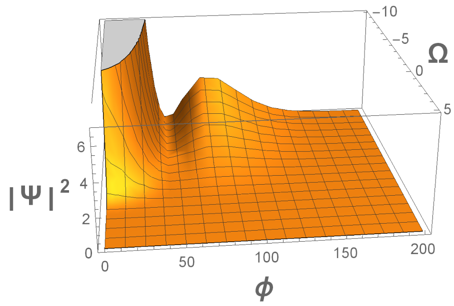
1. Radiation epoch, $\omega_X = \frac{1}{3}$, $\alpha = 2, \rightarrow \beta = \frac{4}{3} \rightarrow \gamma = \frac{2}{3}$.

When we choose the radiation case, (46) is written as

$$|\Psi|^2 = \psi_0^2\, e^{Q\Omega}\, \mathbb{E}^2_{\frac{4}{3}}\left(-z^2\right)\, K^2_{\frac{Q}{2}}\left[\frac{\mu}{\hbar}e^{\Omega}\right], \quad z = \frac{\mu}{3\sqrt{2}\hbar^{\frac{2}{3}}}\phi^{\frac{2}{3}}. \tag{47}$$

In the following Figure 1, we take the probability density (47); in the first and second Figures, and for better viewing in the plots, we take the constant $\psi_0 = \frac{1}{10}$, and in the third Figure the value becomes 1. In all Figures, the behavior of the probability density, in both variables (Ω, ϕ), has the appropriate decadent behavior. The range of the variable equals to $\phi \in [0, 3000]$, $[0, 200]$, and $[0, 40]$, respectively.





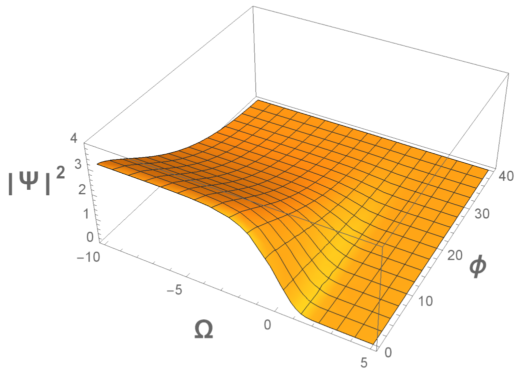


Figure 1. In the radiation era, we plot the Equation (47), considering different values in the order parameter Q $= -1, 1, 0$, from top to bottom, respectively. We consider the value for the parameter $\mu = 0.5$; we discard other values in the Q parameter.

2. Solution to $\omega_X = \frac{2}{3}, \alpha = \frac{5}{4}, \rightarrow \beta = \frac{5}{3} \rightarrow \gamma = \frac{5}{6}$.

   The probability density of the wave function becomes (here, $z = \left(\frac{5}{4\sqrt[4]{2}}\right)^{\frac{1}{3}} \frac{\sqrt{5}\mu}{6\sqrt{2}\hbar^{\frac{5}{6}}}\phi^{\frac{5}{6}}$)

$$\Psi^2 = \psi_0^2\, e^{Q\Omega}\mathbb{E}^2_{\frac{5}{3}}(-z^2)\, K^2_{Q}\left[\frac{\mu}{\hbar}e^{\frac{1}{2}\Omega}\right]. \tag{48}$$

In the Figure 2, we take the probability density (48); in the first and second Figures, and for better viewing in the plots, we take the constant $\psi_0 = \frac{1}{\sqrt{10}}$, and in the third Figure the value becomes 1. In all Figures, the behavior of the probability density, in both variables (Ω, ϕ), has the appropriate decadent behavior, and it presents an oscillatory behavior when $\omega_X \rightarrow 1$, since that is the behavior according to the Equation (44). Only for $Q = -1$, the probability density has a moderate increase in the direction where the scalar field evolves.

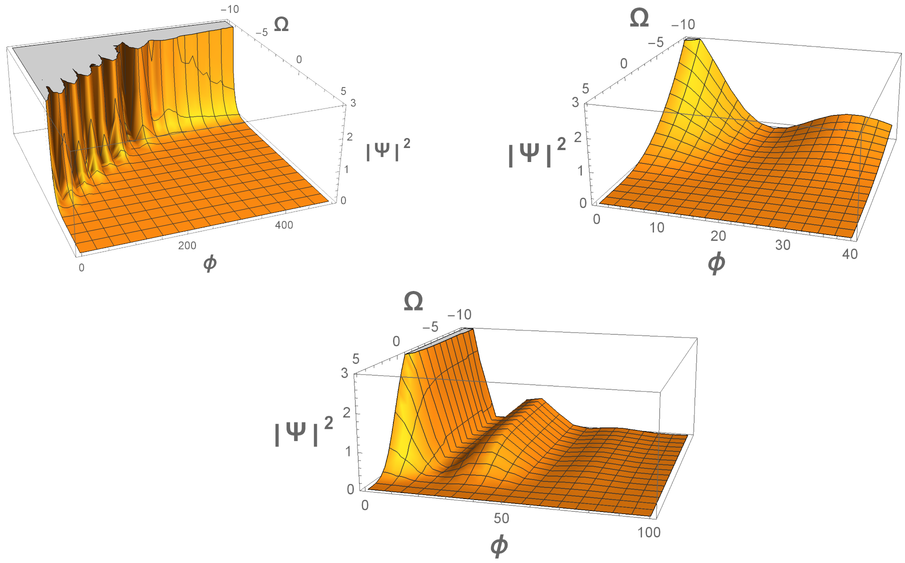


Figure 2. In the radiation-like era, we plot different combinations of the Equation (48), considering different values in the order parameter $Q = -1, 0, 1$, from top to bottom, respectively. We consider the value for the parameter $\mu = 0.5$; we discard other values in the Q parameter.

3. Dust era, $\omega_X = 0$, $\alpha \to \infty$; thus, $\beta = 1 \to \gamma = \frac{1}{2}$. In the dust case, the solution for the scale factor becomes

$$\mathcal{A} = A_0 \, e^{\frac{Q\Omega}{2}} \, Z_\nu \left[\frac{\mu}{\hbar} \sqrt{\pm 1} e^{\frac{3}{2}\Omega} \right], \qquad \nu = \pm \frac{Q}{2}. \tag{49}$$

In this case, the fractional differential Equation (41) for the scalar field is reduced to the first-order differential equation (for both signs in μ^2)

$$\frac{d\mathcal{B}_\mp}{d\phi} \mp \left(\frac{\alpha}{2^{\alpha-1}} \right)^{\frac{1}{2\alpha-1}} \frac{\mu^2}{24\hbar} \mathcal{B}_\mp = 0, \qquad \to \qquad \mathcal{B}_\mp = \beta_\mp \, e^{\pm \left(\frac{\alpha}{2^{\alpha-1}} \right)^{\frac{1}{(2\alpha-1)}} \frac{\mu^2}{24\hbar} \Delta\phi}.$$

Then, the probability density of the wave function becomes

$$\Psi^2 = \psi_0^2 \begin{cases} e^{\left(Q\Omega + \left(\frac{\alpha}{2^{\alpha-1}} \right)^{\frac{1}{(2\alpha-1)}} \frac{\mu^2}{12\hbar} \Delta\phi \right)} J^2_{\frac{Q}{3}} \left[\frac{\mu}{\hbar} e^{\frac{3}{2}\Omega} \right] \\ e^{\left(Q\Omega - \left(\frac{\alpha}{2^{\alpha-1}} \right)^{\frac{1}{(2\alpha-1)}} \frac{\mu^2}{12\hbar} \Delta\phi \right)} K^2_{\frac{Q}{3}} \left[\frac{\mu}{\hbar} e^{\frac{3}{2}\Omega} \right] \end{cases} \tag{50}$$

In the following Figure 3, we present the behavior of the probability density Ψ^2 by using the Equation (50) and taking the values for the order parameter $Q = -1, 0, 1$, because with these values, the probability density presents a structure well-defined for this era. In some of them, one structure did not appear; thus, we gave it a profile for the probability density for particular values in the scalar field. In these cases, the behavior of our Universe is quite selective in this formalism. Additionally, we can notice that the probability density has a moderate increase in the direction where the scalar field evolves. Similar results were reported in other formalisms [58–60].

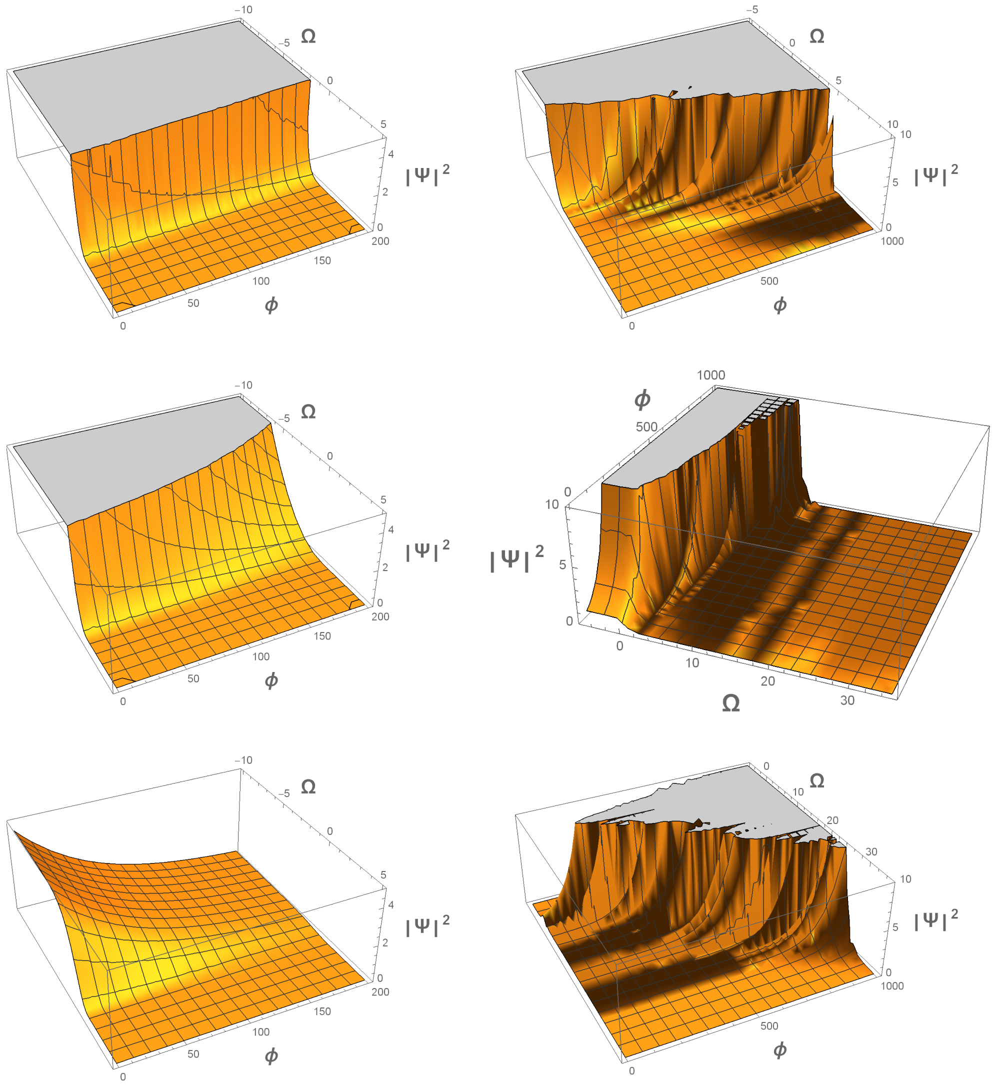


Figure 3. In the dust era, we have the corresponding solution $\mathcal{B}_+$ with the modified Bessel function K_ν and $\mathcal{B}_-$, the ordinary Bessel function of the Equation (50), considering different values in the order parameter $\mathrm{Q} = -1, 0, 1$, from top to bottom, respectively. We consider the value for the parameter $\mu = 0.5$; we discard other values in the Q parameter.

4. inflation such as $\omega_X = -\frac{1}{3}$, $\alpha = -1$; thus, $\beta = \frac{2}{3} \to \gamma = \frac{1}{3}$.
   For this particular case, (46) is written as

$$\Psi^2 = \psi_0^2\, e^{Q\Omega}\mathbb{E}^2_{\frac{2}{3}}(-z^2)\; K^2_{\frac{Q}{4}}\left[\frac{\mu}{\hbar}2e^{\Omega}\right],; \tag{51}$$

   however, the argument in the Mittag–Leffer function is complex, being

$$z = (0.6873648184993014 - 0.396850262992049841)\frac{\mu}{6\hbar^{\frac{1}{3}}}\phi^{\frac{1}{3}}, \tag{52}$$

   and the corresponding graph of the probability density can be made based on this function, taking the Re[z] or Im[z] parts.

5. inflation such as $\omega_X = -\frac{2}{3}$, $\alpha = -\frac{1}{4}$; thus, $\beta = \frac{1}{3} \to \gamma = \frac{1}{6}$.
   For this particular case, (46) is written as

$$\Psi^2 = \psi_0^2\, e^{Q\Omega}\, \mathbb{E}^2_{\frac{1}{3}}(-z^2)\; K^2_{\frac{Q}{5}}\left[\frac{\mu}{\hbar}e^{\frac{5}{2}\Omega}\right]. \tag{53}$$

and the argument of the Mittag–Leffer function, in this case, is equal to the previous case, complex

$$z = (0.5946035575013603 - 1.0298835719535588I)\frac{\mu}{6\hbar^{\frac{1}{6}}}\phi^{\frac{1}{6}}. \quad (54)$$

In a general way, the behavior of the probability density for both inflation-like scenarios is similar, in the Re[z] or Im[z] parts, over a wide range of values in the scalar field, as it appears in Figure 4. For the behavior for both inflation-like cases in the value of Ω, the behavior is appropriate.

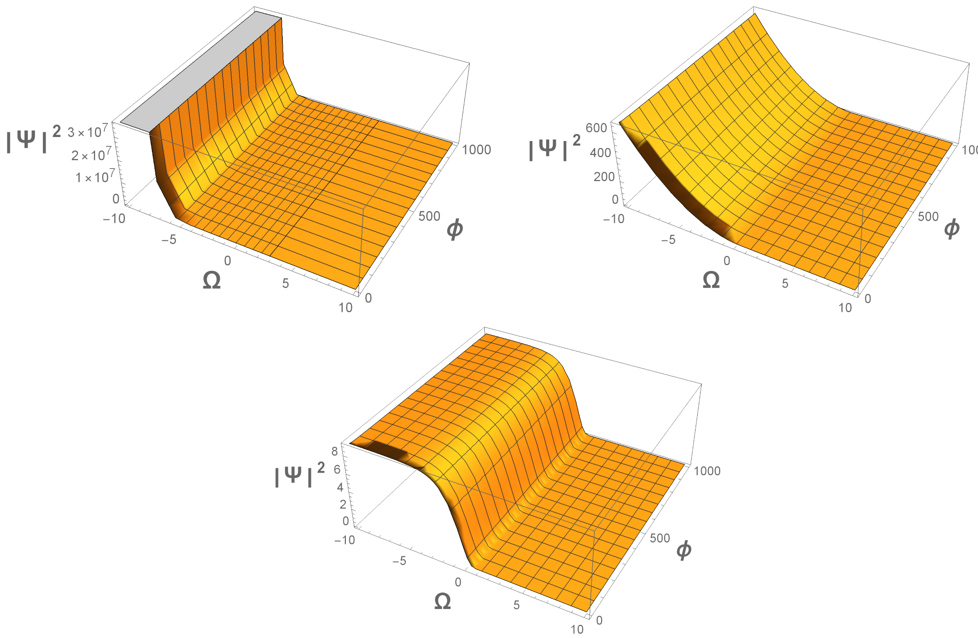


Figure 4. In the second case of the inflation-like scenario, corresponding to Equation (53), the behavior for the probability density in terms of the Mittag–Leffler function with complex values in its argument z is shown, for $Q = -1, 0, 1$. From top to bottom, respectively, we consider the value for the parameter $\mu = 0.5$. In these graphs, we use the Re[z] only; however, the plots with Im[z] are similar.

4. Final Remarks

There are different formalisms to incorporate fractional derivatives to cosmology. One of them starts from the variational principle of the action of general relativity with a fractional kernel; another way is starting from a particular configuration, for example, the FLRW model, and changing the ordinary derivatives with fractional one [14–20]. Unlike the previous formalisms, in the present work, we employed an action that contains a Lagrangian and a fractional parameter; in this way, equations of non-integer order in cosmology are obtained.

Unlike the previous formalism, in the present work, we employed a barotropic equation with perfect fluid for the energy momentum tensor in the K-essence scalar field into the Lagrangian and Hamiltonian formalism, obtaining the momentum of the scalar field with fractional numbers. However, the momentum of the scale factor appeared in the usual way. We obtained the classical solutions for different scenarios in the Universe, which are similar to those which were obtained for standard matter 16 years ago, see ref. [35] in Equations (6) and (34) in the time τ. In this sense, we can introduce the idea that the kinetic energy of the scalar field should falsify the standard matter by employing the K-essence formalism. In the quantum regime, we found a fractional differential equation for the scalar field, where the Mittag–Leffler function is the novel solution in many scenarios with real or complex values in its argument z. With this in mind, we visualized two alternatives in our analysis; the first one is within the traditional expectation over the behavior of the probability density, where the best candidates for quantum solutions are those that have a damping behavior with respect to the scale factor, which appear in all scenarios under our study, without saying anything about the scalar field. The other alternative scenario is when we keep the scale factor scenario, and we consider the values of the scalar field as significant in the quantum regime, appearing in various scenarios in the behavior of the Universe. This is mainly in those where the Universe shows huge behavior, for example, in the inflation-like scenario, see Figure 4 and the actual epoch, Figure 3 or Figure 5, where the scalar field appears as a background. In other words, the interpretation of probability density of the unnormalized wave function, is given when we demand that Ψ does not diverge when the scale factor A (or Ω) goes to infinity, and the scalar field is arbitrary. However, the evolution with the scalar field is now important in this class of theory and others, as it appears in some stages of evolution of our Universe, intended to serve as a a background for the evolution of the Universe in the classical world. The quantum regime appears with big values in the corresponding figures (see the corresponding (3), (5) and (4) plots). However, it is interesting to mention that in the radiation-like scenario, Figures 1 and 2, this behavior over the scalar field is less significant in the formation of atoms and close to the stiff matter scenarios, where an oscillatory behavior takes place in the scalar field. We briefly illustrate the main results in this work.

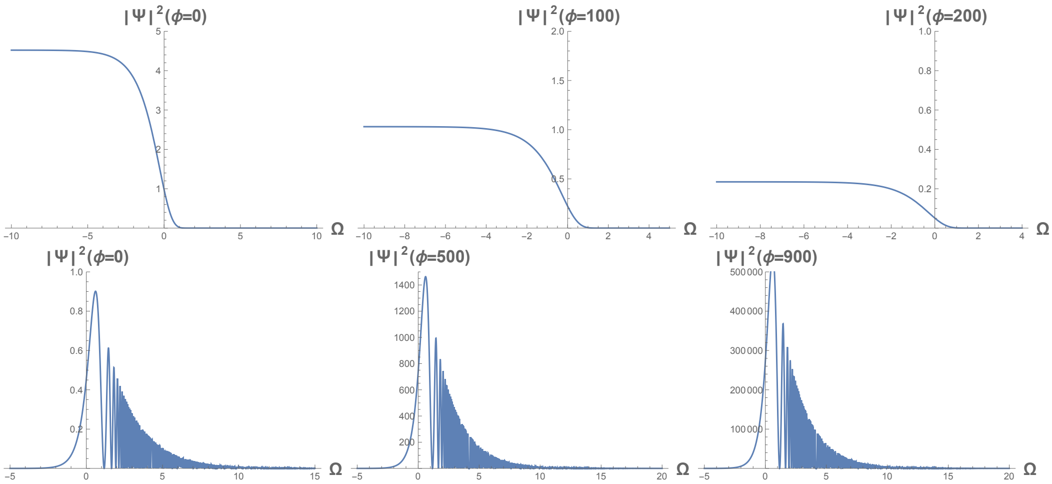


Figure 5. These graphs represent a break-off for the defined value in the scalar field ϕ, for the case in the factor ordering Q = 1, taking into account both solutions (50) given in Figure 3. The first line corresponds to the solution with the modified Bessel function, and the second line employs the ordinary Bessel function. This behavior appears in a similar way for different values in the factor ordering parameter, $Q = -1, 0, 1$. We can see that the values in the amplitude of the probability density for big values in the scalar field, have very large growth, acting as a background in the classical level.

1. Using the K-essence formalism in a general way, applied to the Friedmann–Lemaβ̂tre–Robertson–Walker cosmological model, we found the Hamiltonian density in the scalar field momenta raised to a power with non-integers. This produces in the quantum scheme a fractional differential equation in a natural way, such as in this variable with order $\beta = \frac{2\alpha}{2\alpha-1}$, where $\alpha \in (-1, \infty)$, which was solved for different scenarios of our Universe.
2. We found in the classical scheme that the time evolution τ of the scale factor for ordinary matter was found 16 years ago by one of us; this time, behavior is reproduced in the K-essence formalism, see Equation (36) in this work, which is consistent with the result obtained in the ref. [35], Equations (6) and (34), with ordinary matter.
3. In the quantum regime, the novel solution at the fractional differential equation in the scalar field was found in terms of the Mittag–Leffler function, with a real or complex argument, and we can see that this function appears in several scenarios of

our Universe in this work. This function is reported in particular work dealing with different disciplines of cosmology.

4. In one of our analyses presented on the probability density, we considered the values of the scalar field as significant in the quantum regime, appearing in various scenarios in the behavior of the Universe; mainly in those where the Universe has huge behavior. For example, in the inflation-like scenario and the actual epoch, where the scalar field appears as a background, the quantum regime appears with big values, but it presents a moderate development in other scenarios with a different ordering parameter Q.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Obtaining Equations of Motion

We take the variation over the fields components in the action for K-essence theory coupled with gravity

$$S = \int \sqrt{-g}[R + f(\phi)\mathcal{G}(X)]d^4x, \tag{A1}$$

where R is the Ricci scalar, g is the determinant to the metric, $f(\phi)$ is a function of the scalar field, and $\mathcal{G}[X]$ is a functional depending of the kinetic energy $X(\phi, g^{\mu\nu}) = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$.

The variation of the fields $(g^{\mu\nu}, \phi)$ in the action (A1) becomes

$$\begin{aligned}
\delta S &= \int \delta\left[\sqrt{-g}R\right]d^4x + \int \delta\sqrt{-g}[f(\phi)\mathcal{G}(X)]d^4x \\
&+ \int \sqrt{-g}[\delta f(\phi)\mathcal{G}(X) + f(\phi)\delta\mathcal{G}(X)]d^4x, \\
&= \int \sqrt{-g}\, G_{\lambda\theta}\delta g^{\lambda\theta}d^4x + \int \sqrt{-g}\frac{1}{2}[-f(\phi)\mathcal{G}(X)]g_{\lambda\theta}\delta g^{\lambda\theta}d^4x + \\
&+ \int \sqrt{-g}\left[\frac{\partial f(\phi)}{\partial\phi}\delta\phi\mathcal{G}(X) + f(\phi)\frac{\partial\mathcal{G}(X)}{\partial X}\delta X\right]d^4x,
\end{aligned} \tag{A2}$$

where the variation of the functional $\mathcal{G}(X)$ is over the kinetic energy

$$\begin{aligned}
\delta X(\phi) &= -\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\delta g^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\delta\phi\nabla_\nu\phi - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\delta\phi, \\
&= -\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\delta g^{\mu\nu} - \nabla^\nu\phi\nabla_\nu\delta\phi;
\end{aligned}$$

introducing into the last equation, we have

$$\begin{aligned}\delta S &= \int \sqrt{-g}\left[G_{\lambda\theta} - \frac{1}{2}f(\phi)\left(\mathcal{G}(X)g_{\lambda\theta} + \frac{\partial \mathcal{G}(X)}{\partial X}\nabla_\mu\phi\nabla_\nu\phi\right)\right]\delta g^{\lambda\theta}d^4x \\ &+ \int \sqrt{-g}\left[\frac{\partial f(\phi)}{\partial\phi}\delta\phi\mathcal{G}(X) + f(\phi)\frac{\partial\mathcal{G}(X)}{\partial X}\{-\nabla^\nu\phi\nabla_\nu\delta\phi\}\right]d^4x.\end{aligned} \tag{A3}$$

However, we know that the total derivative

$$\begin{aligned}\nabla_\nu(f(\phi)G_X\nabla^\nu\phi\delta\phi) &= \frac{df(\phi)}{d\phi}\nabla_\nu\phi\nabla^\nu\phi\, G_X\delta\phi + f(\phi)G_{XX}X_{,\nu}\nabla^\nu\delta\phi + f(\phi)G_X\nabla^\nu_\nu\phi\delta\phi \\ &+ f(\phi)G_X\nabla^\nu\phi\nabla_\nu\delta\phi.\end{aligned} \tag{A4}$$

Thus,

$$\begin{aligned}-f(\phi)G_X\nabla^\nu\phi\nabla_\nu\delta\phi &= \frac{df(\phi)}{d\phi}\nabla_\nu\phi\nabla^\nu\phi\, G_X\delta\phi + f(\phi)G_{XX}X_{,\nu}\nabla^\nu\delta\phi + f(\phi)G_X\nabla^\nu_\nu\phi\delta\phi \\ &- \nabla_\nu(f(\phi)G_X\nabla^\nu\phi\delta\phi),\end{aligned} \tag{A5}$$

and reinserting into (A3), we have

$$\begin{aligned}\delta S &= \int \sqrt{-g}\left[G_{\lambda\theta} - \frac{1}{2}f(\phi)\left(\mathcal{G}(X)g_{\lambda\theta} + \frac{\partial\mathcal{G}(X)}{\partial X}\nabla_\mu\phi\nabla_\nu\phi\right)\right]\delta g^{\lambda\theta}d^4x \\ &+ \int\sqrt{-g}\left\{\frac{\partial f(\phi)}{\partial\phi}\delta\phi\mathcal{G}(X) + \frac{df(\phi)}{d\phi}\underbrace{\nabla_\nu\phi\nabla^\nu\phi}_{-2X}\, G_X\delta\phi + f(\phi)G_{XX}X_{;\nu}\nabla^\nu\phi\,\delta\phi\right. \\ &\left. + f(\phi)G_X\nabla^\nu_{;\nu}\phi\delta\phi - \nabla_\nu(f(\phi)G_X\nabla^\nu\phi\delta\phi)\right\}d^4x, \\ &= \int\sqrt{-g}\left[G_{\lambda\theta} - \frac{1}{2}f(\phi)\left(\mathcal{G}(X)g_{\lambda\theta} + \frac{\partial\mathcal{G}(X)}{\partial X}\nabla_\mu\phi\nabla_\nu\phi\right)\right]\delta g^{\lambda\theta}d^4x \\ &+ \int\sqrt{-g}\left\{\frac{df(\phi)}{d\phi}[\mathcal{G}(X) - 2XG_X] + f(\phi)[G_{XX}X_{,\nu}\nabla^\nu + G_X\nabla^\nu_\nu\phi]\right\}\delta\phi d^4x,\end{aligned} \tag{A6}$$

where we have eliminated the integral over the total derivative.

The variation over the scalar field gives the equation of motion for this field, being

$$\frac{df(\phi)}{d\phi}[\mathcal{G}(X) - 2XG_X] + f(\phi)[G_{XX}X_{,\nu}\nabla^\nu + G_X\nabla^\nu_\nu\phi] = 0, \tag{A7}$$

which corresponds to Equation (15). For obtaining the Einstein field-like equations, we take the variation on the metric $g^{\mu\nu}$,

$$G_{\mu\nu} = \frac{1}{2}f(\phi)\left[\nabla_\mu\phi\nabla_\nu\phi\frac{\partial\mathcal{G}(X)}{\partial X} + g_{\mu\nu}\mathcal{G}(X)\right], \tag{A8}$$

where the energy-momentum tensor becomes

$$T_{\mu\nu}(\phi) = +\frac{1}{2}f(\phi)\left[\nabla_\mu\phi\nabla_\nu\phi\frac{\partial\mathcal{G}(X)}{\partial X} + g_{\mu\nu}\mathcal{G}(X)\right], \tag{A9}$$

and considering the energy-momentum tensor of a barotropic perfect fluid for the scalar fields

$$T_{\mu\nu}(\phi) = (\rho + P)u_\mu(\phi)u_\nu(\phi) + P\, g_{\mu\nu}, \tag{A10}$$

we have that the pressure P and the energy density ρ of the scalar fields become

$$P(\phi) = \frac{1}{2} f(\phi)\mathcal{G}, \qquad \rho(\phi) = \frac{1}{2} f\left[2X\frac{\partial \mathcal{G}}{\partial X} - \mathcal{G}\right], \tag{A11}$$

the four-velocity becomes $u_\mu u_\nu = \frac{\nabla_\mu \phi \nabla_\nu \phi}{2X}$ and the barotropic index ω_X is

$$\omega_X = \frac{f(\phi)\mathcal{G}}{f\left[2X\frac{\partial \mathcal{G}}{\partial X} - \mathcal{G}\right]}. \tag{A12}$$

Appendix B. Obtaining the Equations of Motion with Particular Metric

We have rewritten the line element in the time $\tau = Ndt = \prime$,

$$\begin{aligned} ds^2 &= -N(t)^2 dt^2 + A^2(t)\left[dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\right], && \text{(A13)} \\ &= -d\tau^2 + A^2(\tau)\left[dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\right], && \text{(A14)} \end{aligned}$$

where the metric element $g_{\tau\tau} = -1$ implies that $\Gamma^\tau_{\tau\tau} = 0$ and $\Gamma^j_{j\tau} = \Omega'$, $j = r, \theta, \phi = 1, 2, 3$.

When the function $f(\phi)$ is constant, the Equation (15) is reduced to

$$\mathcal{G}_X \phi^{;\nu}_{;\nu} + \mathcal{G}_{XX} X_{;\nu} \phi^{;\nu} = 0; \tag{A15}$$

using the metric (A14), we obtain that the different parameters into the Equation (A15) are

$$\begin{aligned} X &= \frac{1}{2}(\phi')^2, \quad \rightarrow \quad (\phi')^2 = 2X, \qquad X' = \phi'\phi'', \quad \rightarrow \quad \phi'' = \frac{X'}{\phi'} \\ \phi^{;\nu}_{;\nu} &= \phi^{,\nu}_{,\nu} + \Gamma^\nu_{\nu\rho}\phi^\rho = \phi'' + \left(\Gamma^\tau_{\tau\tau} + \Gamma^1_{1\tau} + \Gamma^2_{2\tau} + \Gamma^3_{3\tau}\right)\phi' = \phi'' + 3\Omega'\phi'. && \text{(A16)} \end{aligned}$$

Thus, the Equation (A15) is rewritten as

$$\mathcal{G}_X(\phi'' + 3\Omega'\phi') + \mathcal{G}_{XX} X'\phi' = \mathcal{G}_X\left(\frac{X'}{\phi'} + 3\Omega'\phi'\right) + \mathcal{G}_{XX} X'\phi' = 0; \tag{A17}$$

multiplying by ϕ', we have

$$[\mathcal{G}_X + 2X\mathcal{G}_{XX}]X' + 6\Omega' X\mathcal{G}_X = 0, \tag{A18}$$

where we had used the previous relations; this equation correspond to (24).

Dividing between XG_X the last equation, we have

$$\begin{aligned} &\frac{X'}{X} + 2\frac{\mathcal{G}_{XX}}{\mathcal{G}_X} X' + 6\frac{A'}{A} = \frac{d}{d\tau}\left(LnX + Ln\mathcal{G}^2_X + LnA^6\right) \\ &= \frac{d}{d\tau} Ln\left(A^6 X\mathcal{G}^2_X\right) = 0, \quad \rightarrow \quad A^6 X\mathcal{G}^2_X = \eta = constant, && \text{(A19)} \end{aligned}$$

obtaining that

$$X\mathcal{G}^2_X = \eta A^{-6}, \tag{A20}$$

which is the Equation (25).

When $\mathcal{G} = X^\alpha$ and substituting into (A20), we have $\alpha^2 X^{2\alpha-1} = \eta A^{-6}$ with $X = \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2$, obtaining for the scalar field ϕ the equation

$$\frac{d\phi}{d\tau} = \sqrt{2}\left[\left(\frac{\eta}{\alpha^2}\right)^{\frac{1}{2(2\alpha-1)}} A^{-\frac{3}{2\alpha-1}}\right] = \sqrt{2}\left[\left(\frac{\eta}{\alpha^2}\right)^{\frac{1}{2(2\alpha-1)}} e^{-\frac{3}{2\alpha-1}\Omega}\right], \tag{A21}$$

whose solution in the time τ is

$$\Delta\phi = \sqrt{2}\left(\frac{\eta}{\alpha^2}\right)^{\frac{1}{2(2\alpha-1)}} \int A^{-\frac{3}{2\alpha-1}} d\tau = \sqrt{2}\left(\frac{\eta}{\alpha^2}\right)^{\frac{1}{2(2\alpha-1)}} \int e^{-\frac{3}{2\alpha-1}\Omega} d\tau, \quad \text{(A22)}$$

Appendix C. Equivalence between Lagrangian Densities

The canonical Lagrangian density $\mathcal{L}_{canonical}(q_i, \Pi_i, t)$ (32) in gravitation theories is obtained from the usual Lagrangian density $\mathcal{L}(q_i, \dot{q}_i, t)$ (29), rewritten the velocities $\dot{q}_i$ in term of the momenta $\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ to the corresponding coordinate field q_i. With this procedure, the canonical Lagrangian density appear directly written as a Lagrangian density in constrained systems, where the Lagrangian multiplier is the lapse function $N(t)$, being the corresponding gauge parameter in this theory. This is equivalent to using the canonical transformation where the hamiltonian density is $\mathcal{H} = \Pi_j \dot{q}^j - \mathcal{L}$. However, from this point of view, in this expression the $\mathcal{H}$ must be interpreted as $\mathcal{H} = N\mathcal{H}_{canonical}$ where the lapse function N appears as a lagrangian multiplier. In the following we realize this calculation, employing the usual canonical transformation. We have the momenta

$$\Pi_\Omega = \frac{12}{N} e^{3\Omega}\dot{\Omega}, \quad \rightarrow \quad \dot{\Omega} = \frac{N}{12} e^{-3\Omega}\Pi_\Omega, \quad \text{(A23)}$$

$$\Pi_\phi = -\left(\frac{1}{2}\right)^{\alpha} \frac{2\alpha}{N^{2\alpha-1}} e^{3\Omega}\dot{\phi}^{2\alpha-1}, \quad \rightarrow \quad \dot{\phi} = -N\left[\frac{2^\alpha}{2\alpha} e^{-3\Omega}\Pi_\phi\right]^{\frac{1}{2\alpha-1}}, \quad \text{(A24)}$$

substituting into the canonical transformation between the Hamiltonian density and Lagrangian density

$$\mathcal{H} = \Pi_j \dot{q}^j - \mathcal{L}$$

$$\begin{aligned}
\mathcal{H} &= \Pi_\Omega\dot{\Omega} + \Pi_\phi\dot{\phi} - e^{3\Omega}\left[6\frac{\dot{\Omega}^2}{N} - \left(\frac{1}{2}\right)^{\alpha}(\dot{\phi})^{2\alpha}N^{-2\alpha+1}\right] \\
&= \Pi_\Omega\left(\frac{N}{12}e^{-3\Omega}\Pi_\Omega\right) + \Pi_\phi\left(-N\left[\frac{2^\alpha}{2\alpha}e^{-3\Omega}\Pi_\phi\right]^{\frac{1}{2\alpha-1}}\right) \\
&\quad -e^{3\Omega}\left[\frac{6}{N}\left(\frac{N}{12}e^{-3\Omega}\Pi_\Omega,\right)^2 - \left(\frac{1}{2}\right)^{\alpha}\left(-N\left[\frac{2^\alpha}{2\alpha}e^{-3\Omega}\Pi_\phi\right]^{\frac{1}{2\alpha-1}}\right)^{2\alpha} N^{-2\alpha+1}\right] \\
&= Ne^{3\Omega}\Pi_\Omega^2\left(\frac{1}{12} - \frac{1}{24}\right) - Ne^{-\frac{3\Omega}{2\alpha-1}}\left(\frac{2^{\alpha-1}}{\alpha}\right)^{\frac{1}{2\alpha-1}}\Pi_\phi^{\frac{2\alpha}{2\alpha-1}} \\
&\quad +Ne^{-\frac{3\Omega}{2\alpha-1}}\frac{1}{2\alpha}\left(\frac{2^{\alpha-1}}{\alpha}\right)^{\frac{1}{2\alpha-1}}\Pi_\phi^{\frac{2\alpha}{2\alpha-1}} \\
&= N\frac{e^{-3\Omega}}{24}\Pi_\Omega^2 - Ne^{-\frac{3\Omega}{2\alpha-1}}\left(\frac{2^{\alpha-1}}{\alpha}\right)^{\frac{1}{2\alpha-1}}\left[\frac{1}{1-2\alpha}\right]\Pi_\phi^{\frac{2\alpha}{2\alpha-1}} \\
&= N\frac{e^{-3\Omega}}{24}\Pi_\Omega^2 - Ne^{-\frac{3\Omega}{2\alpha-1}}\frac{2\alpha-1}{2\alpha}\left(\frac{2^{\alpha-1}}{\alpha}\right)^{\frac{1}{2\alpha-1}}\Pi_\phi^{\frac{2\alpha}{2\alpha-1}} \\
&= N\frac{e^{-\frac{3\Omega}{2\alpha-1}}}{24}\left(e^{-\frac{6(\alpha-1)}{2\alpha-1}\Omega}\Pi_\Omega^2 - \frac{12(2\alpha-1)}{\alpha}\left(\frac{2^{\alpha-1}}{\alpha}\right)^{\frac{1}{2\alpha-1}}\Pi_\phi^{\frac{2\alpha}{2\alpha-1}}\right), \quad \text{(A25)}
\end{aligned}$$

corresponding to Equation (33).

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