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Novel Free Differential Algebras for Supergravity

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Abstract: We develop the theory of Free Integro-Differential Algebras (FIDA) extending the powerful technique of Free Differential Algebras constructed by D. Sullivan. We extend the analysis beyond the superforms to integral- and pseudo-forms used in supergeometry. It is shown that there are novel structures that might open the road to a deeper understanding of the geometry of supergravity. We apply the technique to some models as an illustration and we provide a complete analysis for D = 11 supergravity. There, it is shown how the Hodge star operator for supermanifolds can be used to analyze the set of cocycles and to build the corresponding FIDA. A new integral form emerges which plays the role of the truly dual to 4-form $F^{(4)}$ and we propose a new variational principle on supermanifolds.

Keywords: supergravity; supergeometry; free differential algebras

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1. Ingredients

A new integral form has been discovered, which extends beyond existing Free Differential Algebra (FDA). [1] This finding confirms the super geometric nature of supergravity and provides a unique perspective on the subject. The extended algebra is renamed Free Integro-Differential Algebra (FIDA).

As is pointed out by different authors [2–11], the superspace nature of supersymmetric and supergravity models is intimately related to the supergeometric structure due to vielbeins and gravitinos. Their supersymmetry and diffeomorphism transformations can be recast in a beautiful geometric framework known as the *rheonomic* approach. This allows us to use the powerful technique of Cartan calculus, such as the exterior differential, the contraction operator, the Lie derivates, etc., and to compute some cohomologies for flat or curved supermanifolds. It turns out that the Chevalley–Eilenberg cohomology is usually non-trivial, and it can be conveniently understood in terms of free differential algebras. The elements of that algebra are usually higher-degree forms and they are additional degrees of freedom in the supergravity field spectrum besides the vielbein, the spin connection, and the gravitinos. There is a vast literature on the argument, which we refer to for details and applications [12–19].

As discussed in several works (see for example [20–24]), it has been shown that there are other sectors of cohomology for supergravity and supersymmetric theories, which play an important role in supergravity theory. It has been discovered, that in integral form cohomology (see [25,26] for the precise definition) there are additional cohomology classes. They are expected because of the Hodge duality discovered in [27,28]. Those are cohomologies in the sector of integral forms and pseudoforms. In the presence of supermanifolds, the exterior bundle is not sufficient to describe the complete geometry and must be supplemented by the sector of integral forms. Those forms can be integrated into the supermanifold and explicitly constructed in terms of the delta function of commuting 1-forms and their derivatives. Therefore, in the present work, we discuss whether the techniques developed in [10,29] can be adapted to this new framework, when integral form cohomologies are present, and what it implies.

It is shown that once the FDA for the superform sector has been constructed, using the ring structure of forms and the module structure of integral forms, the integral form sector is removed by adding suitable potentials. However, not completely. It is proven in Section 1.2 the potentials needed for superforms are insufficient to compensate for all integral forms. Indeed, at least one requires a maximal picture to define the FIDA. It could happen that one integral form with the maximal-picture is not enough since the introduction of a maximal-picture potential might introduce new cohomology classes with higher pictures, and for that, one needs one more potential.

What is the role of the maximal-picture potential (and occasionally also the additional one where new integral form cohomologies emerge)? There are two aspects to be discussed: (1) Does it change the physical spectrum of the theory? (2) What is the role of this additional integral form? It is shown in Section 3.3 that, indeed, there are no additional degrees of freedom, and the new integral form is related to the original spectrum of the theory. Concerning its role, we have to recall that to write down an action, we have to integrate over the entire supermanifold, and this can be achieved with an integral form. The presence of a naïve integral form in the spectrum tells us that we must build the integral form to construct a consistent action and its equations of motion [20–24]. One can relate this non-trivial integral form (or its potential) as a reflection of the existence of the Berezianian (see [30]), and the fact that there is a single non-trivial integral form seems to indicate that the Berezianian is a truly essential ingredient in the supergravity realm.

In Section 1, we list and discuss the ingredients needed for the analysis and the general theory. In Section 2, we give some examples, starting from a toy example to a non-Abelian coset manifold example. In Section 3, we apply the construction to D = 4, 6, 11 models, and we construct the complete FIDA using the Hodge duality. New cocycles are shown and the relations among them are discussed. In Section 4, we write some conclusions and open issues on some delicate mathematical questions we are not able to discuss in the present work.

1.1. *Free Differential Algebras (FDAs)*

Given a Lie supergroup $\mathcal{G}$, we denote its Lie algebra by $\mathcal{L}_G$ with n bosonic generators T_a and m fermionic generators Q_α. Associated with each generator, we introduce the Maurer–Cartan (MC) forms (V^a, ψ^α), as follows: having chosen an element g of the supergroup $\mathcal{G}$, we compute an element of the Lie algebra, and we expand it in terms of the generators as

$$g^{-1}dg = V^a T_a + \psi^\alpha Q_\alpha \tag{1}$$

(in this paper, we will follow the Einstein convention, which involves summing on repeated indices). The MC forms satisfy the MC equations

$$\begin{aligned} dV^a &= f^a{}_{\alpha\beta}\psi^\alpha \wedge \psi^\beta + f^a{}_{bc}V^b \wedge V^c\,, \\ d\psi^\alpha &= f^a{}_{b\beta}V^b \wedge \psi^\beta\,. \end{aligned} \tag{2}$$

where $f^a{}_{\alpha\beta}, f^a{}_{\alpha\beta}$, and $f^a{}_{b\beta}$ are the structure constants satisfying the Jacobi identities. The differential d is the Chevalley–Eilenberg differential, and its nilpotency follows from Jacobi identities. If the bosonic generators T^a corresponds to the translation generators on a supermanifold and Q^α the supersymmetry generators, then the MC forms (V^a, ψ^α) represent the supervielbein associated with the supermanifold. The MC forms (V^a, ψ^α) carry form number equal to one and they are anticommuting and commuting, respectively, for the wedge product

$$V^a \wedge V^b = -V^b \wedge V^a\,, \quad V^a \wedge \psi^\alpha = \psi^\alpha \wedge V^a\,, \quad \psi^\alpha \wedge \psi^\beta = \psi^\beta \wedge \psi^\alpha\,. \tag{3}$$

For convenience, we use the notation $E^A = (V^a, \psi^\alpha)$, and collectively we denote by f^A_{BC} the structure constants where $A = (a, \alpha)$ runs over all indices. In terms of E^A, the commutation properties are summarized in

$$E^A \wedge E^B = (-1)^{(|A|+1)(|B|+1)} E^B \wedge E^A \tag{4}$$

where $|A| = 1$ for V^a and $|A| = 0$ for ψ^α.

On the space of forms $\Omega^{(p)}(\mathcal{SM}, \mathbb{R})$ with trivial coefficients, one can compute the Chevalley–Eilenberg (CE) cohomology at every form number. For a general discussion on this point for super Lie algebras, we refer to [25,26]. The CE cohomology classes (labeled by I with form degree p_I) are expressed in terms of the MC forms E^A as follows:

$$\Omega^{(p_I)}_I = \Omega_{I,A_1\dots A_{p_I}} E^{A_1} \wedge \dots \wedge E^{A_{p_I}}, \tag{5}$$

such that

$$d\Omega_I^{(p_I)} = 0, \quad \Omega_I^{(p_I)} \neq dA_I^{(p_I-1)} \tag{6}$$

where the coefficients $\Omega_{I,A_1\dots A_{p_I}}$ are constant.

The *free differential algebra* (FDA) extension $\mathcal{L}'_G$ of a Lie algebra $\mathcal{L}_G$ (studied in [4,10,29,31]) enlarges the set of MC forms E^A to include a new set of p-forms $\{A_I^{(p_I)}\}$ associated with each Chevalley–Eilenberg cohomology classes in Equation (5) such that

$$\begin{aligned} dE^A + \frac{1}{2} f^A{}_{BC} E^B \wedge E^C = 0 \\ dA_I^{(p_I-1)} + \Omega_I^{(p_I)} = 0 \end{aligned} \tag{7}$$

It is clear that $\Omega_I^{(p_I)}$ differing by exact pieces $d\Phi_I^{(p_I-1)}$ lead to equivalent FDA's, via the redefinition $A_I^{(p_I-1)} \to A_I^{(p_I-1)} + \Phi_I^{(p_I-1)}$. The whole procedure can be repeated on the free differential algebra $\mathcal{L}'_G$, which now contains the set of p-forms $\{E^A, A_I^{(p_I-1)}\}$. To proceed, one must calculate the cohomology using the latest set of forms and consider any new cocycles that may arise, as

$$\Omega^{(q_{I'})}_{I'} = \Omega_{I',A_1\dots A_r}{}^{I_1\dots I_s} E^{A_1} \wedge \dots \wedge E^{A_r} \wedge A_{I_1}^{(p_{I_1}-1)} \wedge \dots \wedge A_{I_s}^{(p_{I_s}-1)} \tag{8}$$

such that $q_{I'} = r + \sum_{I \in \mathcal{I}} p_I$, where $\mathcal{I}$ is the set of indices I of the potentials $A_I^{p_I}$ appearing in the cocycle $\Omega_{I'}^{(q_{I'})}$. The coefficients $\Omega_{I',A_1\dots A_r}{}^{I_1\dots I_s}$ are constant. Then, as above, one introduces new potential $A_{I'}^{(q_{I'}-1)}$ satisfying Equation (6). If such a polynomial exists, the FDA of Equations (7) can be further extended to $\mathcal{L}''_G$ with $\{E^A, A_I^{p_I-1}, A_{I'}^{q_{I'}-1}\}$. Of course, one would like to know if the procedure stops after a finite number of steps or continues indefinitely.

Computing the Hilbert–Poincaré series associated with the CE cohomology, one can explicitly test whether the introduction of new potentials $A_I^{(p-1)}$ trivializes the cohomology. This can be easily achieved by adding the contribution to the Hilbert–Poincaré series of the new potentials and checking if the complete Hilbert–Poincaré series equals 1. The Hilbert–Poincaré series will be reviewed in the next section, but the general procedure has already been discussed in [32].

As for ordinary Lie algebras, a dynamical theory based on FDA is obtained by introducing non-vanishing curvature for MC forms and all potentials $\{E^A, A_I^{p_I-1}, A_{I'}^{q_{I'}-1}\}$ of the FDA, see [10] for a precise discussion and several interesting examples.

The rewriting of the FDA's in terms of larger Lie (super)algebras, by expressing the p-forms with $p > 1$ as products of 1-form fields involving new fields, has been considered already in [1] for $d = 11$ supergravity. Recent developments of this idea can be found in [33–35].

1.2. Free Integro-Differential Algebras (FIDAs)

Besides the super form sector of the theory represented by the spaces $\Omega^{(p|0)}(\mathcal{SM},\mathbb{R}) \equiv \Omega^{(p)}(\mathcal{SM},\mathbb{R})$ (with $p \geq 0$), we are also interested in the integral form sector of the theory, which we denote by $\Omega^{(p|m)}(\mathcal{SM},\mathbb{R})$ (with $p \leq n$). To define the latter, we introduce the symbol $\delta(\psi^\alpha)$ with the distribution-like property

$$\psi^\alpha \wedge \delta(\psi^\alpha) = 0\,, \quad \alpha \text{ is not summed} \tag{9}$$

and the commutation relations

$$\begin{aligned} &\delta(\psi^\alpha)\wedge\delta(\psi^\beta) = -\delta(\psi^\beta)\wedge\delta(\psi^\alpha) \\ &V^a \wedge \delta(\psi^\beta) = -\delta(\psi^\beta)\wedge V^a \\ &\psi^\alpha\wedge\delta(\psi^\beta) = \delta(\psi^\beta)\wedge\psi^\alpha\,, \quad \alpha\neq\beta \end{aligned} \tag{10}$$

According to the first equation, the maximal number of $\delta(\psi^\beta)$ is equal to the fermionic dimension of the supermanifold $\mathcal{SM}$. We assign to every single delta a degree (imported from string theory jargon) known as *picture* number, therefore the maximal picture is n. Under coordinate changes, the symbol $\delta(\psi^\beta)$ does not transform as a tensor [2] and generally is not a globally defined quantity. Nonetheless, the product of all $\delta(\psi^\beta)$ transform as a density. In particular, the integral form

$$\mathrm{Vol}^{(n|m)} = V^1\wedge\cdots\wedge V^n\wedge\delta(\psi^1)\wedge\cdots\wedge\delta(\psi^m) \tag{11}$$

transforms as a Berezinian section of the supermanifold, as can be easily checked by using the above properties.

We define

$$\iota_{\partial_\alpha}\delta(\psi^\beta) \equiv \frac{\partial}{\partial\psi^\alpha}\delta(\psi^\beta) = \delta'(\psi^\beta)\delta^\beta_\alpha \tag{12}$$

where $\iota_{\partial_\alpha}\psi^\beta = \delta^\beta_\alpha$ is the usual pairing between an odd vector field ∂_α, and its dual form ψ^β. $\delta'(\psi^\beta)$ denotes the first derivative with respect to the argument of the delta function. Likewise, $\delta^{(g)}(\psi^\beta)$ is the g-derivative with respect to the argument of the delta function. For $\delta^{(g)}(\psi^\beta)$, the distribution-like property (integration by parts)

$$\psi^\beta\delta^{(g)}(\psi^\beta) = -\delta^{(g-1)}(\psi^\beta)\,, \quad \beta \text{ is not summed} \tag{13}$$

holds. Notice that, according to Equation (13), $\delta^{(g)}(\psi^\beta)$ carries the $(-g)$-form degree, and $\delta(\psi^\alpha)$ carries no form degree. Acting with the differential d on these distribution-like expressions, we use the chain rules

$$d\Big(\delta^{(g)}(\psi^\beta)\Big) = d\psi^\beta\wedge\delta^{(g+1)}(\psi^\beta), \tag{14}$$

and therefore, the Chevalley–Eilenberg differential can be extended to act on those expressions.

Generically, a form $\Omega^{(p|q)}$ with p form degree and q picture (number of deltas) on a supermanifold can be expressed in its local forms as

$$\Omega^{(p|q)} = \Omega^{(p|q),g_1\dots g_q}_{[a_1\dots a_r](\alpha_1\dots\alpha_s)[\beta_1\dots\beta_q]}V^{a_1}\wedge\dots\wedge V^{a_r}\wedge\psi^{\alpha_1}\dots\wedge\psi^{\alpha_s}\wedge\delta^{(g_1)}(\psi^{\beta_1})\dots\wedge\delta^{(g_q)}(\psi^{\beta_q}) \tag{15}$$

where the first indices $[a_1\dots a_r]$ are anti-symmetrized, the second set $(\alpha_1\dots\alpha_s)$ is symmetrized, and the third set $[\beta_1\dots\beta_q]$ is anti-symmetrized. We also attach the indices $g_1\dots g_q$ to the coefficients of $\Omega^{(p|q)}$ to label the order of derivatives on the deltas. The total picture equals q and the total form number is $p = r + s - \sum_{i=1}^{q} g_i$. The two extreme cases $\Omega^{(p|0)}$ and $\Omega^{(p|m)}$ are known as super forms and integral forms.

Given two forms $\Omega^{(p|q)}$ and $\Omega^{(p'|q')}$, we can multiply them as follows

$$\Omega^{(p+p'|q+q')} = \begin{cases} \Omega^{(p|q)} \wedge \Omega^{(p'|q')} & \text{if } q+q' \leq m \\ 0 & \text{if } q+q' > m \end{cases} \tag{16}$$

Notice that due to Equation (10), if the argument of two delta's in $\Omega^{(p|q)}$ and $\Omega^{(p'|q')}$ is the same, their product vanishes (exactly in the same way as for two differential forms V^a).

The differential d changes the form number p, for $\Omega^{(p|q)}$,

$$d : \Omega^{(p|q)} \longrightarrow \Omega^{(p+1|q)} \tag{17}$$

but it does not change the picture number q. To change the latter, one needs *picture changing operators* (PCOs) $\mathbb{Y}^{(0|1)}$, and it is defined as follows: given an anticommuting smooth function Ξ (e.g., $\Xi = \theta^\alpha$, namely the coordinate function along the fermionic coordinate θ^α on $\mathcal{SM}$) on the supermanifold, we set

$$\mathbb{Y}^{(0|1)} = \Xi\delta(d\Xi) \tag{18}$$

where $d\Xi$ is the differential of Ξ. $\mathbb{Y}^{(0|1)}$ carries no form degree, but the picture equals one. In addition, because of distribution-like properties (9) and (12)–(14), $\mathbb{Y}^{(0|1)}$ is closed and not exact. The PCO is thoroughly discussed in [20,21], wherein the connection with the Poincaré dual form of the immersion of a bosonic submanifold into the supermanifold is demonstrated.

The PCO $\mathbb{Y}^{(0|1)}$ that raises the picture by one unit and acts multiplicatively on $\Omega^{(p|q)}$

$$\Omega^{(p|q)} \longrightarrow \Omega^{(p|q+1)} = \mathbb{Y}^{(0|1)} \wedge \Omega^{(p|q)} \tag{19}$$

which satisfies $d\left(\mathbb{Y}^{(0|1)} \wedge \Omega^{(p|q)}\right) = \mathbb{Y}^{(0|1)} \wedge d\Omega^{(p|q)}$. Notice that $\Omega^{(p|q+1)}$ could vanish if the delta in $\mathbb{Y}^{(0|1)}$ is one of the delta in $\Omega^{(p|q)}$. The maximal picture PCO $\mathbb{Y}^{(0|m)}$ is obtained by taking the wedge product of $\mathbb{Y}^{(0|1)}$ along all possible MC 1-forms ψ^α or by choosing a set of m-anticommuting functions Ξ^α, and we set

$$\mathbb{Y}^{(0|m)} = \bigwedge_{\alpha=1}^{m} \Xi^\alpha \delta(d\Xi^\alpha) \tag{20}$$

This PCO is an integral form $\Omega^{(0|m)}$, and it is an element of the cohomology of the differential d: $H^{(0|m)}(d, \mathbb{R})$. For a comprehensive look at various examples, please refer to the citation [25]. Additionally, the citation [30] can be consulted for a thorough mathematical derivation. The latter source explains the distinction between the differential d that operates on superforms and d_K (known as the Koszul differential) that operates on the integral form. However, for practical purposes, this difference may not be significant in computational applications.

On the other hand, we can construct the lowering PCO as follows:

$$\Omega^{(p|q)} \longrightarrow \Omega^{(p|q-1)} = \mathbb{Z}^{(0|-1)}\Omega^{(p|q)} = \left[d, \Theta\left(\iota_{\hat{X}}\right)\right]\Omega^{(p|q)} \tag{21}$$

where $\hat{X}$ is an odd vector field, and $\Theta\left(\iota_{\hat{X}}\right)$ is the Heaviside (step) function, which can be conveniently defined using the integral representation. If $\hat{X} = \hat{X}^\alpha \partial_\alpha$, then $\iota_{\hat{X}}\psi^\alpha = \hat{X}^\alpha$ and

$$\Theta(\iota_{\hat{X}}) = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{e^{-it\iota_{\hat{X}}}}{t + i\epsilon} dt \tag{22}$$

We will come back to this PCO in the forthcoming sections.

In this discussion, our focus will be on the two opposite scenarios: superforms with no picture and integral forms with the highest picture number. For these cases, we can simplify the expressions using Equation (15).

$$\begin{aligned}\Omega^{(p|0)} &= \Omega^{(p|0)}_{[a_1\dots a_r](\alpha_{r+1}\dots\alpha_p)} V^{a_1}\dots V^{a_r}\psi^{\alpha_{r+1}}\dots\psi^{\alpha_p} \\ \Omega^{(p|m)} &= \Omega^{(p|m),(\beta_{r+1}\dots\beta_r)}_{[a_1\dots a_{p+r}]} V^{a_1}\dots V^{a_{p+r}}\iota_{\beta_1}\dots\iota_{\beta_r}\bigwedge_{\alpha=1}^{m}\delta(\psi^\alpha) \end{aligned} \tag{23}$$

where $\iota_\beta = \iota_{\partial_\beta}$, and the latter is defined in Equation (12).

We can utilize the Hodge star operator, as defined in the sources [27,28], which we have conveniently revised for the reader's understanding. Given a form $\omega(x,\theta,V,\psi)$, considered as a generalized function of the supervielbein (V^a,ψ^α) and of the coordinates (x^a,θ^α) of the supermanifold $\mathcal{SM}$ its Hodge dual is written in terms of a Fourier transform

$$\star\omega(x,\theta,V,\psi) = i^{r^2-n^2} i^l \int_{\mathcal{SM}'} e^{i(v^a\eta_{ab}V^b + p^\alpha\lambda C_{\alpha\beta}\psi^\beta)}\omega(x,\theta,v,p)[d^n v d^m p] \tag{24}$$

where r is the number of V's, l is the number of ψ's, n is the bosonic dimension, and $\mathcal{SM}'$ is the dual superspace whose fundamental coordinates are (v^a,p^α) (respectively, anticommuting and commuting). The symbol $[d^n v d^m p]$ denotes the Berezin integral over v^a and the Riemann–Lebesgue integral over p^α. The metric g to which $\star$ is related is given by the tensor

$$g = \eta_{ab}V^a\otimes V^b + C_{\alpha\beta}\psi^\alpha\otimes\psi^\beta \tag{25}$$

where η_{ab} is a symmetric matrix and $C_{\alpha\beta}$ is an antisymmetric matrix (for the case $D=4$ this matrix concides with the charge conjugation matrix). The function $\omega(x,\theta,v,p)$ is the function obtained from $\omega(x,\theta,V,\psi)$ by substituting $V\to v$ and $\psi\to p$, leaving the supermanifold coordinates x,θ untouched.

Using the Hodge dual operator, $\Omega^{(p|0)}$ and $\Omega^{(p|m)}$ are related as follows

$$\star\Omega^{(p|0)} = \Omega^{(n-p|m)} \tag{26}$$

and satisfy the following identity

$$\Omega^{(p|0)}\wedge\star\Omega^{(p|0)} = \prod_{a=1}^{n}V^a\prod_{\alpha=1}^{m}\delta(\psi^\alpha) \equiv \mathrm{Vol}^{(n|m)} \tag{27}$$

and the right-hand side $\mathrm{Vol}^{(n|m)}$ represents the Berezinian top form on the space of MC forms, as discussed in Equation (11). If the superalgebra is a matrix superalgebra, $\mathrm{Vol}^{(n|m)}$ is just the superdeterminant. In paper [25,26] it is proven that for each cohomology class (cocycles) in the superform sector $H^{(p|0)}$ there exists a corresponding integral form cocycle $H^{(n-p|m)}$. This corresponds to the usual Poincaré duality in the case of conventional manifolds.

Now, according to the FDA techniques, for each cocycle of $\omega_I^{(p|0)}\in H^{(p|0)}$, (where I is a label for the cocycle) one introduces a new potential such that

$$dA_I^{(p-1|0)} = \omega_I^{(p|0)} \tag{28}$$

to trivialize the cohomology. This can be carried out for all cocycles in $H^{(p|0)}$ leading to a set of potentials $A_I^{(p-1|0)}$, however, the cocycles are related by a ring structure

$$\omega_I^{(p_I|0)}\wedge\omega_J^{(p_J|0)} = \sum_K C^K_{IJ}\omega_K^{(p_I+p_J|0)} \tag{29}$$

with integer coefficients C_{IJ}^K, and therefore there might be overcounting. For that, one needs to introduce a basis of potentials $A_I^{(p-1|0)}$ solving the equations

$$A_I^{(p_I-1|0)} \wedge dA_J^{(p_J-1|0)} = \sum_K \tilde{C}_{IJ}^K A_K^{(p_I+p_J-1|0)} . \tag{30}$$

where $\tilde{C}_{IJ}^K$ are related to C_{IJ}^K by taking into account a reshuffling of the potentials [3]. This structure selects a linear basis of potentials to be used in the FDA together with the MC forms E^A.

Then, one has to compute the new extended cohomology with additional potentials to see whether new cocycles emerge (an extended analysis for $D = 4, 6, 10, 11$ models with extended superspace has been performed in [32]). The procedure can be iterated until we obtain empty cohomology and the complete free differential algebra is built $(V^a, \psi^\alpha, A_I^{(p-1)})$.

As proven in [25,26] for each superform cocycle $\omega_I^{(p|0)}$ there is a corresponding integral form cocycle $\omega_I^{(p|m)}$ (Poincarè duality), and the best way to establish the isomorphism is using the Hodge dual operator as

$$\star\omega_I^{(p_I|0)} = \omega_I^{(n-p_I|m)} \tag{31}$$

If we suppose that $d\omega_I^{(n-p|m)} \neq 0$, it follows

$$d\omega_I^{(n-p|m)} = d(\star\omega_I^{(p_I|0)}) = \Sigma_I^{(n-p_I+1|m)} \tag{32}$$

for a $(n - p_I + 1|m)$ form $\Sigma_I^{(n-p_I+1|m)}$. We act again with the Hodge dual to obtain

$$d^\dagger\omega_I^{(p_I|0)} = \star d \star \omega_I^{(p_I|0)} = \Lambda_I^{(p_I-1|0)} \tag{33}$$

Together with the closure $d\omega_I^{(p_I|0)} = 0$ and the usual definition of the Laplace–Beltrami operator $\Delta = dd^\dagger + d^\dagger d$, we can write Equation (33) as

$$\omega_I^{(p_I|0)} = d\left(\Delta^{-1}\Lambda_I^{(p_I-1|0)}\right) \tag{34}$$

where Δ^{-1} is a Green function of the Laplace–Beltrami differential $\Delta = dd^\dagger + d^\dagger d$. Therefore, if $\Lambda_I^{(p_I-1|0)}$ is non-vanishing, then $\omega_I^{(p_I|0)}$ is exact, which contradicts the hypotheses that $\omega_I^{(p_I|0)}$ since it is a cocycle. This proof may not be applicable in general, but it can be completed for super Lie algebras in the specific case where the differential is the CE differential d and metric g in Equation (25) is invariant.

The integral cocycles denoted as $\omega_I^{(n-p_I|m)}$ do not constitute a ring. However, they form a module concerning the ring structure (29). This is evident from the Equation (35):

$$\omega_I^{(p|0)} \wedge \omega_J^{(p'|m)} = \sum_K D_{IJ}^K \omega_K^{(p+p'|m)} \tag{35}$$

When a differential cocycle $\omega_I^{(p|0)}$ is wedged with an integral cocycle $\omega_J^{(p'|m)}$, the resulting cocycle is also integral. Therefore, it can be expanded using integer coefficients D_{IJ}^K. It is important to note that the picture number remains unchanged while the form numbers are added. Using the potentials $A_I^{(p-1|0)}$ for the differential cocycles, we can write Equation (35) as follows:

$$dA_I^{(p-1|0)} \wedge \omega_J^{(p'|m)} = d\left(A_I^{(p-1|0)} \wedge \omega_J^{(p'|m)}\right) = \sum_K D_{IJ}^K \omega_K^{(p+p'|m)} \tag{36}$$

which tells us that for non-vanishing coefficients D^K_{IJ}, the introduction of the potentials $A_I^{(p-1|0)}$ allows us to remove all integral cocycles except one! Indeed, the integral form $\omega_I^{(p_I|m)}$ with the lowest form number cannot be obtained by the module structure (35), and therefore the introduction of the potentials $A_I^{(p_I-1|0)}$ is not enough to construct the completely free differential algebra. Then, for the lowest form-number integral form, say $\omega_0^{(p_0|m)}$, we need a novel potential $A_0^{(p_0-1|m)}$ such that

$$\omega_0^{(p_0|m)} = dA_0^{(p_0-1|m)} \tag{37}$$

Then, finally, the FIDA is spanned by the generators

$$\left(V^a, \psi^\alpha, A_I^{(p_I-1|0)}, A_0^{(p_0-1|m)}\right) \tag{38}$$

There are some remarks:

1. Notice that among the integral cocycles, in the case of unimodular superalgebras, we always have the highest form integral cocycle

$$\star 1 = \omega^{(n|m)} = \mathrm{Vol}^{(n|m)} \tag{39}$$

which is closed and not exact. This class becomes trivial once we have introduced the potentials $A_I^{(p_I-1|0)}$ for the differential forms. Indeed, using Equation (27), we have

$$\mathrm{Vol}^{(n|m)} = d\left(A_I^{(p-1|0)} \wedge \star\omega_I^{(p|0)}\right) = d\left(A_I^{(p-1|0)} \wedge \omega_I^{(n-p|m)}\right) \tag{40}$$

for any I running over the set of independent cocycles. Then, having introduced all potentials one has left with two cocycles: the trivial constant $\omega_0^{(0|0)}$ and the lowest integral form $\omega_0^{(p_0|m)}$. It may happen that $p_0 = 0$, and therefore $\omega_0^{(p_0|m)} = \mathbb{Y}^{(0|m)}$; that is, it coincides with the product of all PCO's. Trivializing the latter, introduce an integral-form potential such that $\mathbb{Y}^{(0|m)} = dA^{(-1|m)}$, which can be used to check the consistency between differential form and integral form cocycles.

2. There might be the possibility (see for example the $OSp(1|2)$ discussed in the forthcoming section) that $A_0^{(p_0-1|m)}$ is not enough to complete the FIDA. Indeed, if $p_0 + m$ is even, the integral form $\omega_0^{(p_0|m)}$ is even and its potential $A_0^{(p_0-1|m)}$ is odd. Therefore, the product

$$\omega_0^{(2p_0-1|2m)} = A_0^{(p_0-1|m)} \wedge \omega_0^{(p_0|m)} \tag{41}$$

is a cohomology class. Indeed, $d\omega_0^{(2p_0-1|2m)} = \omega_0^{(p_0|m)} \wedge \omega_0^{(p_0|m)} = 0$ because of Equation (16). Notice that we assumed that the picture carried by $A_0^{(p_0-1|m)}$ is different from that carried by $\omega_0^{(p_0|m)}$. Then, finally, we can remove $\omega_0^{(2p_0-1|2m)}$ by adding a further potential $A_0^{(2p_0-2|2m)}$. The role of those new potentials is not fully understood, and it can be studied by analyzing its dynamics. A similar role has been played by the B field associated with the 7-cocycle in $D = 11$ supergravity [1]. From a kinematical point of view, the B field is introduced to construct the FDA but, it turns out to be dynamically irrelevant.

1.3. Hilbert–Poincaré Series

One effective way to calculate cohomologies is by computing the Hilbert–Poincaré series/polynomial. There are various methods for approaching this, including utilizing localization theorems, the Molien–Weyl formula, or the enumeration of invariants under certain isometry groups. By using these techniques, we can determine the number of independent cocycles, their form number, the picture number, and their parity. This serves as a helpful checking strategy for explicit computations.

We briefly review the definition of Hilbert–Poincaré series and Poincaré polynomials. For X a *graded* vector space with direct decomposition into p-degree homogeneous subspaces given by $X = \bigoplus_{p\in\mathbb{Z}} X_p$, we call the formal series

$$\mathbb{P}_X(t) = \sum_p (\dim X_p)(-t)^p \tag{42}$$

the *Hilbert–Poincaré series* of X. Notice that we have implicitly assumed that X is a of *finite type*, i.e., its homogeneous subspaces X_p are finite-dimensional for every p. The unconventional sign in $(-t)^p$ takes into account the *parity* of X_p, which takes values in $\mathbb{Z}_2$ and it is given by $p \bmod 2$: this will be particularly useful in supergravity where commuting and anticommuting variables are needed. If also $\dim X$ is finite, then $\mathbb{P}_X(t)$ becomes a polynomial $\mathbb{P}_X[t]$, called the *Poincaré polynomial* of X. The evaluation of the Poincaré polynomial at $t=1$ yields the so-called *Euler characteristics* $\chi_X = \mathbb{P}_X[t=1] = \sum_p (-1)^p \dim X_p$ of X. If we assume that the pair (X, d) is a differential complex for X a graded vector space and $d : X_p \to X_{p+1}$ for any p, then the cohomology $H_d^\bullet(X) = \bigoplus_{p\in\mathbb{Z}} H_d^p(X)$ is a graded space, where $b_p(X) \equiv \dim H_{dR}^p(X)$ is the p-th Betti number of X. The Poincaré polynomial of X, which is defined by the Euler–Poincaré formula, can be expressed as follows:

$$\mathbb{P}_X[t] = \sum_p b_p(M)(-t)^p \tag{43}$$

This polynomial serves as the generating function of the Betti numbers of X. Calculating $\mathbb{P}_X[t]$ can be a challenging task, but there are techniques available to make it easier. For example, localization methods or the Molien–Weyl formula can be used. The Molien–Weyl formula, which is discussed in [37–40], can derive all Betti numbers by utilizing the "telescopic nesting" property (for more information, see [32]). Although we do not provide the details of the Hilbert–Poincaré series/polynomial computation in this paper, we present results for various cases in the forthcoming sections. Even if the notion of Betti numbers is originally related to the topology of a certain manifold or topological space, by extension, in this paper we will call *Betti numbers* the dimensions of any cohomology space valued in a field; in particular, we will call p-th Betti numbers of a certain Lie (super)algebra the dimension of its Chevalley–Eilenberg p-cohomology group $b_p(\mathfrak{g}) = \dim H_{CE}^p(\mathfrak{g})$, so that the Hilbert–Poincaré series of the Lie (super)algebra $\mathfrak{g}$ is the generating function of its Betti number

$$\mathbb{P}_{\mathfrak{g}}(t) = \sum_p b_p(\mathfrak{g})(-t)^p. \tag{44}$$

Notice that we used the notation $\mathbb{P}(t)$ on purpose: indeed, as we shall see, the Chevalley–Eilenberg cohomology $H_{CE}^\bullet(\mathfrak{g})$ is not in general finite dimensional for a generic Lie superalgebra $\mathfrak{g}$. In our framework, it is convenient to introduce a second grading (picture number). In that case, the space is said to be *bigraded vector space* $X = \sum_{p,q\in\mathbb{Z}} X^{p,q}$, then the gradation $X = \sum_r X^r$ given by

$$X^r = \sum_{p+q=r} X^{p,q} \tag{45}$$

is called the *induced total gradation*. One can write a *double Hilbert–Poincaré series*

$$\mathbb{P}_X(t, \tilde{t}) = \sum_{p,q} (-t)^p (-\tilde{t})^q \dim X^{p,q} \tag{46}$$

which, in any case, allows easier identification of cohomological classes (see, e.g., [25,26] where double Hilbert–Poincaré series have been used to select different type cohomologies). In the forthcoming section, we use the second grading $\tilde{t}$ to count the picture.

2. Examples

In this section, we analyze in detail the following examples: a toy model, an Abelian-group manifold, a non-Abelian group manifold, and a coset model. All these models are interesting examples of the theory discussed above. The charge/scale assignment is performed to preserve the Maurer–Cartan equations, or equivalently, to commute with the differential d. In the Abelian cases, the assignment is t^2 to bosonic 1-forms and t to fermionic 1-forms, in non-Abelian cases, the assignment corresponds to the form number. We refer to [25,32] for an extended and complete discussion on charge/scale assignments.

2.1. Toy Model Example

We consider the bosonic 1-forms T and b and the fermionic 1-form ψ with the Maurer-Cartan equations

$$dT = -2T \wedge b + \psi^2 \,, \qquad d\psi = \psi \wedge b \,, \qquad db = 0 \,. \tag{47}$$

(this example is taken from [10] page 802, example 2). It is a solvable super-Lie algebra. We can proceed as follows: we introduce a 0-form ϕ, such that $b = d\phi$. Then, we can rewrite the MC as follows:

$$T = e^{2\phi}(dx + \theta d\theta) \,, \qquad \psi = e^{\phi} d\theta \,, \qquad b = d\phi \,. \tag{48}$$

We have introduced the new coordinates (0-forms) x, θ which describes the superline. The solvable super-Lie algebra can be thought of as the gauging of the scale invariance. The gauge field of the scale invariance is ϕ.

The cohomology is easily computed in terms of x, θ, ϕ.

$$\begin{aligned} \omega^{(0|0)} &= 1 \,, & \omega^{(1|0)} &= b \,, \\ \omega^{(0|1)} &= (dx + \theta d\theta)\, \delta'(d\theta) \,, & \omega^{(1|1)} &= d\phi \wedge (dx + \theta d\theta)\, \delta'(d\theta) \,, \end{aligned} \tag{49}$$

but they can easily be reconverted in terms of the original

$$\omega^{(0|0)} = 1 \,, \qquad \omega^{(1|0)} = b \,, \qquad \omega^{(0|1)} = T\, \delta'(\psi) \,, \qquad \omega^{(1|1)} = b \wedge T\, \delta'(\psi) \,, \tag{50}$$

with the relations

$$\omega^{(0|0)} \wedge \omega^{(1|1)} = \omega^{(1|0)} \wedge \omega^{(0|1)} \tag{51}$$

Let us compute the Poincaré polynomial for the cohomology

$$\mathbb{P}(t, \tilde{t}) = (1 - t) + (t - t^2)\tilde{t} = (1 - t)(1 + t\tilde{t}) \tag{52}$$

where the first equation has the following meaning: $(1 - t)$ is the cohomology of the superforms $\omega^{(0|0)}$ and $\omega^{(1|0)}$, and $(t - t^2)\tilde{t}$ is the cohomology of the integral forms $\omega^{(0|1)}$ and $\omega^{(1|1)}$. The second equality means $(1 - t)$ is the cohomology of b times $(1 + \tilde{t})$, which is the Poincaré polynomial of the CE cohomology of the supertranslation on a super line $(dx + \theta d\theta, d\theta)$, which scale as t^2 and t [25,26].

As is well known, there is another important operator to be used: the rising PCO Z. This changes the picture, and it is written as $Z = [d, \Theta(\iota_D)]$, where D is the vector field dual to ψ, namely $\iota_D \psi = 1$. Then, we have

$$Z(\omega^{(0|1)}) = d\Big(\Theta(\iota_D) T\, \delta'(\psi)\Big) = d\Big(\frac{T}{\psi^2}\Big) = \frac{-2t \wedge b + \psi^2}{\psi^2} + 2\frac{t \wedge b}{\psi^2} = 1\,,$$
$$Z(\omega^{(1|1)}) = d\Big(\Theta(\iota_D) b \wedge T\, \delta'(\psi)\Big) = d\Big(\frac{b \wedge T}{\psi^2}\Big) = \frac{b \wedge \psi^2}{\psi^2} = b\,, \tag{53}$$

so, it correctly maps cohomologies into cohomologies. Notice that it reduces the picture, but does not change the form number. In addition, we have to underline that in the intermediate step, we generated some *inverse* forms (also known as Large Hilbert Space, see for example [41]) and they are converted into conventional superforms.

Let us now discuss the FIDA. According to [10], in the presence of new cohomology classes, one introduces new forms to cancel those cocycles. In particular, here we have two new forms: a 0-form $b = d\phi$, and we introduce the $(-1|1)$ form $A^{(-1|1)}$ to cancel the cocycle $\omega^{(0|1)}$ as

$$dA^{(-1|1)} = \omega^{(0|1)}\,. \tag{54}$$

This implies also

$$d\Big(-A^{(-1|1)} \wedge b\Big) = \omega^{(1|0)} \wedge \omega^{(0|1)} = b \wedge T\, \delta'(\psi)\,, \tag{55}$$

We also have an additional interesting equation, which is

$$Z(dA^{(-1|1)}) = dZ(A^{(-1|1)}) = Z(\omega^{(0|1)}) = 1 \tag{56}$$

The first equality follows from $[d, Z] = 0$. The second equality is the definition of $ddA^{(-1|1)}$. The last equality is due to Equation (53). This implies

$$dZ(A^{(-1|1)}) = 1\,. \tag{57}$$

Namely, even the constants are not in the cohomology.

Before proceeding, we still have to check whether there are other possible cocycles. We note that $A^{(-1|1)}$ has odd parity (its d-variation is even), therefore there are no powers of $A^{(-1|1)}$. In addition, it carries picture +1. One could explore the possibility that this is a new picture, but Equation (53) seems to say that the picture is the same as of $\delta(\psi)$. This implies that we cannot consider combinations of the form $A^{(-1|1)} \wedge \delta(\psi)$. Since we have already explored all possibilities independent of $A^{(-1|1)}$, we are left with

$$\Omega = A^{(-1|1)} \wedge (\alpha(\psi) + \beta(\psi) b + \gamma(\psi) b + \rho(\psi) t \wedge b) \tag{58}$$

where $\alpha(\psi), \ldots, \rho(\psi)$ are polynomials of ψ. The closed forms turn out to be exact

$$\Omega_C = \alpha' A^{(-1|1)} \psi^2 + 2\beta' A^{(-1|1)} \psi^2 \wedge b = d\Big(\alpha' A^{(-1|1)} e^{-2\phi} T + \beta' A^{(-1|1)} \psi^2\Big) \tag{59}$$

where α', β' are numbers. This implies that there are no other cohomology classes in the present extended algebra. Thus, the free differential algebra is given by

$$T, \psi, \phi, A^{(-1|1)} \tag{60}$$

We can still consider $Z(A^{(-1|1)})$ as an inverse form and it automatically implies that even the constants are exact.

2.2. Abelian Group Manifold Example: U(1 | 1)

Let us consider the explicit example of $U(1|1)$. The MC equations read

$$dU = 0, \qquad dW = -\psi^+\psi^-, \qquad d\psi^+ = U\psi^+, \qquad d\psi^- = -U\psi^- \,. \tag{61}$$

The superform cohomology is given by

$$\omega^{(0|0)} = 1\,, \qquad \omega^{(1|0)} = U\,, \qquad \omega^{(p|0)} = 0\,, p > 1 \tag{62}$$

This means that only the Abelian factor, whose associate MC form is U, is in the cohomology of superforms. The cohomology among integral forms read

$$\omega^{(p|2)} = 0, p \leq 1\,, \qquad \omega^{(1|2)} = W\delta(\psi^+)\delta(\psi^-)\,, \qquad \omega^{(2|2)} = UW\delta(\psi^+)\delta(\psi^-)\,, \tag{63}$$

and the last expression corresponds to the Berezinian

$$\mathcal{B}er = U \wedge W \wedge \delta\big(\psi^+\big) \wedge \delta\big(\psi^-\big) \,. \tag{64}$$

The two classes in $\omega^{(1|0)}$ and $\omega^{(1|2)}$ are dual, via the Berezinian complement duality

$$U \wedge \star U = \mathcal{B}er\,; \tag{65}$$

notice that they live in two distinct (though quasi-isomorphic) complexes. In the present case, the Hilbert–Poincaré polynomial is given by

$$\mathbb{P}(t, \tilde{t}) = (1 - t)(1 - t\tilde{t}^2) \tag{66}$$

Each term corresponds to Equations (62) and (63); in particular, we have $(1 - t)$ correspond to $\omega^{(0|0)}$ and $\omega^{(1|0)}$, while $(1 - t)(-t\tilde{t}^2) = -t\tilde{t}^2 + t^2\tilde{t}^2$ correspond to $\omega^{(1|2)}$ and $\omega^{(2|2)}$.

Notice that by setting $\tilde{t} = 1$ we recover the bosonic subgroup $U(1) \times U(1)$ and the Hilbert–Poincaré polynomial is just the product of the polynomial for each Abelian factor. On the other hand, if we set $\tilde{t} = t$ we obtain the polynomial $(1 - t)(1 - t^3)$ which is the Poincaré polynomial of $U(2)$, and finally setting $\tilde{t} = 0$ we obtain $(1 - t)$ which is, according to Fuks theorem [42], just the superform cohomology, and it corresponds to a $U(1)$.

By following the constructive methods of FDAs (see, e.g., [10]), we have to introduce new forms to the Lie superalgebra to trivialize the CE cohomology classes. In particular, in order to compensate for the class $\omega^{(1|0)} = U$, one has to introduce an even $(0|0)$-form ω_0 such that

$$d\omega_0 = U\,. \tag{67}$$

We should now re-evaluate the CE cohomology of the FDA generated by $U, W, \psi^1, \psi^2, \omega_0$. Again, we will investigate both superforms and integral forms. It is not difficult to show that the only non-trivial cohomology group among superforms is H^0: the new closed objects introduced by ω_0 are of the form

$$\omega^{(p+1|0)} = \omega_0 U \wedge \omega^{(p|0)}\big(\psi^+, \psi^-\big)\,, \,\forall p \geq 0\,, \tag{68}$$

where $\omega^{(p|0)}(\psi^+, \psi^-)$ is a degree p polynomial in the variables ψ^+, ψ^-. If $p = 0$, one has

$$\omega_0 U = \frac{1}{2} d\Big[\omega_0^2\Big]\,. \tag{69}$$

If $p \geq 1$, by using the relations

$$\omega_0 U \wedge (\psi^+)^p = \frac{1}{p} d\left[\omega_0 \wedge (\psi^+)^p - \frac{1}{p}(\psi^+)^p\right] ,$$
$$\omega_0 \wedge U \wedge (\psi^-)^p = -\frac{1}{p} d\left[\omega_0 \wedge (\psi^-)^p - \frac{1}{p}(\psi^-)^p\right] , \forall p \geq 1 , \tag{70}$$

together with Equations (61) and (67), it is easy to show that any superform as Equation (68) is exact. This can be easily checked by modifying the Hilbert–Poincaré polynomial: the introduction of the new form ω_0 implies that

$$\mathbb{P}(t, \tilde{t}) \mapsto \frac{\mathbb{P}(t, \tilde{t})}{(1-t)} = (1 - t\tilde{t}^2) \tag{71}$$

Let us move to integral forms: the first important remark involves the class $\mathcal{B}er$. The introduction of the $(0|0)$-superform ω_0 implies that $\mathcal{B}er$ is exact:

$$\mathcal{B}er = U \wedge W \wedge \delta(\psi^+) \wedge \delta(\psi^-) = d\left[\omega_0 W \wedge \delta(\psi^+) \wedge \delta(\psi^-)\right] . \tag{72}$$

This fact has as a direct implication the failure of the "Berezinian complement duality", which then does not hold for the FDA $\mathfrak{g}'$. Analogously, notice that also the form $\mathcal{B}er \otimes \omega_0^p$, $p \geq 0$, is exact:

$$\omega_0^p U \wedge W \wedge \delta(\psi^+) \wedge \delta(\psi^-) = \frac{1}{p+1} d\left[\omega_0^{p+1} W \wedge \delta(\psi^+) \wedge \delta(\psi^-)\right] , \forall p \geq 0 . \tag{73}$$

Notice that adding ω_0 still defines a cohomology space, since it does not involve the form U (hence, the introduction of ω_0 does not spoil the non-exactness). This indicates that despite U and $\iota_U \mathcal{B}er$ being dual (in the sense of Equation (65)), they can not be compensated by a single new term in the FDA. Moreover, it is not difficult to prove that ω_0 does not introduce new cohomology classes among integral forms and among superforms. Really, the new closed objects introduced by ω_0 (except for Equation (73)) are of the form

$$\omega^{(2-p|2)} = \omega_0 U \wedge \omega(\iota_+, \iota_-) \delta(\psi^+) \wedge \delta(\psi^-) , \tag{74}$$

where $\omega^{(-p|0)}(\iota_+, \iota_-)$ is a (formal) degree p polynomial in the variables ι_+, ι_-. The exactness of terms as Equation (74) is easily seen by considering Equations (61) and (67) and the relations

$$\omega_0 U \wedge \iota_+^p \delta(\psi^+) \wedge \delta(\psi^-) = -\frac{1}{p} d\left[\omega_0 \iota_+^p \delta(\psi^+) \wedge \delta(\psi^-) + \frac{1}{p} \iota_+^p \delta(\psi^+) \wedge \delta(\psi^-)\right] ,$$
$$\omega_0 U \wedge \delta(\psi^+) \wedge \iota_-^p \delta(\psi^-) = \frac{1}{p} d\left[\omega_0 \delta(\psi^+) \wedge \iota_-^p \delta(\psi^-) - \frac{1}{p} \delta(\psi^+) \wedge \iota_-^p \delta(\psi^-)\right] . \tag{75}$$

We now want to trivialize the remaining cohomology class among integral forms. We introduce then a $(0|2)$-integral form $\eta^{0|2}$ so that

$$d\eta^{0|2} = W \wedge \delta(\psi^+) \wedge \delta(\psi^-) . \tag{76}$$

This new generator does not introduce new cohomology classes. Since the new generator $\eta^{0|2}$ is commuting, we have to multiply the Hilbert–Poincaré polynomial with $1/(1 - t\tilde{t}^2)$, which cancels the remaining factor of Equation (71). In this way, we have completed the FIDA for this model.

2.3. Non-Abelian Group Manifold Example: OSp(1 | 2)

If we consider the super Lie-algebra $\mathfrak{osp}(1|2)$, there are the MC forms V^a, ψ^α with the MC equations

$$dV^a = \psi\gamma^a\psi + (V \wedge V)^a\,, \qquad d\psi^\alpha = V_a(\gamma^a\psi)^\alpha\,. \tag{77}$$

We have shown that there are the following cocycles:

$$\omega^{(0|0)} = 1\,, \qquad \omega^{(3|0)} = \frac{1}{2}\psi\gamma_a\psi V^a + \frac{1}{3!}(V \wedge V \wedge V)\,, \tag{78}$$

$$\omega^{(0|2)} = \frac{1}{2}(V \wedge V)^{ab}\iota\gamma_{ab}\iota\delta^2(\psi) + \delta^2(\psi)\,, \qquad \omega^{(3|2)} = \frac{1}{3!}(V \wedge V \wedge V)\delta^2(\psi)\,. \tag{79}$$

where we have displayed in the first line the superform cocycles and, in the second line, the integral form cocycles. In addition, we have to recall that there are pseudoforms obtained by acting with $Z_{(\alpha)}$. There are two PCO $Z_{(\alpha)}$ associated with the two directions in the fermionic space. Then, we have

$$\omega^{(0|1)}_{(\alpha)} = Z_{(\alpha)}(\omega^{(0|2)})\,, \qquad \omega^{(3|1)}_{(\alpha)} = Z_{(\alpha)}(\omega^{(3|2)})\,, \tag{80}$$

By consistency we have that $Z_{(1)}Z_{(2)}\omega^{(0|2)} = \omega^{(0|0)}$ and $Z_{(1)}Z_{(2)}\omega^{(3|2)} = \omega^{(3|0)}$. The complete set of cohomologies is easily described by the Hilbert–Poincaré polynomial

$$\mathbb{P}(t, \tilde{t}) = (1 - t^3)(1 + \tilde{t})^2\,. \tag{81}$$

Now, we can construct the FDA for the present model. At first, we introduce a 2-form $B^{(2|0)}$ to cancel the cocycle $\omega^{(3|0)}$. This has a twofold effect, it cancels $\omega^{(3|0)}$, but it also cancels $\omega^{(3|2)}$ as follows

$$\omega^{(3|2)} = d\Big(B^{(2|0)} \wedge \omega^{(0|2)}\Big) \tag{82}$$

We still have one additional cocycle to cancel: $\omega^{(0|2)}$. This can be achieved by introducing the -1-form potential with picture number $+2$ as

$$dA^{(-1|2)} = \omega^{(0|2)} \tag{83}$$

to compensate for the cocycle $\omega^{(0|2)}$ (which is the PCO $\mathbb{Y}$). Notice that the ring/module structure implies also

$$d\Big(A^{(-1|2)} \wedge \omega^{(3|0)}\Big) = \omega^{(3|2)} \tag{84}$$

which means again that the volume form $\omega^{(3|2)}$ is written in terms of the additional $A^{(-1|2)}$ and the superform $\omega^{(3|0)}$. This is consistent with Equation (82) since $A^{(-1|2)} \wedge \omega^{(3|0)} = B^{(2|0)} \wedge \omega^{(0|2)} + d(\text{exact})$. As discussed in Section 3.1, we have still one cocycle of the form

$$\omega^{(-1|4)} = A^{(-1|2)} \wedge \omega^{(0|2)} \tag{85}$$

which requires a new potential $B^{(-2|4)}$ to complete the FIDA. Still, at the moment we do not have a physical interpretation of those new ingredients, and they play a role algebraically.

In addition, we can act with the PCO $Z_{(\alpha)}$ on $A^{(-1|2)}$ to obtain

$$d\Big[Z_{(\alpha)}\Big(A^{(-1|2)}\Big)\Big] = \omega^{(0|1)}_{(\alpha)} \tag{86}$$

Namely, the new forms

$$A^{(-1|1)}_{(\alpha)} = Z_{(\alpha)}\left(A^{(-1|2)}\right), \tag{87}$$

cancel the cocycle of the pseudo-form type, but they are not new ones. So, we conclude by observing that the FIDA is generated by

$$V^a\,,\ \ \psi^\alpha\,,\ \ B^{(2|0)}\,,\quad A^{(-1|2)}\,,\ \ ,B^{(-2|4)}\,, \tag{88}$$

A remark: Notice there is an additional form $A^{(-1|0)} = Z_{(1)}Z_{(2)}\left(A^{(-1|2)}\right)$, which has an interesting behaviour of $A^{(-1|0)}_{(\alpha)}$: $d\left[Z_{(1)}Z_{(2)}\left(A^{(-1|2)}\right)\right] = \omega^{(0|0)} = 1$, which implies that in the present case, even the constants are exact forms. This is in a complete analogy with the superstring Hilbert space: the cohomology (vertex operators) of the BRST superstring charge selects the physical states on the small Hilbert space, but in the Large Hilbert Space (where the zero mode ξ_0 is introduced) the BRST cohomology is empty. Not even the constants are considered cohomology classes, if one admits inverse forms. Notice that, once we have $A^{(-1|0)}$, every closed form can be made exact. In addition, we should observe that $A^{(-1|0)}$ is an inverse form, and this is a completely new ingredient in the framework. Its role has to be understood in this new framework.

2.4. Coset Manifold Example: OSp(1|4)/SO(1,3)

We consider the case with $D = 4$ and $N = 1$ described by the supercoset manifold $OSp(1|4)/SO(1,3)$. It has four bosonic dimensions and four fermionic dimensions. The vielbeins V^a are obtained by the natural identification of Maurer–Cartan forms of $\mathfrak{osp}(1|4)$

$$V^{\alpha\beta} = \gamma^{\alpha\beta}_a V^a + \gamma^{\alpha\beta}_{ab}\omega^{ab} \tag{89}$$

in terms of Dirac matrix decomposition where ω^{ab} is the spin connection of $SO(1,3)$. The vielbeins and the spin connection satisfy the MC equations

$$\begin{aligned} R^{ab} &\equiv d\omega^{ab} + \omega^{ac}\wedge\omega_c{}^b = V^a\wedge V^b + \frac{1}{2}\bar\psi\gamma^{ab}\psi\,,\\ \rho &\equiv d\psi + \frac{1}{4}\omega^{ab}\gamma_{ab}\psi = \frac{i}{2}V^a\gamma_a\psi\,,\\ T^a &\equiv dV^a + \omega^a{}_b V^b = \frac{i}{2}\bar\psi\gamma^a\psi\,. \end{aligned} \tag{90}$$

The supervielbeins ψ^α are in the Majorana representation of $SO(1,3)$.

The study of the cohomology classes of the coset $OSp(1|4)/SO(1,3)$ can be easily performed knowing the Chevalley–Eilenberg classes of $OSp(1|4)$ and those of $SO(1,3)$ (using the Hochshild–Serre spectral sequence). We list them in terms of Poincaré polynomials

$$\begin{aligned} \mathbb{P}_{OSp(1|4)}(t,\tilde t) &= (1-t^3)(1-t^7)(1+\tilde t^4)\\ \mathbb{P}_{SO(1,3)}(t) &= (1-t^3)^2\\ \mathbb{P}_{OSp(1|4)/SO(1,3)}(t,\tilde t) &= \frac{(1-t^4)(1-t^8)(1+\tilde t^4)}{(1-t^4)^2} = (1+t^4)(1+\tilde t^4) \end{aligned} \tag{91}$$

where t counts the form degree, and $\tilde t$ counts the picture degree. For the last equation we have used the theorem by Greub–Vanstone–Halperin which allows us to write the Poincaré polynomial for coset spaces, but also we have studied at each picture number the different cohomologies.

Notice that in the case of the supergroup $OSp(1|4)$, the factor $(1+\tilde{t}^4)$ indicates that there are two sets of classes: the superforms and the integral forms. Explicitly, we have the four classes

$$\begin{aligned} \omega^{(0|0)} &= 1\,, & \omega^{(4|0)} &= \bar{\psi}\gamma_{ab}\psi V^a V^b + \epsilon_{abcd} V^a V^b V^c V^d\,, \\ \omega^{(0|4)} &= \delta^4(\psi) + V^a V^b \bar{\iota}\gamma_{ab}\iota\delta^4(\psi)\,, & \omega^{(4|4)} &= \epsilon_{abcd} V^a V^b V^c V^d \delta^4(\psi)\,. \end{aligned} \tag{92}$$

with the relation $\omega^{(0|0)} \wedge \omega^{(4|4)} = \omega^{(4|0)} \wedge \omega^{(0|4)}$. They are all closed and not exact. In particular the class $\omega^{(0|4)}$ is to be identified with the PCO which can be used to compute the action in superspace (see [43]). To simplify the description of the result it is convenient to use a chiral/antichiral decomposition and we make the cosmological constant λ explicit

$$\begin{aligned} R^{\alpha\beta} &\equiv d\omega^{\alpha\beta} + \omega^{\alpha\beta}\epsilon_{\gamma\delta}\omega^{\delta\beta} = \lambda^2 V_2^{\alpha\beta} + \lambda\psi^\alpha\psi^\beta\,, \\ R^{\dot\alpha\dot\beta} &\equiv d\omega^{\dot\alpha\dot\beta} + \omega^{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\delta}\omega^{\dot\delta\dot\beta} = \lambda^2 V_2^{\dot\alpha\dot\beta} + \lambda\bar\psi^{\dot\alpha}\bar\psi^{\dot\beta}\,, \\ \rho^\alpha &\equiv d\psi^\alpha + \frac{1}{4}\omega^{\alpha\beta}\epsilon_{\beta\gamma}\psi^\gamma = \lambda V^{\alpha\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}\,, \\ \rho^{\dot\alpha} &\equiv d\bar\psi^{\dot\alpha} + \frac{1}{4}\omega^{\dot\alpha\dot\beta}\epsilon_{\dot\beta\dot\gamma}\bar\psi^{\dot\gamma} = \lambda V^{\alpha\dot\alpha}\epsilon_{\alpha\beta}\psi^\beta\,, \\ T^{\alpha\dot\alpha} &\equiv dV^{\alpha\dot\alpha} + \omega^{\alpha\gamma}\epsilon_{\gamma\delta}V^{\delta\dot\alpha} + \omega^{\dot\alpha\dot\gamma}\epsilon_{\dot\gamma\dot\delta}V^{\dot\delta\alpha} = \bar\psi^{\dot\alpha}\psi^\alpha\,. \end{aligned} \tag{93}$$

By setting $\lambda = 0$, we reproduce the flat MC equations and the Lorentz subgroup decouples from the MC equations (actually it plays a role as a semidirect product of $\mathfrak{siso}(1,3)$ of the super-Poincaré). Notice that the spin connection ω^{ab} is split into a self-dual and anti-self dual part $\omega^{\alpha\beta}, \omega^{\dot\alpha\dot\beta}$. In the same way, we have the self-dual and anti-self dual curvatures $R^{\alpha\beta}, R^{\dot\alpha\dot\beta}$ and the combinations $V_2^{\alpha\beta}$ and $V_2^{\dot\alpha\dot\beta}$. Using the cosmological constant, we can redefine the scaling of the different coordinates as $[V] = t^2$, $[\psi] = t$, $[\omega] = t^0$ and $[\lambda] = t^{-2}$. This scaling commutes with Equation (93). Now we can express the cohomology classes on a new basis as follows

$$\begin{aligned} \omega^{(0|0)} &= 1\,, & &\longleftrightarrow\ 1 \\ \omega^{(4|0)} &= \left(V_2^{\alpha\beta}\psi_\alpha\psi_\beta + V_2^{\dot\alpha\dot\beta}\bar\psi_{\dot\alpha}\bar\psi_{\dot\beta}\right) + \lambda V^4\,, & &\longleftrightarrow\ t^6 \\ \omega^{(0|4)} &= \lambda^{-1}\delta^4(\psi) + \left(V_2^{\alpha\beta}\iota_\alpha\iota_\beta\delta^4(\psi) + V_2^{\dot\alpha\dot\beta}\bar\iota_{\dot\alpha}\bar\iota_{\dot\beta}\delta^4(\psi)\right), & &\longleftrightarrow\ t^2\tilde{t}^4 \\ \omega^{(4|4)} &= V^4\delta^4(\psi)\,. & &\longleftrightarrow\ t^8\tilde{t}^4 \end{aligned} \tag{94}$$

The constant λ is placed in such a way as to respect the ring structure $\omega^{(4|0)} \wedge \omega^{(0|4)} = \omega^{(4|4)}$. Notice that compared with the flat case, only two classes of each sector (superforms, and integral forms) survive the curvature of the space. In particular, the cocycle $\omega_2^{(3)}$ discussed above disappears from the cohomology. This is consistent with the fact that the cohomology discussed is the equivariant Chevalley–Eilenberg cohomology of the coset space.

In the present case, the FIDA is easily built. We have to add an anticommuting $(3|0)$-form $A^{(3|0)}$ which scales as t^6, and an anticommuting $(0|4)$-form $A^{(-1|4)}$ which scales as $t^2\tilde{t}^4$, such that

$$dA^{(3|0)} = \omega^{(4|0)}\,, \quad dA^{(-1|4)} = \omega^{(0|4)} \tag{95}$$

and in turn, we also have to add two commuting potentials $B^{(2|0)}$ and $B^{(-2|4)}$, which scale as t^8 and $t^4\tilde{t}^8$. This renders the cohomology trivial as can be seen by using the Poincaré polynomial

$$P^{FIDA}_{OSp(1/4)/SO(1,3)} = (1+t^6)(1+t^2\tilde{t}^4)\frac{(1-t^6)}{(1-t^{12})}\frac{(1-t^2\tilde{t}^4)}{(1-\tilde{t}^4 t^8)} = 1 \tag{96}$$

The complete analysis of psuedoforms will be deferred to future publications.

3. Hodge Dual Operator, Dual Cocycles and Harmonic Cocycles

3.1. D = 4

The computation for the cocycles has been discussed and performed in [32]. We use here the same notations and conventions.

Let us use the Hodge dual construction: given a $(p|0)$-form in the superspace $\mathbb{R}^{(4|4)}$, $\omega^{(p)}(V,\psi,\bar{\psi})$. We can calculate its Hodge dual by the formula

$$\star\omega^{(p)}(V,\psi,\bar{\psi}) = \# \int e^{i(V^{\alpha\dot{\beta}}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\sigma^{\beta\dot{\beta}} + \psi^{\alpha}\epsilon_{\alpha\beta}b^{\beta} + \bar{\psi}^{\dot{\alpha}}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{b}^{\dot{\beta}})}\omega^{(p)}(\sigma,b,\bar{b})[d^4\sigma d^2 b d^2\bar{b}] \tag{97}$$

where $\sigma, b, \bar{b}$ are auxiliary variables needed to define the Hodge dual operation on a given form. The coefficient # is computed by choosing the signature of the superspace, requiring the idempotency of the star operation (see [27,28]) and it is irrelevant for the present purposes. Then, we have

$$\begin{aligned}
\omega_0^{(4|4)} &= \star 1 && = V^4\delta^4(\psi)\,, && \longleftrightarrow \quad t^8\tilde{t}^4 \\
\omega_1^{(1|4)} &= \star\omega_1^{(3)} && = V_3^{\alpha\dot{\alpha}}\iota_\alpha\bar{\iota}_{\dot{\alpha}}\delta^4(\psi)\,, && \longleftrightarrow \quad -t^4\tilde{t}^4 \\
\omega_2^{(0|4)} &= \star\omega_2^{(4)} && = V_2^{\alpha\beta}\iota_\alpha\iota_\beta\delta^4(\psi)\,, && \longleftrightarrow \quad t^2\tilde{t}^4 \\
\omega_3^{(0|4)} &= \star\omega_3^{(4)} && = V_2^{\dot{\alpha}\dot{\beta}}\bar{\iota}_{\dot{\alpha}}\bar{\iota}_{\dot{\beta}}\delta^4(\psi)\,, && \longleftrightarrow \quad t^2\tilde{t}^4 \\
\omega_4^{(-1|4)} &= \star\omega_4^{(5)} && = V^{\alpha\dot{\alpha}}\iota_\alpha\bar{\iota}_{\dot{\alpha}}\delta^4(\psi)\,, && \longleftrightarrow \quad -\tilde{t}^4 \\
\omega_5^{(0|4)} &= \star\omega_5^{(4)} && = \delta^4(\psi)\,, && \longleftrightarrow \quad \tilde{t}^4\,,
\end{aligned} \tag{98}$$

where ι_α and $\bar{\iota}_{\dot{\alpha}}$ are the derivatives of the Dirac delta's $\delta^4(\psi)$, with respect to their arguments. The scale $\tilde{t}$ is assigned to every single delta $\delta(\psi)$. Notice that all forms are closed except $\omega_4^{(-1|4)}$ which gives

$$d\omega_4^{(-1|4)} = \psi^\alpha\bar{\psi}^{\dot{\alpha}}\iota_\alpha\bar{\iota}_{\dot{\alpha}}\delta^4(\psi) = 4\,\delta^4(\psi) = 4\,\omega_5^{(0|4)} \tag{99}$$

by integration by parts. Therefore, cohomology in that sector is represented by the Poincaré polynomial

$$\mathbb{P}_{N=1}(t) = (2t^2 - t^4 + t^8)\tilde{t}^4 \tag{100}$$

which is exactly the Poincaré dual expression to

$$\mathbb{P}_{N=1}(t) = (1 - t^4 + 2t^6) \tag{101}$$

Notice that, differently from the usual cohomologies, the Poincaré duality cannot be established in the same complex of differential forms, but it has to be searched into the complex of integral forms. The overall $\tilde{t}^4$ stands for the picture equal to 4 of each term.

The complete Poincaré polynomial has the following form:

$$\mathbb{P}_{N=1}(t) = (1 - t^4 + 2t^6) + (2t^2 - t^4 + t^8)\tilde{t}^4 \tag{102}$$

One might also consider a partial Hodge dualization as follows:

$$\star_C\omega^{(p)}(V,\psi,\bar{\psi}) = \# \int e^{i(V^{\alpha\beta}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\sigma^{\beta\dot{\beta}} + \psi^{\alpha}\epsilon_{\alpha\beta}b^{\beta})}\omega^{(p)}(\sigma,b,\bar{\psi})[d^4\sigma d^2 b] \tag{103}$$

where only the dual of ψ is considered. $\star_C$ stands for chiral Hodge dual.

In that case, we have the following expressions

$$
\begin{aligned}
\omega_0^{(4|2)} &= \star_C 1 &&= V^4\delta^2(\psi)\,, &&\longleftrightarrow\ t^8\tilde{t}^2\\
\omega_1^{(3|2)} &= \star_C\omega_1^{(3)} &&= V_3^{\alpha\dot\alpha}\iota_\alpha\bar\psi_{\dot\alpha}\delta^2(\psi)\,, &&\longleftrightarrow\ -t^6\tilde{t}^2\\
\omega_2^{(0|2)} &= \star_C\omega_2^{(4)} &&= V_2^{\alpha\beta}\iota_\alpha\iota_\beta\delta^2(\psi)\,, &&\longleftrightarrow\ t^2\tilde{t}^2\\
\omega_3^{(4|2)} &= \star_C\omega_3^{(4)} &&= V_2^{\dot\alpha\dot\beta}\bar\psi_{\dot\alpha}\bar\psi_{\dot\beta}\delta^2(\psi)\,, &&\longleftrightarrow\ t^6\tilde{t}^2\\
\omega_4^{(1|2)} &= \star_C\omega_4^{(5)} &&= V^{\alpha\dot\alpha}\iota_\alpha\bar\psi_{\dot\alpha}\delta^2(\psi)\,, &&\longleftrightarrow\ -t^2\tilde{t}^2\\
\omega_5^{(0|2)} &= \star_C\omega_5^{(4)} &&= \delta^2(\psi)\,, &&\longleftrightarrow\ \tilde{t}^2\,,
\end{aligned}
\tag{104}
$$

Notice that we have computed the different cocycles with respect to only the variables V and ψ. Accordingly, we have computed the form degree and the picture number.

By computing the closure of the different expressions we obtain

$$
\begin{aligned}
d\omega_0^{(4|2)} &= 0\,,\\
d\omega_1^{(3|2)} &= \omega_3^{(4|2)}\,,\\
d\omega_2^{(0|2)} &= \omega_4^{(1|2)}\,,\\
d\omega_3^{(4|2)} &= 0\,,\\
d\omega_4^{(1|2)} &= 0\,,\\
d\omega_5^{(0|2)} &= 0\,,
\end{aligned}
\tag{105}
$$

and this implies that there are only two cohomology classes $\omega_0^{(4|2)}$ and $\omega_5^{(0|2)}$ (the chiral volume form and the chiral PCO). This can be expressed in terms of the Poincaré polynomial as follows:

$$
\mathbb{P}_{N=1}(t) = 2(1+t^8)\tilde{t}^2 \tag{106}
$$

where factor 2 stands for the chiral and the antichiral representations.

3.2. D = 6

The computation for the cocycles has been discussed and performed in [32]. We use here the same notations and conventions.

In this section, we introduce the Hodge dual operator for the flat supermanifold $(6|16)$ underlying the model $N = (4,0)$. We recall the supervielbeins $(V_{\alpha\beta}, \psi_\alpha^A)$ are respectively anticommuting and commuting 1-forms and we introduce the Hodge dual operator $\star$. For that, we need a metric on the supermanifold space $(\eta^{[\alpha\beta][\delta\gamma]}, \eta_{AB}^{\alpha\beta})$ to construct scalar products, then one needs dual variables $(\sigma_{\alpha\beta}, b_\alpha^A)$ (in the same representation of $(V_{\alpha\beta}, \psi_\alpha^A)$), and finally, given a form $\omega(V,\psi)$, we set

$$
\star\omega(V,\psi) = \#\int \omega(\sigma,b)e^{i(\sigma_{\alpha\beta}\eta^{[\alpha\beta][\delta\gamma]}V_{\delta\gamma}+b_\alpha^A\eta_{AB}^{\alpha\beta}\psi_\beta^B)}[d^6\sigma d^{16}b] \tag{107}
$$

where $\omega(\sigma,b)$ is the original form $\omega(V,\psi)$ where we substitute the supervielbeins in terms of the dual variables. The coefficient # is needed to implement the idempotency: $\star^2 = 1$.

In the vectorial representation (since it is an $SO(6)$ representation), we have η^{ab} using the vector indices $a,b = 1,\dots,6$, or written in the spinorial representation (antisymmetric tensor of $SU(4)$), it reads $\eta^{[\alpha\beta][\delta\gamma]} = \epsilon^{\alpha\beta\delta\gamma}$. In the spinorial representation for ψ_α^A however, there is no such metric. This is due to the well-known properties of the $SU(4)$ group. Therefore, $\eta_{AB}^{\alpha\beta}$ does not exist. This means the Hodge dual operation $\star$ is not invertible, and therefore it is not well-defined. However, we can restrict the action of the Hodge dual operator on the bilinear expressions in the spinors, where we can define a well-defined action of the Hodge dual operation.

Note that since the R-symmetry is $Usp(4)$ for $N = (4,0)$ and $SU(2)$ for $N = (2,0)$, the only invariant expressions are built with the antisymmetric tensors $\mathbb{C}^{AB}$ or ϵ^{ABCD} (in the case of $USp(4)$). Since only the spinor fields ψ^A_α carry those indices, we have that the bilinear expressions are always antisymmetric in the spinorial indices $\alpha, \beta, \dots$. Therefore, we can set the Hodge dual operation of those bilinears as follows (that can be also deduced by an integral formula as above by introducing dual variables $b_{\alpha\beta\gamma}$)

$$\star\left(\psi^{[A}_{[\alpha}\psi^{B]}_{\beta]}\right) = V^6 \epsilon_{\alpha\beta\gamma\delta}\mathbb{C}^{AR}\mathbb{C}^{BS}\iota^\gamma_R\iota^\delta_S\delta^{(16)}(\psi) \tag{108}$$

which is a $(4|16)$ integral form. Notice that the contraction $\iota^\alpha_A = \partial/\partial\psi^A_\alpha$ removes the one-degree form, and it carries the opposite representation with respect to ψ^A_α. The Hodge dual is defined such that

$$\psi^{[R}_{[\gamma}\psi^{S]}_{\delta]} \wedge \star\left(\psi^{[A}_{[\alpha}\psi^{B]}_{\beta]}\right) = \epsilon_{\alpha\beta\gamma\delta}\mathbb{C}^{AR}\mathbb{C}^{BS}V^6\delta^{(16)}(\psi) \tag{109}$$

where the integral form $\omega_0^{(6|16)} = V^6\delta^{(16)}(\psi)$ is the Berezinian of the supermanifold. Decomposing the supervielbeins along the curved coordinates (x^m, θ^A_μ) of the supermanifold, we have

$$V^a = E^a_m dx^m + E^{a,\mu}_A d\theta^A_\mu\,, \qquad \psi^A_\alpha = E^A_{\alpha,m}dx^m + E^{A,\mu}_{\alpha,B}d\theta^B_\mu\,, \tag{110}$$

where $(E^a_m, E^{a,\mu}_A, E^A_{\alpha,m}, E^{A,\mu}_{\alpha,B})$ are superfields in the coordinates (x^m, θ^A_μ) and

$$\omega_0^{(6|16)} = V^6\delta^{(16)}(\psi) = \mathrm{Sdet}\begin{pmatrix} E^a_m & E^{a,\mu}_A \\ E^A_{\alpha,m} & E^{A,\mu}_{\alpha,B}\end{pmatrix} d^6x\,\delta^{(16)}(d\theta)\,. \tag{111}$$

The matrix in the superdeterminant is a supermatrix with dimension $(6+16)\times(6+16)$. So, the definition of the Hodge dual corresponds to the definition of the Berezinian complement. Given a superform $\omega^{(p)}$, its Hodge dual is an integral form $\omega^{(6-p|16)} = \star\omega^{(p)}$ such that

$$\omega^{(p)} \wedge \star\omega^{(p)} = F(x^m, \theta^A_\mu)V^6\delta^{(16)}(\psi) \tag{112}$$

where $F(x^m, \theta^A_\mu) = ||\omega^{(p)}||^2$.

Let us now consider the case $N = (2,0)$ and $N = (4,0)$ with the classes $\omega_1^{(4)}, \omega_2^{(3)}, \omega_3^{(6)}$ and $\omega_4^{(7)}$ computed in [32]. Their Hodge duals read

$$\begin{aligned}
\omega_1^{(2|16)} = \star\omega_1^{(4)} &= \frac{1}{6!}V^6\epsilon^{ABCD}\eta^{ab}\bar{\iota}_{[A}\gamma_a\iota_{B]}\,\bar{\iota}_{[C}\gamma_b\iota_{D]}\delta^{16}(\psi)\,,\\
\omega_2^{(3|16)} = \star\,\omega_2^{(3)} &= \frac{1}{5!}\epsilon^{abcdef}V_bV_cV_dV_eV_f\mathbb{C}^{AB}\bar{\iota}_{[A}\gamma_a\iota_{B]}\delta^{16}(\psi)\,,\\
\omega_3^{(0|16)} = \star\,\omega_3^{(6)} &= \delta^{16}(\psi)\,,\\
\omega_4^{(-1|16)} = \star\,\omega_4^{(7)} &= V^a\mathbb{C}^{AB}\bar{\iota}_{[A}\gamma_a\psi_{B]}\delta^{16}(\psi)\,,\\
\omega_0^{(6|16)} = \star\omega_0^{(0)} &= V^6\delta^{16}(\psi)\,.
\end{aligned} \tag{113}$$

The indices over the new integral forms denote the form degree and the picture number. Now, we can apply the differential operator d and we use the MC equations to obtain

$$\begin{aligned}
d\omega_1^{(2|16)} &= \omega_2^{(3|16)}\,, \qquad & d\omega_2^{(3|16)} &= 0\,,\\
d\omega_4^{(-1|16)} &= \omega_3^{(0|16)}\,, \qquad & d\omega_3^{(0|16)} &= 0\,.
\end{aligned} \tag{114}$$

In the case of $N = (2,0)$, the invariant $\omega_1^{(2|16)} = 0$ and the invariant $\omega_2^{(3|16)}$ is a cohomology class in the integral form sector, together with the class $\omega_0^{(6|16)}$ given in Equation (110).

The polynomial $\mathbb{P}_{N=(2,0)}$ is

$$\mathbb{P}_{N=(2,0)} = -(1-t^4)t^8\tilde{t}^{16} \tag{115}$$

where $\tilde{t}^{16}$ counts the picture number of the integral forms. The term $t^{12}\tilde{t}^{16}$ corresponds to the Berezinian $\omega_0^{(6|16)}$ while $-t^8\tilde{t}^{16}$ corresponds to the class $\omega_2^{(3|16)}$. The signs respect the statistic of the cohomology classes.

Finally, notice that acting with the Hodge dual on Equation (114), we had

$$\begin{aligned} \star d \star \omega_1^{(4)} &= \omega_2^{(3)}\,, \qquad \star d \star \omega_2^{(3)} = 0\,, \\ \star d \star \omega_4^{(7)} &= \omega_3^{(3)}\,, \qquad \star d \star \omega_3^{(3)} = 0\,. \end{aligned} \tag{116}$$

Combining these equations with the closure of the cocycles, and using the conventional Laplace–Beltrami $\Delta = dd^\dagger + d^\dagger d$ and $d^\dagger = \star d \star$ we obtain

$$\begin{aligned} \Delta\omega_1^{(4)} &= \omega_1^{(4)}\,, \quad \Delta\omega_2^{(3)} = \omega_2^{(3)}\,, \\ \Delta\omega_3^{(6)} &= \omega_4^{(7)}\,, \quad \Delta\omega_4^{(7)} = \omega_4^{(7)}\,, \end{aligned} \tag{117}$$

In the case $N = (2,0)$, since $\omega_1^{(4)} = 0$, there are two cohomology classes ω_2^3 and $\omega_3^{(3|16)}$ and they satisfy

$$\Delta\omega_2^{(3)} = 0\,, \quad \Delta\omega_3^{(3|16)} = 0\,. \tag{118}$$

This shows that the cocycles that are cohomology classes are also harmonic in the usual sense, while those classes that are not in the cohomology are eigenforms of the Lapalce-Beltrami operator with a non-zero eigenvalue. This corresponds to the Hodge theorem for superforms.

In the present context, in the case of $N = (2,0)$ and $N = (2,2)$, we can apply the procedure to identify new forms to add to the theory. Again, we use the conventions and the results of [32].

In the case of $N = (2,0)$, we have the form cocycles $\omega_0^{(0)}, \omega_2^{(3)}, \omega_3^{(6)}, \omega_4^{(7)}$ and the integral cocycles $\omega_2^{(3|16)}, \omega_3^{(0|16)}, \omega_4^{(-1|16)}, \omega_0^{(6|16)}$. Computing the cohomology, we are left with

$$\omega_0^{(0)}, \omega_2^{(3)}\,, \qquad \omega_2^{(3|16)}, \omega_0^{(6|16)} \tag{119}$$

represented by the polynomial (putting together forms and integral forms)

$$\mathbb{P}_{N=(2,0)}(t) = (1-t^4)(1-t^8\tilde{t}^{16})$$

Now, to follow the FDA technique, we cancel the $\omega_2^{(3)}$ cocycle by adding the $A^{(2)}$ form (which scales as t^4, according to our conventions) such that

$$dA^{(2)} = \omega_2^{(3)}\,, \tag{120}$$

Notice that the newborn $A^{(2)}$ carries no representation of the R-symmetry and Lorentz group. It is a 2-form and therefore it is a commuting field with respect to the wedge product. Notice that we can form wedge products between forms and integral forms, and we can immediately observe that

$$d\left(A^{(2)} \wedge \omega_2^{(3|16)}\right) = \omega_0^{(6|16)}\,, \tag{121}$$

namely, also the cohomology class $\omega_0^{(6|16)}$ is trivialized! This is rather striking: the Berezinian class $\omega_0^{(6|16)}$ becomes an exact form, and it drops out from the cohomology. Notice that also the scales match correctly: $A^{(2)} \wedge \omega_2^{(3|16)}$ scales as $t^{12}\tilde{t}^{16}$ as the Berezinian $\omega_0^{(6|16)}$.

Did we completely trivialize the cohomology? Let us check it by using the Poincaré polynomial. To compute the polynomial for the FDA, we just divide $\mathbb{P}_{N=(2,0)}(t)$ by the contribution of $A^{(2)}$, namely

$$\mathbb{P}_{N=(2,0)}(t) \longrightarrow \mathbb{P}^{FDA}_{N=(2,0)}(t) = \frac{(1-t^4)(1-t^8\tilde{t}^{16})}{(1-t^4)} = (1-t^8\tilde{t}^{16}) \tag{122}$$

which clearly shows that the cohomology related to $(1-t^4)$ is removed, but there is still a remainder. Indeed, we still have one cohomology class around: $\omega_2^{(3|16)}$. Let us follow the FDA prescription and introduce the integral form $A^{(2|16)}$ with scales $t^8\tilde{t}^{16}$ such that

$$dA^{(2|16)} = \omega_2^{(3|16)}\,, \tag{123}$$

which finally cancels the last cohomology class. Notice that it seems the natural ingredient: it is a 2-integral form to be compared with the conventional 2-form introduced in Equation (120), but they are not related. This is crucial, since it appears that the new FDA requires new ingredients never seen before. Finally, observe that $A^{(2|16)}$ is again a commuting, invariant tensor, and therefore we can apply the computation of the Poincaré polynomial as above

$$\mathbb{P}^{FDA}_{N=(2,0)}(t) \longrightarrow \mathbb{P}^{FIDA}_{N=(2,0)}(t) = \frac{(1-t^8\tilde{t}^{16})}{1-t^8\tilde{t}^{16})} = 1 \tag{124}$$

which signifies that we have trivial cohomology and we were able to construct a complete and consistent FDA. Notice that it is also true

$$d\left(A^{(2|16)} \wedge \omega_2^{(3)}\right) = \omega_2^{(3|16)}\,, \tag{125}$$

but that implies that $\left(A^{(2|16)} \wedge \omega_2^{(3)}\right)$ differs from $\left(A^{(2)} \wedge \omega_2^{(3|16)}\right)$ by exact terms, indeed we have that

$$A^{(2|16)} \wedge \omega_2^{(3)} + A^{(2)} \wedge \omega_2^{(3|16)} = d\left(A^{(2)} \wedge A_2^{(2|16)}\right) \tag{126}$$

as a consistency check.

Two important remarks:

(1) Notice that we have treated $A^{(2)}$ as a commuting quantity. Indeed, we have multiplied the Poincaré polynomial by $1/(1-t^4)$. The expansion of which leads to

$$\frac{1}{1-t^4} = 1 + t^4 + t^8 + t^{12} + \ldots\,, \quad \longleftrightarrow 1\,, A^{(2)}\,, A^{(2)} \wedge A^{(2)}\,, A^{(2)} \wedge A^{(2)} \wedge A^{(2)} \;\ldots \tag{127}$$

which implies that we have to take any power of $A^{(2)}$. That can be understood if we admit that this form is expanded on the basis of superforms that allow any power of them.

(2) What about $A^{(2|16)}$? Again, we have adopted the pragmatic point of view: we have considered as a commuting quantity, and therefore we admit any power of it. Nevertheless, this clashes with the picture number. It carries picture number 16 and therefore, we cannot admit any more powers of it. We can conceive a way out by introducing new

commuting spinorial 1-form η in the game as in Fré-D'Auria algebra and the related pictures $\delta(\eta)$ [1,33,34].

3.3. D = 11

The computation of cocycles for 11d supergravity can be done using the Molien–Weyl formula and it gives [4]

$$\mathbb{P}_{11d}(t) = 1 + t^6 \tag{128}$$

where the t^6 stands for $\omega^{(4)} = \bar{\psi}\Gamma_{ab}\psi V^a V^b$. There is only one cocycle in the present sector and the result is consistent with the literature [1]. To construct the FDA, we have to add a 3-form $A^{(3)}$, which scales with t^6 such that

$$dA^{(3)} = \omega^{(4)} \tag{129}$$

Then the resulting Poincaré polynomial becomes

$$\mathbb{P}_{11d}(t, t^2) = 1 + t^6 \longrightarrow P^{FDA}_{11d}(t, t^2) = (1 + t^6)(1 - t^6) = 1 - t^{12} \tag{130}$$

This means that the FDA is not complete. Indeed, we see immediately, that there is a new cohomology class

$$\omega^{(7)} = A^{(3)} \wedge \omega^{(4)}_3 - \bar{\psi}\Gamma_{a_1\dots a_5}\psi V^{a_1} \dots V^{a_5} \tag{131}$$

To cancel that class, one needs to introduce a further commuting potential $B^{(6)}$ such that $dB^{(6)} = \omega^{(7)}$. Therefore, the final expression for the Poincaré polynomial is

$$P^{FDA}_{11d}(t, t^2) = (1 + t^6)(1 - t^6) = 1 - t^{12} \longrightarrow P^{FDA}_{11d}(t, t^2) = \frac{(1 + t^6)(1 - t^6)}{1 - t^{12}} = 1 \tag{132}$$

Notice that in this expression, the factor $1/(1 - t^{12})$ takes into account the infinite series of the powers $(B^{(6)})^k$.

The relevant Fierz identity is

$$(\bar{\psi}\Gamma^{ab}\psi)(\bar{\psi}\Gamma_a\psi) = 0\,. \tag{133}$$

and bi-spinor decomposition is

$$\psi_\wedge\bar{\psi} = \frac{1}{32}\left(\Gamma_a(\bar{\psi}\Gamma^a\psi) - \frac{1}{2}\Gamma_{ab}(\bar{\psi}\Gamma^{ab}\psi) + \frac{1}{5!}\Gamma_{a_1\dots a_5}(\bar{\psi}\Gamma^{a_1\dots a_5}\psi)\right) \tag{134}$$

We recall that there are two interesting forms written in terms of 1-forms ψ^α and vielbeins V^a:

$$\omega^{(4)} = \bar{\psi}\Gamma_{ab}\psi V^a V^b\,, \qquad \omega^{(7)} = \bar{\psi}\Gamma_{a_1\dots a_5}\psi V^{a_1} \dots V^{a_5}\,, \tag{135}$$

They satisfy the following equations

$$d\omega^{(4)} = 0\,, \quad d\omega^{(7)} = -\frac{1}{2}\omega^{(4)}_3 \wedge \omega^{(4)}_3\,. \tag{136}$$

The second equation is a consequence of the Fiersz identities

$$\bar{\psi}\Gamma_{a_1\dots a_5}\psi\bar{\psi}\Gamma^{a_5}\psi = \bar{\psi}\Gamma_{[a_1a_2}\psi\bar{\psi}\Gamma_{a_3a_4]}\psi$$

Let us consider the Hodge dual of those superforms

$$\begin{aligned}\omega^{(7|32)} = \star\omega^{(4)} &= V^{a_1}\dots V^{a_9}\epsilon_{a_1\dots a_9 b_1 b_2}\,\bar{\iota}\Gamma^{b_1b_2}\iota\delta^{32}(\psi)\,,\\ \omega^{(4|32)} = \star\omega^{(7)} &= V^{a_1}\dots V^{a_6}\epsilon_{a_1\dots a_6 b_1\dots b_5}\,\bar{\iota}\Gamma^{b_1\dots b_5}\iota\delta^{32}(\psi)\,,\end{aligned} \tag{137}$$

where $\bar{\iota}\Gamma^{b_1b_2}\iota = \frac{\delta}{\delta\bar{\psi}}\Gamma^{b_1b_2}\frac{\delta}{\delta\psi}$ are the derivatives with respect to the argument of the delta functions. Therefore, they act by integration by parts. In particular, if we compute the wedge product of ω_4 with $\star\omega_4$ (and analogously for ω_7), we obtain the volume form

$$\omega^{(4)}\wedge\star\omega^{(4)} = V_1\dots V_{11}\delta(\psi_1)\dots\delta(\psi_{32})\,,\quad \omega^{(7)}\wedge\star\omega^{(7)} = V_1\dots V_{11}\delta(\psi_1)\dots\delta(\psi_{32})\,. \tag{138}$$

Notice that the first one has degrees $(7|32)$ (due to the presence of nine vielbeins and two derivatives), while the second one has degree $(4|32)$. Notice that both are closed

$$\begin{aligned} d\star\omega^{(4)} &= 9\,(\bar{\psi}\Gamma^{a_1}\psi)V^{a_2}\dots V^{a_9}\epsilon_{a_1\dots a_9 b_1 b_2}\,\bar{\iota}\Gamma^{b_1b_2}\iota\delta^{32}(\psi)\\ &= 9\,\mathrm{tr}(\Gamma^{a_1}\Gamma^{b_1b_2})V^{a_2}\dots V^{a_9}\epsilon_{a_1\dots a_9 b_1 b_2}\,\delta^{32}(\psi) = 0\,,\\ d\star\omega^{(7)} &= 6\,V^{a_2}\dots V^{a_6}\epsilon_{a_1\dots a_6 b_1\dots b_5}\,\bar{\iota}\Gamma^{b_1\dots b_5}\iota\delta^{32}(\psi)\\ &= 6\,\mathrm{tr}(\Gamma^{a_1}\Gamma^{b_1\dots b_5})V^{a_2}\dots V^{a_9}\epsilon_{a_1\dots a_6 b_1\dots b_5}\,\delta^{32}(\psi) = 0 \end{aligned} \tag{139}$$

and they vanish because of the trace between the gamma matrices. On the other hand, if we compute the Hodge dual of $d\omega^{(7)}$, we obtain

$$\star d\omega^{(7)} = -\frac{1}{4}V^{a_1}\dots V^{a_7}\epsilon_{a_1\dots a_7 b_1\dots b_4}\bar{\iota}\Gamma^{b_1b_2}\iota\bar{\iota}\Gamma^{b_3b_4}\iota\delta^{32}(\psi) \tag{140}$$

Using the Fiersz identities, we can recast the derivatives as follows:

$$\star d\omega^{(7)} = -\frac{1}{4}V^{a_1}\dots V^{a_7}\epsilon_{a_1\dots a_7 b_1\dots b_4}\bar{\iota}\Gamma^{b_1b_2b_3b_4b_5}\iota\bar{\iota}\Gamma_{b_5}\iota\delta^{32}(\psi) \tag{141}$$

Then we can compute the differential

$$d\star d\omega^{(7)} = -\frac{7}{4}\bar{\psi}\Gamma^{a_1}\psi\dots V^{a_7}\epsilon_{a_1\dots a_7 b_1\dots b_4}\bar{\iota}\Gamma^{b_1b_2b_3b_4b_5}\iota\bar{\iota}\Gamma_{b_5}\iota\delta^{32}(\psi) \tag{142}$$

The integration by part produces two different structures: one vanishes because of the usual trace of gamma matrices, but the second structure gives the expression

$$d\star d\omega^{(7)} = -\frac{7}{2}\star\omega^{(7)} \qquad\Longrightarrow\qquad \star\, d\star d\omega^{(7)} = -\frac{7}{2}\omega^{(7)} \tag{143}$$

then finally it leads (together the vanishing of $d\star\omega_7 = 0$, to the equations

$$\Delta\omega^{(7)} = -\frac{7}{2}\omega^{(7)}\,,\qquad \Delta\omega_3^{(4)} = 0\,. \tag{144}$$

The second equation follows from $d\omega_4 = 0$. Then, we found that those forms satisfy a (massive) Laplace–Beltrami equation(as discussed in [44]. This is an indication that a Hodge theory may be established for supermanifolds).

One question arises. Can one compute the dual forms from the Molien-Weyl formula? For that, we need to change the plethystic exponential for ψ as follows (for details see [32])

$$PE\left[\frac{1}{t}\chi_{32}(z_1,\dots,z_5)\right] = \bar{t}^{32}\prod_{i=1}^{32}\frac{1}{(1-\chi_{32,i}1/t)}\,, \tag{145}$$

where the parameter t counts the degree of forms. This is to count the number of derivatives ι_{D_i} (which technically is the contraction along an odd vector field D_i) acting on $\bigwedge_{i=1}^{32}\delta(\psi^i)$. The factor $\tilde{t}^{32}$ counts the picture number. Thus, we finally obtain the result

$$\mathbb{P}_{\star 11d}(1/t,u) = \frac{(1-u)\Big(u\Big(1-\frac{u^4}{t^6}-\frac{u^6+u^4+u^2}{t^4}-\frac{u^8+u^4+1}{t^2}+\big(u^2+u+1\big)\big(u^6+u^3+1\big)u+1\Big)\Big)}{\frac{1}{1-t^4}}$$

which leads to

$$\mathbb{P}_{\star 11d}(1/t,t^2) = -(1+t^6)t^{16}\tilde{t}^{32} = -(t^{16}+t^{22})\tilde{t}^{32} \tag{146}$$

which represent the two classes

$$\begin{aligned} \omega^{(7|32)} &= \star\omega^{(4)} = V^{a_1}\dots V^{a_9}\epsilon_{a_1\dots a_9 b_1 b_2}\,\bar{\iota}\Gamma^{b_1 b_2}\iota\delta^{32}(\psi) \\ \omega^{(11|32)} &= \star 1 = V^1\dots V^{11}\delta^{32}(\psi)\,. \end{aligned} \tag{147}$$

The overall sign in Equation (146) correctly detects the parity of those classes. Note that it does not appear in the cohomology the dual form $\star\omega^{(7)}$. Indeed, it can be easily shown that this form is exact

$$\begin{aligned} \star\omega^{(7)} &= d\omega^{(3|32)}\,, \\ \omega^{(3|32)} &= V^{a_1}\dots V^{a_7}\epsilon_{a_1\dots a_7 b_1 b_4}\,\bar{\iota}\Gamma^{b_1 b_2}\iota\bar{\iota}\Gamma^{b_3 b_4}\iota\delta^{32}(\psi) \end{aligned} \tag{148}$$

Then, finally, we can compute the FDA of 11d superspace in both sectors: superforms and integral forms. The complete Poincaré polynomial is

$$\mathbb{P}_{11d}(t) = (1+t^6) - (1+t^6)t^{16}\tilde{t}^{32} = (1+t^6)(1-t^{16}\tilde{t}^{32}) \tag{149}$$

Therefore, following the prescription of the FDA, we have to add the 3-form $A^{(3)}_6$ and the 6-form $B^{(6)}_{12}$ such that

$$\begin{aligned} dA^{(3)}_6 &= \omega^{(4)}_6 \\ dB^{(6)}_{12} &= \omega^{(7)}_{12} - A^{(3)}_6\wedge\omega^{(4)}_6 \end{aligned} \tag{150}$$

which implies, at the level of the Poincaré polynomial

$$\mathbb{P}_{11d}(t) = (1+t^6)(1-t^{16}\tilde{t}^{32}) \longrightarrow P^{FDA}_{11d}(t) = (1-t^{16}\tilde{t}^{32}) \tag{151}$$

Therefore we need to compensate the last cocycle by adding an integral potential. We add the $(6|32)$ form such that

$$dB^{(6|32)} = \omega^{(7|32)} \tag{152}$$

Then, the spectrum of the FDA is given by $A^{(3)}, B^{(6)}, B^{(6|32)}$.

The 7-cocycle $\omega^{(7)}$ plays a crucial role in the analysis of p-branes, making it intriguing that it arises solely from algebraic analysis, as demonstrated here. In their work cited as [1], the authors aimed to incorporate $B^{(6)}$ as an original degree of freedom of the theory. However, it was found that its Lagrangian formulation, as presented in [1], is not appropriate. Its significance can only be grasped from a non-perturbative perspective, such as in its coupling to M5 branes.

On the other hand the $B^{(6|32)}$ might jeopardize the interpretation introducing new degrees of freedom (DOFs). However, if we set

$$F^{(4)} = dA^{(3)} = \star F^{(7|32)} = \star dB^{(6|32)} \tag{153}$$

we finally related the DOFs of the four form $F^{(4)}$ of the CJS supergravity with those of the new integral potential $B^{(6|32)}$. The question of whether this equation can be derived by an action principle is not known at the moment, but the existence of pseudoforms in picture 16 might be crucial. To verify Equation (153), we list the components of $F^{(4)}$

$$F^{(4)} = F^{(4)}_{a_1\dots a_4} V^{a_1}\dots V^{a_4} + F^{(4)}_{a_1\dots a_3\alpha_1} V^{a_1}\dots V^{a_3}\psi^{\alpha_1} + \dots + F^{(4)}_{\alpha_1\dots\alpha_4}\psi^{\alpha_1}\dots\psi^{\alpha_4} \tag{154}$$

while for the dual form

$$\begin{aligned} F^{(7|32)} &= \Big(\mathcal{F}_{a_1\dots a_7} V^{a_1}\dots V^{a_7} + \mathcal{F}^{\alpha_1}_{a_1\dots a_8} V^{a_1}\dots V^{a_8}\iota^{\alpha_1} + \\ \dots &\quad + \mathcal{F}^{\alpha_1\dots\alpha_4}_{a_1\dots a_{11}} V^{a_1}\dots V^{a_{11}}\iota^{\alpha_1}\dots\iota^{\alpha_4}\Big)\delta^{(32)}(\psi) \end{aligned} \tag{155}$$

Imposing Equation (153) we can fix the components of $F^{(7|32)}$ in terms of those of $F^{(4)}$ as follows

$$\begin{aligned} \mathcal{F}_{a_1\dots a_7} &= \epsilon_{a_1\dots a_7}{}^{b_1\dots b_4} F^{(4)}_{b_1\dots b_4} \\ \mathcal{F}^{\alpha}_{a_1\dots a_8} &= \epsilon_{a_1\dots a_8}{}^{b_1\dots b_3} C^{\alpha\beta} F^{(4)}_{b_1\dots b_3\beta} \\ \vdots \quad & \vdots \quad \vdots \\ \mathcal{F}^{\alpha_1\dots\alpha_4}_{a_1\dots a_{11}} &= \epsilon_{a_1\dots a_{11}} C^{\alpha_1\beta_1}\dots C^{\alpha_4\beta_4} F^{(4)}_{\beta_1\dots\beta_4} \end{aligned} \tag{156}$$

where $C^{\alpha\beta}$ is the charge conjugation matrix.

Equation (153) can be derived by an action as follows

$$S = \int_{\mathcal{M}^{(11|32)}} \left(\frac{1}{2} F^{(7|32)} \wedge \star F^{(7|32)} - F^{(4)} \wedge F^{(7|32)} + \mathcal{L}^{(11|0)}(V, \psi, A^{(3)}) \wedge \mathbb{Y}^{(0|32)}\right) \tag{157}$$

where $\mathcal{M}^{(11|32)}$ is the supermanifold on which we integrate the Lagrangian $\mathcal{L}^{(11|0)}(V, \psi, A^{(3)})$ (given in [10] without the zero form $f_{a_1\dots a_4}$) and $\mathbb{Y}^{(0|32)}$ is the PCO operator which allows us to convert the Lagrangian into an integral form. The PCO $\mathbb{Y}^{(0|32)}$ depends upon the vielbein and the gravitino field ψ. It is closed and not exact. Introducing the 3-form $A^{(3)}$, we can compute the equations of motion with respect to $A^{(3)}$ and $F^{(7|32)}$ as follows

$$\begin{aligned} F^{(7|32)} - \star F^{(4)} &= 0 \\ dF^{(7|32)} + \left(\frac{\delta}{\delta A^{(3)}}\mathcal{L}^{(11|0)}(V, \psi, A^{(3)})\right) \wedge \mathbb{Y}^{(0|32)} &= 0\,. \end{aligned} \tag{158}$$

where the derivative with respect to $A^{(3)}$ is the Euler-Lagrangian derivative. Since $\mathbb{Y}^{(0|32)}$ is independent of $A^{(3)}$ and, therefore, it can be extracted from the derivatives. In addition, since $\mathbb{Y}^{(0|32)}$ is closed, we can perform integration by parts in computing the functional derivatives of the action. The first equation implies the identifications of the d.o.f.'s of $F^{(4)}$ with those of $F^{(7|32)}$. Inserting this result into the first equation, we obtain

$$d \star F^{(4)} + \left(\frac{\delta}{\delta A^{(3)}}\mathcal{L}^{(11|0)}(V, \psi, A^{(3)})\right) \wedge \mathbb{Y}^{(0|32)} = 0\,. \tag{159}$$

Acting the inverse PCO Z (which is also closed and not exact), we obtain

$$d\,Z\Big(\star dA^{(3)}\Big) + \frac{\delta}{\delta A^{(3)}}\mathcal{L}^{(11|0)}(V, \psi, A^{(3)}) = 0\,. \tag{160}$$

(at the moment assume that Z acts only on $\mathbb{Y}^{(0|32)}$, but there should be more difficult cases). The expression is an $(8|0)$ form. We notice that with the integral form $F^{(7|32)}$, the auxiliary zero form $f_{a_1\dots a_4}$ is no longer needed.

4. Conclusions and Outlook

In the present work, we have completed the analysis of the supergravity cocycles for different models extending the FDA beyond the superforms to integral forms and pseudoforms. We give general arguments of the structure of the FIDA, but we do not provide a complete mathematical analysis which might be very interesting along the lines of [17]. We use the Hodge dual operator to complete some of the results obtained in previous work [32], and we also plan to work the pseudo-form sector for those theories. A dual construction along [45–47] will be certainly interesting in order to develop the geometrical understanding of the underlying supergeometry and application to more general supermanifolds is a target for future works.

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Notes

1. In the original paper [1], R. D'Auria and P. Fré referred to *commutative semi free differential algebras* as "Cartan integrable systems". In nowadays literature, the semi free differential algebras used in supergravity are misnamed as "FDA"s. We keep the same terminology in the present work.
2. Under a generic infinitesimal transformation $\psi^\alpha \to \psi^\alpha + \sum_{\beta\neq\alpha} \delta\Lambda^\alpha_\beta \psi^\beta$, where $\delta\Lambda^\alpha_\beta$ are the infinitesimal parameters of the transformation, $\delta(\psi^\alpha)$ transforms as

$$\delta(\psi^\alpha) \to \delta(\psi^\alpha + \sum_{\beta\neq\alpha} \delta\Lambda^\alpha{}_\beta \psi^\beta) = \delta(\psi^\beta) + \sum_{\beta\neq\alpha} \delta\Lambda^\alpha{}_\beta \psi^\beta \delta'(\psi^\alpha) + \dots$$

where the Taylor expansion is used. The formal Taylor series is taken as a definition for the transformation properties of a single delta and, since the derivatives of delta are easy to deal with, there is a consistent method to handle with single-delta expressions (see for example [36]).

3. Expressing $\omega_I^{(p_I|0)}$ in terms of the potentials $A_I^{(p_I-1|0)}$, we have $d(A_I^{(p_I-1|0)} \wedge dA_J^{(p_J-1|0)} - \sum_K \tilde{C}^K_{IJ} A_K^{(p_I+p_J-1|0)}) = 0$, which implies that $A_I^{(p_I-1|0)} \wedge dA_J^{(p_J-1|0)} = \sum_K \tilde{C}^K_{IJ} A_K^{(p_I+p_J-1|0)} + f_{IJ}\omega^{(p_I+p_J-1|0)}$, but if all $\omega^{(p_I+p_J-1|0)}$ are replaced by $dA^{(p_I+p_J-2|0)}$, we can redefine the contributions on the left hand side of Equation (30) by exact terms.
4. The computation is presented in a separate paper [32].

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