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Inequality in the Distribution of Wealth and Income as a Natural Consequence of the Equal Opportunity of All Members in the Economic System Represented by a Scale-Free Network

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Abstract: The purpose of this work is to examine the nature of the historically observed and empirically described by the Pareto law inequality in the distribution of wealth and income in an economic system. This inequality is presumed to be the result of unequal opportunity by its members. An analytical model of the economic system consisting of a large number of actors, all having equal access to its total wealth (or income) has been developed that is formally represented by a scale-free network comprised of nodes (actors) and links (states of wealth or income). The dynamic evolution of the complex network can be mapped in turn, as is known, into a system of quantum particles (links) distributed among various energy levels (nodes) in thermodynamic equilibrium. The distribution of quantum particles (photons) at different energy levels in the physical system is then derived based on statistical thermodynamics with the attainment of maximal entropy for the system to be in a dynamic equilibrium. The resulting Planck-type distribution of the physical system mapped into a scale-free network leads naturally into the Pareto law distribution of the economic system. The conclusions of the scale-free complex network model leading to the analytical derivation of the empirical Pareto law are multifold. First, any complex economic system behaves akin to a scale-free complex network. Second, equal access or opportunity leads to unequal outcomes. Third, the optimal value for the Pareto index is obtained that ensures the optimal, albeit unequal, outcome of wealth and income distribution. Fourth, the optimal value for the Gini coefficient can then be calculated and be compared to the empirical values of that coefficient for wealth and income to ascertain how close an economic system is to its optimal distribution of income and wealth among its members. Fifth, in an economic system with equal opportunity for all its members there should be no difference between the resulting income and wealth distributions. Examination of the wealth and income distributions described by the Gini coefficient of national economies suggests that income and particularly wealth are far off from their optimal value. We conclude that the equality of opportunity should be the fundamental guiding principle of any economic system for the optimal distribution of wealth and income. The practical application of this conclusion is that societies ought to shift focus from policies such as taxation and payment transfers purporting to produce equal outcomes for all, a goal which is unattainable and wasteful, to policies advancing among others education, health care, and affordable housing for all as well as the re-evaluation of rules and institutions such that all members in the economic system have equal opportunity for the optimal utilization of resources and the distribution of wealth and income. Future research efforts should develop the scale-free complex network model of the economy as a complement to the current standard models.

Keywords: wealth; income; inequality; scale-free network; Planck distribution; Pareto law; Gini coefficient; equal opportunity; Pareto index

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1. Introduction

The wealth inequality among members of various contemporary societies and even societies going back to medieval times was studied systematically in the late 19th and early 20th centuries by the Italian engineer, economist, and sociologist Vilfredo Pareto based on available data (Pareto 1925; Kelley 1971). Pareto's analysis indicated that inequality occurred invariably in all societies with no exceptions, although there was a relatively small degree of variability of wealth distribution among societies This wealth distribution is described by a power law, which is also known as the Pareto law in economic theory (Pareto Principle 2024). Income inequality was also found to obey the same law. A rough enunciation of the Pareto law states that 20% of the population owns 80% of the wealth and is abbreviated as the 80-20 law (Newman 2006). Power laws have been found empirically to occur in a variety of diverse phenomena such as number of brush fires, the magnitude of earthquakes, the size of cities, contribution to taxation and the worldwide web traffic. A power law is self-similar over a wide range of magnitudes and produces outcomes completely different from a normal or Gaussian law. This fact explains the frequent breakdowns of sophisticated financial instruments, which are modeled on the assumption that a Gaussian relationship is appropriate to, for example, stock price movements. The 80-20 law is indicative of all those phenomena that obey power law probability distributions, although the actual figures, i.e., 80% and 20%, can vary by plus or minus 10% in each case.

Recently, the French economist Thomas Piketty stated that wealth is unequally distributed in every political system from capitalism to communism, but he provided no explanation for it (Piketty 2014). The notion of wealth inequality as an established fact of a society has been around for a long time. The "Mathew Effect" as it is known in the field of sociology in modern times is but one manifestation of this notion (Kaufmann and Stützle 2017). The origin of the Mathew Effect is the statement in the Gospel of Mathew 25:29 in connection with the Parable of the Talents: "For unto everyone that hath shall be given and shall have abundance; but from him that hath not shall be taken away even that which he hath" (Coogan 2007). The uneven distribution of wealth even among the citizens of the Athenian Democracy at its peak in the middle of the 5th century BCE was commented upon by Plato, Demosthenes, and Xenophon a century later (Sargent 1925). The number of servants employed by a citizen in a variety of uses such as domestic, commercial, agricultural, and silver mining operations would indicate the rank of a citizen within the Athenian economy at the time with the wealthiest individual, Nikias, having 1000 servants along with 100 talents (one talent equals 26 kg of silver) of property, while a poor citizen would have at least one servant in his household (Aristophanes in his play *Ploutos*). According to Plato everybody with more than 50 servants in his household was considered as being wealthy. Xenophon was left rich by his father with 52 servants and 14 talents of overall property. Some 6% of the wealthiest Athenians owned 32% of the land and had a wealth valued from a high of 100 talents to a low of 3 talents.

Returning to the present time, Piketty's work sparked a significant discussion, with some favorable opinions and others critical and occasionally dismissive. The American economist Paul Krugman, winner of the Nobel Prize in Economics in 2008, called Piketty's work as "the most important book of the year (in 2014)—and possible the decade". Piketty has also been described as a new Karl Marx and placed in the same league as the economist John Maynard Keynes. One of the constructive criticisms of Piketty's work expressed by two German authors, the economics journalist Stephan Kaufmann and the political scientist Ingo Stützle, is that Piketty takes economic inequality for granted as being part of not only capitalist societies, but other economic systems as well (Kaufmann and Stützle 2017). This criticism, which is a valid one, has prompted the development of this work, i.e., the analytical basis for the justification of Piketty's assumption that "wealth inequality is inscribed into the economic development of any kind".

We may also mention two other interesting points regarding the accumulation of wealth. First, Benjamin Franklin, one of the Founding Fathers of the American Republic, observed in the late 18th century that in addition to hard work and discipline, the grace of the Creator was necessary for financial success (Franklin 1757). More recently, the Italian researcher Alessandro Pulchino, along with two other colleagues at the University of Catania, has determined based on computer simulations that above all other measurable elements such as hard work, education, talent, intelligence, and so on, the element of "luck or chance" is the unknowable and unpredictable element necessary for one to acquire wealth (The MIT Business Review 2018). That is to say, the wealthiest people are not necessarily the most talented, but rather the luckiest! And while we cannot predict who the "lucky" ones are going to be, we will try to understand how "luck" or inequality is built into the economic system.

The confirmation of the empirical Pareto distribution across multiple socio-economic systems and time periods would suggest that inequality is real and ubiquitous. The question then becomes whether inequality in wealth and income distributions exists because (a) inequality is imposed by certain actors in the economic system to their advantage or (b) inequality is there by design, i.e., as part of the universe that we live in, and is therefore unavoidable or (c) perhaps a combination of both, i.e., human imposition and natural design. To answer these questions, we develop an analytical model of an economic system that leads to the derivation of Pareto's law or distribution. This derivation from first principles would then allow us to answer three fundamental questions as follows: (a) understand the tenets of the validity of the Pareto law in an economic system; (b) appreciate why inequalities in wealth and income occur; and (c) determine how far off an economic system is from optimal wealth and income distributions to implement meaningful and consequently effective corrective policies.

2. Methodology

We consider and describe an economic system comprised of actors with certain properties. We can visualize this economic system as a scale-free complex network comprised of nodes, i.e., actors, and links interconnecting the nodes, i.e., interactions between the actors (Barabási 2014). The choice of this model, i.e., a scale-free complex network of nodes and links, constitutes a departure from the traditional analytical models of the economy as will be discussed in some length later in this work and is based on the empirical observation that it must include the Pareto law, also described mathematically as a power law (Pareto Principle 2024). Power laws are characterized as representing scale-free systems that are ubiquitous in nature (Newman 2006). A scale-free complex network comprised of nodes and links should therefore provide an appropriate model of an economic system (Barabási 2014). The objective of the analytical model is to find the probability distribution function of wealth and income among these actors. This is necessitated by the fact that we cannot predict which actor will acquire certain amount of wealth or income. Thus, a statistical analysis resulting into the probabilistic distribution of the wealth or income of the actors in the economic system is the best we can do. We use the term "wealth" in the remainder of this section for simplicity, but the model is equally well applicable to income as well as other economic entities.

2.1. Methodology Process

The properties of the economic system under consideration are as follows:

(a) The economic system under consideration consists of N actors. The number of actors N is assumed to be fixed, although this condition will be relaxed later in the analysis. The N actors are assumed to be independent of one another within the economic system, i.e., each actor has an equal probability as any other to attain any degree of wealth. However, no actor is identical to another from the point of view of accumulating wealth (The MIT Business Review 2018; Barabási 2014). Consequently, we assign to each actor (node) a so-called fitness factor η that describes the ability of that actor to interact (be linked) with other actors in the economic system (Barabási 2014). Moreover, each interaction (link) is identical to any other. That is to say, the number of interactions (links) of an actor to all other actors determines the amount of wealth of that actor. Lastly, each of the N actors in the economic system must have some finite (non-zero) wealth to be able to interact with any other actor in the system and be part of it.

(b) The total wealth of the economic system is U. It is finite and constant over the period of examination of the state of the system. It is comprised of the sum of the wealth (number of links) of each individual actor in the system denoted by U_i with i varying from 1 to N: $U = \Sigma\, N_i\, U_i$, i = 1, 2, . . . N. U_i is considered to be a stochastic or random variable in our model.

(c) The state of the system is in a dynamic equilibrium. Equilibrium means that the economic system has attained the most likely, i.e., the one with the highest probability, configuration in terms of the distribution of wealth levels (number of links) among its N actors (nodes) under the prevailing constraints of a fixed number of actors and fixed amount of total wealth. Dynamic implies that the economic system is free to attain a different equilibrium as the total wealth and/or the number of actors change over time.

(d) The wealth is considered to have a minimum value of "h" which can be the unit or "quantum" of currency, say, a dollar in the case of the United States, such that the wealth of each actor is expressed as follows: $U_i = u_i \cdot h$, where u_i will be an integer number varying from 1 to a very large number. Wealth is thus not infinitely divisible, but it has a minimum value of h. Consequently, the minimum amount of wealth of any actor in the economic system is at least h[1]. Moreover, this value h represents the minimum possible amount of wealth exchange (existence of one link) between any of the actors[2].

(e) Each of the N actors (nodes) can be in (connected to) any of several possible wealth elements (number of links) G, under the constraint of fixed total wealth U, whereby G could be as large as the ratio of U/h since h is the minimum wealth amount (link size). In other words, G is the number of wealth elements (links), which must be divided over the N actors. It is clear then that G can be less than, equal to, or larger than N.

(f) The number of sequences of the N actors (nodes) with a particular number of wealth elements (number of links) G is designated by an index i, whereby the total wealth U is viewed as being divided up in a succession of small ranges $U_i + \Delta U_i$ with ΔU_i equal to or greater than h. We thus describe the range of values of G by stating the number of possible wealth elements (links) G_i for each Ni sequence of actors and a fixed wealth incremental amount (link size h).

(g) Any of the G wealth elements (number of links) for each of the N actors is accessible to all of them, i.e., there are no exclusions of any actors from any wealth elements. Thus, G_i represents the number of possible wealth elements (number of links) out of the total wealth levels G associated with the actor (node) $N_{i,}$ because the number of available actors N, implies that a value of G_j may be 0, i.e., no one of the N actors is found in that wealth element, while another value of the wealth element G_k may be found in more than one actor. In fact, any one of the N actors may contain all the G wealth elements at the exclusion of all other actors as distinct possibility.

This economic system described above, as already indicated, represents essentially a scale-free complex network comprised of actors/nodes and wealth elements/links (Barabási 2014). We already noted that the empirical Pareto distribution is a power law, and it must be an integral part of the analytical model of the economic system. Consequently, the proposed model of the economic system should be described by a scale-free network, whose structure is intrinsically representative of a power law, rather than a random network (Barabási 2014). The structure of a random network versus that of a scale-free network is shown schematically in Figure 1.[3] The power law structure of a scale-free network as opposed the random (Gaussian) structure of a random network should be apparent.

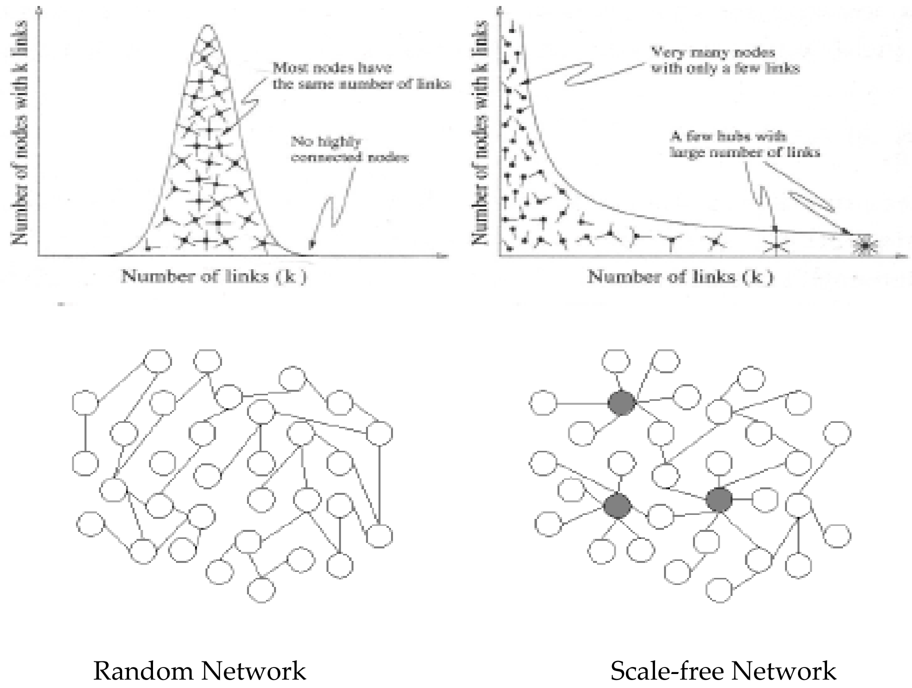


Figure 1. Schematic comparison of the distribution of links (k) in random networks where most nodes have the same number of links and scale-free networks where a few nodes can have a large number of links, resulting in the formation of hubs (Barabási 2014).

Such a scale-free system has been described in analogy to a quantum physical system in Statistical Thermodynamics[4] (Planck 1900; Bianconi and Barabási 2001). The quantum physical system in Statistical Thermodynamics consists of quantum particles indistinguishable from one another and non-interacting with one another (photons) in a dynamic equilibrium in a black body (a cavity that contains the entire physical system). As already alluded to, the indistinguishability of physical particles translates into no distinction or preferential treatment of any actor (node) in the economic system over another, i.e., equal opportunity for all the actors. The physical system has at any given time a total energy E determined uniquely by the temperature T of the cavity at that time[5]. The system consists of G photons (total number of links). These photons are distributed among, i.e., occupying, N radiation or energy states (nodes). Each energy state N is occupied by m photons, each having an energy E_m, while the number m is an integer varying from 0 to G and designating the degree of occupancy of that state commensurate with its energy level[6]. Consequently, if an energy state contains zero (m = 0) photons at a given moment in time, it contributes nothing to the total energy E and as far as the system is concerned it does not exist at that moment. The sum of the energies (m E_m) of the photons over all the N energy states in the physical system will be equal to its total energy E. As already indicated, it has been established that a scale-free network of nodes and links can be modeled via a Planck/Bose–Einstein quantum thermodynamic system, where (a) the nodes of the network correspond to the N energy states of the physical system and (b) the links of the network correspond to the number G of quantum particles that occupy its energy states as shown schematically in Figure 2 (Barabási 2014; Planck 1900; Bianconi and Barabási 2001). Each node (actor) is characterized by a fitness η to ensure that each node (actor) acquires links (wealth elements) at a different rate (Barabási 2014; Bianconi and Barabási 2001). The fitness feature represents the innate skills of each actor in the economic system. The fitness is associated with the energy level (links or wealth elements) $\varepsilon = (-1/\beta) \ln \eta$ of a node (actor) in the equivalence of the quantum physical particles system to the complex network (economic system) and will not be invoked any further in this model. The parameter β plays the role of inverse temperature in the physical system and inverse wealth in the economic system (Barabási 2014; Bianconi and Barabási 2001).

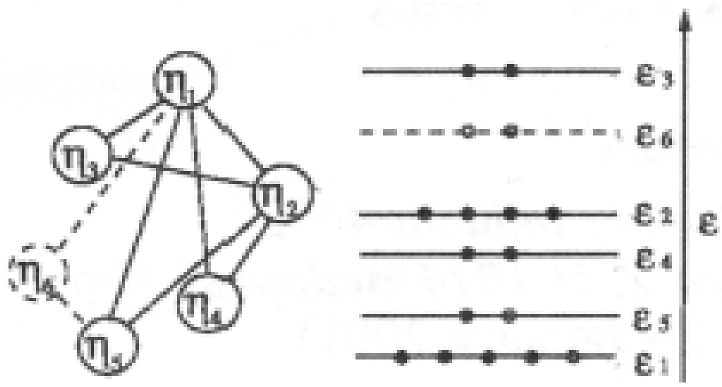


Figure 2. Schematic illustration of the mapping between an economic system comprised of actors and wealth elements and represented as a network comprised of nodes (actors) and links (wealth elements) and a statistical thermodynamic system of indistinguishable and non-interacting gas of photons (links—wealth elements) distributed in different energy states (nodes—actors)—a node (actor) is characterized by a fitness η and has a varying number of wealth elements (photons—links) occupying the different energy states ε (actors—nodes) of the physical system (complex network—economic system). The dashed node and links signify an actor addition to the network system (Barabási 2014).

Returning to the economic system of wealth distribution under development and its analytical representation as a scale-free network of nodes and links, we can now map it to the Planck/Bose–Einstein distribution, where the actors (nodes) of the economic system are the energy states in the physical system and the wealth amount, i.e., number of wealth units or elements, of each actor (number of links associated with that node) is reflected by the number of physical particles in the respective energy state, i.e., the particular actor. This mapping is shown schematically in Figure 3. The objective is to calculate the wealth probability distribution function of the N actors (nodes) with a total wealth U distributed among G wealth elements (links). We also expect to analytically derive the empirical form of the Pareto distribution and by doing so to gain a fundamental understanding of the observed occurrence of wealth distribution and inequality. We apply the well-known statistical quantum thermodynamics derivation of the photon gas system, while employing the symbols of the economic system as described above (Planck 1900). As we have already discussed, the economic system is such that there are no restrictions as to how many actors N_i can be in any wealth element G_i, i.e., none, one, two, and so on up to all of them. Each wealth element G_i is considered as a cell into which anywhere from 0 to N actors can fit. Moreover, G can be smaller than, equal to, or larger than N. The distribution of G_i wealth elements over N_i actors can only take place in a finite, well-defined number of ways. To determine the total number of these different ways, we can allocate the N actors into the G wealth elements, and we consider all the possible sequences of numbers for any given N and G. We also note that two sequences are different if they contain the same numbers but in a different order (Planck 1900).

ECONOMIC SYSTEM HIERARCHICAL STOCHASTIC		**SCALE-FREE COMPLEX NETWORK**		**PLANCK STATISTICAL MECHANICS SYSTEM**
Actors	→	**Nodes**	→	**Resonators/Energy Levels**
Wealth Elements /States	→	**Links**	→	**Quantum Particles/Photon Gas**

Figure 3. Schematic of the mapping of the economic system into a scale-free complex network and its modeling as a Planck photon gas physical system.

As an example, if N = 10 and G = 100, then we may have as one possibility the following outcome of wealth elements (links) allocation among the actors (nodes) shown in Table 1 (Planck 1900). It should be remembered that in the network configuration of the economic system, the actors are represented by nodes and the number of wealth elements by the number of links connected to each node. In the example of Table 1, Actor 7 (node, energy level) has 20 wealth elements (links, photons) and Actor 4 (node, energy level) has 0 wealth elements (links, photons)[7].

Table 1. Illustrative example of a wealth distribution allocation of wealth elements G_i in an actor ensemble N_i for a number of actors (N = 10) and wealth elements (G = 100).

Actor N_i	1	2	3	4	5	6	7	8	9	10
Wealth Element G_i	7	28	11	0	9	2	20	4	4	5

2.2. Methodology Findings

Based on combinatorial analysis, the total number of different ways (permutations) we can allocate the sequence of N actors (energy levels—nodes) into G wealth elements (photons—links) for the economic system under consideration as a scale-free complex network, in analogy to the statistical thermodynamics system, is given below (Planck 1900)[8]:

$$\Omega_{N,G} = \Pi\ (N_i + G_i - 1)!/(N_i - 1)!\ G_i! \text{ for all } i = 1, 2, \ldots, N \quad (1)$$

The probability $P_{N,G}$ of finding the economic system in the condition specified by the N and G would be proportional to $\Omega_{N,G}$:

$$P_{N,G} = K\ \Omega_{N,G} \qquad \text{where K is a constant} \quad (2)$$

The entropy $S_{N,G}$ of the system of N actors with G wealth elements would be equal to the natural logarithm of the probability $P_{N,G}$ in accordance with Boltzmann's elucidation of the connection of the second law of thermodynamics of the macroscopic property of entropy and the microscopic state of the system (Planck 1900; Schrödinger 1989; Kittel 2004; Tolman 1979):

$$S_{N,G} = -\ln(P_{N,G}) \quad (3)$$

First, we note that N and G are much greater than 1. Second, we can rewrite factorials by employing the Sterling approximation: $x! = x^x/e^x$. Thus, we obtain the following (Planck 1900):

$$\Omega_{N,G} = -\Sigma \ln((N_i + G_i)^{Ni+Gi}/N_i^{Ni}\ G_i^{Gi}\ ) \quad (4)$$

We obtain for the entropy of the economic system under consideration

$$S_{N,G} = -\Sigma\ (((N_i + G_i) \ln(N_i + G_i)) - N_i \ln N_i - G_i \ln G_i) + \ln K \text{ for all } i = 1, 2, \ldots, N \quad (5)$$

Next, we determine the most probable condition of the economic system under a fixed number of actors N and fixed amount of total wealth U (Schrödinger 1989; Tolman 1979; Kafri and Kafri 2013). The most probable condition is obtained when $S_{N,G}$ becomes a maximum under the aforementioned two constraints (fixed number of actors—nodes/energy states; and total wealth—number of links/photons). We also write the total wealth U and total number of actors N as follows:

$$U = h\ \Sigma\ N_i\ (U_i/h) \text{ and } N = \Sigma\ N_i \qquad \text{for } i = 1, 2, \ldots, N \quad (6)$$

where U_i/h is a stochastic variable, say, x that describes wealth and represents the continuous version of the discrete variable u_i mentioned earlier.

In order to find the maximum of the entropy $S_{N,G}$ under the two constraints of fixed N and fixed U, we employ the method of Lagrange undetermined multipliers (Planck 1900; Tolman 1979; Kafri and Kafri 2013). We maximize the function

$$f(N_i) = S_{N,G} + \gamma\, N + \beta\, U = \tag{7}$$

$$-\Sigma\, (((N_i + G_i)\, \ln(P_i + G_i)) - N_i\, \ln N_i - G_i\, \ln G_i) + \ln k + \gamma\, \Sigma\, N_i + \beta h\, \Sigma\, N_i\, (U_i/h)) \tag{8}$$

by setting $\delta f(N_i) = 0$, where γ and β are the two undetermined multipliers reflecting the two constraints of fixed N and fixed U in the economic system. Thus, we obtain the following:

$$\Sigma\, ((-\ln(N_i + G_i) + \ln(N_i)) + \gamma + \beta\, h\, (U_i/h))\, \delta N_i = 0 \text{ for } i = 1, 2, 3, \ldots \tag{9}$$

Since the variations δN_i can be arbitrary, this equation can be satisfied if all its coefficients are zero for any i. Thus, we have the following equation for the distribution of N_i:

$$N_i = G_i/(\exp\, (\gamma + \beta\, h\, (U_i/h)) - 1) \text{ for } i = 1, 2, \ldots, N \tag{10}$$

where N_i represents, as already indicated, a discrete function expressing the effect of varying the assignment or distribution of actors N_i (nodes or energy levels) to the different groups of possible G_i wealth elements or levels (links or photons).

We will next determine the functional dependency of G_i on the stochastic variable U_i. As we have already indicated U and N are very large such that the possible wealth increments $U_{(i+1)} - U_i = \Delta U_i$ are relatively small. We thus regard the total wealth U as divided into a succession of small wealth ranges $U_i + \Delta U_i$ with a corresponding number of wealth states G_i in the ΔU_i wealth range. In most cases, the wealth range will be near-enough continuous so that its description can be regarded as given by an expression of the following form (Tolman 1979):

$$G_i = f(U_i)\, \Delta U_{i/}h \text{ for } i = 1, 2, \ldots, N \tag{11}$$

where $f(U_i)$ is a continuous function of the wealth Ui introduced to describe the distribution density of the wealth elements. Moreover, the distribution density f(U) is known empirically to be relatively constant when $U \to \varepsilon\, U$, such that $f(\varepsilon U) = f(U)\, q(\varepsilon)$ (Feynman 1998). This equation has the unique solution: $f(U) = A\, U^n$, where A is a constant and n is a number (Feynman 1998). Moreover, the distribution density f(U) must approach zero as U becomes large. Thus, the number n must be negative. Since all the actors N_i in the system have equal access to the available wealth U_i independently of the value U_i, then the functional form of $f(U_i)$ would be proportional to a power of the wealth $U_{i:}$ as follows:

$$f(U_i) = k\, (Ui/h)^{-\delta} \text{ for } i = 1, 2, \ldots, N \text{ and k a dimensionless constant, } \delta > 0 \tag{12}$$

where δ is a parameter greater than zero, and h is the already introduced constant of wealth, i.e., minimum value or quantum of wealth[9] (Planck 1900; Tolman 1979; Feynman 1998). The h parameter, i.e., the minimum or quantum of wealth (income) amount as already introduced, is employed to ensure that f(U) and therefore G is a number, while U has the units of wealth. Moreover, the negative sign in the exponent δ is introduced to ensure that as U_i gets larger the number of states G_i diminishes, i.e., fewer states are available for a larger amount of wealth. This latter outcome is to be expected naturally as being descriptive of a scale-free system and is also confirmed from observation, starting with Pareto as already discussed (Pareto 1925). Thus, the functional dependency of G_i on U_i is written as follows:

$$G_i = k\, [(1/h)\, (U_i)]^{-\delta}\, (\Delta U_{i/}h) \text{ for } i = 1, 2, \ldots, N \tag{13}$$

Since the wealth increments ΔUi are very small compared to U itself, we can express Equation (10) in conjunction with Equation (13) in a continuous differential form as follows (Tolman 1979):

$$dN(x) = k\, x^{-\delta}\, dx/(\exp(\gamma + \beta\, h\, x) - 1)] \text{ for } x \geq 1 \tag{14}$$

where the discrete variable $(U_i/h))$ has been replaced with the continuous stochastic variable x. We would note that in Equation (14) we have the following: (a) the stochastic variable x is dimensionless; (b) the minimum value of x is 1 referring to any of the actors with the minimum wealth h; and (c) dN represents the number of actors with a wealth x in the range from x to x + dx in the economic system under consideration being in a state of equilibrium (maximum entropy).

The three constants γ, β, and δ that appear in Equation (14) are to be regarded as parameters, whose values depend on the characteristics of the economic system being in an equilibrium condition. Moreover, we note that the parameters γ and δ represent numbers, i.e., have no physical unit, while the parameter β has the unit of inverse wealth, i.e., the inverse of the unit (metric) of h. It is important for the purposes of this work to be able to calculate the values of these parameters. Consequently, we will develop certain relationships and derive their optimal numerical values later in this work. We may characterize the three parameters γ, β, and δ at this time as follows based on their significance in the analytical model. First, we note from Equation (8) that if the number of actors N in the system is not fixed, i.e., actors are allowed to enter or exit the system, then the parameter γ vanishes, i.e., becomes zero, because the fixed actor constraint in the Lagrange multiplier maximization of the entropy $S_{N,G}$ in Equation (8) disappears (Tolman 1979). That is to say, the parameter γ is descriptive of the "variability" of the actor number N of the economic system (nodes or occupied energy levels). We may note that in Planck's black body radiation distribution, the number of photons is not fixed, hence the value of the parameter γ is equal to zero (Planck 1900). However, if the number of particles is fixed, we obtain the Bose-Einstein distribution. In both types of systems, the total energy available is always fixed. In either case, any number of photons or particles, i.e., links or wealth can occupy an energy level, i.e., node or actor. Likewise in our economic model, any number of actors can have the same amount of wealth or income, i.e., be in the same wealth or income state. Second, as we will see shortly the ratio of the total wealth U to the number of actors N in the system is equal to $1/\beta$. Consequently, the parameter $1/\beta$ is associated with "the equilibrium or average wealth" of the actors in the economic system, given that in the physical system of photons this factor is proportional to the temperature of the physical system, which in turn is indicative of the average energy of the constituent photons[10] (Planck 1900; Bianconi and Barabási 2001). Thus, the higher the value of $1/\beta$, the more wealth elements (links) G are available in the economic system. Lastly, we note from Equation (13) that the parameter δ describes the density distribution of wealth elements G for a total wealth level U (Tolman 1979; Feynman 1998). We may note that while in a physical system the parameter δ is associated with the "dimensionality" of space, in an economic system the parameter δ is associated with of the degree of complexity of the hierarchical structure of the stochastic process ascribing wealth (links) out of a given total amount to the actors (nodes) in the system[11].

We note that if we integrate Equation (14) from x to infinity, we obtain the number of actors N(x) with wealth equal to or greater than x in the system. Moreover, if we integrate Equation (14) for x from one to infinity, we obtain the total number of actors N in the economic system. We thus have the following:

$$N(x) = \int_x^\infty dN \ \text{ for } x \geq 1 \tag{15}$$

$$N = \int_1^\infty dN \tag{16}$$

The calculation of the integrals in Equations (15) and (16) cannot be obtained in a closed form in most cases. Instead, an infinite series can be obtained (Tolman 1979; Zwillinger 2012). The ratio N(x)/N then gives the proportion of actors in the economic system with wealth equal to or higher than x. We also note that by analogy to the previous two equations we have the following:

$$\mathrm{U(x)} = h\int_x^{\infty} x\,dN \text{ for } \mathrm{x} \geq 1 \tag{17}$$

$$\mathrm{U} = h\int_1^{\infty} x\,dN \tag{18}$$

where U(x) is the wealth of the N(x) actors in the system and U is the total wealth of the system.

As is shown in Appendix A, Equations (16) and (18) can be written as an infinite sum of exponential integrals, also known as Schlömilch functions, as follows:

$$\mathrm{N} = \mathrm{k}\sum_1^{\infty} \exp\,(-\mathrm{n}\gamma)\mathrm{E}\delta(\mathrm{n}\beta\mathrm{h}) \text{ for } \delta = 1,2,3,4 \text{ and } \mathrm{n} = 1,2,3,4\ldots \tag{19}$$

$$\mathrm{U} = \mathrm{kh}\sum_1^{\infty} \exp(-\mathrm{n}\gamma)\mathrm{E}\delta - 1(\mathrm{n}\beta\mathrm{h}) \text{ for } \delta = 1,2,3,4 \text{ and } \mathrm{n} = 1,2,3,4\ldots \tag{20}$$

where the exponential integral or Schlömilch function $E_\delta(n\beta h)$ is defined as follows:

$$\mathrm{E}\delta(\mathrm{n}\beta\mathrm{h}) = \int_1^{\infty} (\exp\,(-\mathrm{n}\beta\mathrm{hx})/\mathrm{x}\delta)\mathrm{dx} \text{ for } (\beta\mathrm{h}) > 0, \delta = 1,2,3,\ldots,\mathrm{n} = 1,2,3,\ldots \tag{21}$$

We note from the definition of the exponential integral function $E_\delta(n\beta h)$ per Equations (21) and (20) that the parameter δ is required to have a minimum value of one[12]. Thus, we obtain

$$\delta \geq 1 \tag{22}$$

We note that per Equation (12), the parameter δ must be a positive number, while Equations (19) and (20) indicate that δ is an integer with a minimum value of δ = 1 per Equation (22).

We also note that for values of the argument (γ + β h x) greater than about 2.5, the denominator in Equation (14) can be approximated by exp((γ + β h x), i.e., eliminate the (−1) term, with an error of about 9% (reduction in the value of dN/dx at the lower end of the wealth value x). This approximation results in retaining only the first terms in the series for N and U per Equations (19) and (20) as discussed in Appendix A. This result is also confirmed by the numerical values of the exponential function $E_\delta(n\beta h)$ for $\delta < 3$ and $n\beta h < 3$ (Zwillinger 2012; NTIS 2022). Two other important relationships of the exponential integral function that will be useful in this work are as follows (NTIS 2022):

$$(\delta - 1)\, E_\delta(x) + x\, E_{\delta-1}(x) = \exp(-x) \tag{23a}$$

$$d/dx\, \{E_\delta\, (x)\} = -E_{\delta-1}\, (x) \tag{23b}$$

We note that the exponential integral function $E_\delta(n\beta h)$ decreases monotonically with an increasing value of δ as well as increasing values of the argument (nβh) as is also discussed in Appendix B (Zwillinger 2012; NTIS 2022). Thus, the ratio U/N is proportional to the ratio of two exponential integrals or Schlömilch functions as follows: $U/N \propto E_{\delta-1}(\beta h)/E_\delta(\beta h)$. The maximum value of this ratio would signify the optimal value of the wealth distribution U among the N actors and would occur for[13]

$$\delta = 2 \text{ optimal value of this parameter in the economic system} \tag{24}$$

We proceed next to calculate the values of βh and γ considering a typical range for the latter parameter and with δ = 2, the optimal value. From Equations (19) and (20), we obtain the following:

$$N = k \exp(-\gamma)\, E_2(\beta h)\, [1 + \exp(-\gamma)\, E_2(2\beta h)/E_2(\beta h) + \ldots] \tag{25}$$

$$U = h\, k \exp(-\gamma)\, E_1(\beta h)\, [1 + \exp(-\gamma)\, E_1(2\beta h)/E_1(\beta h) + \ldots] \tag{26}$$

For simplicity, we designate the quantities in Equations (25) and (26) described by the sum of infinite series as follows:

$$S_2\, (\gamma, (n\beta h)) = [1 + \exp(-\gamma)\, E_2(2\beta h)/E_2(\beta h) + \ldots] \tag{27}$$

$$S_1\, (\gamma, (n\beta h)) = [1 + \exp(-\gamma)\, E_1(2\beta h)/E_1(\beta h) + \ldots] \tag{28}$$

The numerical values of the $E_1(\beta h)$ and $E_2(\beta h)$ have been summarized in Table 2 for a range of values for (βh) obtained from published data (NTIS 2022). It is obvious that the value of γ would affect the value of (βh) and vice versa. As already indicated, the value of the parameter γ is expected to approach zero as the number of actors N in the system fluctuates, i.e., is not constant. Moreover, we note that as the value of γ approaches "two," the ratio of the two series described by Equations (27) and (28) converges rapidly to one (see also Appendix A). Consequently, we obtain

$$U/N = 1/\beta = h\, (E_1(\beta h)/E_2(\beta h))\, (S_1\, (\gamma, (n\beta h))/S_2\, (\gamma, (n\beta h)) \tag{29}$$

Thus, to determine the range of the value of (βh) we need to solve the following two equations applicable to the range of the value of the parameter γ (from zero to infinity):

$$\beta\, h = E_2(\beta h)/E_1(\beta h) \text{ for } \gamma \geq 2 \tag{30a}$$

$$\beta\, h = (E_1(\beta h)/E_2(\beta h))\, (S_1\, (\beta h))/S_2\, ((\beta h)) \text{ for } \gamma = 0 \tag{30b}$$

Equations (30a) and (30b) suggest that (β h) < 1 since the value of the exponential function for a given parameter x (=β h) decreases for an increasing value of the index δ (Zwillinger 2012). Employing the data in Table 2 (see also Appendix A), we have numerically calculated the following optimal range for the value of the parameter (βh) depending on whether the number of actors N in the economic system is fixed or variable (actors enter and leave the system):

$$(\beta\, h) = 0.609 \quad \text{for a fixed number of actors} \ldots\ldots \tag{31a}$$

$$(\beta\, h) = 0.664 \quad \text{for a variable number of actors} \ldots\ldots \tag{31b}$$

Table 2. Numerical values of exponential integral functions of E_1 (βh) and E_2(βh) for various values of (βh).

Bh	E_1 (βh)	E_2 (βh)	E_2 (βh)/E_1 (βh)
0.2	1.22265	0.57420	0.46964
0.4	0.70238	0.38932	0.55436
0.6	0.45438	0.27618	0.60782
0.8	0.31060	0.20084	0.64665
1.0	0.21938	0.14850	0.67691
2.0	0.04890	0.03775	0.77198
3.0	0.01305	0.01064	0.81545
4.0	0.00378	0.00320	0.84656
5.0	0.00015	0.00060	4.00000

We are to interpret a large value of γ corresponding to a relatively smaller number of fixed actors N in the system, while a decreasing number of γ signifies an increasing number of actors up to the value of γ equal to "zero" signifying an infinitely large, i.e., essentially variable, or unknown, number (Tolman 1979). In the remainder of this analysis, we will consider the least restrictive option represented by a variable number of actors, thereby setting $\gamma = 0$ (Planck 1900).

We note that the ratio N(x)/N as defined by Equations (15) and (16) is equal to the so-called survival distribution function (SDF) of the wealth of the actors in the economic system, i.e., the probability of actors having wealth equal to or higher than x. The survival probability function is defined as the complimentary of the cumulative distribution function (CDF) by

$$\text{SDF} = 1 - \text{CDF} = 1 - \Phi(x) \tag{32}$$

where $\Phi(x)$ is the cumulative distribution function (CDF) of the wealth in the economic system. We also know that the probability density function (PDF) designated here by $\phi(x)$ is defined by the differential of the cumulative distribution function:

$$\phi(x) = d[\Phi(x)\}\ dx \tag{33}$$

Thus, we obtain from Equations (14)–(16) and (20) the following analytical expression of the normalized probability density function of the distribution of the number of actors with a certain wealth x in the economic system under consideration:

$$\phi(x) = (k/N)\ x^{-\delta}/(\exp\ (\beta\ h\ x) - 1)) =$$

$$= (1/\sum_{1}^{\infty} E_2\ (n\beta h))(x - \delta/(\exp(\beta hx) - 1))\ldots x \geq 1, n = 1,2,3\ldots\ldots \tag{34}$$

This equation is the exact economics analogue of the famous Planck black body radiation law in thermodynamics (Planck 1900; Tolman 1979). The probability density function f(x) is a monotonically decreasing function with increasing x. It has a maximum value at x = 1 and tends to zero as x increases to infinity. The shape of the probability density function f(x) would depend on the numerical values of the parameters δ and (βh). The best or optimal values of these parameters are given by Equations (24) and (31).

The cumulative distribution function $\Phi(x)$ is obtained by integrating Equation (34) from 1 to x as follows:

$$\Phi(x) = \int_1^x dx\ \phi(x) =$$

$$= \int_1^x dx(1\sum_{1}^{\infty} E_2(n\beta h))(x - \delta/(\exp(\beta hx) - 1)) \text{ for } 1 \leq x \leq \infty \tag{35}$$

and a series with infinite terms would result as the integral of $(x^{-\delta}/(\exp\ (\beta\ h\ x) - 1))$ cannot be obtained in a closed form. We can, however, simplify and obtain an upper bound for the functions $\phi(x)$ and $\Phi(x)$ and at the same time demonstrate that both Equations (34) and (35) are equivalent to a power law in that limit. To this end, we expand the exponential function in the denominator of Equation (34) in a power series as follows (Zwillinger 2012):

$$(\exp\ (\beta\ h\ x) - 1) = ((1 + (\beta\ h\ x)/1! + (\beta\ h\ x)^2/2! + (\beta\ h\ x)^3/3! + \ldots) - 1) =$$

$$= (\beta\ h\ x)\ (1 + (\beta\ h\ x)/2! + (\beta\ h\ x)^2/3! + \ldots\ldots) \tag{36}$$

We substitute Equation (36) into Equation (34) and we obtain for $\phi(x)$ the following expression:

$$\phi(x) = (1/\sum_{1}^{\infty} E_2(n\beta h))(x - \delta - 1/((\beta h)((1 + (\beta hx)/2! + (\beta hx)2/3! + \ldots))), x \geq 1 \tag{37}$$

We note that the infinite series in the denominator of Equation (36) has a minimum value for x = 1. We thus have the following:

$$(1 + (\beta\ h)/2! + (\beta\ h)^2/3! + \ldots) \leq (1 + (\beta\ h\ x)/2! + (\beta\ h\ x)^2/3! + \ldots) \text{ for } x \geq 1 \tag{38}$$

Moreover, we can write the infinite sum on the left-hand side of Equation (37) in a closed form:

$$(1 + (\beta\ h)/2! + (\beta\ h)^2/3! + \ldots) = (\exp(\beta\ h) - 1)/(\beta\ h) \tag{39}$$

Thus, we obtain from Equations (36)–(39) the upper bound of the probability density function $\phi(x)$ as well as its maximum value $\phi(x)_{max}$ that occurs at x = 1 as follows:

$$\phi(x) \leq (1/\sum_{1}^{\infty} E_2(n\beta h))(x - \delta - 1/((\exp(\beta h) - 1) \text{for} x \geq 1 \tag{40a}$$

$$\phi(x = 1)\max = (\frac{1}{\sum_1^\infty E_2}(n\beta h))((\exp(\beta h) - 1) \tag{40b}$$

The upper bound of the cumulative distribution function $\Phi(x)$ is then obtained by integrating Equation (40a) from x = 1 to x. Thus, we obtain

$$\Phi(x) = \int_1^x dx\ \phi(x) \leq (1/\sum_{1}^{\infty} E_2(n\beta h))((\exp(\beta h) - 1)\delta)(1 - x - \delta) \text{ for } 1 \leq x \leq \infty \tag{41}$$

We would note that the function $\Phi(x)$ is normalized by definition. This implies that the quantity $(\sum_1^\infty E_2(n\beta h))\ ((\exp(\beta\ h) - 1)\ \delta)$ must be equal to one. Consequently, we obtain

$$1/(\sum_{1}^{\infty} E_2(n\beta h))(\exp(\beta h) - 1) = \delta \tag{42}$$

However, the approximate numerical value of the left-hand side quantity in Equation (42) is $(1/(0.43718 \times 0.942547)) = 2.4268$, while the right-hand side value is 2.000. This result is to be expected and is consistent with one or another or both small approximations employed in the preceding calculations per Equations (40a) and (41)[14]. We thus obtain the following results:

$$\phi(x) \cong (1/\sum_{1}^{\infty} E_2(n\beta h))(x - \delta - 1/((\exp(\beta h) - 1) = \delta/x\delta + 1 \text{ for } x \geq 1 \tag{43a}$$

$$\phi(x = 1)\max = (1/\sum_{1}^{\infty} E_2(n\beta h))((\exp(\beta h) - 1) = \delta \tag{43b}$$

$$\Phi(x) \cong (1/\sum_{1}^{\infty} E_2\ (n\beta h))((\exp(\beta h) - 1)\delta)(1 - x - \delta) = 1 - x - \delta \text{ for } 1 \leq x \leq \infty \tag{43c}$$

Equations (34), (40a) and (43a) for the probability density function $\phi(x)$ and Equations (35), (41) and (43c) for the cumulative distribution function $\Phi(x)$ of the actors, each having a certain wealth, in an economic system indicate that we have arrived at a most important result as follows: **In an economic system modeled as a scale-free complex network comprised of a varying number of actors with a given total amount of wealth and with each actor having equal access to this wealth, then the distribution of this wealth among these actors in the state of dynamic equilibrium (maximum entropy) follows a Planck-type law that has as an upper bound the empirically valid Pareto power law in such a system. Moreover, the parameters in the Pareto power law have a certain meaning within the economic system with significance as summarized in Table 3.**

Table 3. Interpretation of the parameters of the empirical pareto law based on a complex network of actors (nodes or energy states) having equal access but varying fitness to acquire certain wealth (number of links or physical particles) in a state of maximum entropy.

Empirical Pareto Law	Scale-Free Network System
Xo = 1 minimum value of variable x	**Xo = 1** normalized value of the variable x to the quantum of wealth (income) **h** that an actor must have to be part of the system
$\boldsymbol{\alpha > 1}$ Pareto index	$\boldsymbol{\delta \geq 1}$ signifies the degree of allocation of links (wealth) to a node (actor) in the economic system governed by a stochastic process; at $\delta = 1$ one node has all the links, and at $\delta = \infty$ every node has only one link. $\boldsymbol{\delta = 2}$ optimal value of the parameter.
$\boldsymbol{\alpha\, Xo/(\alpha - 1)}$ expected value	$\boldsymbol{1/\beta}$ denotes the average allocation of wealth (income) per actor in the economic system with $(\beta h) < 1$ optimally; $\boldsymbol{1/\beta}$ is analogous to the temperature in a physical system and thus is a measure of entropy.
	$\boldsymbol{\gamma \geq 0}$ parameter varying from zero to infinite associated with the number of actors in the economic system $\boldsymbol{\gamma = 0}$ number of actors large and variable (implicit assumption in Pareto's Law)

3. Discussion of the Results

A few observations are in order based on the modeling of the economic system in a market economy as a stochastic system comprised of wealth (or income) levels and actors treated as random variables.

First, the treatment and modeling of an economic system as a complex network that is consistent with quantum physics allows naturally for the occurrence of a varying number of actors, i.e., nodes or energy levels, and the ability for the total wealth or income, i.e., number of total links or quantum particles to increase or decrease. The quantum system of an ensemble of particles such as photons has potentially an infinite number of levels of energy states. But such states are only relevant so long as they are occupied by a particle (Planck 1900; Kittel 2004; Tolman 1979). Thus, states, i.e., actors, appear or disappear to the extent they occupy an energy level. Moreover, the parameters of a quantum system such as its total energy can fluctuate unlike a classical physical system (Kittel 2004; Tolman 1979; Prigogine 1980; Prigogine and Stengers 1984). These fluctuations then are allowed in the economic system described by a complex network/quantum physical system analogue leading to the change among others in the total wealth (income). Thus, the economic system strives to a state of maximum entropy, but continuously occurring wealth (income) fluctuations can move the system into a different state of equilibrium and of maximum entropy. We may also mention that in analogy to the quantum system of a gas of photons having a ground state with infinite energy, known as *Null Punkt Energie* or Zero-Point Energy (Planck 1900; Kittel 2004; Tolman 1979), then wealth (income) can enter or dissipate into the "ground" state of the economic system having in essence infinite wealth (income) potential. This feature of the economic system can then account for the observed changes in its total wealth (income) as well, i.e., creation or destruction of it.

Second, each actor in the economic system as a complex network has equal probability to wealth or income and has the freedom to act independently of all the other actors in the system. Under these conditions the economic system tends naturally to a state of equilibrium, i.e., to the most likely state of existence. This state is represented by a Planck-type law distribution having as an upper bound the distribution discovered empirically by Pareto over one hundred years ago by analyzing actual wealth and income data of various communities for which such information existed. Essentially, the derivation of the empirical Pareto's law via the Planck-type law distribution from first principles allows us to elucidate as to why a power law occurs in economics and shed light on the critical importance of equal opportunity as a key characteristic of that law.

Third, it is important to note that equilibrium does not mean a static system but rather a dynamic one, in the sense that the assignment of certain amount of wealth (income, etc.) is not to any particular actor but rather statistical in nature, since all actors are indistinguishable, i.e., cannot be differentiated, as far as the economic system is concerned, and thus signifies a statistical outcome within the economic system. This model of the economy or any model for that matter as already discussed cannot predict which actor will end with more wealth (income, etc.), but rather that certain percentage of people within the economic system will. Moreover, the modeling of the economic system in analogy to a quantum system in the physical world allows for small continuous fluctuations in the amount of total wealth (income) which are going to occur like those in the total amount of energy in a physical/thermodynamic system.

Fourth, the actors in the economic system, represented by a scale-free complex network in accordance with empirical evidence, by having equal access to wealth (income, etc.) available in the system, i.e., having equal opportunity, end up with a highly skewed distribution, whereby a few actors end up with a high portion of the wealth and the majority of actors with a lesser portion of it, i.e., obtaining unequal outcome. This outcome would seem contradictory to the expectation that given equal access, all else being equal, that the wealth (income, etc.) would be divided equally among the members of the community. However, upon reflection we note that there is only one state within the economic system where everybody would have the same amount of wealth (income, etc.), but there are a huge number of states with uneven distribution that would be favored statistically. Moreover, a system with the one state of equal distribution (outcome) must remain static for it must remain always in that state. While that state of wealth (income, etc.) distribution is possible, it cannot occur in a naturally dynamic system such as the economy[15]. This realization also clearly shows that it is essentially impossible to have an economic system where every member has the exact same amount of wealth (income). We may also note that the Planck-type distribution of wealth (income, etc.) of the economic system, which is obviously quite complex, suggests that the economic system is hierarchical in nature, i.e., fewer nodes (actors) have many more links (wealth or income). Such a system is far more stable than a non-hierarchical system, because of the constraints imposed by the higher levels of the hierarchy to the lower levels, i.e., more of the wealth (income, etc.) controlled by a small number of actors vs. less wealth controlled by many actors (Simon 1996; Oyama 2000). However, excessive control would also lead to stagnation as fewer actors (nodes) control more wealth (links) to the ultimate limit of one actor controlling all the wealth[16]. Such a state is also static and unstable.

Fifth, the Planck-type law distribution of wealth (income, etc.) within an economic system offers the most efficient allocation of that resource, i.e., wealth, income, etc., among the members comprising that system, because it is the one that maximizes the entropy of the system. Since entropy is classically related to negative (lack of) information (Shannon and Weaver 1949; Cover and Thomas 1991), it may be counterintuitive that less information would lead to a most optimal or efficient outcome. A simple thought experiment can in fact clear any confusion. In an economic system comprised of N actors and a total wealth U, an equal distribution of wealth would indicate that each actor would have U/N wealth. This state confers the maximum amount of information, i.e., the wealth of every actor is exactly known, and possess the minimum amount entropy, i.e., the economic system is perfectly ordered. The only way for the economic system to stay in that condition is to never change, i.e., remain static. But this is an impossibility in the real world, as a force akin to Adam Smith's invisible hand, would drive the actors to act whereby the distribution of wealth becomes unequal, the entropy increases to a maximum and the economic system reaches a stable, i.e., optimal, state. Thus, we conclude the following: (a) maximum entropy leads to a hierarchical economic system, which is stable; (b) information is related to constraint, i.e., the higher the constraint on any system, economic or otherwise, the more static the system becomes—static is equivalent to stagnant, i.e., everything is known about the system, i.e., all actors have an exactly known amount of wealth or income, and should not be confused with stable and dynamic that allows for a continual change; and (c) a highly ordered system such as the one where all actors have equal amount of wealth (income, etc.) is improbable, has the lowest entropy and is unstable because the actors comprising the system acting independently will increase the entropy through a fluctuation of the total wealth U that would increase over time[17]. Moreover, the economic system will not remain forever in a particular state of equilibrium because the number of actors will also change with time, i.e., actors leave and are added, resulting in a different wealth (income, etc.) distribution. The change in number of actors may lead to the creation of additional wealth (income, etc.) in the economic system. This is equivalent to the increase in energy in a physical system, which would then decrease the entropy of the economic system and bring it into another state of equilibrium (Von Bertalanffy 1969; Schrödinger 1944). Moreover, the quantum physics understanding of entropy suggests that fluctuations of system parameters such as wealth and income can be and are as important as average values of these parameters (Prigogine 1980; Prigogine and Stengers 1984). We will return to this point shortly.

Sixth, the probability density function described by the Planck-type law per Equation (34) provides the most accurate distribution of wealth (income, etc.) across the entire spectrum of wealth (income, etc.) vis-à-vis the power law derived empirically by Pareto over one hundred years ago to represent extreme wealth situations (Acemoglu 2015)[18]. However, we have also shown that the empirical Pareto-type power law per Equation (43a) can be obtained from the exact analytical Planck law from Equation (34) with a minimal simplifying assumption per Equation (37). The derivation of the Pareto law from first principles then allows us to understand its fundamental implications regarding the distribution of wealth and income in the economic system.

Seventh, the scale-free complex network model of the economic system makes no differentiation between wealth and income such that the preceding conclusions are equally applicable to income as they are to wealth. Moreover, the analytical model predicts an optimal value for the Pareto index that can guide policymakers to reduce and/or eliminate external biases on opportunity equality that would tend to exacerbate the outcome inequality.

We would also note parenthetically that the application of statistical physics models in economics and more specifically in the description of financial systems has been expanding in recent decades (Mantegna and Stanley 2000). The motivation has been the realization that economic systems are complex by nature and cannot be described accurately by employing a reductionist approach to their study. A similar realization occurred among physicists over a hundred years ago attempting to describe the behavior of physical systems comprising a very large number (Avogadro Number) of particles. This led to the development of statistical thermodynamics (Schrödinger 1989; Khinchin and Gamow 1949; Prigogine 2017). And while at that same time the empirical discovery of a power law (the Pareto law) in the description of wealth occurred, it is only in recent times that concepts and models borrowed from physics have been applied to economics to describe the relevant effects. The application of power laws in the management of organizations as a relevant concept has also emerged recently (Andriani and McKelvey 2009).

4. Theoretical Implications

An examination of the findings of this work vis-à-vis the current understanding of the Pareto-type distributions of wealth and or income will be carried out next. To this end, we consider and compare first both the exponential distribution, which is often compared to a power law, as well as the standard empirical Pareto power law[19]. An exponential probability density function of the stochastic/random variable x with rate λ has a PDF $f(x) = \lambda \exp(-\lambda x)$ and a CDF $F(x) = 1 - \exp(-\lambda x)$ for $x > 0$ as shown in Figure 4 (Exponential

Distribution 2022)[20]. The Pareto distribution of a random variable x with index α and a minimum value x_m has a PDF $f(x) = \alpha\ (x_m)^a/(x)^{a+1}$ and a CDF $F(x) = 1 - (x_m)^a/(x)^a$ for $x > x_m$ as shown in Figure 5. We may also note that the Pareto distribution can be derived from an exponential distribution through the transformation of the random variable y into x via $y = y_m \exp(x)$ for $x > 0$. This can be shown as follows: $F(y) = 1 - (y_m)^a/y^{\,a+1} = 1 - (y_m)^a/(y_m \exp(x))^a = 1 - (\exp - (\alpha\, x))$, the latter being the exponential function CDF of the random variable x with rate α.

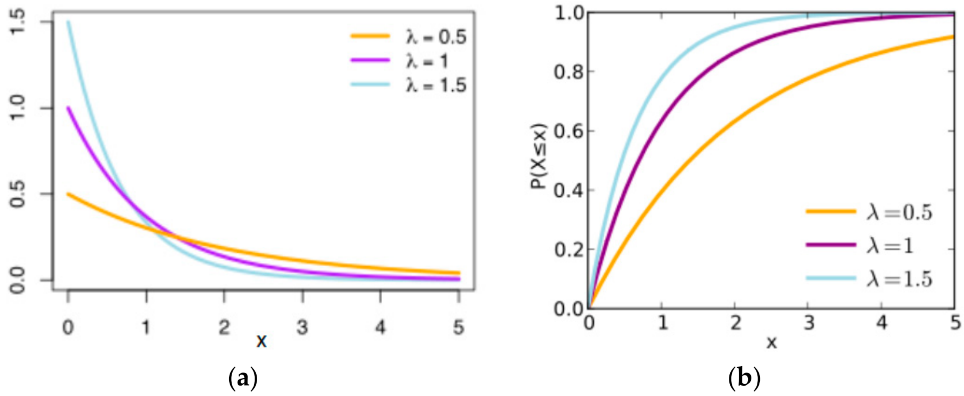


Figure 4. Exponential distribution of the random variable x with rate λ: (**a**) Probability density function (PDF); and (**b**) Cumulative density function (CDF).

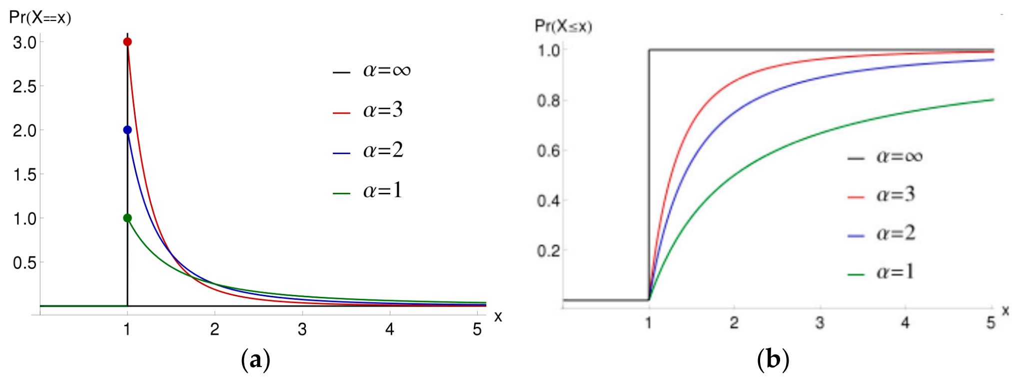


Figure 5. Pareto distribution of the random variable x with index α and a minimum value of the random variable of $x_m = 1$: (**a**) Probability density function (PDF); (**b**) Cumulative density function (CDF).

As has been already indicated, the Pareto distribution is found empirically in social sciences, including economics as already mentioned (Pareto Principle 2024; Pareto Distribution 2022)[21]. In this light, the Pareto distribution is most often presented in terms of its survival density function (SDF), which is defined as the complementary CDF, i.e., the function (1 − CDF). This would be a function of the form $(y/y_m)^{-\alpha}$, where the variable y varies from y_m to infinity. The value of the SDF is initially one and declines to 0 as y increases from the minimum value y_m to infinity. We may note that at first sight the Pareto distribution may seem to have much in common with the exponential distribution, which has a survival function equal to $1 - F(x) = F(x) = \exp(-\lambda\, x)$ (Exponential Distribution 2022). However, the SDF of the Pareto distribution declines much more slowly than that of the exponential distribution and has a so-called "fat tail" as the value of the random variable x increases (Pareto Distribution 2022). As such, the Pareto distribution represents wealth and income distributions well for most of the actors in the system as the value of the random variable increases. On the other hand, the Pareto distribution does not represent wealth or income distribution as well for small values of the random variable, i.e., a relatively larger number of actors with exceedingly low wealth or income. This is where the application of a Planck-type probability density function per Equation (34) could be advantageous to the standard Pareto-type power law.

Most importantly, however, by comparing the Planck-type PDF derived in this article per Equation (34) as well as the resulting power law per Equation (43) to the standard Pareto PDF and CDF, we can elucidate the significance of the empirical parameters appearing in the latter, thereby providing an insight as to their meaning as descriptors of an economic system. Thus, we gain the following insight into the meaning of the parameters appearing in the empirical Pareto power law, also summarized in Table 3.

(I) The minimum value of the calibrated stochastic variable (x/x_m) being equal to one $(x/x_m = 1)$ is indicative of the fact that in the economic system, no actors can exist with no wealth or income (see Figure 5).

(II) There is a parameter designated by the symbol x_m to calibrate the stochastic variable x such that (x/x_m) has no physical units. We conclude that $x_m = h$ and describes the absolute minimum amount of wealth an actor possesses. Thus, $x_m = h$ is identical to the so-called quantum of wealth (income, etc.). In other words, the "quantum" nature of any economic system is an integral feature and represents a finite quantity, i.e., a quantity that is not zero.

(III) The so-called Pareto index α is identified to be identical to the parameter δ in the Planck distribution, the former describing a generic scale-free system and the latter indicating the dimensionality of the economic space, which is essentially a measure of the complexity (number of eigenstates in a quantum system) of the economic system. Since power laws describe self-organizing systems, one concludes that what we call here the dimension of the economic system, in analogy to a physical system, is tantamount to the degree of the self-organization of the economic system and more specifically of the degree of the hierarchical structure of the economic system as a scale-free complex network.

(IV) The power law describes an economic system that has a variable number of actors in it, whereby actors enter and exit the system at will. Moreover, one concludes based on the Planck distribution that a higher average wealth (income, etc.) per actor described by the parameter (βh) is obtained in an economic system with varying number of actors vs. one with a fixed number. In other words, an economic system has by natural design a variable number of actors so that the average wealth per actor is maximized.

(V) While the empirical Pareto law allows for a variable index $\alpha > 1$ and cannot determine if the index has an optimal or desired value, the derived Planck distribution indicates that for an economic system, the Pareto index has an optimal value which is equal to 2 $(\alpha = \delta = 2)$. Optimal here is equated to the value that the parameter δ can attain in the analytical model such that the most efficient distribution of wealth and of income occurs characterized by the maximum entropy, i.e., stability, of the economic system. As already indicated, this realization can guide policymakers to devise policies that approach the optimal distribution of wealth and of income while ensuring equal opportunity to all.

We know that by definition the cumulative distribution function F(x) expresses the cumulative share of wealth (income, etc.) of the population having wealth (income, etc.) less than or equal to x. Since the CDF is a strictly increasing function, we can obtain the inverse function x(F) and divide it by the total wealth U to obtain a function of the cumulative share of the wealth $L(F) = x(F)/U$ as a function of the cumulative share of the population possessing it. This is called the Lorenz function as shown in Figure 6 (Lorenz Curve 2024). The Lorenz function was developed in 1905 by Max O. Lorenz to represent inequality in the wealth distribution.

Thus, the Lorenz function represents the portion L(F) of the cumulative wealth (income, etc.) of the population (y-axis) that is cumulatively earned by the corresponding portion F of the population (x-axis). The Lorenz function has as a parameter the Pareto index α and given that $\alpha = \delta$ it will also have as a parameter the Planck index (δ), even though the shape of the curves between the starting point (0, 0) and end point (1, 1) will be slightly different for the Pareto and Planck distributions.

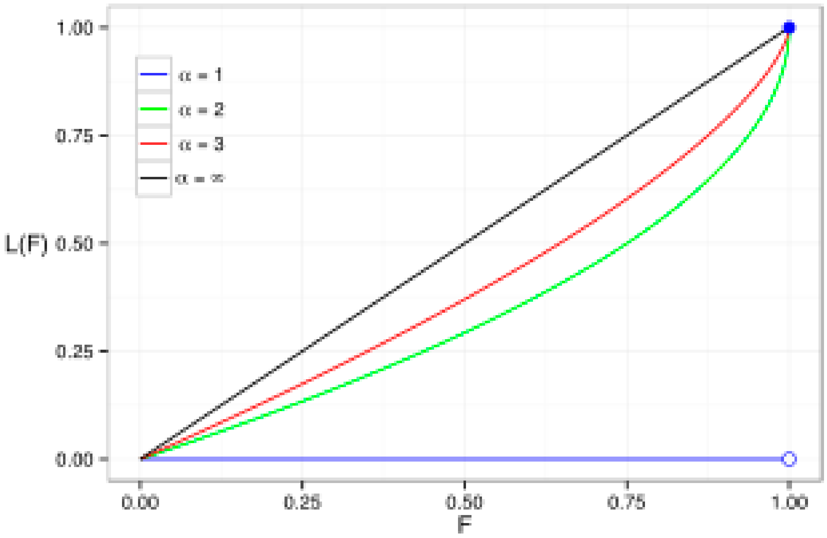


Figure 6. Lorenz function for several Pareto distributions with index α. The case α = ∞ (45-degree line) corresponds to a perfectly equal distribution (G = 0) and the case α = 1 (horizontal line) corresponds to complete inequality (G = 1). The case α = 2 corresponds to the optimal allocation of wealth and income based on the Planck distribution derived herein.

Associated with the Lorenz function is the Gini coefficient, G, a well-known parameter in economic theory proposed by Corrado Gini in 1912 to describe the inequality of wealth and income in an economic system and applied to nations as a whole or to social groups starting in the 1970s. The Gini coefficient is usually defined mathematically based on the Lorenz function and shows the proportion of the total income of the population (y-axis) that is cumulatively earned by the bottom x of the population as illustrated in Figure 6 (Gini Index 2019; Gini Coefficient 2024). The line at 45 degrees in Figure 6 thus represents a perfect equality of wealth or income and reflects a Pareto index approaching infinity. The Gini coefficient can then be thought of as the ratio of the area that lies between the line of equality and the Lorenz curve over the total area under the line of equality. The Gini coefficient is a number varying from 0 to 1 or typically 0 to 100%, the lower number indicting total equality, i.e., all actors in the economic system having equal wealth or income and 1 or 100% indicating total inequality, i.e., one actor holding all the wealth or the income in the economic system.

The value of the Gini coefficient is related to the Pareto index α or the parameter δ in our complex network model of the economy by the following equation:

$$G = 1/(2\alpha - 1) = 1/(2\delta - 1) \tag{44}$$

Thus, Equation (44) allows for the calculation of the Gini coefficient of wealth or of income in an aggregate economic system for which the empirical Pareto distribution for wealth or income has been ascertained via the collection and analysis of appropriate economic data. Consequently, the Gini coefficient provides a straightforward and easy-to-understand measure of the prevailing inequality of wealth and income in an economic system. Of course, a fundamental implicit assumption is that an economic system such as the economy of a nation should strive toward total equality of wealth and income, i.e., attain a Gini coefficient as close to zero as possible. As this work has shown, however, this is an impossibility in any economic system. We will discuss this issue in the next section, where we examine and compare actual data relating to inequality of wealth and income pertaining to several national economies.

In concluding this section, we may mention that there are proposals by several economists to improve the structure of the Gini coefficient to more effectively measure inequality of wealth and income. Amartya Sen, the 1998 Noble prizewinner in Economics, has proposed to define the Gini coefficient as half of the relative mean absolute difference, which is mathematically equivalent to the Lorenz curve definition (Sen 1997). The mean absolute difference is the average absolute difference of all pairs of the population, and the relative mean absolute difference is the mean absolute difference divided by the average in order to normalize for scale. Deaton Angus, the 2015 Nobel prizewinner in Economics, has proposed an alternative analytical expression for the Gini coefficient more suitable to assess inequality among individuals and households instead of national economies (Deaton 1997). Notwithstanding these and other proposals of redefining the Gini coefficient, its traditional definition is adequate for the discussion of the results of this work.

5. Status of Inequality in Wealth and in Income Globally

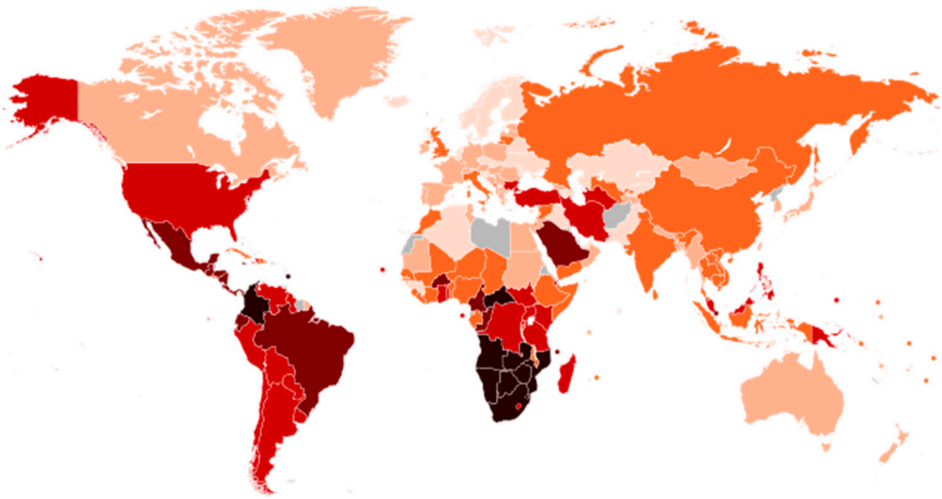
In this section, we will compare economic data from around the globe to ascertain the degree of inequality in wealth and income. The readily available calculated values of the Gini coefficient for several national economies will be considered as a good measure of wealth and income inequality. The results of this work for an economic system represented by a complex network with variable number of actors, show that the obtained Planck-type distribution having as an upper bound the Pareto distribution has an optimal value of $\alpha = \delta = 2$ for the Pareto–Planck index. Consequently, we can calculate the corresponding optimal Gini coefficient for the economic system. Under these conditions the optimal value of the corresponding Gini coefficient will be G ($\alpha = \delta = 2$) = 0.333. The Gini coefficient of 0.333 would then reflect the most equitable or optimal state of either wealth or income distribution possible among the members of any economic system according to this analysis. This conclusion is, of course, contrary to an expected Gini index of zero to denote absolute equality in the distribution of wealth and income. We may also add that the expectation of an ideal zero Gini index is not based on any vigorous economic theory, but rather in philosophical arguments of fairness (Deaton 2013).

It is instructive to compare this result regarding the optimal value of the Gini coefficient to publicized values of the Gini coefficient of various national economies. First, the global income Gini coefficient has been estimated by several researchers as ranging from 0.654 to 0.684, i.e., well above the "optimal" figure suggested by our model (UNDP 2021). Second, the global income of the world 20% (top quintile of the population) has been calculated at a high of 81.2% in prevailing currency exchange rates to a low of 69.5% in power purchase parity rates (Ortiz and Cummins 2011). Moreover, the evolution of the global Gini coefficient for income over the past two hundred years, as calculated by several researchers, is shown in Table 4 (Milanovic 2009; The World Bank Group 2016).

Table 4. The evolution of the income Gini coefficient (GF) globally in the last two hundred years.

Year	GF	Year	GF	Year	GF
1820	0.43	1950	0.64	1998	0.74
1850	0.53	1960	0.64	2002	0.71
1870	0.56	1980	0.66	2003	0.72
1913	0.61	1988	0.80	2008	0.70
1929	0.62	1993	0.76	2013	0.65

All these global income and wealth distribution figures are consistent with the 80-20 Pareto principle. Thus, if 20% of the world population controls 80% of all income, then the income Gini coefficient is about 76% ($\alpha = \delta = 1.16$)[22]. Along the same lines then, 1% of the world's population would own 50% of all wealth and the wealth Gini coefficient would be 76%. In fact, according to certain claims, 1% of the world's population owns just over 50% of the world's wealth (Treanor 2015). Lastly, we have reproduced in Figures 7 and 8 maps of the respective global income and global wealth Gini coefficients (World Bank Group 2022; Hechler-Fayd'herbe 2019). We can safely conclude that the current global inequality of income and wealth distribution is well beyond, i.e., far exceeds, what an optimal distribution of income and of wealth inequality would be as an unavoidable occurrence in an economic system, where all participants have had equal opportunity. And while the optimal value for the Gini coefficient for income and wealth should be at 0.333 according to this work, the actual Gini coefficient is well above 0.60 for income and well above 0.75 for wealth in most of the world.



Color Legend: Above 50 Between 45 and 50 Between 40 and 45 Between 35 and 40 Between 30 and 35 Below 30 No data

Figure 7. Map of the income Gini coefficient by country for the time period mostly between 2015 and 2022 based on data from the World Bank and other sources—Created by Alice Hunter (World Bank Group 2022).

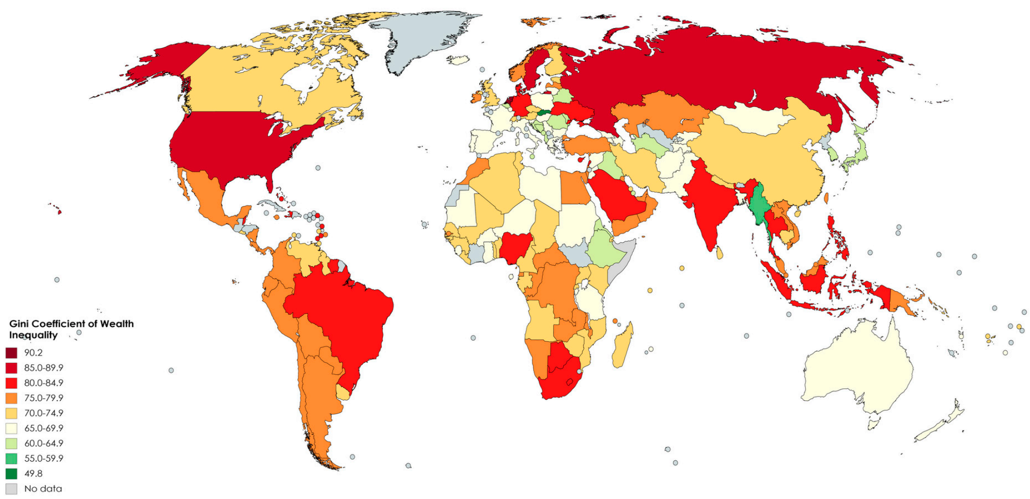


Figure 8. Map of the wealth Gini coefficient by country showing values ranging from a high of 0.850–0.899 (dark red) to a low of 0.550–0.599 (light green) based on data derived from the Wealth Handbook by Credit Suisse in 2019 (Hechler-Fayd'herbe 2019).

The Gini coefficients for wealth and income of the four largest economies in the world are considered next with their respective values summarized in Table 5 (UNDP 2021). It is instructive to note the large difference in inequality between wealth and income in all four of these economies as well. Perhaps this is due to the difference in the rate of return on capital vs. that on income as suggested by Thomas Piketty, who maintains that the long-term rate of the former is 4 to 5% per year vs. 1 to 2% per year for the latter (Piketty 2014)[23]. However, this assertion masks in our view the root cause of the much higher inequality in wealth vs. that of income that we will address in the next section.

Table 5. Wealth and income Gini coefficients as well as the income percentage of the top quintile of the population in the four largest economies in the world in 2005 calculated from economic data.

Country	Gini Coefficient		Top Quintile Percent
	Wealth	**Income**	**Income**
United States	0.801	0.464	50.1
Japan	0.547	0.319	35.7
Germany	0.671	0.311	38.5
China	0.550	0.449	47.8

The data in Table 5 suggest that the income distributions in Japan and Germany are very near the optimal levels, having a Pareto index of 2.067 and 2.107, respectively, while those in the USA and China are at a Pareto index of 1.578 and 1.614, respectively. Wealth inequality is, on the other hand, further removed from the optimal level standing at a low Pareto index of 1.124 in the USA to a high, but still suboptimal, index of 1.409 and 1.414 for Japan and China, respectively, and an intermediate index of 1.125 for Germany. It seems that governments of advanced and/or larger economies have a better ability to affect income inequality through taxation and social welfare measures. In Table 5, we also give the percent income for the top 20% of the population for each of the same four countries. We note that these income figures are better than the global average. Incidentally, the proverbial 80-20 Pareto principle for either wealth or income requires an index $\alpha = \delta = 1.16$ and would result in a Gini coefficient of about 0.76, which is off by more than a factor of two of the optimal value of 0.333. We may also conclude that an "80-20" distribution of wealth or income is far removed from the optimal one ($\alpha = \delta = 2$), which should result in the top quintile (20%) of the population having almost 45% of the wealth or 45% of income or almost half as much as the proverbial 80% amount for wealth (Newman 2006). But as we have already indicated, the global situation is even more skewed toward higher wealth inequality, if it is true as reported that 50% of the wealth is controlled by just 1% of the population (Treanor 2015).

We may also note that the income Gini coefficient is calculated on a market basis as well as on an after-tax and transfers basis[24]. Thus, for the United States, the country with the largest population among OECD countries, the pre-tax income Gini coefficient was 0.49, and the after-tax income Gini coefficient was 0.38, in 2008–2009. The OECD averages for the total population in OECD countries were 0.46 for the pre-tax income Gini coefficient and 0.31 for the after-tax income Gini coefficient (OECD 2012). Taxes and social spending in OECD countries significantly lower the effective income inequality. In general, European countries, especially Nordic and Continental welfare states, achieve lower levels of income inequality than other countries. Wealth inequality is, of course, more difficult to affect in just about any economy given that inheritance laws along with entrenched rules and institutions appear to play a crucial part in the observed unequal outcomes.

6. Policy Implications and Future Research Recommendations

The inequality in income distribution and even more so in wealth distribution observed around the globe has been addressed by several distinguished economists in the past several decades, including prominently among others Amartya Sen and Angus Deaton, both winners of the Nobel prize in Economics in 1998 and 2015, respectively (Sen 1997; Deaton 2013). The motivation of this work was also based on the quest to understand why inequality occurs and what is driving the occurrence of the empirical Pareto distribution in income and wealth. The conclusion of this quest has been two-fold: (a) Inequality in outcome, i.e., the distribution of wealth and income, is unavoidable even when equal opportunity is given to all members of the economic system; and (b) the prevailing global inequality in the distribution of income and of wealth far exceeds what should be expected in an economic system where everybody has equal opportunity. This realization then can be instrumental in developing policies that can effectively reduce the degree of inequality of outcome, although they can never eliminate it as we have shown in this work[25].

Traditionally, it has been assumed, unfortunately incorrectly, that inequality of outcome can be eliminated altogether via progressive policies such as taxation and transfer payments. Such policies attempt to affect the inequality outcome but fail to effectively address a root cause of the exacerbated inequality outcome, which is the apparent lack of equality of opportunity. In other words, taxation, the transfer of payments, and other social welfare approaches treat the symptoms but fail to address the cause of the problem. For example, in the OECD countries, as we discussed in the previous section, the pre-tax Gini coefficient across the entire population is 0.46 and the after-tax is reduced to 0.31. In both instances, the Gini coefficient is suboptimal from the ideal 0.333 figure. Inequality of opportunity increases the inequality of income outcome at one end and then extreme taxation and the transfer of payments penalizes success at the other end. It would be far more efficient for any government to ensure the equality of opportunity by investing resources to improve education, health, and housing among other options to all members in the economic system such that these members would have an equal opportunity to reach the optimal income outcome possible. As for wealth inequality, the oft-proposed heavy taxation to reduce it would appear to be quite ineffective, given that the Gini coefficient would have to be almost halved to obtain the optimal value of 0.333 from the typical values prevailing in most of the world today. Redistributing wealth from rich to poor will not solve the excessive state of wealth inequality, the poor will not become rich, and a certain number of people in the economic system may lose interest in creating the wealth that benefits the entire society (Deaton 2013). Thus, a thorough examination of the rules and institutions in a society that favor the accumulation of excessive wealth by certain members of the society to the detriment of the society as a whole needs to be carried out. Only then can successful policies be implemented to ensure equal opportunity to all members of the economic system.

Regarding future research, economists should further develop the scale-free complex network model as an alternative or a complement to current models of the economy. In this light, we would like to briefly discuss the statistical scale-free network economic model employed in this work vis-à-vis the standard formal model of the economy that is referred to as the Arrow–Debreu model (Arrow-Debreu 2023; Geanakoplos 2004; General Equilibrium Theory 2023; Kirman 1997). The latter model describes in an analytical fashion the tenets of the general equilibrium theory as enunciated originally by Léon Walras in the last quarter of the 19th century (Kirman 1997). The Arrow–Debreu model is comprised of a finite set of autonomous actors, characterized as either consumers or producers each taking decisions independently and interacting only through the market that establishes the price system. The actors in the Arrow–Debreu model of the economy take no account of the consequences of their own actions on the state of the aggregate system, nor do they anticipate the consequences of the actions of other actors in the system. The consumer actors maximize their utility, while the producer actors maximize their profit, thereby obtaining a market equilibrium state, which is characterized as being Pareto-efficient (General Equilibrium Theory 2023). Pareto-efficient or Pareto-optimal is a state where no action or allocation of resources is available that makes one actor better off without making another actor worse off. It is within this structure that the basic theorems of welfare economics, which form the basis of the recommendations of the free market, have been developed. However, it has been observed over a long period of time that once actors recognize that they have even minimal market power they can distort the price system such that the market no longer provides an efficient solution for the allocation of resources (Kirman 1997). The ability of certain actors to have an influence on economic outcomes is referred to as "imperfect competition". The scale-free complex network model of the economy, on the other hand, is based on the empirical Pareto or power-law distribution of wealth or income, does not differentiate among actors, all of whom are allowed to interact with one another, and describes the nature of the interactions among these actors as evidenced by the number of links connecting each actor to all other actors in the system. The increase or decrease in these links represents the transfer of wealth (or income) from one actor any other one. The interactions in the network model of the economy can occur in different ways among the actors of the system. The actors can learn over time from their previous experiences, whereby the interactions modify continuously the relation between actors as well as the relation of each actor to the entire system and the entire system to its constituent actors (Kirman 1997). An interesting observation is that the Walrasian description of the economy that is presented analytically in the Arrow–Debreu model is associated with a minimal amount of information needed to achieve an efficient outcome (Kirman 1997). However, direct interactions between actors involve the passage of a great deal of information. To attain a fully optimal situation, each actor in the model must consider the actions of all the other actors to reach equilibrium. This invokes the choice of optimal strategies by each actor given the strategies of all the other actors to attain the so-called Nash equilibrium obtained in full-blown non-cooperative game theory models. The complex scale-free network model of the economy circumvents these limitations of the Walrasian Arrow–Debreu model by describing the interactions of the actors within the economic system in a statistical fashion invoking the concepts of statistical thermodynamics (Barabási 2014; Kirman 1997). Perhaps, there will be a way in the future to incorporate scale-free complex networks into the Arrow–Debreu model.

Most important, however, is the realization motivated from the renaissance of networks in physics and mathematics that the power of complex networks can be instrumental in everything from company structure to the marketplace such that understanding network effects is becoming the key to survival in a rapidly evolving new economy.

7. Concluding Remarks

An analytical model for calculating the probability distribution of wealth (or income) of an economic system comprised of a very large number of actors N, all having equal access to the wealth, and a certain amount of wealth (or income) U available to these actors has been developed. This model employs a complex network comprised of nodes (actors) and links (wealth or income) and is analogous to a physical model developed over 100 years ago by Max Planck to calculate the black body radiation of an ensemble of photons in thermal equilibrium, the latter obtained by maximizing the entropy of the system, i.e., the natural logarithm of all the available states under the aforementioned constraints of the number of actors (energy levels/nodes) and wealth or income (number of particles/links). This model shows that the resulting distribution of the most efficient allocation of the total wealth or income U among the N actors has a probability density comprised of the product of a power function and an exponential function of the random variable $x = U/h$, where h is the quantum of wealth or income such as the unit of currency and the parameter (β h), which represents the average wealth (income, etc.) per actor in the economic system. This probability density function, which we call the Planck distribution, is quite analogous in form and features to its counterpart black body radiation law in physics. Two other parameters that enter naturally into the analytical model are designated by γ and δ and are associated with the variability of the population N and the hierarchical structure in the economic system, respectively. The parameter γ can vary from zero to infinity, with higher numerical values corresponding to smaller populations and a value of zero signifying an economic system where the number of actors N is no longer fixed. The parameter δ is the measure of the complexity of the economic system and has an optimal value of two. The empirical Pareto power law can be readily obtained from the thus-derived Planck distribution as its upper bound. This then leads to an understanding of the functional form of the Pareto power law and the "economics" meaning of its parameters. For example, the Pareto index is identified with the parameter δ of the degree of hierarchical complexity of the economic system. This realization would suggest an optimal Gini coefficient of 0.333 for wealth and income. A comparison of this optimal Gini coefficient value to actual data for several countries shows that the wealth Gini coefficient is typically twice as high as the optimal value. The corresponding actual income Gini coefficient, on the other hand, is typically higher than the optimal value and in other instances lower than the optimal value due to actions such as taxation and the transfer of payments attempting to reduce the inequality of outcome. However, such attempts try to deal with the symptoms of the problem rather than attempt to deal with the cause, which is the lack of equality of opportunity.

This work also answers conclusively the question posed in the introduction, namely, why wealth and income will always be distributed unequally among the members of an economic system even under the best conditions where each member has equal access to the total available wealth and income. While the analytical model does not favor wealth over income, the observed higher inequality of wealth over income is explained based on the observed return rate on wealth being historically higher than that on income. However, the true explanation, i.e., the reason why the observed return on wealth is higher than that on income, is likely due to the historical difference of long-term policies such as inheritance laws favoring the former over the latter. Subsequent work should focus on explaining based on this model the differences observed empirically between wealth and income. On the other hand, an income Gini coefficient less than the optimal value, due mainly to taxation and social spending policies to counteract the perceived inequality, does not represent the best utilization of applied monetary resources either. Thus, the key policy mechanism aimed at reducing income as well as wealth inequality should focus on attaining the optimal value within an economic system, i.e., the national economy. This is tantamount to implementing policies to ensure that all actors have equal opportunity in the economic system.

The most important conclusion of this work is that the naturally occurring inequality in the distribution of wealth and of income is inevitable and cannot be eliminated by any economic, social, or other policy. It is simply a natural consequence of how the natural world is set up and it must be recognized as such. Economic systems, however, should and can strive to develop policies that do not skew inequality above or below the optimal level, but rather lead to the optimal wealth or income inequality within those systems by ensuring truly equal opportunity and access to all the members of the economic system. Policies promoting education and health among the members of the society as well as re-evaluation of rules and institutions promoting wealth accumulation would ensure equal opportunity, thereby being much more effective to reduce the current inequality in income and in wealth vs. taxation and social welfare measures for the optimal distribution of income and wealth within the economic system! It is also suggested that the analytical modeling of an economic system as a scale-free complex network be given serious consideration by economists as an alternative or as a complimentary model to the current Arrow–Debreu model of the economy.

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Appendix A

Starting with Equation (14), we can rewrite it as follows:

$$dN(x)/dx = f(x) = k\, x^{-\delta}/(\exp(\gamma + \beta\, h\, x) - 1)] \text{ for } x \geq 1 \tag{A1}$$

In order to simplify the presentation, we set the following:

$$y = \exp(-(\gamma + \beta\, h\, x)) \tag{A2}$$

We can then write

$$f(x) = k\, x^{-\delta}\, y/(1 - y) = k\, x^{-\delta}\, y\, (1 + y + y^2 + y^3 + \ldots.) \tag{A3}$$

where we have expanded the term $1/(1 - y)$ into an infinite series, given that $y < 1$, because γ and (βh) are non-negative parameters and x is a variable equal or greater than one (Zwillinger 2012).

We thus can calculate the values of N and U per Equations (16) and (18) and the definition of $E_\delta(n\beta h)$ per Equation (20) as follows:

$$\begin{aligned} N &= \int_1^\infty dN = \int_1^\infty dx(\mathrm{hkx} - \delta(y + y2 + y3 + \ldots.)) = \\ &= k\{\exp(-\gamma)E\delta(\beta h) + \exp(-2\gamma)E\delta(2\beta h) + \exp(-3\gamma)E\delta(3\beta h) + \ldots.\} = \\ &= k\sum_1^\infty exp(-n\gamma)E\delta(n\beta h) \text{ with } n = 1, 2, 3, 4 \ldots \end{aligned} \tag{A4}$$

$$\begin{aligned} U &= h\int_1^\infty x\, dN = \int_1^\infty dx(\mathrm{hkx} - \delta + 1(y + y2 + y3 + \ldots.)) = \\ &= hk\sum_1^\infty \exp(-n\gamma)E\delta - 1(n\beta h) \text{ with } n = 1, 2, 3, 4 \ldots \end{aligned} \tag{A5}$$

We also know by definition that $U/N = 1/\beta$ and thus we have the following from (A4) and (A5) with $\delta = 2$:

$$(\beta h) = (E_2(\beta h)\, S_2\, (\gamma, (n\beta h))/(E_1(\beta h)\, S_1\, (\gamma, (n\beta h)) \tag{A6}$$

where we have designated the following:

$$\begin{aligned} &\sum_1^\infty \exp(-(n-1)\gamma)E1(n\beta h) = E1(\beta h)S1(\gamma, (n\beta h)), S1(\gamma, (n\beta h)) = \\ &= [1 + \exp(-\gamma)E1(2\beta h)/E1(\beta h) + \ldots.] \end{aligned} \tag{A7}$$

$$\begin{aligned} &\sum_1^\infty \exp(-(n-1)\gamma)E2(n\beta h) = E2(\beta h)S2(\gamma, (n\beta h)), S2(\gamma, (n\beta h)) = \\ &= [1 + \exp(-\gamma)E2(2\beta h)/E2(\beta h) + \ldots.] \end{aligned} \tag{A8}$$

We note that in Equation (A1) for values of the argument (γ + β h x) greater than about 2.5, the numerical value of the quantity exp((γ + β h x) is > 12.182 vs. that of {exp((γ + β h x)} − 1) which would be > 11.182. Consequently, the error by eliminating the (−1) term is less than 9% (reduction in the value of dN/dx at the lower end of the wealth value x). This approximation results in simplifying Equation (A1), if approximate closed-formed functions for N(x) and U(x) are desired.

Table A1. Calculated values of the {E_1(βh) S_1(γ, (nβh)} and {E_2(βh) S_2(γ, (nβh)} for selected values of the parameters γ and (βh) for up to nine terms n.

2Term No	**γ = 2, (βh) = 0.609**		**γ = 0, (βh) = 0.664**	
N	**{E_1(βh) S_1(γ, (nβh)}**	**{E_2(βh) S_2 (γ, (nβh)}**	**{E_1(βh) S_1 (γ, (nβh)}**	**{E_2(βh) S_2 (γ, (nβh)}**
1	0.02262	0.01375	0.406573	0.251123
2	0.00046	0.00032	0.163122	0.111953
3	0.00001	0.00000	0.048900	0.037750
4	0.00000	0.00000	0.025239	0.023395
5	0.00000	0.00000	0.009968	0.007490
6	0.00000	0.00000	0.003780	0.003200
7	0.00000	0.00000	0.001366	0.001471
8	0.00000	0.00000	0.000088	0.000800
9	0.00000	0.00000	0.000000	0.000000
Sum 1–9	0.02309	0.01407	0.658415	0.437182
Ratios	0.01407/0.2309 = 0.609		0.437182/0.658415 = 0.664	

We next calculate the values of the functions S_1 (γ, (nβh)) and S_2 (γ, (nβh)) for selected values of γ and (βh). Two sets of values are selected as follows: {γ = 2, (βh) = 0.609} and {γ = 0, (βh) = 0.664}. The values of the Schlömilch functions E_1(nβh) and E_2(nβh) are obtained from interpolation of the data in Table 2. Up to nine terms (n = 9) are calculated as the values of the functions E_1(nβh) and E_2(nβh) diminish very rapidly with an (nβh) value greater than five. The results of these calculations are given in Table A1.

Appendix B

The exponential integral function E_δ(nβh) is shown graphically for four values of the parameter δ = α in Figure A1 (Bocquet 2024). These four numerical values of the index "a" include 0, 0.5., 1 and 2. As already indicated, the value of a = 0 results in a trivial form of a simple exponential integral. Moreover, a negative value of the index "a" results in an ill-behaving, i.e., non-defined, function that diverges to infinity as the argument x increases. Lastly and although the function E_δ(nβh) = E_α(x) is defined for the index a = δ being an integer, numerical calculations can be executed for a non-integer index such as a = 0.5 as shown in Figure A1. The important observation from Figure A1 is that the numerical values of the function E_δ(nβh) decrease monotonically as either one or both a = δ and x = nβh increase.

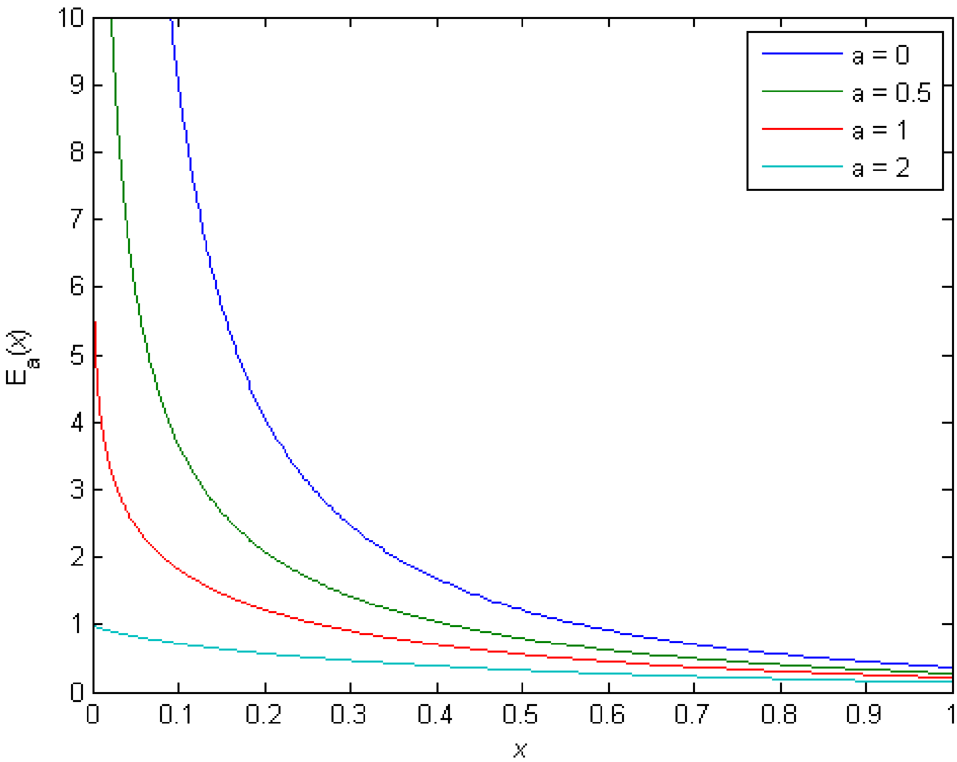

Figure A1. Graphical representation of the exponential integral function E_a(nβh) with a = δ and with x = nβh per Equations (19) and (20).

Notes

1 Thus, every actor N in the economic system is considered to have a non-zero, i.e., finite amount of wealth, because the state of an actor with "zero" wealth would be tantamount to that actor not participating in the economic system. Likewise, there are no actors with negative wealth because borrowing from physics such actors would be bound, i.e., inaccessible, or unable to participate in the economic system.

2 Thus, h is equivalent to the existence of one link between any two nodes in the network system. Consequently, if an actor (node) has, say, m links to other actors (nodes), then the wealth of that actor (node) will be described analytically as (m·h). See also (a) above.

3 Most quantities in nature can be described by random networks that display the familiar bell-curve distribution, which has a peak defining a scale and exponentially decaying tails. A scale-free network is described by a power law distribution, which does not have a peak, and hence it lacks scale, and comprises a continuously decreasing curve, albeit at a much lower rate than an exponential. A distinguishing feature of a power law distribution is that it describes many small events that co-exist with a few large ones. For example, the height of humans is described by a random or Gaussian (bell curve) distribution while the GDP of nations follows a power law distribution.

4 This would correspond to what is described in Statistical Quantum Thermodynamics as the Bose–Einstein statistics for particles with finite mass and integral spin. The original derivation of the Bose–Einstein statistics for photons (zero mass and integral spin) is known as the Planck statistics or black body radiation law.

5 The physical system is in a dynamic thermal equilibrium at temperature T, the term dynamic signifying that if and as T changes with time the system attains a new state of equilibrium consistent with the corresponding change in the total energy of the system. The cavity at temperature T is also designated as the thermal bath of the system.

6 From quantum mechanics, each photon can have an energy E_n = h (1/2 + n ν_N), n_i = 0, 1, 2... with h being the quantum of action (Planck's constant) and v_N the photon fundamental frequency associated with the radiation modes N. The existence of Planck's constant h signifies that energy is delivered in discrete packets (quanta). The existence of energy levels or states in a physical system has been established experimentally and the theory attempts to explain these observations.

7 In the example of Table 1, the actor N_i = 4 has zero wealth elements–links and consequently does not contribute to the economic system in that configuration. However, there will be statistically several other configurations, where this actor will have a non-zero wealth elements or links.

8 The calculation of the number of permutations is essentially equivalent to the calculation of the possible ways of allocating G identical balls into N urns, where each urn is allowed to contain 0, 1, 2,... and up to G balls. This number is N·(N + 1)·(N + 2)...(N + G − 1)/1·2·3...G = (N + G − 1)!/(N−1)! G! as indicated by Equation (1).

9 Even if we had selected the exponent in Equation (12) to be a parameter without the negative sign, we would have concluded then that this parameter should be a negative number to be consistent with the observation that the number of actors decreases monotonically as the associated wealth increases.

10 In a physical system such as, for example, photons in a black body cavity in equilibrium, the parameter β is found to be inversely proportional to the equilibrium temperature T of the system. The equilibrium temperature of the physical system is in turn proportional to the average energy per particle of the physical system. Total wealth in the economic system would be the counterpart of total energy in the physical system. However, we should keep in mind the equivalence between an economic system and a physical system, whereby the actors correspond to occupied energy states and the wealth of any actor is represented by the number of links associated with that actor, whereby links correspond to particles, i.e., photons, in the physical system.

11 Dimensionality here refers to the multiplicity of the states vs. wealth and has nothing to do with the dimensions of the physical space, although in the physical system of photons and other particles the dimension of the physical space affects the value of the corresponding δ parameter, because a higher space dimension, say three versus two, allows for more possible states. Dimensionality in an economic system is indicative of its hierachical structure and refelcts the free-scal nature of it where a small numnber of noded have a lot of links and most nodes have few links.

12 The function $E_\delta(n\beta h)$ for $\delta = 0$ is well defined and is equal to a simple exponential function.

13 The maximum value of the U/N ratio is obtained for $\delta - 1 = 1$ or $\delta = 2$.

14 These approximations include the closed form upper bound simplification of $\Phi(x)$ per Equation (38) vis-à-vis an infinite-term series expression with additional lower (more negative) terms in powers of the variable x and the linearized interpolations in the numerical calculations of the functions $E_1(n\beta h)$ and $E_2(n\beta h)$ or the inclusion of only a small number of terms in the infinite series representing these functions. If we were to include a very large number of terms (theoretically infinite in the sum $\sum_1^\infty E_2(n\beta h)$), then its numerical value should increase from the value of 0.43718 of its first nine terms to 0.53048. Obviously, the convergence of this sum is very slow as the data in Table A1 suggest.

15 The number of states where only one actor has all the wealth and everybody else has no wealth is much higher than one and is equal to the number N of members of the system. That is to say, the likelihood of one actor controlling all the wealth is N times higher than the likelihood of all actors having equal wealth so long as each actor has equal access to the available wealth (income, etc.). And of course, the number of possible states is the highest when all actors have equal probability/oppotunity of having access to the available wealth. This results in the most equitable or optimal distribution of wealth among all the actors.

16 An economic system where in the limit one actor controls all the wealth, although having N times higher probability to occur that a system where all the actors have equal wealth, is still unstable and can only be static to remain in that state. Unstable means far from equilibrium, i.e., maximum entropy. In physics such a state can occur under certain conditions and is dscribed as a Bose-Einstein condensate.

17 The total wealth U is known/determined within plus or minus of the finite (non-zero) minimum amount of wealth or quantum of wealth h, i.e., it is not exactly U. The same is true for the income.

18 Pareto determined empirically, i.e., by examining relevant data from a variety of sources going back to the Middle Ages, that the number of people N_x with wealth higher than x could be modeled as a power law: $N_x = A\, x^{-a}$, where A is a proportionality constant. If the total population is N_o and the minimum wealth is x_o, we then have: $N_x/N_o = (x/x_o)^{-a}$. We can normalize this expression by putting $x_o = 1$ and call the ratio x/x_o the relative wealth S. Then the ratio $X = N_x/N_o$ describes the proportion of the population that has a (relative) wealth greater than x and represents essentially the Pareto distribution survival function. The parameter "α" has a numerical value equal to or greater than one and is called the Pareto index. Since the respective areas under the curve $F(x) = (x)^{-a}$ for x varying (a) from one to infinity and (b) from x to infinity are equal to $1/(\alpha - 1)$ and to $(x)^{1-a}/(1/(\alpha - 1))$, the ratio of the latter to the former, designated by S, is equal to $(x)^{1-a}$. Since $x = X^{-1/\alpha}$ we have $S = X^{1-1/\alpha}$, which represents the proportion of wealth S for the proportion in the population X. Therefore, we deduce that the value of the Pareto index α is given by the equation: $\alpha = \ln(X)/(\ln(X) - \ln(S))$. Thus, for the 80-20 rule, i.e., 20% of the population (X) control 80% of the wealth (S), the corresponding value of α is calculated to be 1.16 from the preceding equation.

19 We use the symbol F(x) for the CDF instead of the symbol $\Phi(x)$ in the remainder of this section because of the customary representation of such functional expressions.

20 The exponential distribution allows for zero wealth (x = 0) and would describe the number of people with wealth higher than x by the equation: $N_x/N_o = \exp(-\lambda\, x)$ where N_o is the total population and "λ" is the rate of the distribution. It underestimates the wealth of fewer wealthier actors (large x) compared to the Pareto distribution. This can be readily seen if we compare the wealth ratio of populations N1 and N2 at their respectively different wealth levels, say, x_1 and x_2. We have for the exponential distribution: $\ln(N_1/N_2) = -\lambda\,(x_i - x_2)$. For the Pareto distribution we have $\ln(N_1/N_2) = -\alpha \ln(x_i/x_2)$.

21 Most probability distributions have well defined means, variances, and higher-order moments. The exponential distribution, for example, with rate λ has a mean of $1/\lambda$ and a variance of $1/\lambda^2$. For such distributions, outcomes far from the mean are very rare. The Pareto distribution, on the other hand, with an index α has a mean of $\alpha\, x_m/(\alpha - 1)$ for $\alpha > 1$ and a variance of $\alpha\,(x_m)^2/(\alpha - 1)^2\,(\alpha - 2)$ for $\alpha > 2$ and has infinite values for the mean and the variance for values of the index lower than those indicated. Distributions such as the Pareto one, on the other hand, have more common outcomes far from the mean and are described as

having a "fat tail". Thus, the Pareto distribution is well suited to describe empirically outcomes far from the mean such as the observed occurrence of very wealthy or high-income actors in the economy. This is essentially the gist of a power law.

22 The percent of wealth (income) S controlled by the top X percent of the population is given by the Pareto equation $S = X^{(1-1/\alpha)}$ such that for $\alpha = 2$ and $X = 0.20$ we obtain $S = 0.447$. In other words, in an economic system where all the actors/particpiants have equal opportunity, 20% of the them ought to control 44.7% of wealth (income) as the optimal distribution of the available income or wealth. The observed distribution of wealth in particular of 80% or even higher for 20% of the actors/population suggests the lack of equal opportunity or access to the economic system. For S = 0.80 and X = 0.20, the value of α or δ would be 1.16, i.e., well below the optimal vaue of $\alpha = \delta = 2$.

23 In our analytical model this difference can be accounted for by employing two different values in the parameter γ, one for wealth and one for income, such that two respective values for the parameter (βh) result. However, the justification for this differentiation at the present moment is an empirical observation based on the difference between the disribution of wealth and of income. Alternatively, the fitness η of the actors can vary between wealth and income such that the "scale-free phase" of the complex network can be modified into a "fit-get-rich phase" whereby certain nodes (actors) accumulate all the wealth, i.e., the fittest prevail. However, fitness can be viewed as an externally imposed condition to obtain certain outcome. In that instance then the actors no longer have equal opportunity, i.e., they are differentiated. This differentiation then would lead to excessive inequality, which is observed globally, particularly as it regards wealth. See also (Bianconi and Barabási 2001).

24 The Gini coefficient on income, sometimes referred to as the pre-tax Gini coefficient, is calculated on income before taxes and transfers, and it measures inequality in income without considering the effect of the taxes and social spending already in place in a country. The Gini coefficient on disposable income, sometimes referred to as the after-tax Gini coefficient, is calculated on income after taxes and transfers.

25 Absolute equality (all actors have equal wealth or income) as well as absolute inequality (one actor has all the wealth or income) are impossible states to attain and maintain (static economic system), although the latter has a higher probability to occur than the former.

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