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An Integrated Methodological Approach for Documenting Individual and Collective Mathematical Progress: Reinventing the Euler Method Algorithmic Tool

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Abstract: In this paper we advance a methodological approach for documenting the mathematical progress of learners as an integrated analysis of individual and collective activity. Our approach is grounded in and expands the emergent perspective by integrating four analytic constructs: individual meanings, individual participation, collective mathematical practices, and collective disciplinary practices. Using video data of one small group of four students in an inquiry-oriented differential equations classroom, we analyze a 10 min segment in which one small group reinvent Euler's method, an algorithmic tool for approximating solutions to differential equations. A central intellectual contribution of this work is elaborating and coordinating the four methodological constructs with greater integration, cohesiveness, and coherence.

Keywords: collective; individual; coordinated analysis; practices; Euler's method

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1. Introduction

In this paper we advance a methodological approach for documenting the mathematical progress of learners as an integrated analysis of individual and collective activity. Our approach is grounded in the emergent perspective [1] and its expanded interpretive framework [2] and further develops a methodology for integrating its analytic constructs. Our primary research goal in this paper is to offer researchers an approach for coordinating individual and collective analyses to gain explanatory and descriptive power, with the intention to further understand the interactive process by which individual and collective mathematical progress is made. Our epistemological stance is that learning mathematics is a human activity of mathematizing, "social through and through" [1] (p. 181). We aim to account for the complexity of mathematical progress, in this case the reinvention of an algorithmic tool for approximating solutions to differential equations, by accounting for not only the mathematical meanings that individual students form as they participate in and interact with others in classroom activities, but also the mathematical reasoning that is developing and living in the classroom collective space. We use the phrase "mathematical progress", as opposed to, say, learning, as an umbrella term that reasonably applies to both individuals and the collective. Our use of the term "progress" includes both accomplishments and changes, both of which can occur over shorter or longer periods of time. In this way, progress can be demonstrated on a single task, as is the case here, or across multiple class sessions. Further details are provided in the Methodology section.

In [2], we offered four constructs through which to delineate and map out mathematical progress: two individual constructs and two collective constructs. That paper served to introduce those four constructs for the purpose of expanding the original interpretative framework for the emergent perspective; it also detailed methodological steps for carrying out the analyses separately for each construct, with suggested *possibilities* for coordinating those analyses. We saw that paper as “a first step in developing a more robust theoretical-methodological approach to analyzing individual and collective mathematical progress”, and we anticipated that “future work will more carefully delineate methodological steps needed to carry out the various ways in which analyses using the different combinations of the four constructs can be coordinated” [2] (p. 279). The purpose of this theoretical–methodological report is to follow up on that promise and provide a thoroughly coordinated and integrated analysis of collective and individual progress. The methodological advance presented here, therefore, is the weaving of four constructs, as opposed to our prior work, which reported each construct’s analysis one after the other.

To demonstrate the coordination of these four constructs, we analyze a very productive 10 min small group episode that occurred on the second day of class in a differential equations course, in which the small group essentially reinvented Euler’s method, which is an algorithmic tool for approximating solutions to first order ordinary differential equations. As such, this mathematically rich and productive episode offers an intriguing opportunity to examine the individual and collective progress of this one small group and thereby provides a paradigmatic example of how to carry out an analysis that interweaves individual and collective mathematical progress. In so doing, we contribute to “the need for multiple empirical techniques to advance, support, and constrain important claims about developmental processes related to ongoing classroom activity” [3] (p. 299).

2. Theoretical Background

The foundational perspective for our line of inquiry is the emergent perspective and an accompanying interpretive framework [1]. The emergent perspective, which is a version of social constructivism, coordinates a social perspective on collective activity with a constructivist perspective on the reasoning of individual students and has been profitably leveraged by a wide range of scholars (e.g., [4–7]). Common across this work is a coordination of individual and collective mathematical progress through the coordination of two constructs, one for the individual and one for the collective.

The work reported here builds on and extends the coordination of individual and collective progress from the original interpretative framework [1] to include coordination across not just two constructs but four constructs (two for the collective and two for the individual). Inspiration for creating an expanded interpretative framework came from more recent theorizing about the value of coordinating different perspectives [8], the nature and forms of classroom participation [9,10], and the emphasis on engaging undergraduate students in disciplinary practices in inquiry-oriented mathematics classrooms (e.g., [11]).

An expanded interpretative framework for the emergent perspective is shown in Figure 1, with four constructs for coordinating individual and collective progress shown in the bottom row [2]. Social and sociomathematical norms, which appear in the first and second rows under the social perspective of the expanded interpretative framework, are foundational constructs for delineating the classroom microculture and are taken to be reflexively related to an individual’s beliefs [12]. Because our analysis focuses on a 10 min episode in one small group on the second day of class (and not on longitudinal classroom data that would admit analyses of social and sociomathematical norms), we focus our analysis within the bottom row of our expanded interpretive framework.

To further highlight the woven approach we take to report our integrated analysis and to highlight the foundational influence of symbolic interactionism [13] in the emergent perspective, we created the image shown in Figure 2. As we moved from our prior work [2] to this analysis, we chose to change the wording of the four constructs to that in Figure 2 for three reasons: (1) we wanted to highlight the unit of analysis (collective versus individual), (2) we wanted to open up collective practices to include normative reasoning that occurs over a shorter time frame within a single group of students, and (3) we believe that using mathematical meanings (versus conceptions) is more consistent with a focus on interactionism [13]. Thus, in this paper, our four constructs, shown in Figure 2, for coordinating mathematical progress are: collective disciplinary practices, collective mathematical practices, individual participation, and individual meanings. The imagery of interlocking puzzle pieces is intended to convey the centrality of interaction and connection between the four constructs that is central to our analysis for characterizing individual and collective mathematical progress.

Social Perspective		**Individual Perspective**	
Classroom social norms		Beliefs about own role, others' roles, and the general nature of mathematical activity	
Sociomathematical norms		Mathematical beliefs and values	
Disciplinary practices	Classroom mathematical practices	Participation in mathematical activity	Mathematical meanings

Figure 1. An expanded interpretive framework.

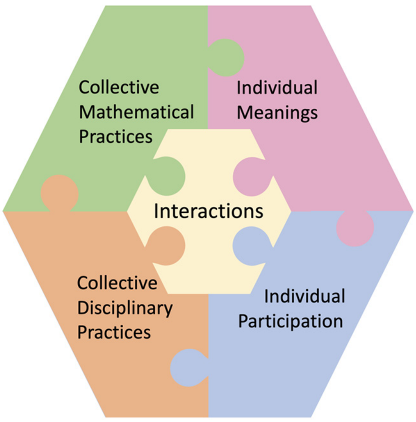

Figure 2. Interconnected constructs.

Symbolic interactionism, which is at the core of the emergent perspective, has three basic premises: a person acts towards things on the basis of the meanings that the person holds for those things, the meanings of those things arise out of the human interaction that one has with others, and the person modifies these meanings through a process of interpretation when encountering those things [13]. According to Blumer,

> Interaction in human society is characteristically and predominantly on the symbolic level; as individuals acting individually [or] collectively. . . they are necessarily required to take account of the actions of one another as they form their own action. They do this by a dual process of indicating to others how to act and of interpreting the indications made by others. . . by virtue of symbolic interaction, human group life is necessarily a formative process and not a mere arena for the expression of pre-existing factors [13] (p. 10).

Thus, interactions are central in mathematical progress. It is precisely for this reason that the center puzzle piece of Figure 2 is "Interactions", for it is the interactions that constitute the substance and glue allowing for the weaving and connecting of the four analytic constructs. These facets of the phenomena under investigation are interdependent and, in many ways, inextricable from one another. This position is consistent with early writings regarding the emergent perspective and interactions that occur within small group work [14]. More specifically, "Students' constructions have an intrinsically social aspect in that they are both constrained by the group's taken-as-shared basis for communication and contribute to its further development". Thus, learning opportunities within small group work "can arise for children as they mutually adapt to each other's activity and attempt to establish a consensual domain for mathematical activity" [14] (p. 26). As a consequence, all four constructs are in dialogue with each other throughout our analysis.

The construct of *individual mathematical meanings* is how we examine what ideas and interpretations individual students develop in their mathematical work. Our view of an individual's mathematical meanings draws on symbolic interactionism, which holds that meaning is not inherent and intrinsic to a concept, but rather takes form "in the process of interaction between people. The meaning of a thing *for a person* grows out of the ways in which other persons act toward the person with regard to the thing" [13] (p. 4, emphasis added). Accordingly, Bauersfeld defined understanding as "the active construction of such meanings and references supported by the social interaction within the culture... These ascriptions of meaning are necessarily specific to the perceived situation" [15] (p. 274). Thus, to analyze the mathematical meanings that an individual brings to bear and progressively develops in their mathematical work with respect to a particular concept, we document the individual's mathematical enacted, developed, and enlarged ways of communicating that concept. This does not happen statically or in isolation; rather:

> Symbolic interactionism emphasizes the interpretative process involved in the development of meaning as an individual responds to, rather than simply reacts to, another's actions. The response is based on the meaning an individual ascribes to another's actions. It is the emphasis on the interpretation that distinguishes symbolic interactionism [16] (p. 135).

Within a small group or whole class discussion, a student interprets others' mathematical utterances by ascribing meaning to those utterances, which may or may not be the meaning that the other person held or aimed to communicate. This occurs as students enact and embrace various participatory roles in their joint mathematical endeavors.

The construct of *individual participation* is how we examine the ways in which an individual student contributes to the mathematical progress that occurs within group settings. Because meanings are social products arising out of the human interaction that one has with others, we operationalize a student's participation as the ways in which they interact in either small group or whole class settings. This is broadly consistent with Wenger's use of participation within a community of practice to refer to "a process of taking part and also to the relations with others that reflect this process. It suggests both action and connection" [17] (p. 55). In the Methodology section we leverage the work of Krummheuer to operationalize the kind of actions and connections that students embrace. In previous research about advancing mathematical activity at the undergraduate level, we characterize an individual's "progression in mathematical thinking as acts of participation in a variety of different socially or culturally situated mathematical practices" [18] (p. 51). Thus, the interactions between an individual's mathematical meanings for a concept both shape and are shaped by their participation in the various activities and goals of the collective communities of which they are members, both at the local, proximal level and the broader disciplinary level.

Collective mathematical practices are normative ways of reasoning that emerge in a particular community as its members solve problems, represent their ideas, share their thinking with others, etc. This intentionally broad construct includes specific meanings for concepts, use of strategies and procedures, interpretations of symbols, use of tools and underlying metaphors, and even gestures [4]. Normative ways of reasoning were introduced in the original interpretive framework for the emergent perspective as a way to call to the fore the mathematical progress that comes to be taken as self-evident by members of the classroom community [1]. Asserting that a mathematical idea is normative, or, in other words, functions-as-if-shared [19], does not mean that everyone in the group has identical ways of reasoning. Instead, it means that "particular ideas or ways of reasoning are functioning in classroom discourse *as if* everyone has similar understandings, even though individual differences in understanding may exist" [2] (p. 263, emphasis added).

Moreover, the *as if* part of function-as-if-shared is consistent with Cobb et al.'s point that "we cannot observe mathematical practices directly any more than we can directly observe the meanings that individual students' measuring activities had for them" [4] (p. 148). We take a similar stance for our inferences about individual meanings expressed through discourse.

A *collective disciplinary practice* refers to broad stroke ways in which mathematicians typically go about doing mathematics. The following disciplinary practices are among those core to the activity of professional mathematicians: defining, algorithmatizing, computing, symbolizing, modeling, conjecturing, and proving [2,18,20–22]. We consider this construct to be at a local group level but with a researcher eye towards the broader practices of mathematics. Note that disciplinary practices are not characterized with an a priori framing but rather are manifested in an emergent way in the classroom.

To relate the two collective-level constructs, we note that not all collective mathematical practices are easily or sensibly characterized in terms of a disciplinary practice. This is because collective mathematical practices capture the emergent, content-specific aspects of student reasoning, whereas a disciplinary practice analysis reflects the kinds of activities that are, in the eyes of the researchers, central to the students' engagement in that disciplinary practice.

3. Setting, Participants, and Additional Background

The data for this study come from a semester-long classroom teaching experiment (following the methodology described in [23]) in a differential equations course conducted at a medium-sized public university in the United States. We selected a 10 min small group episode from the second day of class because of its potential to illustrate the coordination of all four constructs as students in this small group reinvented an algorithmic tool to approximate solution functions. There were four students in this group: Liz, Deb, Jeff, and Joe (all names are pseudonyms).

There were 29 students in the class, all of whom were either pursuing a degree in engineering, mathematics, or science and had previously taken differential and integral calculus. Class met four days per week for 50 min class sessions for a total of 15 weeks. The classroom teaching experiment was part of a larger design-based research project that explored ways of building on students' current ways of reasoning to develop more formal and conventional ways of reasoning. In a previous analysis of this same classroom, researchers analyzed the collective mathematical progress of the first 22 days of instruction and identified six groupings of ideas that function-as-if-shared [24]. The second grouping of ideas focuses on Euler's method and included the following six ideas that functioned-as-if-shared:

1. Using a recursive process yields graphs that are approximations, not exact,
2. The smaller time change increments that are taken in the method, the more accurate the prediction/graph,
3. The exact solution functions use instantaneous rates of change where the approximation graph comprises linear stretches of rates of change,
4. The graph of the approximation goes up as the time increments are chosen to be smaller (during portions where the solution is concave up),
5. The initial slope at time zero is tangent to the exact solution, and
6. The initial slope is the same no matter if the approximation is over a 1-year time period or a half-year time period.

This prior analysis documents that the Euler algorithm was indeed a powerful tool for these students to make warranted claims about the algorithm (e.g., the recursive process yield approximations, smaller step sizes result in more accurate predictions) as well as reasoned comparisons between different approximations and between the exact solution and approximations (e.g., the initial slope at time zero is tangent to the exact solution). As powerful and thorough as this analysis is, it leaves open the question of how this algorithmic tool initially came to be. The integrated individual and collective mathematical progress presented here completes this storyline by providing a compelling analysis of the origins of this algorithmic tool for these students.

As is often the case in classroom teaching experiments, the instructor is part of the research team; in this case, the first author of this manuscript was the instructor. The research project team made use of the instructional design theory of realistic mathematics education (RME) to inform the design of the curriculum and instruction. Central to RME, and one of the goals of the instructor, is the design of instructional sequences that challenge learners to organize key subject matter at one level to produce new understanding at a higher level [25]. In this process, new mathematics (for the learners) emerges through a bottom-up process of guided reinvention (e.g., [22,26–29]). Another explicit intention of the instructor was to initiate and sustain productive social norms for interactive discussion. In particular, the instructor had a goal to create a learning environment in which students felt safe in explaining their reasoning, however tentative, and were obligated to listen to and to try and understand the reasoning of others.

Data for this paper are the video and transcript of Liz, Deb, Jeff, and Joe working together, uninterrupted by the teacher or other students, through a task in their small group on the second day of class. On the first day of class and followed up on the second day of class, students, both in small groups and in whole class discussion, created what they saw as reasonable population versus time graphs for a situation in which the population had unlimited resources and reproduced continuously. Graphs depicting constant rate of change were offered by the instructor as a possibility, but rejected by the students because as the population increased, so would the rate of change. Instead, students argued for exponential growth. The instructor then used their graphs as an opportunity to introduce the rate of change equation $dP/dt = 3P$ as a differential equation that was consistent with their graphs and reasoning that the rate of change would increase as the population increased. Students then worked on the task shown in Figure 3 for approximately 10 min in their small groups. Recall that this was only the second day of class and students had not been introduced to any analytical, numerical, or graphical techniques for analyzing differential equations.

Consider the following rate of change equation, where P(t) is the number of rabbits at time *t* (in years): $dP/dt = 3P(t)$ or in shorthand notation $dP/dt = 3P$. Suppose that at time $t = 0$ we have 10 rabbits (think of this as scaled, so we might actually have 1000 or 10,000 rabbits). Figure out a way to use this rate of change equation to approximate the future number of rabbits.

- At $t = 0.5$ and $t = 1$.
- At $t = 0.25$, $t = 0.5$, $t = 0.75$, and $t = 1$

Figure 3. The task.

4. Methodology

Our qualitative methodological approach began with the first author making a first pass through the coding for all four constructs. We then proceeded along three phases of work, using the preliminary coding as background and comparison. Each phase was carried out collaboratively among the three authors, as opposed to approaches in which different individuals code and then check for consistency. Our collaborative approach to analysis draws on the method of interaction analysis [30]. Interaction analysis is an empirical approach for investigating humans as they interact. In this approach, collaborative analysis "is particularly powerful for neutralizing preconceived notions on the part of researchers and discourages the tendency to see in the interaction what one is conditioned to see or even wants to see" [30] (p. 44). Interaction analysis has a guiding assumption that knowledge is not only fundamentally social in origin and use but is also situated in particular social environments; thus "expert knowledge and practice are seen not so much as located in the heads of individuals but as situated in the *interactions* among members of a particular community engaged with the material world" [30] (p. 41, emphasis added). As such, interaction analysis is highly compatible with the emergent perspective and symbolic interactionism.

In the first phase of our analysis, we immersed ourselves in the data with cycles of watching the video, reading transcripts, and taking notes of relevant gestures, pauses, or other non-verbal interactions among the four students. We organized the transcript according to talk turns. Each time the speaker changed, a new turn number was assigned. In the second phase of the analysis we created a table with columns for turn number and speaker, transcript, and four additional columns, one for each construct. Each talk turn was assigned the corresponding line number with the table. We also had a catch-all notes column to record emerging insights or reflections. We then embarked on coding the transcript per the four constructs, returning to the video and initial coding as needed. Pragmatically, one construct was foregrounded at a time, but each was always there in the table and hence we occasionally noted a code for one of the other constructs. We then examined the coding across columns for relationships among the constructs. All three authors discussed and refined the coding as a team for all four constructs until consensus was reached.

In the third phase of the analysis, we told the story of the students' collective and individual mathematical progress. Our guiding question for this phase was: In what ways did each construct influence or relate to the others? The imagery of the interlocking puzzle pieces from Figure 2 kept our focus on the interactions among constructs. To address this guiding question and create the storyline, we looked for notable milestones or shifts in focus in students' mathematical progress and divided the transcript into four segments accordingly. We then wove together the story of that progress in terms of the interrelated constructs. A perhaps useful metaphor for the third phase is that of a weaver, who takes separate colors of threads and creates a pattern within a multi-colored blanket.

We next provide details on how we coded for each construct. Depending on the construct, we used a combination of a priori codes and open coding [31].

5. Collective Mathematical Practices

Our first round of coding focused on the small group's collective mathematical progress. Consistent with others who have documented collective mathematical practices (e.g., [5,6,32]), we used Toulmin's Model of Argumentation [33]. Briefly stated, Toulmin delineated six different aspects that comprise the structure of an everyday argument (Claim, Data, Warrant, Backing, Qualifier, and Rebuttal). Numerous researchers in mathematics education have used this framework to study mathematical argumentation (e.g., [34–36]). However, here we use Toulmin analysis of student argumentation to document and justify our claims about the progress of students' ways of reasoning [5,24].

We began by coding the transcript for all Claims, Data, Warrants, Backing, Qualifiers, and Rebuttals. Claims are assertions that a speaker(s) put forth as true, and the Data are evidence to support that Claim. Typically, the Data consist of facts or procedures that lead to the conclusion that is made. A Warrant speaks to the connection between the Data and Claim. The Data–Claim–Warrant constitute the core of the argument. A Backing provides further support for the core argument. A Qualifier speaks to the degree or level of confidence in the Claim, and a Rebuttal is an objection that offers conditions under which the Claim is invalid. In order to count as an argument, we required there to be at least a Data–Claim pair (i.e., Claims with no supporting justification were not counted as arguments). In total, we identified 16 different arguments in the 10 min small group work. The majority of these are detailed in the next main section.

Next, we operationalized the small group's collective mathematical progress by documenting mathematical ways of reasoning that functioned-as-if-shared for this small group. In this case, the collective mathematical progress is evidenced by the emergence of ways of reasoning that function-as-if-shared. To document collective mathematical practices, we used of the methodology detailed in [19] (we refer readers to this chapter for details on the method). This involved collaboratively analyzing the 16 arguments for instances in which any of the following criteria were met: when the Backings and/or Warrants for a particular Claim are initially present but then drop off (Criterion 1); or when a Claim shifts to Data or Warrant in a subsequent argument (Criterion 2). In total, we identified the following four mathematical ways of reasoning that functioned-as-if-shared:

- The initial rate of change is 30,
- 30 means 30 more rabbits in one year,
- The change in the population is 0.5 times the rate of change,
- The new population is the old population plus the change in the population.

6. Individual Participation

In order to document how individual students contribute to the mathematical progress occurring in their community of practice, we leverage aspects of Krummheuer's framework of *production design* and *recipient design* roles [9,10]. This framework, which is compatible with the emergent perspective, grows out of and in response to the theoretical stance of learning-as-participation. Krummheuer claimed that learning-as-participation was largely still in the abstract as a sensitizing concept, leaving a gap regarding methodological tools when attempting to operationalize the stance to characterize empirical data. Thus, Krummheuer developed the production and recipient framework to "describe different roles of participation in everyday mathematics classroom situations" [10] (p. 82) with the aim to substantiate and operationalize aspects of classroom interactions that can be interpreted with the view of learning-as-participation. In this paper, we adopt and adapt four participation roles from within his framework and offer four novel roles that grew out of our analysis of this data set.

Krummheuer operationalized participation as two main types of interaction with the developing mathematics within the community of practice: production and reception (we did not make use of the recipient design roles in the present analysis because the data set did not capture the nuances necessary for that type of analysis). The four roles within the production design framing aim to characterize the originality of a person's utterance in a classroom setting. Krummheuer saw two main threads through which one can delineate originality of an utterance: semantically (i.e., according to the content) and syntactically (i.e., according to the formulation), which lead to the following categorizations:

- Author: the participant is responsible for both the content and formulation of an utterance,
- Relayer: the participant is not responsible for the originality of neither the content nor the formulation of an utterance,
- Rephraser: the participant attempts to express the content of a previous utterance in his/her own words,
- Leverager: the participant takes part of the content of a previous utterance and attempts to express a new idea (Krummheuer's original terms for Rephraser and Leverager were Spokesman and Ghostee, respectively. We changed the terms of these two categories to fit with what we felt was a better match for our interpretation of the categories' intended meaning and to use a term without implied gender).

Although these formulations of production roles are useful in delineating variations in how individuals in our data set were participating in mathematical activity, we found them to be insufficient to account for an additional way in which students participated in the mathematical discourse of their small group. In particular, our open coding indicated the need for a new category of participation role—that of *facilitator design*. More specifically, we identified four different ways in which these students facilitated the flow of ideas in their small group:

- Focuser: the participant directs the group's attention to a particular mathematical issue,

- Elicitor: the participant attempts to bring out another's idea,
- Checker: the participant seeks agreement or sensibility of an utterance,
- Summarizer: the participant pulls ideas together.

In order to carry out the analysis of individuals' participation within their small group, we considered each of the 64 talk turns to determine how it functioned in terms of either production design or facilitator design.

Progress in individual participation can be evidenced in two ways. First, the ways in which individual students participate are intertwined with the collective mathematical progress, and hence accomplishments in collective mathematical practices are inseparable from accomplishments in individual participation. Second, the method allows for examination of possible changes in how individuals participate (e.g., an increase in Authoring roles).

7. Individual Meanings

To analyze the progress in mathematical meanings that an individual brings to bear in their mathematical work with respect to a particular concept, we document the individual's mathematical enacted, developed, and enlarged uses of that concept as expressed in their discourse with others. In our data set, progress most frequently occurred as accomplishment: data sets over longer time spans would also admit an analysis of changes in an individual's meanings.

In any data set in which students are doing mathematics there will likely be several concepts at play, each of which might have multiple meanings for the students. Our first step in identifying relevant concepts is to do what some refer to as an a priori analysis. Although we did not formally do this for this analysis, we did start by grounding ourselves in the mathematics that students would likely leverage and develop to solve the task at hand. Rate of change in general and dP/dt in particular for this task were, perhaps not surprisingly, identified as key concepts for reinventing the Euler method algorithm. To state in slightly more general terms, the concepts and their associated meanings that we analyzed had to be both present in the data and relevant for students to make progress.

Next, we turned to the literature [36–40] to gain insight into the wealth of meanings that characterize a student's mathematical thinking for that concept. While our coding made use of findings and insights from the literature, we very intentionally did not limit ourselves to these existing meanings and instead allowed ourselves to engage in open coding as well. Our literature-informed open coding led to the following codes for the different meanings of dP/dt that students engaged:

- as an instantaneous rate of change,
- as an indicator of the initial rate of change,
- as a ratio of two changes,
- as an indicator for how the population is changing,
- as a unit ratio,
- as a unit ratio that can decremented,
- as a quantity that can take on different values.

8. Collective Disciplinary Practices

The construct of collective disciplinary practices can be used to analyze many different mathematical practices. For this paper our analysis focuses on the practice of algorithmatizing, which we define as the practice of creating and using algorithms. This choice was made because the mathematical progress of this small group work entails students engaging in the first steps of creating an algorithm. These first steps lay the relational [41] foundation for how to use P values and dP/dt values to approximate a future population value. An expert will recognize students' work as Euler's method, although the students do not as of yet know that what they are reinventing is in fact Euler's method.

Through a process of open coding we went through the data looking for aspects of students' mathematical thinking that refer to values and relationships between these values that students were creating and using in their first steps in creating this algorithm. More specifically, students' creation of the Euler method algorithm tool involved the following:

- engaging in goal-directed activity,
- isolating attributes,
- forming quantities,
- creating and verbalizing relationships between quantities,
- investigating the generality of the procedure,
- expressing these relationships symbolically.

The first of these, engaging in goal-directed activity [3], might occur in many different disciplinary practices. In our data this came to the fore when students acted in ways that focused the group on particular aspects of the quantities and their relationships that would be important for building the algorithm. Our use of quantity is compatible with Thompson, who defines a quantity as an attribute with a measurable magnitude [42]. To build the algorithm students needed to isolate attributes of the quantities. Students expressed various meanings for dP/dt and leveraged these differently to create and explain steps in building their algorithm. Mathematical progress, therefore, is as a form of accomplishment when students isolate attributes, form quantities, and create and verbalize relationships between quantities. The quantities included P, change in P, and changing dP/dt values. The relationships described how these quantities were interconnected and could be used to compute values of these quantities at later points in time. Towards the end of their small group work, students started investigating the generality of the procedure, and later in class and in a subsequent class students worked on expressing these relationships symbolically. For extended data sets a researcher would be able to document changes to the collective disciplinary practice of algorithmatizing (e.g., formalizing the method, improving the approximation method).

9. A Coordinated Analysis through Four Constructs

In this section we exemplify a coordinated analysis of individual meanings, individual participation, collective mathematical practices, and collective disciplinary practices. The 10 min small group work is divided into four segments: Getting started, Making first step progress, Deciding to repeat, and Using smaller time increments. We separate the analysis into these four parts to reflect the small group's slight shifts in focus as they worked on the task and reinvented the Euler method algorithmic tool. To visually highlight the four constructs and to suggest the weaving that is created by the integrated individual and collective analysis, we capitalize the first letter of argumentation elements (e.g., Data), italicize participation roles (e.g., *focuser*), underline particular meanings (e.g., unit ratio), and bold algorithmatizing activities (e.g., **isolating attributes**).

9.1. Segment 1—Getting Started

The first four minutes of small group work consists of 19 talk turns and the production of four arguments. We start by detailing Argument 1, as this constitutes the basis for the small group's subsequent work and hence is the lynchpin for what follows. The totality of Argument 1 occurs over the first 12 talk turns (see Figure 4) and involves contributions from three of the four students. During this interaction, the group determines, with justification, what the initial rate of change is. In addition to using this argument in our analysis of collective mathematical practices, we weave together various individual participation roles, individual meanings that emerge, and the start of the students' collective disciplinary practice of algorithmatizing. Because people often explicate the motivation or inspiration before stating a Claim, it is not unusual for arguments to unfold in a manner such that a Warrant or Backing comes before the Claim and Data. Moreover, parts of an argument are often repeated, and a complete argument may develop over multiple lines. All of this is the case in Argument 1. The transcript in Figure 4 includes our argumentation coding, where the coded portion of each utterance is shown in regular font followed by the specific element of the argument (e.g., (**C1**) stands for the Claim in Argument 1) and non-coded portions are shown in gray font.

1 Liz	To find out the rate of change initially, at that point in time, when time equals zero. [Joe: What's that again? What's the first part?] I would plug in the population of rabbits for P to determine the rate of change, when, initially, just at the instant, like initially, what's the rate of change when time equals zero (**W1**). So if we had a graph, it's kind of like what we were just talking about, we are trying to determine the rate of change when this time is equal to zero (**B1**). [Liz sketches an exponential curve and points her pen at the point where the curve crosses the y-axis]
2 Joe	Oh ok. This is where 10 rabbits at zero (**D1**).
3 Liz	What do you think? [Deb: I kind of know so I want to let you guys. Liz: Ok, don't tell me, don't tell me anything. Hide it. Deb: While I rethink how it went]
4 Liz	Oh ok, so I get the rate of change at time, initially, the instantaneous rate of change would be 30 (**C1**). Did I multiply it right? [5 second pause while she looks at Joe and Jeff]
5 Liz	And then I guess the simple …
6 Joe	Are we trying to figure out what P is?
7 Liz	Okay, well this [the symbol dP/dt] (**D2**) is the change in the population over the change in time (**C2**).
8 Jeff	Right.
9 Liz	Okay, and this 3 I'm taking as being [Jeff: the constant] the constant or whatever you call the growth rate [Jeff: Growth rate, right]. And this is really, is really, P of t, right? or the given, right? or the given population at any given point of time t [Jeff: Right], but this is just short hand notation for it. So I thought, well, if we know the population is ten when our time equals zero (**D1**) [Jeff: Right], can we plug in the 10 for P(t) population at time zero and find out initially what the rate of change is (**W1**)? [Jeff: I see what you're saying.]
10 Joe	It would be 10 = 3 t zero
11 Liz	Times 10
12 Jeff	Okay I see so it would be 30 (**C1**).

Figure 4. Initial transcript for Segment 1, with coding indications for Arguments 1–2.

In attending to individual student participation within Argument 1, we see three of the students take on different production roles. Liz is the *author* for three components of the argument (Claim, Warrant, and Backing) because she is the first one to articulate the respective ideas (lines 1, 4, and 9). Jeff (12) plays the role of *relayer* for the Claim (which is that the initial rate of change is 30). Recall that a *relayer* is someone who restates a previously articulated idea. This is significant in this case because it provides evidence that Jeff shares and agrees with the Claim that Liz made. Jeff gives similar confirmation on what we coded as Liz's Warrant (see (9 Liz) in Figure 3). Joe, for his part, is also an *author* because he is the first to articulate the Data (line 2), although in what we coded as Argument 3 his interpretation of what the 10 means is incorrect and inconsistent with his groupmates' interpretation. Deb, who has seen this task before, intentionally pulls herself out of the discussion and hence allows space for her groupmates to develop their own ideas while she privately works the problem. Deb rejoins the discussion, however, in the next segment.

In attending to individual mathematical meanings within lines 1–12, we see two different meanings articulated for dP/dt. In particular, in the Claim for Argument 1, Liz plugs in the initial population value of 10 into the rate of change equation and obtains 30, which she interprets to be the instantaneous rate of change. She says "the instantaneous rate of change is 30" while sketching an exponential graph and pointing her pen to where the graph intersects the vertical axis. In calculus, the notion of instantaneous rate of change is often associated with the slope of the tangent line at a point, and this might be how Liz interprets the 30; however, we do not have strong evidence for this graphical interpretation of instantaneous rate of change. A second meaning that emerges for dP/dt is the change in the population over the change in time, which we coded as a ratio of two changes. This discrete, ratio-based interpretation is authored by Liz (line 7) and is the Claim for Argument 2, which she provided in response to a question from Joe, "Are we trying to figure out what P is?"

With an eye towards the small group's collective disciplinary practice, the development of Argument 1 is also illustrative of the group's participation in the disciplinary practice of algorithmatizing. In particular, we see students contributing to two aspects of creating an algorithm: **engaging in goal-directed activity** (i.e., focusing on finding the initial rate of change) and **isolating attributes** (i.e., how to interpret the initial rate of change). **Engaging in goal-directed activity** coincides with the participation role of *focuser,* and **isolating attributes** coincides with articulating two different meanings for dP/dt, as the initial instantaneous rate of change and as a ratio of two changes. Recall that a *focuser* is one of four new facilitator roles that we found necessary to add to Krummheuer's [10] set of production and recipient roles in order to capture newly identified participation roles.

We define a *focuser* as someone who directs others' attention toward a specific goal or activity. Liz takes on this role in line 1 when she directs her and her groupmate's attention "To find out the rate of change initially, at that point in time, when time equals zero". With this statement Liz sets out a **goal-directed activity** of finding the initial value of dP/dt, which an expert will see as a specific case of a component of the more general Euler's algorithmic tool. This goal is realized with Liz and Jeff agreeing that the initial value of dP/dt is 30. Joe also acts as a *focuser* when in turn 6 he poses the following question to his group: "Are we trying to figure out what P is?" With this question, Joe focuses his group on what attribute of the problem situation they are trying to determine, P or dP/dt. Joe's question prompts Liz to articulate the meaning for dP/dt, namely as a ratio of two changes. As we show in Segment 2, this is a powerful meaning for this group that enables them to make use of **isolating attributes of dP/dt** to figure out a general approach to compute the change in population, which is a necessary component of the Euler method algorithm. Moreover, the two acts of the *focuser* lead to the argument's Claim of actually finding the initial rate of change and interpreting this value as a ratio of two changes, respectively.

As noted previously, in (12) Jeff acts as a *relayer* by restating the Claim that the initial rate of change is 30, which is followed by Liz (13) asking her groupmates, "30, I mean does that make sense?" This first instance of anyone in this group "checking in" with their peers about their thinking gave rise to our identifying another new facilitator role, that of *checker*. Acting as a *checker* can serve multiple functions in a group. For example, it can lead to coherence among groupmates and build confidence in their ideas, it can open a space for someone to ask a clarifying question, and it can be an opportunity for someone to disagree. In this case, Liz acting as *checker* gives rise to Jeff indicating agreement, "Yeah, that makes sense" (14), and for Joe to offer a counter argument, which we coded as Argument 3. In this argument Joe asserts (incorrectly) that 10 is actually equal to 3P(t). Joe's incorrect Claim functions productively in the small group because it gives rise to Liz putting forth what we coded as Argument 4, shown in Figure 5, in which she counters Joe's interpretation of what the "10" refers to.

16 Liz	Well 10 is actually the population **(D4)** so I'm taking that that has to be the population at time t. This is actually. I don't think it's [the 10] **(C4)** telling us how the population is changing, which would be dP/dt. **(W4)**

Figure 5. Transcript and Toulmin coding for Argument 4.

As a whole, Argument 4 relates again to the algorithmatizing aspect of **isolating attributes** because it reasserts the meaning of 10 as the initial population and puts forth a new meaning for the quantity dP/dt, namely as an indicator for "how the population is changing" (16 Liz). Joe does not object to Liz's argument; in the video, he appears to think quietly about what has been said. Thus far, the meanings for dP/dt include as an instantaneous rate of change, as a ratio of two changes, and as an indicator for how the population is changing. Developing meaning(s) for attributes that figure prominently in their mathematical work will serve to ground their reinvention of Euler's method as a product of their own reasoning and sense making.

Next, Liz and Jeff act as *focusers* in that they each ponder what now to do with the 30. In particular, Liz (17) says, "So if we have that [initial rate of change is 30], the question is how can we use that to help us figure out the population after, say, a half year has elapsed?" and Jeff (18) says, "how would we work time into the equation to get the next, uh, population or change in population?" As before, this particular facilitator role of *focuser* promotes students' **goal-directed activity** toward creating an algorithm. At this point, Deb (19) joins the discussion and says to the group, "That is exactly what I did".

9.2. Segment 2—Making First Step Progress

The next two and a half minutes of small group work consists of 21 talk turns and the production of six arguments. During this segment, the small group makes considerable progress on creating an algorithm, including the surfacing of two new meanings for rate of change, engaging a range of participant roles, and establishing two ways of reasoning that function-as-if-shared within their small group. As we did in Segment 1, we provide the transcript that contains our argumentation coding (see Figure 6).

20 Deb	We have the 30 (**D5**).
21 Liz	We have the 30 to work with (**D5**), so couldn't we say we don't [5 second pause]
22 Deb	[Pointing to Liz's paper] You said the population is 10, right (**D1**)?
23 Liz	um hm.
24 Deb	So three times ten would give us our rate of change (**D5**).Say 0.5 years passes, and this is our rate of change. Then we'll take that 0.5 times the rate of change (**W5**) which will give us what [slight pause looks up to Jeff and Joe], the new amount of rabbits (**C5**) plus the old amount of rabbits. (**C6**)
25 Liz	So the old amount of rabbits is ten (**D8**)
26 Deb	Am I making sense?
27 Jeff	I think so, so that would be 25 (**C7**), is that what you're saying?
28 Liz	Okay I think I get what you're saying. Ok, so like we're at time zero and we have 10 rabbits, and supposedly the rate of change, well not supposedly, we're saying that the rate of change is 30 (**D8**) [Jeff: Yeah, for the] at time zero. So it's going to grow at a rate of, I don't know if I'm going to say this right, at 30 rabbits per year? [looks up at Deb] (**C8**)
29 Deb	Right. [Liz: Ok] So we'll have 30 more rabbits. (**D9**)
30 Liz	But we only want to go a half a year (**W9**). [Liz looks at everyone]
31 Deb	So it'll be 0.5 times 30 (**W9**), [Jeff: Right] which is 15 (**C9**).
32 Liz	And so we're really not figuring out the rate of change we're figuring…Well this is the rate of change and we're using the rate of change to figure out the number of rabbits we are going to increase by in half a year (**B5**).
33 Deb	Well your, the new population [Joe cuts her off]
34 Joe	Well this doesn't make sense to me because if t is 0 (**D10**) then we have 0 (**C10**). But you said when t is zero we have 10 (**R**ebuttal to Argument 1). (Note that his assertion is incorrect)
35 Liz	I think it just means initially we have 10 (**R**ebuttal to C10).
36 Joe	Well if you use this [points to Liz's paper] and you times zero you have zero rabbits (**D10**) [Liz: But we're] Or the rate of change would be 0, that's right (**C10**).
37 Liz	Well actually we're going to multiply it by a half a year (**B5**, continuation of turn 32).

Figure 6. Initial transcript for Segment 2.

The segment begins with Deb and Liz acting as *relayers,* noting that they "have the 30 to work with" (20 Deb, 21 Liz), where the 30 is in the initial rate of change. Recall that a *relayer* is someone who repeats a previous comment. In this case, Liz and Deb repeat what was previously in Argument 1 a Claim that needed to be justified. In saying that they "have the 30 to work with", Liz and Deb position the initial rate of change as a known fact. Within Segment 1, deciding that the initial rate of change is 30 was a Claim that had to be justified; now it shifts to Data in Argument 5, which is consistent with how Liz and Deb position 30 as a known fact. Thus, per Criterion 2 of the collective mathematical progress analysis there is empirical evidence that 30 being the initial rate of change functions-as-if-shared. In addition to the collective mathematical progress that Argument 5 accomplishes, Deb's and Liz's contributions to Argument 5 verbalize the amount of change that occurs over the first half year. This is a significant step forward in algorithmatizing because these contributions are **isolating attributes**, in particular the amount of change that occurs over a half year (we are figuring “out the number of rabbits we are going to increase by in half a year” (32 Liz)).

Moreover, in the midst of being an *author* of an argument, Deb enacts the facilitator role of *checker*. In particular, she asks her groupmates, “Am I making sense?” Simply asking this question does not in and of itself justify charactering Deb as a *checker*. Instead, her groupmates have to interpret this question as a genuine request, which is indeed the case here. Deb’s *checker* move affords the emergence of the various meanings that her groupmates are developing for rate of change, one of which leads to another way of reasoning that functions-as-if-shared.

In line 24 Deb follows up her statement “The new amount of rabbits” with “plus the old amount of rabbits”. Although this is a compound claim, we only include the first part in Argument 5 because the Data, Warrant, and Backing all focus solely on the amount of change after a half year. However, the second part of what Deb says (“plus the old amount”), which we coded as the Claim for Argument 6, is significant for Jeff, for he responds by saying, “I think so, so that would be 25, is that what you’re saying?” Jeff’s response to Deb is one in which he acts a *leverager* because he takes part of the content of a Deb’s argument (“the new amount of rabbits plus the old amount of rabbits”, which an expert would recognize as the first iteration of Euler’s method) and mentally computes and says out loud the next population value, which is something that Deb has not provided. In this way, Jeff contributes to the group’s algorithmatizing efforts by **forming quantities**, in particular the amount of rabbits after a half year.

Liz, rather than bring out the computation, in line 28 elaborates on what is for her the meaning of the rate of change. Similar to Argument 5, the Data that Liz use (line 28) are that the initial rate of change is 30, and she uses these Data to make a new Claim (30 can be interpreted as 30 more rabbits in one year). With this comment Liz acts as a *rephraser* because she expresses Deb’s idea in her own words. This results in her expressing for the first time in their group a particular meaning for dP/dt, namely as the amount of rabbits that will accumulate over a one-year time period, what we refer to <u>as a unit ratio</u>.

Interpreting rate of change as the amount of change that happens over a unit time increment is consistent with what Thompson refers to as speed length [43]. Speed length refers to a meaning for rate in which a person conceptualizes, say, 30 miles per hour as the distance traveled in one hour. Similarly, we see here the initial rate of change of 30 means for Liz the increase in the number of rabbits over a unit time interval. Hence, we refer to this meaning of rate as a unit ratio. The expression of this comes with the Qualifier (“I don’t know if I am going to say this right”) and a non-verbal *checker* role (looking to Deb and to her other groupmates as if to seek affirmation).

The previous exchanges between the groupmates illustrate well the weaving of argumentation, algorithmatizing, various participation roles, and meaning-making threads. In particular, as a response to Deb’s argument in which the Euler method algorithm begins to emerge and the interpretation of 30 as the initial rate of change functions-as-if-shared, we see the groupmates take on the participation roles of *relayer*, *author*, *checker, leverager*, and *rephraser*; we also see the interactive constitution of a new meaning for rate of change, that of <u>unit ratio</u>. The collective progress in recognizing 30 as the initial rate of change (which functions-as-if-shared in Argument 5) and the emergence of the <u>unit ratio</u> meaning of 30 are building blocks for reinventing the Euler method algorithm because it begins to **isolate and quantify** the amount of rabbits that needs to be added to the initial amount of rabbits. All of this illustrates well the interplay between collective and individual mathematical progress.

As the discussion continues Deb (line 29) says, “So we’ll have 30 more rabbits”, and Liz follows by saying, “But we only want to go a half a year”. With this contribution, Liz acts as a *focuser* in that she directs her group’s attention to the need to know the population increase over a half year, not over a full year. In so doing Liz introduces a refinement to the unit ratio meaning of rate of change, namely as a <u>unit ratio that can be decremented</u>. In refining this meaning of rate of change, she furthers their algorithmatizing progress by focusing their attention on the change that happens over a half year. This occurs through Argument 9 in which the Claim is that the change over a half year is 15. The Data for this Claim are that there are 30 more rabbits over one year. Thus, since the Data for this new Claim were previously the Claim in Argument 8 per Criterion 2 there is evidence that the unit ratio meaning for rate of change now functions-as-if-shared. As such, this also functions as additional evidence for the collective algorithmatizing progress because it is now a mathematical truth for this small group that 30 is the amount that would be accumulated over one year.

Next, in line 32 Liz acts as *summarizer* by offering a Backing for the core of Argument 5. "And so we're really not figuring out the rate of change we're figuring... Well this is the rate of change and we're using the rate of change to figure out the number of rabbits we are going to increase by in half a year". At this point, Joe, who has been largely a silent participant, speaks up and expresses some confusion over what the initial population is at time zero. In particular, Joe (line 34) says, "Well this doesn't make sense to me because if t is 0 then we have 0. But you said when *t* is zero we have 10". Joe's statement here harkens back to the very first argument in which his groupmates worked with the problem's given statement that the value of the initial population is 10. At the time, Joe apparently agreed with this when he contributed to the following Data to Argument 1, "Oh, ok. This is where 10 rabbits at zero". However, now by examining Joe's comments in line 10 ("It would be 10 = 3 t zero") and in line 15 ("Wouldn't 10 = 3P(t)?") we see that Joe is referring to 10 as the initial rate of change. Liz proceeds to correct him and says, "Well 10 is actually the population".

Thus, it appears that the entire time that his groupmates were making progress, Joe was not part of this progress. In particular, although our analysis establishes that two ideas function-as-if-shared in the group up until this point, lines 10 and 15 indicate that Joe was not part of this collective mathematical progress. Such a finding is wholly consistent with what is meant by collective progress. Collective progress does not mean that every member of the collective shares identical meanings. It means that the group functions "as if" everyone is in agreement. Indeed, throughout the first episode and up until this point in the second episode, Liz, Jeff, and Deb have acted "as if" Joe also was using 10 as the initial population.

After countering Joe's Claim that the initial population is zero, Liz in line 37 continues with her Backing for Argument 5. "Well actually we're going to multiply it by a half year". With this, Liz puts the finishing touch on reinventing the first step of Euler's method (i.e., algorithmatizing). In particular, Liz verbalizes how to find the increase in rabbits after a half year has elapsed (**isolating attributes and forming quantities**).

Next, Deb in line 38 follows up on what Liz just said and acts as both author and summarizer by offering an explanation of her approach, which we coded as Argument 6 (see Figure 7).

38 Deb	This is what I did. First I looked at the fact that this is a rate of change equation. So this is telling me how many rabbits are being produced every year (**W6**). So If I know 3 times the original population is produced every year, then I have 3 times 10 is produced every year (**D6**). But I want to know how many is produced in 0.5 years. So I know how many rabbits are produced per year, so if I multiply that by 0.5 then I'll know how many more rabbits have been produced. So I take that new number that I get and add it to the old population. (**C6** started in 24 and continues in 40)
39 Joe	It's like compound interest almost. (**B6**)
40 Deb	Uh huh, so then I have the new population of rabbits. (**C6** continuation from 38)

Figure 7. Transcript containing Argument 6.

In this final Argument in Episode 2, Joe, who previously was out of step with his groupmates, now contributes to the group's progress by adding a Backing (line 39) to the core of the argument authored by Deb (lines 38, 40), likening the process to that of compound interest. This contribution suggests he follows the argument put forth by Deb. Within Argument 6, Deb leverages as Data the meaning of rate of change as a unit ratio, which previously was a Claim made by Liz. Hence, per Criterion 2 there is repeated evidence that the unit ratio meaning for rate of change functions-as-if-shared. Finally, Deb (38) acts as *summarizer* for the group when she contributes to the disciplinary practice of algorithmatizing (**creating and verbalizing relationships between quantities**) by stating in words the heart of Euler's method algorithm ("multiply that [the number of rabbits produced per year] by 0.5... and add it to the old population").

9.3. Segment 3—Deciding to Repeat

The next part of the small group work consists of 19 talk turns and five arguments (see Figure 8). Two of these arguments (Arguments 11, 12) are key to building the algorithmic tool in terms of students determining how and why to repeat parts of the algorithm to make it iterative. As such, the algorithmatizing activity of the students is focused on **creating and verbalizing relationships between quantities**. As students solidify earlier ideas and build new ideas, several different participation roles appear. In addition, this solidifying process allows students to use various meanings for the rate of change dP/dt to describe the creation of relationships between quantities.

41 Deb	**Then I find the one with my new rate of change (W11), so I just take that population and put it in for P (D11). And that is 3 times whatever that is (C11).** I mean that's what you guys were saying anyway.
42 Liz	Yeah I get it, do you guys get what Deb is saying?
43 Jeff	**Yeah you get 25 (C7) and then you get 55 [sic].**
44 Deb	**I think we should make a chart like he did. [showing her table to Jeff] But this would be your equation. This would be your 0.5, and then rabbits per year, and that will be your new amount of rabbits that's been added on, add that to your old amount of rabbits, and you'll get your new population (B11).**
45 Jeff	**I think you can go dp/dt=30, actually you can do it like this. dt will be 0.5, and then you add that to the old (D12) and then you do it again for the next one (C12).**
46 Deb	Exactly. I didn't see all that. I had to write it out.
47 Jeff	When you said it that's what popped into my head.
48 Deb	But that's what you guys were saying anyway.
49 Liz	What do you have right there? [Deb: shows Liz her paper]
50 Deb	**You take your old population, or your old rate of change which is really like rabbits per year [Liz: Ok] say, we already know it's 30 [sic] rabbits per year. How much time that has passed? 0.5 time has passed. Because 0.5 amount of time goes passed,0.5 times this will give me how many new rabbits I have accomplished (D13]) So I take the new amount of rabbits I have and add it to the old amount of rabbits I have which is 30 [Liz: ok] and that will give me the new population. (C13) Liz: Ok, so here's] And now I know the new population I know the new rate of change (C14) because I took the new population and put it in right here (D14) and I have the new rate of change. (C14)**

Figure 8. Initial transcript for Segment 3.

At the beginning of this segment Deb (line 41) authors the argument that the new rate of change is 3 times the new population. Deb serves as the *author* for Claim, Data, and Warrant and *relayer* for the Backing. Liz (line 42) functions as a *checker*, making sure that everyone is on board, while Jeff (line 43) tests his understanding by acting as a *relayer* of his previous Claim in Argument 7 that P at 0.5 is 25, although he then states, "and then you get 55" (sic) instead of calculating the new dP/dt of $3 \times 25 = 75$.

Liz and Jeff's contributions lead to Deb adding a Backing to Argument 11. This Backing uses the earlier Claim from Argument 5 that the change in P, "the amount of rabbits added on", is 0.5 times the rate of change and the Claim from Argument 6 that the new population is the old population plus the new amount "added on". In this way, these two statements have moved from Claim to Backing, satisfying Criterion 2, so that each of these statements are now functioning-as-if-shared.

This transition of the earlier Claims from Arguments 5 and 6 to interpretations that function-as-if-shared sets the stage for the generalization of these relationships needed to create the iteration of the algorithm. Following from Deb's statement in (line 44), Jeff (line 45) is the *author* of a new Claim (Argument 12) and a *relayer* of the ideas contained in the Data for this Claim. The Claim of Argument 11 starts the iterative process by pointing to the new rate of change at 0.5. The Claim for Argument 12 states that the entire cycle of calculations should be repeated with the Data for Argument 12 highlighting the previous calculation of multiplying dP/dt by 0.5 (Claim of Argument 5) and then adding the result to the old P (Claim of Argument 6).

Deb (line 46) agrees with Jeff's Claim, saying, "Exactly. I didn't see all that. I had to write it out". Liz (line 49) then serves as an *elicitor* to further the clarification of the argument, "What do you have right there? [indicating Deb's paper]". Deb (line 50) then serves as a *relayer*, restating the previous calculations but with an emphasis on enacting them after "0.5 time has passed". Deb's explanations in line 50 further serve the algorithmatizing practices of **forming quantities** and **creating and verbalizing relationships between quantities**. The quantities formed and the relationships created leverage Deb's meanings for rate of change. For example, she states in line 50, "You take. . . your old rate of change which is really like rabbits per year" (a unit ratio meaning) and then "0.5 times this will give me how many new rabbits I have accomplished" (a unit ratio that can be decremented meaning). Deb continues, "I took the new population and put it in right here, and I have the new rate of change" (a quantity that can take on different values meaning).

As the conversation continues, Liz (line 51) serves again as an *elicitor* drawing out the reason for Deb to use a different value for dP/dt than they used previously, "And the reason for putting in the new population would be what?" In response, Deb (line 52) continues her explanation of the algorithmatizing by clarifying her meaning for rate of change, "Because now my population is larger [pulls hands apart] and I know the population changes at a constant of 3 times whatever that population is at that moment in time". Here she draws on the fact that dP/dt is a constant rate for each interval, but it changes from one interval to the next based on the size of the population (a quantity that can take on different values meaning).

In line 53, Liz then serves as a *rephraser* (attempting to express Deb's ideas in her own words) and a *summarizer* by pulling together ideas that have been stated earlier. In this way Liz rephrases previously stated ideas about how to find the increase in population and add this to the previous population to obtain the new population. See Figure 9.

53 Liz	Okay, so basically, the reason, cuz the reason, cuz I get you up into the point where you say you want to put in, what I understand is that we found our rate of change initially at time zero and I understand using that to find out what our population is after half a year. If we are expected to grow by 30 rabbits in a year then, in a half a year we grow by 15 rabbits and our new population will be 15 (**D15**).
54 Deb	No no
55 Liz	I mean 25 because 15 plus 10 is 25 (**D15**).

Figure 9. Rephrasing and summarizing.

Jeff (line 56) acts as a *relayer* of his previous idea and a *checker* to see if others agree that his idea is correct, "But then you would have to do it again, start over again?" This seems to be in line with Liz's thoughts as she immediately follows with an agreement and clarification, "Then you start over again. So it's kind of like if you wanted to shift it back and say it's like our new initial population but just a different—So we could label it time equals zero if we wanted to".

Deb acts as a *checker*, "Everybody agree?" Liz states agreement while Jeff and Joe both nod their heads. In this way the group comes to a significant milestone in their algorithmatizing activity. The quantities formed in Segment 2, the change in population and the new population after 0.5 years, are quantities that can be newly established at each 0.5 in that you can "start over again". The idea to "start over again" brings together the meaning of dP/dt as a quantity that can take on different values with the beginnings of the algorithmatizing aspect of **investigating the generality of the procedure.**

9.4. Segment 4—Using Smaller Time Increments

In this last segment, which consists of only five talk turns, the students consider using a time increment of 0.25 instead of 0.5. Deb, who already has this figured out, prompts her groupmates to determine the difference. See Figure 10.

60 Deb	I'm going to make a chart now. I can already, well I'll let you guys determine, what's the difference between when you went from 0.5 to 1 and when you went from 0.25 to 0.5 to 0.75? Because look we're ...
61 Jeff	So, that's, I'm sorry to interrupt you but, if you do it for the 0.5 here it is going to be different than for the 0.5 here (**C16**) because you have to include that other step (**D16**). (Jeff leans over and points at Deb's nascent chart, Joe looks on, and Liz is busy writing down her thoughts from the end of segment 3. See Figure 11).
62 Deb	Right. [Joe: Ok, ok]
63 Joe	Oh, because you have to start at 0.25 (**W16**).
64 Deb	Right, our increments have changed.

Figure 10. Partial transcript for Segment 3.

Here we see Deb act as an *elicitor* when she seeks to bring out her groupmates' ideas ("What's the difference between. . . . . .") in a rather teacher-like way. In response to Deb's question, Jeff and Joe become *authors* for the last argument of this 10 min session. Jeff reaches across to point at the chart Deb is beginning to make (see Figure 11) and makes the Claim that the two population estimates at t = 0.5 "are going to be different". The Data for his Claim are that "you have to include the other step". Including this additional step and recognizing that this now produces a different approximation is a key insight for the collective disciplinary practice of reinventing the Euler method algorithm. In particular, changing the time increment is a first step toward **investigating the generality of procedure** because one aspect of generality is flexibility in the time increment used.

Figure 11. Joe pointing to Deb's chart.

Joe, who earlier had not been on the same page as his groupmates and contributed much less than the others, now makes individual progress by providing a Warrant for Argument 16, "Oh, because you have to start at 0.25". Joe's Warrant resonates with Liz's earlier utterance in line 57 when she summarized, "so it's kind of like if you wanted to shift it back [Joe refers to this as 'start at 0.25'] and say, it's like our new initial population". Thus, although Liz's voice is not present in this exchange (she is busy writing down her thoughts), Joe acts as a *relayer* of her ideas in his Warrant, which embraces the meaning of rate of change as a quantity that can take on different values because with "starting over" comes computing a new dP/dt value. The segment ends with Deb acting as a *summarizer* (again, in a teacher-like way) with her brief statement, "Right, our increments have changed".

Although this episode is brief, we see the interwoven nature among all four constructs. The facilitator roles of *elicitor* and *summarizer* bookend the production of Data–Claim–Warrant whose content push forward the collective disciplinary practice of algorithmatizing by **investigating the generality of the procedure** and engage an important meaning for dP/dt as a quantity that can take on different values. It is at this point that the instructor invites students back to a whole class discussion in which different groups report on their progress. In a subsequent class the instructor invites students to express their algorithm symbolically, which coalesces in expressing this as $y_{next} = y_{now} + \left(\frac{dy}{dt}\right)\Big|_{now} * \Delta t$ and tagging their approach as Euler's method.

10. Discussion and Conclusions

In this paper we further developed a method for documenting the mathematical progress of learners as an integrated analysis of individual and collective activity using the reinvention of an algorithmic tool for approximating solutions as a case example. In so doing, we contribute to a growing body of research that is also working on this thorny methodological and theoretical problem (e.g., [32,44–46]). The method developed here illuminated the intricate connections between and interweaving of our analysis through four constructs. For example, the students' algorithmatizing propelled their mathematical progression and was populated with particular arguments and mathematical meanings. The students participated by stating arguments and meanings (e.g., as *author*, *relayer*) but also by encouraging others to do so and by highlighting important issues (e.g., as *elicitor*, *focuser*). The new facilitator roles of *focuser, elicitor, checker,* and *summarizer* we identified are important both in shaping the in-the-moment and retrospective analysis. It also allows researchers to examine relationships among the constructs, such as which meanings were central to the development of the reasoning that functioned-as-if-shared or to a disciplinary practice's development. For example, in the analysis presented here the unit ratio meaning for rate of change was the catalyst for making progress on reinventing Euler's method algorithmic tool.

We next step back and recap our extended analysis of the four segments to highlight the integrated approach among constructs. In Segment 1 the focus of the initial arguments and algorithmatizing is **goal-directed activity** towards **isolating attributes of dP/dt.** In particular, Claim 1 is the realization that 30 is the initial instantaneous rate of change, and Claim 2 is the recognition of dP/dt as a ratio of two changes. The role of *focuser* propels the **goal-directed activity,** and the role of *checker* allows for the negotiation the students need for **isolating the attributes of dP/dt**. The segment ends with a new meaning for dP/dt, this time in the Warrant for Argument 4 that dP/dt tells how the population is changing. In Segment 2, the small group makes considerable progress on creating an algorithm, including the surfacing of two new meanings for rate of change (as a unit ratio and as a unit ratio that can be decremented), engaging a range of participant roles, and establishing two ways of reasoning that function-as-if-shared within their small group. Two of the arguments in Segment 3 are key to building the algorithm in terms of students determining how and why to repeat parts of the algorithm to make it iterative. As such, the students' algorithmatizing activity is focused on **creating and verbalizing relationships between quantities**. As students solidify earlier ideas and build new ideas, several different participation roles appear. In addition, this solidifying process allows students to use various meanings for the rate of change to describe the **creation of relationships between quantities**. Finally, in Segment 4, the students consider using a time increment of 0.25 instead of 0.5. In this closing segment we see the confluence and synergy among all four analytical constructs. The facilitator roles of *elicitor* and *summarizer* bookend the production of Data–Claim–Warrant whose content push forward the collective disciplinary practice of algorithmatizing by **investigating the generality of the procedure** and engage the meaning as a quantity that can take on different values.

We chose to present each of the four segments by leading with an argument (as opposed to a particular meaning, for example). We did this because it offered an organized and informative way to provide the reader with extended transcript data, and it allowed us to capture the overall gist of what the students discussed. Thus, leading with an argument offered a bird's eye view and the relevant transcript data that could be linked and framed within the various constructs. Moreover, leading with an argument provided a segue into the more integrated and interwoven analysis that foregrounded the other constructs. We came to this approach via trial and error. That is, we experimented with leading with different constructs but found it challenging to tell the story in an efficient and informative way. Reflecting on this experience, we posit that because we are examining mathematical progress, it is useful to start with a construct that drives the mathematics forward, both for individuals and for the group. In our experience, starting the Results section with Toulmin's argument laid out the meanings and building blocks for the algorithm.

10.1. Implications and Limitations

The analysis suggests to us both curricular and pedagogical implications. For instance, precisely because of the productive role that interpreting the meaning of 30 (the initial rate of change) had for this group, changes have been made to the inquiry-oriented differential equations curriculum [47]. The student materials now include the hint to interpret the meaning of the initial rate of change, and the instructor materials now call out explicitly the importance of the unit ratio meaning in the reinvention of Euler's method for making progress on reinventing the algorithm. Pedagogically, we conjecture that the roles within the new facilitator design can be helpful to instructors in their classrooms as they establish expectations for ways that students can meaningfully support each other in their small groups. For example, if an instructor uses named roles for students (e.g., reporter, scriber, spy) one could create a new role, that of "monitor". This student could be charged with ensuring that everyone in their group gets a chance to talk, to have their ideas heard, and that the group is helping each other learn from each other's ideas. This operationalization of "monitor" was inspired by the roles of *elicitor* and *checker* that were identified in our analysis. In fact, one of us has begun integrating this named student role and is experiencing some positive outcomes with it in the classroom.

We also acknowledge that various limitations exist with respect to our methodological approach. To accomplish an integrated analysis of mathematical progress at the individual and collective level, it was imperative to have data in which mathematical progress at both individual and collective levels was evident. Thus, researchers who endeavor to use the methodology offered through our integrated analysis must have access to data for which the four analytical constructs are sensible to use. For example, throughout the 10 min segment analyzed in this paper, the students in the small group explained their mathematical reasoning and tried to make sense of others' reasoning. Indeed, as Gravemeijer stated, classrooms that leverage RME instructional design principles

> encompass the obligations for students to come up with their own solutions, explain and justify their solutions, to try to understand the explanations and solutions of their peers, to ask for clarification when needed, and eventually to challenge the ways of thinking with which they do not agree. The teacher's role is not to explain, but to pose tasks, and ask questions that may foster the students' thinking, and help them in this manner to build on their current understanding and to construe more advanced mathematical insights [48], (p. 220).

This description is consistent with classifications of what it means to teach using inquiry in university mathematics classrooms [49,50]. We find it difficult to imagine, for instance, that a lecture classroom with little to no observable student input would provide the data necessary for the individual and collective analyses described in this paper. Even with a rich data set, researchers may find it challenging to discern what grain size is appropriate for this methodology. In this paper, we analyzed a 10 min small group discussion with no teacher interaction; certainly, whole class discussions over longer time periods would be appropriate to analyze in a similar manner. Such an analysis would allow researchers to document changes in, for example, individual mathematical meanings and participation roles over time.

10.2. Next Steps

We see two extensions of the work presented here. The first centers on who the actors are in the interactions. Our data set did not include the teacher, but we see no reason that one could not use this approach to analyze a data set that includes the teacher, either interacting with students in a small group or leading a whole class discussion. Argumentation is a collective effort, and the teacher may, among other things, contribute to elements of an argument (e.g., provide a Backing or connect students' informal ideas and notation to conventional terminology and notation) as well as offer questions and requests that help students elaborate and/or clarify their reasoning [35]. These kinds of contributions could aptly be captured with any of the four constructs, however, we see particular promise in making use of the individual participation construct as the teacher seeds and supports argumentation and connects students' ideas to more formal and conventional terminology. Our choice to analyze a short episode in which a group of students made considerable progress, even without the intervention of the teacher, was, in our view, a strength because it allowed us to refine and operationalize the method. In several related analyses, we and others (e.g., [2,5,24,32]) did in fact include the teacher, and hence extending the approach presented here to data sets that include the teacher has strong precedent.

The second extension centers on the research questions asked. In this analysis we did not focus on equitable and inclusive participation. However, the level of detail in this approach certainly would admit close examination of who contributed, both in terms of number and nature of contributions, thus making strong connections between mathematical progress and equity. For example, it would be straightforward to examine the number of times that particular students (or the teacher) provided Claims (versus, say, Warrants and Backings) or the number of times that particular individuals acted as *summarizer* or *elicitor*, for example. A focus on equitable and inclusive participation could also examine whose voice and ideas were taken up (or privileged) and by whom.

10.3. Conclusions

In conclusion, the methodological approach presented here provides a nuanced and coordinated analysis of individual and collective mathematical progress. Although the case we presented here focused on an inquiry-oriented course in differential equations, the same methodological approach could be used in inquiry-oriented classrooms in other content areas in the mathematical sciences (e.g., statistics) and even in other disciplines (e.g., chemistry) with appropriate modifications to the framing and terminology used. For example, the collective disciplinary practices will be different in statistics and chemistry and instead of using the phrase "individual mathematical meanings" one would use the phrase "individual statistical meanings" or "individual chemical meanings", but the overall methodological approach remains the same. Thus, regardless of the content or disciplinary focus, the coordination of individual and collective analyses allows researchers to gain greater explanatory and descriptive power, and it also allows researchers to better understand the interactive process by which individual and collective mathematical progress is made.

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