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Review

Techno-Mathematical Discourse: A Conceptual Framework for Analyzing Classroom Discussions

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Abstract: Extensive research has been published on the nature of classroom mathematical discourse and on the impact of technology tools, such as virtual manipulatives (VM), on students' learning, while less research has focused on how technology tools facilitate that mathematical discourse. This paper presents an emerging construct, the Techno-Mathematical Discourse (TMD) framework, as a means for analyzing and interpreting aspects of learning when students use technological representations to mediate mathematical discussions. The framework focuses on three main components: classroom discourse, technology tools, and mathematical tasks. This paper examines each of these components, and then illustrates the framework using examples of students' exchanges while interacting with virtual manipulatives. The TMD Framework has applications relevant to teachers, teacher educators, and researchers concerning how technology tools contribute to discourse in mathematics classrooms. The TMD framework addresses a critical issue in mathematics education, in that classroom teachers and researchers need to understand how technology facilitates classroom interactions and how to best leverage technology tools to enhance students' learning of mathematics.

Keywords: virtual manipulatives; classroom discourse; technology education

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1. Introduction

As part of recent reform efforts in mathematics education, mathematical discourse and use of technology have emerged as key characteristics of high-quality instruction and engaging mathematics. Mathematical discourse enables students to think about "what it means to know mathematics, what makes something true or reasonable, and what doing mathematics entails. It is central to both what students learn about mathematics and how they learn it" [1] (p. 54). When students have opportunities to articulate and discuss mathematical concepts, they develop a deep understanding of those concepts. With its roots in Vygotskian social learning theories, the study of the structure and flow of mathematical discourse in the classroom has generally taken a socio-cultural or socio-linguistic perspective, and many researchers have developed different ways to conceptualize and classify how discourse occurs in classrooms and its impact on learning [2–18].

Technology offers a variety of tools that enhance the learning of mathematics concepts by expanding representational possibilities and amplifying and reorganizing students' approaches to problem solving [19]. For example, virtual manipulatives (VM) [20,21], and gaming applications available on a variety of platforms (e.g., calculators, personal computers, tablets) have the potential to significantly influence the depth to which students understand important mathematics concepts [22]. Due to a variety of factors, implementation of technologies in education settings tends to lag behind the pace at which the technologies are developed [23]. The need to advance technology applications in education by focusing on how the technologies are used in the classroom, and not just on identifying what technologies are available presents a critical issue in mathematics education [24]. To this end, the purposes of this paper are: (a) to introduce the Techno-Mathematical Discourse (TMD) conceptual framework, which models how technology can interact with discourse-rich learning environments; (b) to apply the framework to students' interactions when using technology to learn mathematics; and (c) to discuss implications and suggest further applications of the framework.

2. Techno-Mathematical Discourse Conceptual Framework

The TMD framework considers three components of the learning environment that impact mathematical discussion: classroom discourse, technology tools, and mathematical tasks (see Figure 1). In essence, the TMD framework describes how students use technological representations to mediate discussion while engaging in worthwhile mathematical tasks. Learning takes place in complex and dynamic environments, and many factors influence how students learn, especially when working with technology. What are the classroom expectations for discourse? What affordances for learning are offered by the technology tool in use? What is the nature of the mathematical task in which students are engaged? The TMD framework presents a structure to think about these questions. It emerges from a synthesis of empirical and theoretical research involving these components [25]. The following sections describe how each component contributes to the nature of students' mathematical discourse when using technology.

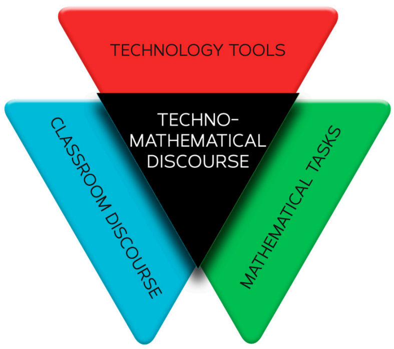


Figure 1. Techno-Mathematical Discourse (TMD) Framework.

2.1. Classroom Discourse

Effective classroom interactions are key to successful learning [26]. Vygotskian notions of learning emphasize discourse and communication as a means to learn new concepts. Describing learning as a socially constructed phenomenon, sociocultural learning theory asserts three major tenets: (a) higher mental processes are determined by how and when they occur; (b) higher mental processes first occur on the social plane (i.e., between people), and then occur on the individual psychological plane; and (c) higher mental processes are mediated by cultural tools and signs (e.g., symbols, speech, and writing). Therefore, students develop understanding as they interact with other individuals through verbal or nonverbal communications or written words [27].

The socially constructed phenomenon of learning is also referred to as commognition—a combination of communication and cognition. In this sense thinking is defined as "the individualized form of the activity of communicating, that is, as communication with oneself" [28] (p. 569). Therefore, in order to deeply understand complex concepts, some form of discussion must take place—even if that conversation occurs within one individual. When considered in the classroom context, rich meaningful communication consists of "interactive and sustained discourses of a dialogic nature between teachers and students aligned to the content of the lesson that addresses specific student issues" [11] (p. 378). In other words, meaningful classroom discourse contributes to students' understanding by promoting effective communication and articulation of thought.

The culture of a classroom and the discourse practices established by the teacher also play considerable roles in shaping classroom mathematical discourse. Sociomathematical norms develop within a classroom and constitute what interactions are valued and what counts as an acceptable mathematical explanation [29]. Through these interactions, students analyze and evaluate the mathematical thinking and strategies of others and deepen their own mathematical understanding. Students must organize and consolidate their mathematical thinking in order to communicate effectively with their classmates and with the teacher [7,11,30–32]. Teachers' discourse practices also influence the nature of classroom discussions. For example, dialogic discourse—involving teacher and students in active communication—tends to be associated with lessons in which students build meaning, explore, and generate hypotheses. On the other hand, univocal discourse—involving teacher communications that require minimal student response—tends to be associated with lessons in which teachers transmit meaning to students, present definitions and procedures, and make applications to individual problems [15]. This idea of teacher- versus student-centrality in whole-class discourse is a common underlying theme in research on classroom discourse [5,8,33–37].

Impact of Classroom Discourse on TMD

It naturally follows that classroom cultures that promote active communication through student-centered discourse will also promote positive TMD. Students are more likely to discuss mathematics concepts when using technology if they are already situated in a classroom with expectations for mathematical discussion. In fact, classroom mathematical discourse is enhanced by the introduction of certain technologies. Discourses associated with technology tools (as described in the following section) tend to be more collaborative, perhaps because students are focused on a common display or screen [2,37,38].

The teacher's role in facilitating mathematical discourse shifts slightly with the introduction of technology. During whole-class discussions, the teacher becomes responsible for orchestrating students' interactions with the technology as well as interactions with each other. Furthermore, the teacher needs to model appropriate discourse practices as students work in small-group collaborations on the computers [13,37]. The dynamic nature of the technology introduces additional elements of interaction beyond those found in tasks not involving the technology. Thus, during these small-group collaboration sessions, the teacher's roles of intervening when necessary, and questioning to extend students' thinking become even more imperative as students work with dynamic technological representations.

2.2. Technology Tools

Over the past few decades, technology has developed new ways to think about and to represent mathematics [39]. These *cognitive technology tools* enhance the learning of mathematics concepts by expanding representational possibilities and by amplifying and reorganizing students' approaches to problem solving [19,40].

With the advancement of computer capabilities, virtual manipulatives have emerged as cognitive technology tools for use in mathematics classrooms. A virtual manipulative is defined as "an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge" [20] (p. 13). Based on this dynamic nature, virtual manipulatives seem to be a combination of *manipulative models* (e.g., base-ten blocks, fraction bars, counting bears), which allow for concrete examples of mathematical relationships and operations and *static pictures*, which provide an image for a learner to internalize [41]. These "computer based renditions of common mathematics manipulatives and tools" [42] (p. 329) provide teachers and students with expanded tools for thinking about mathematics concepts. Recent research supports the use of virtual manipulatives as effective instructional tools [43–47]. Emerging research also indicates that virtual manipulatives on touch-screen platforms, such as iPads may have instructional benefits [48–51].

Computer-based representations vary in their level of cognitive fidelity [52]. Some representations offer manipulative tools that truly reflect the user's actions and choices without dictating solution paths.

Other representations include concept tutorials (with or without manipulative tools) to guide students to a conceptual or procedural understanding of the mathematics. Still, others present an electronic figure, either static or in motion, very similar to a textbook or worksheet. A recent meta-analysis of research on virtual manipulatives [21] identified five affordances offered by specific design features and elements of different virtual manipulatives: focused constraint, creative variation, simultaneous linking, efficient precision, and motivation. Focused constraint describes features of virtual manipulatives that focus students' attention on particular aspects of mathematical objects—aspects that they otherwise may not have paid attention to. The affordance of creative variation "allows students to generate their own representations, encourages creativity and novelty, and prompts experimentation" [21] (p. 43). Simultaneous linking describes the features of many virtual manipulatives that dynamically link different forms of mathematical representation (e.g., graphical, pictorial, symbolic, abstract). Efficient precision refers to features of virtual manipulatives that "provide precise mathematical examples, and create multiple copies of dynamic objects efficiently" [21] (p. 44). Lastly, motivation was identified as an affordance, in that virtual manipulatives have the potential to make learning more enjoyable and to encourage students to persist in problem solving. These affordances, along with varying features of virtual manipulatives, have implications for instructional use.

Impact of Technology Tools on TMD

Through techno-mathematical discourse, technology enhances the communication of mathematical ideas and supports students' learning of mathematics concepts. When learning mathematics concepts with technology in a discourse community, students have access to multiple modalities of mathematical representations. First, technology tools, such as virtual manipulatives, provide dynamic pictorial and symbolic representations of mathematics concepts. Second, the dynamic visual displays serve as common experiences about which students can engage in meaningful classroom discussions incorporating both verbal and gestural (i.e., embodied) interactions [13,53,54]. Students' understanding of mathematical concepts is strengthened when they make connections among representations in pictorial, symbolic, verbal, and embodied modalities [55]. Of course, the strength of TMD is influenced by the affordances of the available technology tools. Technology tools that present students with multiple representations and that have high levels of cognitive fidelity tend to promote students' TMD. Technology tools that guide students step-by-step to pre-determined formulas or representations tend to hinder students' TMD [25].

2.3. Mathematical Tasks

"Worthwhile mathematical tasks", as defined by the National Council of Teachers of Mathematics (NCTM) [1], promote communication, engage students' intellect, develop mathematical understandings and skills, represent mathematics as an ongoing human activity, and embed mathematics in meaningful contexts. For example, instead of having students simply memorize multiplication facts or mathematical vocabulary, worthwhile tasks embed the multiplication facts and vocabulary in "meaningful contexts that help students see the need for definitions and terms as they learn new concepts" [1] (p. 33).

A worthwhile mathematical task is one that engages students' intellect and calls for problem solving and mathematical reasoning. According to Smith and Stein [56], tasks vary in their level of cognitive demand. Tasks with lower levels of cognitive demand involve reproduction of memorized facts and algorithmic procedures with no connection to the concepts underlying the procedures. They have clear solution paths and require no explanation of mathematical thinking beyond a description of the procedure used. Tasks with higher levels of cognitive demand (i.e., worthwhile mathematical tasks) involve multiple solution paths and/or multiple possible solutions. Students must analyze the task and present solutions in multiple representational forms. Smith and Stein note that tasks with higher levels of cognitive demand likely produce anxiety for some students due to the uncertain and unpredictable nature of the problem. This anxiety is a sign of cognitive disequilibrium experienced by students as they come to understand new concepts [57]. The mathematical content and tasks presented in a lesson significantly affect the amount of learning that occurs [58]. By presenting non-routine problems that require students to actively engage in mathematics (as opposed to mindlessly following procedures), worthwhile mathematical tasks represent mathematics as an "ongoing human activity" [1] (p. 33) and provide opportunities for students to make deep connections between mathematical ideas.

Impact of Mathematical Tasks on TMD

In order for rich discussions to take place, students must be presented with tasks that are worth talking about [6,8,12,59]. Therefore, even though a task may incorporate technology tools, if it is not a worthwhile mathematical task, it will not produce the desired TMD, regardless of the affordances offered by technological tools. When worthwhile mathematical tasks make explicit use of the technology tools, the opportunity for rich TMD increases [51–54].

2.4. Links Among Mathematical Discourse, Technology Tools, and Mathematical Tasks

Technological developments constantly emerge presenting opportunities to improve classroom practices and learning. A great deal of research has been conducted in an attempt to verify the usefulness of such technologies. However, research on the role of discourse in technology-based learning settings is less plentiful. From the research that has been conducted, two major themes emerge: (a) the impact of dynamic representations on the content and nature of mathematical discourse; and (b) the impact of computer feedback on student collaborations. These themes are discussed in the following sections.

2.4.1. Impact of Dynamic Representations on Classroom Discourse

Technology has the potential to produce dynamic representations of mathematics concepts. The dynamic nature of these representations has a profound impact on the level of classroom mathematical discourse. For example, Ares, Stroup, and Schademan [53] describe a lesson using networked classroom technology—a wireless network of graphing calculators that collects students' solutions and displays them collectively on a screen at the front of the room. In this particular lesson, students used their calculators to "maneuver an elevator" by determining how many levels it would move up or down in one-second intervals. The collective resulting position-time graphs were then displayed on the front screen. Different tasks throughout the lesson gave specific parameters causing the students to focus on different mathematical relationships (e.g., end on the −2 floor using any combination of movements, the fourth movement must be to go up three floors). The researchers noted that the collective representation encouraged students to interact with each other and comment on the various solutions. Students focused on the mathematics represented dynamically on the visual display and used it as a basis for their mathematical discussions. Additionally, the visual display mediated a shift in the discourse from conceptual to more formal language (e.g., "they all go up at the same time" to "each line has the same slope, so they are all parallel to each other").

Similarly, Sinclair [13] and González and Herbst [54] each report on studies with dynamic interactive geometry software (Geometer's Sketchpad and Cabri Geometry, respectively). In Sinclair's study, students worked in pairs with Geometer's Sketchpad to complete a sequence of tasks on proving congruency (e.g., applications of reflection and rotation). The dynamic nature of the software enabled the students to test conjectures and receive immediate feedback. Just as observed by Ares et al. [53], the students in Sinclair's study used the visual representations to fuel their mathematical discussions. However, these students displayed varying degrees of effectiveness in their discussions. As noted above, they engaged in productive discourse by explaining their thinking and asking thoughtful questions. But at other times, students' discourse actually hindered the development of mathematical ideas. Due to this variation in productivity, Sinclair emphasizes the need for follow-up classroom sessions after time spent in the computer lab to solidify understanding and to ensure that all students have appropriate opportunities to learn the content.

González and Herbst [54] report a more positive view of student discourse when working with dynamic interactive geometry software. Students in this study also completed a sequence of tasks to investigate congruency. However, instead of applying transformations (as in Sinclair's [13] study) these tasks required them to experiment with midpoints and angles. The measuring and dragging features of the Cabri Geometry software enabled students to quickly and accurately assess the results of their experiments. The interactive features of the software tools supported all students' learning in the lesson. In whole-class discussions, advanced students described how they used the tools to prove their conjectures and pointed out new ideas. At the same time, other students who did not fully understand the technical terms for the geometrical relationships could still participate in discussions because of the support of the technological representations. Therefore, this study confirms previous findings that interacting with dynamic representations enables and encourages students to talk deeply about mathematics.

2.4.2. Impact of Computer Feedback on Student Collaborations

The ability for technology to give dynamic feedback to students, either verbally or nonverbally, contributes to the level of classroom mathematical discourse. Studies have shown that valuable visual feedback provided by graphing software programs, among other technologies, prompt productive problem-solving student discourse. For example, Gibbs [38] documented students' attempts to graph particular quadratic functions with varying scales. When the computer-produced graph did not visually match the graphs that students had previously drawn, discussions ensued regarding the discrepancies and how to reconcile them. Likewise, other studies report positive effects on problem-solving discussions as a result of feedback from dynamic computer diagrams [37,54].

Evans et al. [2] conducted a study comparing effects of virtual and physical tangram puzzles on student discourse. Using a multimodal approach (speech, gesture, gaze, and actions) to analyze the discourse of 7- to 8-year-old children, the researchers identified more co-references (i.e., shared reference points) among the students when using the virtual manipulative tangrams. They determined that discourses associated with the virtual manipulatives tended to be of a more collaborative nature, perhaps due to a forced focus on a common screen and having to negotiate control of the mouse. On the other hand, students using the physical tangram pieces had the option to handle the pieces individually without permission from the rest of the group. The focus on a common display to promote active mathematical discourse aligns with previous findings [37,53].

More recently, Anderson-Pence [25] documented differences in fifth-graders' student-student discourse associated with various types of virtual manipulatives. Using a mixed methods multiple case study design, the researcher reported that students' discussions reflected the most robust levels of generalization, justification, and collaboration when using virtual manipulatives that linked multiple mathematical representations and that allowed students to problem solve. This study builds upon previous research by indicating features of virtual manipulatives that encourage quality discourse among students.

3. Applying the TMD Framework

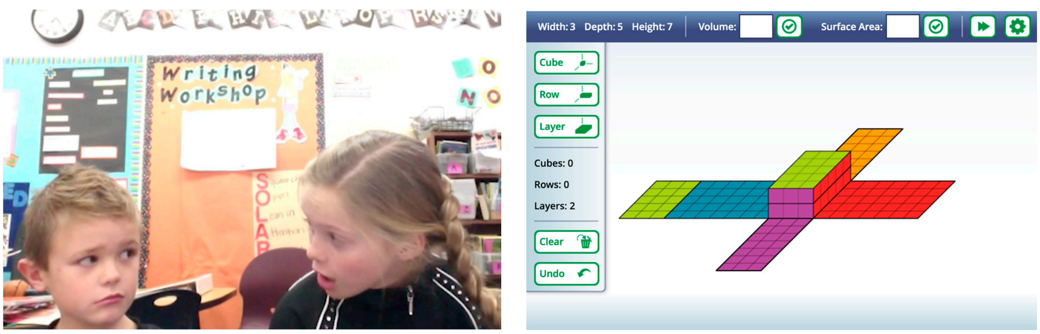
The TMD framework, when applied to the classroom, offers teachers a way to consider different aspects of mathematics instruction integrated with technology. As a teacher makes instructional decisions, he or she may reflect on any of the following questions: Does the technology tool address the target objective of my lesson? What technology tool will best help my students to develop understanding of this target objective? What feedback features of the technology tool are likely to motivate my students to learn and to discuss mathematical ideas? Will the technology tool engage the students in an open-ended task or in a task with one way to arrive at the correct answer? How will the technology tool and/or discussion help students to build fluency with procedures and/or develop understanding of concepts? Does the technology tool and task engage students in meaningful mathematics? Can the technology tool assist in orchestrating mathematical discussion so that students are able to share their thinking? These questions (and many more) influence how teachers design mathematics instruction. Teachers can make effective instructional choices by considering curriculum standards, their own knowledge of their students' academic and personal background, and the components of the TMD framework.

3.1. Examples of TMD in Action

The following excerpts provide evidence of the TMD framework in a classroom setting. These excerpts are drawn from a study in which Anderson-Pence examined students' mathematical discourse while working with the virtual manipulatives [25]. The focus of the study was to examine student-student discourse associated with various types of virtual manipulatives. Three pairs of fifth-grade students participated in nine lessons each. The discussions were video-recorded, transcribed, coded, and analyzed. For each lesson, the classroom teacher began by activating students' prior knowledge, posing intriguing questions, and orienting the students to the virtual manipulative that they would be using in the lesson. After the students had spent some time working with the virtual manipulatives, the teacher facilitated a whole-class discussion focused on students' solutions and what was learned as a result of the activities with the virtual manipulatives. In the three examples presented here, one of the pairs, Colton and Callie (psydonyms), discuss and solve mathematical problems while working with various virtual manipulatives.

3.1.1. Filling Boxes

In the first example, Colton and Callie use a virtual manipulative, Cubes [60], in which they can specify the dimensions of a rectangular prism (see Figure 2). The virtual manipulative then generates a net of that prism, and the students use representations of cubes to "fill the box". In preparation for the students' work with Cubes, the teacher activated students' prior knowledge of length, width, height, and volume and showed a concrete model of a rectangular prism made of cubes. She asked questions such as, "How might we find the volume of this prism? What would be a quick and/or efficient way to count the number of cubes in this prism?" Because this was the students' first experience with Cubes, the teacher also instructed the students on key features and aspects of this particular virtual manipulative (e.g., how to place cubes on the workspace, various ways to clear cubes from the workspace). Finally, the teacher oriented the students to the task sheet that they would use while working with the virtual manipulative and communicated the expectation that students were to work with a partner and solve the problems together. In this exchange, the students use Cubes to look for patterns while determining the volumes of a $3 \times 5 \times 7$ box, of a $5 \times 7 \times 3$ box, and of a $7 \times 3 \times 5$ box (see Table 1).



(**a**) (**b**)

Figure 2. (**a**) Colton and Callie conversing while using Cubes to solve problems; (**b**) Screenshot of Colton and Callie's work on Cubes [60].

Table 1. Transcript of Colton and Callie's exchange using Cubes.

Line	Student	Speaking Turn
65	Callie	Ok, so how many unit blocks to fill the box?
66	Colton	20 . . .
67	Callie	. . . So 6, no that's 5 blocks . . .
68	Colton	(*using the mouse pointer to count the "blue" height of the box*) 1, 2, 3, 4, 5, 6, 7. 20 times 7 is 140. So it should be 140 blocks.
69	Callie	. . . For the whole thing, yeah. 140 blocks.
70	Colton	20 times 7 is 140. I'll just write 140 blocks . . . So change it to 5. What's the next number?
71	Callie	Width 5, depth 7, height 3. (*Colton types second set of dimensions into the virtual manipulatives (VM)*) So 3 times . . .
72	Colton	Wait. Hold on. Just a second. Let's go back to that one. It didn't have 20 on the bottom. We need to go back (*Colton types previous set of dimensions into the VM*).
73	Callie	Yeah it did.
74	Colton	3 . . . 5 . . . 7. Look, (*using mouse pointer to count the dimensions of the base of the first box*) 1, 2, 3, 4, 5 times 3. So it's . . .
75	Both	. . . 15 . . .
76	Colton	. . . times 7 . . . Not 20.
77	Callie	15 . . . 105.
78	Colton	Ok. All right. Now we do this . . . (*Types second set of dimensions into the VM*).
79	Callie	So the next one is width 5, depth 7, height 3. So . . .
80	Both	(*Colton uses mouse pointer as both count aloud the dimensions of the base of the second box*) 1, 2, 3, 4, 5, 6, 7 . . .
81	Callie	. . . times 5.
82	Colton	. . . times (*using mouse pointer to count*) 1, 2, 3, 4, 5. Yep.
83	Callie	So 35 times . . .
84	Colton	(*makes a sweeping motion with the mouse across the "blue" height of the box*) . . . 3. I got . . .
85	Both	. . . 105.
86	Callie	Again! . . . Ok (*reading task*) "What is the volume of a box with width 7, depth 3, . . . "
87	Colton	(*types third set of dimensions into the VM*) It's just changing the numbers up. So I think it will be 105.
88	Callie	(*reading task*) " . . . height, 5." Let's double check to see if it is 105 blocks. So . . .
89	Colton	(*Colton makes sweeping motion with the mouse pointer over the dimensions of the base of the third box*) 1, 2, 3.
90	Callie	So 7 times 3. 21 times . . .
91	Colton	(*examining image without counting with the mouse pointer*) . . . 5?
92	Callie	5 . . . . So . . . 105.
93	Colton	Yep . . . Ok.

This exchange demonstrates how the technology tool supported Colton and Callie in visualizing a single box orientated in three different ways. Instead of just asking them to calculate the volume of each box, this mathematical task required the students to make connections among the three boxes. Therefore, the worthwhile mathematical task provided something worth talking about and the virtual manipulative aided in the communication of their ideas. The technology tool provided a precise and accurate representation of volume that the students used to test ideas and problem solving strategies. As shown in this example, the discourse between Colton and Callie was characterized by collaboration and were efficient problem-solving strategies. Their language (e.g., let's, we, etc.) reflected a joint effort (lines 72, 78, 88). They quickly recognized errors in their work and made the necessary adjustments (line 76). They frequently referred back to the visual display of the virtual manipulative to catch mistakes, verify solutions, and justify their thinking (lines 72, 74, 80, 89). Figure 3 illustrates the TMD framework as applied to this exchange.

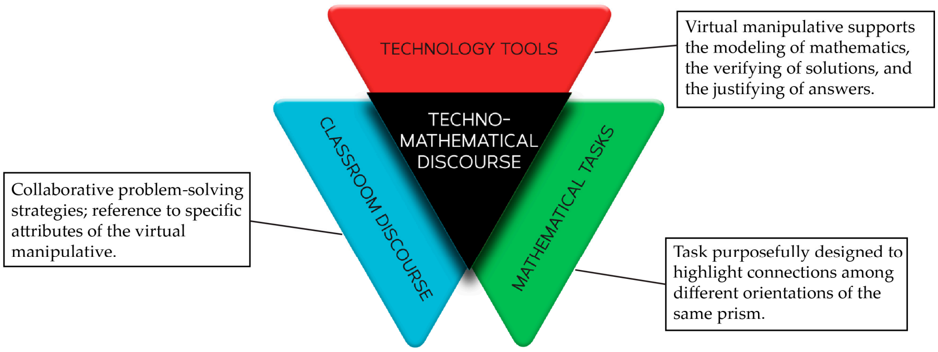
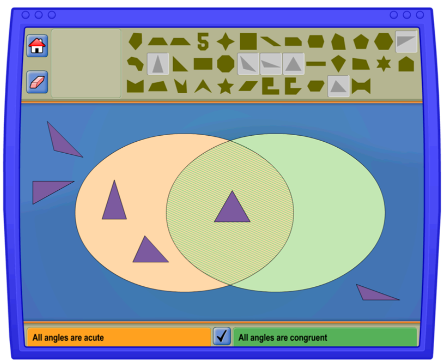


Figure 3. Students' TMD incorporated elements related to the Cubes VM, their discursive exchange, and the mathematical task at hand.

3.1.2. Sorting Triangles

In the second example, Colton and Callie use a virtual manipulative, Shape Sorter [61], which presents multiple shapes that can be sorted by various attributes using a Venn diagram (see Figure 4). Students then "drag and drop" the shapes into the appropriate section of the Venn diagram. A "check answers" feature allows students to verify their answers. In preparation for the students' work with Shape Sorter, the teacher activated students' prior knowledge by having the class identify obtuse, acute, and right angles. She defined key vocabulary words that the students would be using in this lesson (e.g., congruent, adjacent, acute triangle, right triangle, obtuse triangle, isosceles triangle, scalene triangle, equilateral triangle). She invited students to consider questions such as, "What are the characteristics of an acute/right/obtuse triangle? What are the characteristics of an isosceles/equilateral/scalene triangle?" Because this was the students' first experience with Shape Sorter, the teacher also instructed the students on key features and aspects of this particular virtual manipulative (e.g., how to select a rule, how to drag shapes onto the workspace, and how to check and clear their solutions). Finally, the teacher oriented the students to the task sheet that they would use while working with the virtual manipulative and communicated the expectation that students were to work with a partner and solve the problems together. In this exchange, the students use Shape Sorter to identify triangles that fit one or both of two rules: (a) all angles are acute, and (b) all angles are congruent (see Table 2).

(a)



(b)

Figure 4. (**a**) Colton and Callie conversing while using Shape Sorter to solve problems. Here Colton is using his hands to model an acute angle; (**b**) Screenshot of Colton and Callie's work on Shape Sorter [61].

Table 2. Transcript of Colton and Callie's exchange using Shape Sorter.

Line	Student	Speaking Turn
90	Callie	So all angles are acute (*selects "All angles are acute" on VM*). All angles are congruent (*selects "All angles are congruent" on VM*). Ok. So all 7 triangles . . . So this triangle... (*hovers mouse pointer over the first triangle in the set*).
91	Colton	. . . has 2 acute.
92	Callie	All angles are congruent—that's not true. All of the angles are not the same length.
93	Colton	Well it doesn't have all angles acute. So . . . it wouldn't be . . .
94	Callie	So it wouldn't go in either. (*moves the first triangle into the space outside of the Venn diagram*) This one. (*hovers mouse pointer over the next triangle in the set*) It definitely has all acute angles. (*makes a sweeping pointing motion toward the computer screen*) But . . .
95	Colton	Are all the angles congruent?
96	Callie	No. Wait.
97	Colton	No, they're not. Like the acute angles. Like one of them is like bigger and the other ones are smaller (*demonstrating larger and smaller angles with hands and forearms*).
98	Callie	Oh, yeah. 'Cause on this shape . . . (*circles mouse pointer over the triangle in question*).
99	Colton	That would just be in the red.
100	Callie	(*moves the triangle into the left-hand section of the Venn diagram*) This one?
101	Colton	Yeah.
102	Callie	Then next shape. This one . . . (*hovers mouse over the next triangle in the set*).
103	Colton	That's a right angle, (*pointing to triangle on screen with finger*) so . . .
104	Callie	Yeah, so it's not all acute. And it's not congruent. (*moves the triangle to the space outside of the Venn diagram*) So . . . ok. (*hovers mouse pointer over the next triangle in the set*) This one. It definitely does not have all acute angles . . .
105	Colton	Not all acute angles.
106	Callie	(*pointing to screen*) and it's a huge line. It doesn't match up with anything else. (*Colton moves the triangle into the space outside of the Venn diagram*) So it's neither. Next one. (*Colton hovers mouse pointer over the next triangle in the set*) That's an obtuse angle, (*pointing to triangle on screen with finger*) so it's not all acute and it's not all congruent. (*Colton moves the triangle into the space outside of the Venn diagram*).
107	Colton	This one . . . (*hovers mouse pointer over the next triangle in the set*).
108	Callie	All acute, right? And then it's all . . .
109	Colton	. . . congruent.
110	Callie	It's congruent, so . . . (*Colton moves triangle into the center of the Venn diagram*) and then . . . (*Colton hovers mouse pointer over the last triangle in the set*) This one. (*points to next triangle with finger*) All angles are acute.
111	Colton	And . . . I don't think they're all congruent. (*moves triangle into the left-hand section of the Venn diagram*) Check. (*clicks on the checkmark to check solution. VM feedback: all correct*).
112	Callie	Yep. We got them all right.

The exchange illustrates how the technology tools can support students' communication of mathematical ideas. For example, Colton connected his own knowledge of acute and obtuse angles to the angles of the triangles on the virtual manipulative by gesturing with his hands the relative sizes of the angles (line 97). These gestures, in concert with the pictorial representations presented by the virtual manipulative, provided a context through which the students could examine and discuss the characteristics of triangles. By combining the images on the virtual manipulative (line 106) with physical gestures, Colton and Callie effectively communicated their mathematical thinking. This particular virtual manipulative also provided feedback to the students on the accuracy of their solution (lines 111–112). Figure 5 illustrates the TMD framework as applied to this exchange.

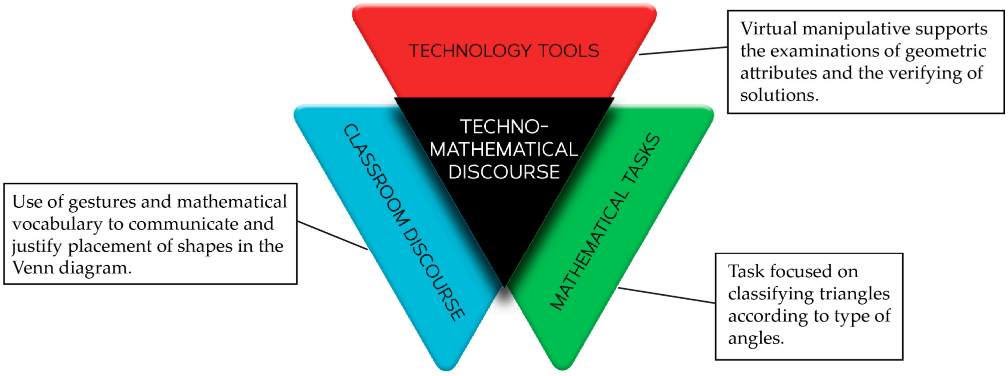
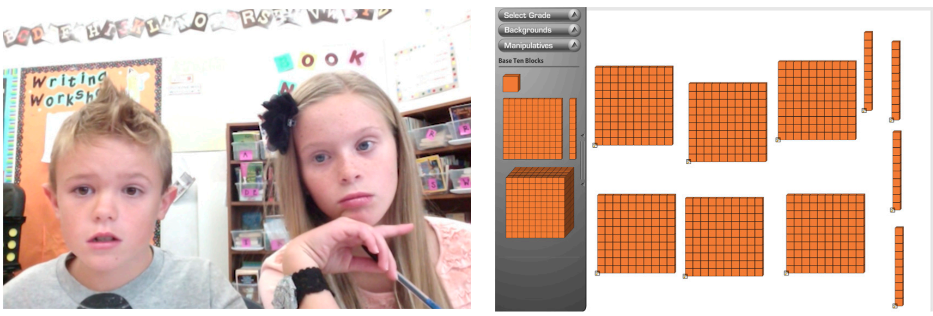


Figure 5. Students' TMD incorporated elements related to the Shape Sorter VM, their discursive exchange, and the mathematical task at hand.

3.1.3. Decimal Division

In the third example, Colton and Callie use the virtual manipulative, Base Ten Blocks [62], in which students can "drag and drop" images of ones-units, tens-rods, hundreds-flats, and thousands-cubes onto a work space (see Figure 6). Each image (with the exception of the ones-units) can be broken up visually into smaller pieces (e.g., a hundreds-flat can be broken up into 10 tens-rods). Likewise, images of the same type may be snapped together. In preparation for the students' work with Base Ten Blocks, the teacher activated students' prior knowledge by posing the problem 84 ÷ 3 and having the class identify two different ways to solve the problem. She then asked the class, "What if we had 8.4 ÷ 3? How does that change the problem? How is dividing decimals similar or different than dividing whole numbers?" She invited a few students' responses, but did not require full understanding at this point of the lesson. Because this was the students' first experience with Base Ten Blocks, the teacher also instructed the students on key features and aspects of this particular virtual manipulative (e.g., how to drag blocks onto the workspace, how to break apart or combine blocks, how to clear blocks from the workspace). Finally, the teacher oriented the students to the task sheet that they would use while working with the virtual manipulative and communicated the expectation that students were to work with a partner and solve the problems together. In this exchange, the students use Base Ten Blocks to solve a story problem involving division of a decimal fraction by a whole number (see Table 3). The story problem that provided the context for this task was:

> Nancy's poster for the school council election covers a space of 6 point 4 square meters. She wants to divide the poster into 4 equal sections for her slogan. How much space will be in each section? Hint: one 10 by 10 square represents 1 square meter. Talk with your partner about how to solve this problem. Write down your answer and explain your thinking.



(**a**) (**b**)

Figure 6. (**a**) Colton and Callie conversing while using Base Ten Blocks to solve problems; (**b**) Screenshot of Colton and Callie's work on Base Ten Blocks [62].

Table 3. Transcript of Colton and Callie's exchange using Base Ten Blocks.

Line	Student	Speaking Turn
45	Colton	So 64 divided by 4 equals what?
46	Callie	Yeah, so we know that we have 6 point 4 square meters. And a 10 by 10 equals one square meter. So if we have six, then we need six 10 by 10s. (*Colton moves 6 hundreds-flats onto the workspace section with mouse*) Ok, then it says, (*reading task*) "Talk with your partner about . . . " So we know that we have 6 whole pieces. But then what about the other 4 . . . ?
47	Colton	We need point 4. So that would be the tens.
48	Callie	Yeah. (*Colton moves 2 tens-rods onto the workspace section with the mouse*)
49	Colton	So . . .
50	Callie	4 (*points to the tens-rods with finger*. *Colton moves two more tens-rods on the workspace section with the mouse*) So ok, then it says, (*reading task*) "Write down your answer and explain your thinking."
51	Colton	So we need to split it into 4 equal sections. So that's . . .
52	Callie	You can do . . .
53	Colton	(*counting hundreds-flats while pointing with the mouse*) 1, 2, 3 . . . One and a half . . . Wait. One and a half [10 by 10 blocks]. Then add one of these (*points to a 1x10 block with finger*). That's 1 point 6 meters. And then . . . Yeah. It would be 1 . . . Wait.
54	Callie	We need 1 point 1 if we take those two. (*pointing to blocks on screen with finger*)
55	Colton	(*moves the triangle into the left-hand section of the Venn diagram*) This one?
56	Callie	Yeah.
57	Colton	And then 1 divided by 4 would be . . . No, 2 divided by 4 would be a half. So you would add another point 5.
58	Callie	So . . .
59	Colton	That would be 1 point . . . 6.
60	Callie	6. Yeah.
61	Colton	So it would be 1 point 6 square meters.

In this example, a worthwhile mathematical task presented students with a contextual word problem and the technology tool provided a way for them to directly model a solution strategy [63]. As in the previous example, Colton and Callie used physical as well as mouse-driven gestures to communicate and justify their ideas. They demonstrated effective problem solving skills (lines 54–57), and worked together to clarify misconceptions and build on each other's ideas. The task also included an expectation that the students talk with each other and justify aloud their solution (lines 46, 50). In this way, the task and tool complemented each other to support the students' TMD. Figure 7 illustrates the TMD framework as applied to this exchange. These examples show that the TMD framework models key factors that contribute to students' discourse when using technology for learning mathematics.

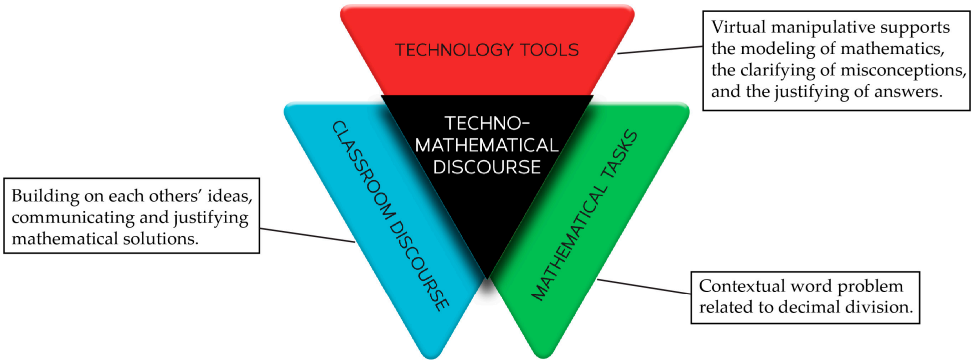


Figure 7. Students' TMD incorporated elements related to the Base Ten Blocks VM, their discursive exchange, and the mathematical task at hand.

4. TMD as a Framework for Classroom Implementation

The TMD framework has implications for classroom teachers and teacher educators. Classroom teachers could consider the components of the framework as they design instructional activities for their students. For example, when preparing to teach equivalent fractions, a teacher may choose to incorporate the Equivalent Fractions virtual manipulative [64] to aid students in developing their conceptual understanding of the topic. Per the TMD framework, this teacher would also need to consider the specific mathematical tasks that he or she would present to the students so that the technology tool may be utilized in an effective manner. Once the mathematical task is developed, the teacher would then plan for specific teaching moves [30] to encourage student-student and teacher-student discourse related to equivalent fractions. Additionally, teacher educators could use the framework as a means to develop pre-service and/or in-service teachers' understanding of how technology, classroom discourse, and mathematical tasks can synergize to enhance mathematics learning in the classroom. For example, after being introduced to the TMD framework, pre-service and/or in-service teachers could watch a video-recorded lesson in which technology was used to teach mathematics, and then identify how the three components of the framework worked together to support students' understanding of the mathematical concepts. Alternatively, teacher educators could present a particular technology tool, such as a virtual manipulative or a calculator, and then have the pre-service and/or in-service teachers develop a lesson plan, including a mathematical task and a plan for facilitating mathematical discussion that could be implemented in the classroom.

5. TMD as a Framework for Research

The TMD framework also has applications for future research on classroom discourse and technology tools. For example, future research can focus on factors related to technology tools, such as students' familiarity with the technology tools, students' perceptions of the technology, differences in platform (e.g., mouse-controlled versus touch-screen devices), or different technological affordances. The examples provided in this paper have only addressed student-student discourse. Because the teacher plays a critical role in classroom discourse, a natural next step for future research would be to describe classroom mathematical discourse between students and teachers. Future research can also be conducted on other classroom discourse factors, such as how TMD is affected by variations in culture, or varying levels of student achievement. Another interesting line of research can examine the influence on TMD of variations in mathematical tasks, such as procedural versus conceptual tasks, specific mathematical domains (e.g., fractions, integers, or place value), or lesson formats (e.g., inquiry- versus direct-instruction). The TMD framework can also be used to examine the assessment of students' learning in technology-based settings. Perhaps, this framework can also enable us to effectively analyze mathematical discourse in online mathematics courses and to develop technology-based design strategies to nurture rich mathematical discourse among online learners. An examination of these factors will deepen the collective understanding of how students interact with each other when engaging in mathematical tasks through the use of technology and how that discourse may be developed effectively.

6. Conclusions

The TMD framework is a tool for examining how technology and mathematical tasks affect discourse in mathematics learning environments. The TMD framework also has the potential to inform teachers' instructional decisions and to guide further research in the areas of classroom mathematical discourse and instruction uses of technology in mathematics classrooms.

The examination of how technology influences classroom instruction and interactions among students is a critical issue in mathematics education. Classrooms, schools, and school districts implement a variety of technology applications in attempts to increase students' mathematical competencies. Applications range from websites accessed through classroom laptops or a computer lab [65] to tablet-based applications [66] to full tutorial systems with built-in tracking and assessment of student progress [67]. Each of these applications has value, but must be used appropriately in order to result in effective learning of mathematics. Given the increasing usage of technology applications in mathematics classrooms, it is critical that teachers, teacher educators, and researchers become knowledgeable of effective and appropriate instructional uses of such technologies, and in particular, how those technologies can be leveraged to enhance students' learning of mathematics. The TMD framework is offered as one way of examining students' mathematical discourse given today's emphasis on technology application and integration. The application of the TMD framework in future research will advance the literature surrounding the use of technology for classroom instruction.

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