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New Exact Solutions with a Linear Velocity Field for the Gas Dynamics Equations for Two Types of State Equations

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Abstract: In this paper, exact solutions with a linear velocity field are sought for the gas dynamics equations in the case of the special state equation and the state equation of a monatomic gas. These state equations extend the transformation group admitted by the system to 12 and 14 parameters, respectively. Invariant submodels of rank one are constructed from two three-dimensional subalgebras of the corresponding Lie algebras, and exact solutions with a linear velocity field with inhomogeneous deformation are obtained. On the one hand of the special state equation, the submodel describes an isochoric vortex motion of particles, isobaric along each world line and restricted by a moving plane. The motions of particles occur along parabolas and along rays in parallel planes. The spherical volume of particles turns into an ellipsoid at finite moments of time, and as time tends to infinity, the particles end up on an infinite strip of finite width. On the other hand of the state equation of a monatomic gas, the submodel describes vortex compaction to the origin and the subsequent expansion of gas particles in half-spaces. The motion of any allocated volume of gas retains a spherical shape. It is shown that for any positive moment of time, it is possible to choose the radius of a spherical volume such that the characteristic conoid beginning from its center never reaches particles outside this volume. As a result of the generalization of the solutions with a linear velocity field, exact solutions of a wider class are obtained without conditions of invariance of density and pressure with respect to the selected three-dimensional subalgebras.

Keywords: gas dynamics equations; state equation; monatomic gas; linear velocity field; inhomogeneous deformation; group analysis; exact solution

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1. Introduction

Group analysis of differential equations is a powerful tool for obtaining exact solutions to nonlinear differential equations [1,2]. The gas dynamics equations have been well studied from the point of view of group analysis within the framework of the program "Submodels" [3]. This became possible thanks to the outstanding scientist Ovsyannikov, who sparked interest in this field of research. As a result of the research, a group classification with respect to the state equation was carried out, and optimal systems of subalgebras [4–9] and submodels [10,11] were obtained for various state equations. In addition, many types of motions were investigated, for example, barochronous [12] and isothermal [13], with a linear velocity field [14–16]. In this paper, new exact solutions with a linear velocity field are sought.

The motion of a continuous medium with a linear velocity field had been studied since the XIX century by Dirichlet and Riemann [17,18]. In their works, they considered motions with the homogeneous deformation of an incompressible fluid. It was assumed that the liquid moves in a force field caused by the mutual attraction of particles according to Newton's law of universal gravitation. Ovsyannikov and Dyson [19,20] independently showed that for a polytropic gas, the system of the gas dynamics equations is reduced to a system of nine ordinary differential equations of the second order. Several first integrals of the system were found. The development of the mathematical theory of these equations were obtained in the works of Andreev [21], Bogoyavlensky [22], Nemchinov [23], etc. Some of the last works were [24–28]. The general form of the solution to the gas dynamics equations with an arbitrary state equation with a linear velocity field with pressure and density depending on time is determined by an autonomous system of ordinary differential equations in [29]. In addition, there is the interesting paper where the author considers a solution possessing linearity with respect to a part of the space coordinates (one or two coordinates) [30].

In order to construct exact solutions, it is necessary to construct submodels, which are the gas dynamics equations written in terms of invariants. The rank of the submodel is the number of independent variables. In this paper, the gas dynamics equations are considered in the case of the special state equation (a pressure equals to the sum of two arbitrary functions of density and entropy) and in the case of the state equation of a monatomic gas. 12-dimensional Lie algebra L_{12} and the 14-dimensional Lie algebra L_{14} correspond to the transformation groups admitted by the gas dynamics equations with specified state equations. All subalgebras of these Lie algebras up to internal automorphisms were listed in optimal systems of nonsimilar subalgebras [6,7]. The constructed submodels and description of particle motion for small-dimensional subalgebras are given in [31–37]. In order to construct submodels, two three-dimensional subalgebras are selected from the Lie algebras L_{12} and L_{14}, and invariant submodels of rank one are constructed. Exact solutions of the gas dynamics equations with a linear velocity field are obtained from the submodels. Particle motion is investigated for the obtained exact solutions.

A different approach is also used to obtain exact solutions. If, initially, for the gas dynamics equations with an arbitrary state equation, a representation of the solution with a linear velocity field is specified, then a classification of submodels according to the state equations is obtained. In this paper, the submodel is selected which generalizes representations of velocities of obtained invariant submodels for the two types of state equations. Exact solutions are obtained and particle trajectories are plotted.

2. Symmetries of Gas Dynamics Equations and Solutions with a Linear Velocity Field

We consider the gas dynamics equations as follows [38]:

$$\begin{gathered}\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + \rho^{-1}\nabla p = 0,\\ \rho_t + (\vec{u}\cdot\nabla)\rho + \rho\nabla\cdot\vec{u} = 0,\\ S_t + (\vec{u}\cdot\nabla)S = 0 \quad \text{or} \quad p_t + (\vec{u}\cdot\nabla)p + \rho a_c^2\nabla\cdot\vec{u} = 0,\end{gathered} \tag{1}$$

where t and $\vec{x}$ are independent variables; $\vec{u}$ is a velocity; p is a pressure; ρ is a density; and S is an entropy, with an arbitrary state equation:

$$p = f(\rho, S). \tag{2}$$

In Equation (1) in the Cartesian coordinate system, we have:

$$\vec{x} = x\vec{i} + y\vec{j} + z\vec{k}, \quad \nabla = \vec{i}\partial_x + \vec{j}\partial_y + \vec{k}\partial_z, \quad \vec{u} = u\vec{i} + v\vec{j} + w\vec{k},$$

where $\vec{i}$, $\vec{j}$, and $\vec{k}$ is the orthonormal basis.

The sound velocity $a_c > 0$ is defined by formula [38]:

$$a_c^2 = \frac{dp}{d\rho}|_{S=const} = f_\rho(\rho, S). \tag{3}$$

The gas dynamics Equation (1) with an arbitrary state Equation (2) are invariant under the action Galilean group extended by uniform dilatation:

$$\begin{aligned}
&\vec{x}' = \vec{x} + \vec{a} \text{ (space translations);}\\
&t' = t + a_0 \text{ (time translation);}\\
&\vec{x}' = O\vec{x}, \vec{u}' = O\vec{u}, OO^T = E, \det O = 1 \text{ (rotations);}\\
&\vec{x}' = \vec{x} + t\vec{b}, \vec{u}' = \vec{u} + \vec{b} \text{ (Galilean translations);}\\
&t' = ct, \vec{x}' = c\vec{x} \text{ (uniform dilatation).}
\end{aligned} \tag{4}$$

The transformation group (4) corresponds to an 11-dimensional Lie algebra L_{11} with the basis generators in the Cartesian coordinate system [3,39]:

$$\begin{gathered}
X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_z,\\
X_4 = t\partial_x + \partial_u, \quad X_5 = t\partial_y + \partial_v, \quad X_6 = t\partial_z + \partial_w,\\
X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v, \quad X_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w,\\
X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u, \quad X_{10} = \partial_t,\\
X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z.
\end{gathered} \tag{5}$$

In the case of special state equation [3]:

$$p = f(\rho) + h(S) \tag{6}$$

the group (4) also includes the following transformation:

$$p' = p + p_0 \text{ (pressure translation).} \tag{7}$$

The Lie algebra L_{11} is extended to a 12-dimensional Lie algebra L_{12}, and the generator

$$Y_1 = \partial_p$$

is added to the basis generators of L_{11} (5). The Lie algebra L_{12} decomposes into the direct sum of two ideals $L_{12} = L_{11} \oplus Y_1$.

In the case of a monatomic gas [3], we have the state equation:

$$p = f(S)\rho^{\frac{5}{3}}. \tag{8}$$

Remark 1. *The system* (1), (2) *admits the equivalence transformation* $S' = K(S)$, *therefore, in what follows, we will assume in* (6), (8) *instead of* $h(S)$, $f(S)$ *just* S *[40].*

For the state Equation (8), the group (4) also includes the following transformations:

$$\begin{gathered}
t' = c_1 t,\ \vec{u}' = \frac{\vec{u}}{c_1},\ \rho' = \frac{\rho}{c_1^3},\ p' = \frac{p}{c_1^5} \text{ (dilatations);}\\
\rho' = g\rho,\ p' = gp,\ S' = \frac{S}{g^{\frac{2}{3}}}\\
\text{(dilatations of thermodynamic parameters of the gas);}\\
t' = \frac{t}{1-ft},\ \vec{x}' = \frac{\vec{x}}{1-ft},\ \vec{u}' = f\vec{x} + (1-ft)\vec{u},\\
\rho' = (1-ft)^3\rho,\ p' = (1-ft)^5 p \text{ (projective transformation);}
\end{gathered} \tag{9}$$

and the Lie algebra L_{11} is extended to a 14-dimensional Lie algebra L_{14}. In this case, generators

$$X_{12} = t^2\partial_t + tx\partial_x + ty\partial_y + tz\partial_z + (x - tu)\partial_u + (y - tv)\partial_v + (z - tw)\partial_w - \\ -3t\rho\partial_\rho - 5tp\partial_p,$$
$$X_{13} = t\partial_t - u\partial_u - v\partial_v - w\partial_w - 3\rho\partial_\rho - 5p\partial_p, \tag{10}$$
$$X_{14} = \rho\partial_\rho + p\partial_p - \tfrac{2}{3}S\partial_S$$

are added to basis generators of L_{11} (5).

A solution with a linear velocity field has the form:

$$\vec{u} = A(t)\vec{x} + \vec{u}_0(t), \tag{11}$$

where $A(t)$ is matrix 3×3, and $\vec{u}_0(t)$ is a 3-dimensional vector. If $\vec{u}_0 = 0$, then (11) is a solution with uniform deformation. If $\vec{u}_0 \neq 0$, then (11) is a solution with an inhomogeneous deformation. It is assumed that the state equation is (2). Many of the solutions with a linear velocity field are obtained from invariants for 4-dimensional subalgebras. In [14], a complete classification of submodels with a solution in the form of a linear velocity field was carried out according to the types of state equations.

3. Exact Solutions of the Gas Dynamics Equations with Special State Equation

We consider the basis generators of a 3-dimensional subalgebra 3.36 from the optimal system of nonsimilar subalgebras of Lie algebra L_{12} [6]:

$$X_1 = \partial_x, \;\; X_3 + X_4 = \partial_z + t\partial_x + \partial_u,$$
$$aX_2 + bX_3 + \gamma Y_1 = a\partial_y + b\partial_z + \gamma\partial_p, \;\; a^2 + b^2 = 1. \tag{12}$$

The coefficient γ is equal to 1 in the case of Lie algebra L_{12} and equal to 0 in the case of Lie algebra L_{11}. For $\gamma = 0$, we have the case of subalgebra 3.44 from L_{11} [41].

The invariants of subalgebra (12) are as follows:

$$t, \;\; u + \frac{b}{a}y - z, \;\; v, \;\; w, \;\; \rho, \;\; p - \gamma\frac{y}{a}, \;\; a \neq 0. \tag{13}$$

The case $a = 0$ gives us a partially invariant submodel of rank 2 and defect 1.

Representation of the invariant solution from (13) has the form

$$u = u_1(t) - \frac{b}{a}y + z, \;\; v = v(t), \;\; w = w(t), \;\; \rho = \rho(t),$$
$$p = p_1(t) + \gamma\frac{y}{a}, \;\; S = S_1(t) + \gamma\frac{y}{a}, \;\; \gamma = 0 \vee 1, \;\; a^2 + b^2 = 1. \tag{14}$$

Substituting (14) into (1) and (6), we obtain the invariant submodel:

$$u_{1t} = \frac{b}{a}v - w, \;\; v_t = -\rho^{-1}\frac{\gamma}{a}, \;\; w_t = 0, \;\; \rho_t = 0, \;\; S_{1t} = -\frac{\gamma}{a}v. \tag{15}$$

The exact solution of the system of Equations (1) and (6) from (14) and (15) is:

$$u = -\frac{b}{a}y + z - \frac{\gamma b}{2a^2\rho_0}t^2 + \left(\frac{b}{a}v_0 - w_0\right)t + u_0,$$
$$v = -\frac{\gamma}{a\rho_0}t + v_0, \;\; w = w_0, \;\; \rho = \rho_0, \;\; p = \gamma\frac{y}{a} + \frac{\gamma^2}{2a^2\rho_0}t^2 - \frac{\gamma}{a}v_0 t - p_0, \tag{16}$$
$$S = \gamma\frac{y}{a} + \frac{\gamma^2}{2a^2\rho_0}t^2 - \frac{\gamma}{a}v_0 t + S_0.$$

On applying Galilean translations (4) with $\vec{b} = (-u_0, -v_0, -w_0)$ and pressure translation (7) to the solution (16), we have that $u_0 = v_0 = w_0 = p_0 = 0$.

The motion of the particles is given by the following equation [38]:

$$\frac{d\vec{x}}{dt} = \vec{u}(\vec{x}, t). \tag{17}$$

Integral curves of the equation (17) are the world lines of particles in space $\mathbb{R}^4(t, \vec{x})$, the projection of which in $\mathbb{R}^3(\vec{x})$ are particle trajectories. From Equations (16) and (17), world lines have the form:

$$x = (aC_3 - bC_2)t + aC_1, \;\; y = -\frac{\gamma}{2a\rho_0}t^2 + aC_2, \;\; z = aC_3. \tag{18}$$

From the generators of subalgebra (12) and Lie equations [1]

$$\frac{\partial \bar{x}}{\partial a} = \xi(\bar{x}), \;\; \bar{x}|_{a=0} = x \tag{19}$$

it follows that:

$$\begin{gathered} \bar{x} = x + \xi; \\ \bar{x} = x + \eta t, \;\; \bar{z} = z + \eta, \;\; \bar{u} = u + \eta; \\ \bar{y} = y + a\zeta, \;\; \bar{z} = z + b\zeta, \;\; \bar{p} = p + \zeta. \end{gathered} \tag{20}$$

Applying the superposition of transformations (20) to Equation (18) with parameters

$$\xi = -aC_1, \;\; \eta = bC_2 - aC_3, \;\; \zeta = -C_2,$$

we obtain:

$$x = 0, \;\; y = -\frac{\gamma}{2a\rho_0}t^2, \;\; z = 0. \tag{21}$$

The trajectories (21) describe Oy axis with $y \leq 0$ for $a > 0$. From this, we can reproduce the following formulas using (20):

$$x = \eta t + \xi, \;\; y = -\frac{\gamma}{2a\rho_0}t^2 + a\zeta, \;\; z = b\zeta + \eta. \tag{22}$$

Jacobian $J = |\partial\vec{x}/\partial\vec{x}_0|$ [41] of transformation (22) is equal to $a \neq 0$, where $\vec{x}_0 = (\xi, \zeta, \eta)$. This means that the world lines of particles do not intersect. The vortex is $\vec{\omega} = (w_y - v_z, u_z - w_x, v_x - u_y) = (0, 1, b/a)$ [38]. The motion of particles is vortex and is restricted by moving plane $y = -\dfrac{\gamma}{2a\rho_0}t^2$ for $p \geq 0$. The pressure is constant along the world line and has the value $p = \gamma\zeta$.

For $\eta = 0$, the trajectories (22) are rays. If the t variable is eliminated in (22) for $\eta \neq 0$, we obtain the following formulas of trajectories:

$$y = -\frac{\gamma(x - \xi)^2}{2a\rho_0\eta^2} + a\zeta, \;\; z = b\zeta + \eta. \tag{23}$$

The trajectories (23) are parabolas in each plane $z = const$, in which the vertex of the parabola is the point $(\xi, a\zeta)$. The parabolas (22) are illustrated in the Figure 1. For subalgebra 3.44 from L_{11}, the world lines of particles are straight lines in a uniformly moving coordinate system [41].

Let the formulas (22) satisfy the initial conditions

$$x(t_0) = x_0, \;\; y(t_0) = y_0, \;\; z(t_0) = z_0,$$

where x_0, y_0, z_0 are local Lagrangian coordinates. In this case, the formulas (22) have the form:

$$x = \left(z_0 - \frac{b}{a}y_0 - \frac{\gamma b t_0^2}{2a^2\rho_0}\right)t + x_0 - t_0 z_0 + \frac{b}{a}t_0 y_0 + \frac{\gamma b t_0^3}{2a^2\rho_0},$$
$$y = -\frac{\gamma t^2}{2a\rho_0} + y_0 + \frac{\gamma t_0^2}{2a\rho_0}, \quad z = z_0. \tag{24}$$

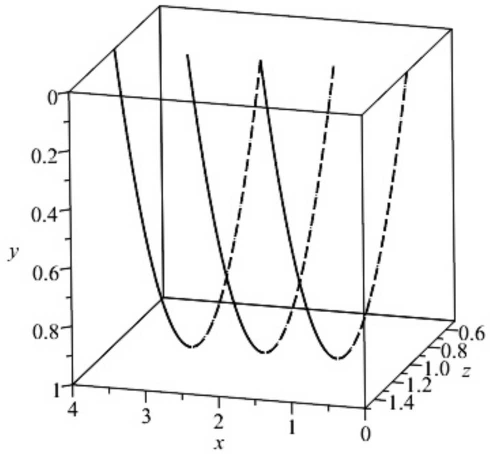

Figure 1. The trajectories (22) with $\gamma = 1$, $a = 1$, $b = 0$, $\rho_0 = \frac{1}{2}$, $\xi = 1;2;3$, $\eta = 1$, $\zeta = 1$. For $t = -1..0$, the trajectories are indicated by a dotted line; for $t = 0..1$, the trajectories are indicated by a solid line.

From now on, we make the following assumption: $b = 0$, then $a = 1$. Let the particles (24) be on the sphere with radius r and centre (x_1, y_1, z_1) at time $t = t_0$:

$$(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2 = r^2. \tag{25}$$

We obtain from Equations (24) and (25) location of the particles

$$x^2 + y^2 + \left((t - t_0)^2 + 1\right)z^2 - 2xz(t - t_0) - 2x_1 x + Ay + Bz = C,$$
$$A = \frac{\gamma(t^2 - t_0^2)}{\rho_0} - 2y_1, \quad B = 2x_1(t - t_0) - 2z_1,$$
$$C = r^2 - x_1^2 - y_1^2 - z_1^2 + \frac{\gamma(t^2 - t_0^2)}{\rho_0}y_1 - \frac{\gamma^2(t^2 - t_0^2)^2}{4\rho_0^2}. \tag{26}$$

Let us rotate the coordinate axes by the α angle

$$x = \tilde{x}\cos\alpha - \tilde{z}\sin\alpha, \quad z = \tilde{x}\sin\alpha + \tilde{z}\cos\alpha. \tag{27}$$

After substituting (27) into (26), we find the angle α equating the coefficient at $\tilde{x}\tilde{z}$ to zero

$$\tan 2\alpha = \frac{2}{t - t_0} \Rightarrow \alpha = \frac{1}{2}\arctan\frac{2}{t - t_0} + \frac{\pi}{2}n, \quad n \in \mathbb{Z}. \tag{28}$$

If $t \to \infty$, then $\alpha \to 0$. Let us choose $\alpha = \frac{1}{2}\arctan\frac{2}{t - t_0}$ from (28), then Equation (26) may be written in the following form:

$$\begin{gathered}
A_1\tilde{x}^2 + y^2 + B_1\tilde{z}^2 + A_2\tilde{x} + B_2\tilde{z} + Ay = C, \\
A_1 = (t-t_0)^2\sin^2\alpha - (t-t_0)\sin 2\alpha + 1, \\
B_1 = (t-t_0)^2\cos^2\alpha + (t-t_0)\sin 2\alpha + 1, \\
A_2 = B\sin\alpha - 2x_1\cos\alpha, \\
B_2 = B\cos\alpha + 2x_1\sin\alpha; \\
\sin\alpha = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{t-t_0}{\sqrt{(t-t_0)^2+4}}}, \quad \cos\alpha = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{t-t_0}{\sqrt{(t-t_0)^2+4}}}, \\
\sin 2\alpha = \frac{2}{\sqrt{(t-t_0)^2+4}}, \quad \cos 2\alpha = \frac{t-t_0}{\sqrt{(t-t_0)^2+4}}, \\
\sin^2\alpha = \frac{1}{2} - \frac{t-t_0}{2\sqrt{(t-t_0)^2+4}}, \quad \cos^2\alpha = \frac{1}{2} + \frac{t-t_0}{2\sqrt{(t-t_0)^2+4}}.
\end{gathered} \tag{29}$$

From Equation (29), the canonical form of a second-order surface has the following form:

$$\begin{gathered}
\frac{\left(\tilde{x} + \dfrac{A_2}{2A_1}\right)^2}{\dfrac{K}{A_1}} + \frac{\left(y + \dfrac{A}{2}\right)^2}{K} + \frac{\left(\tilde{z} + \dfrac{B_2}{2B_1}\right)^2}{\dfrac{K}{B1}} = 1, \\
K = C + \frac{A_2^2}{4A_1} + \frac{B_2^2}{4B_1} + \frac{A^2}{4}.
\end{gathered} \tag{30}$$

Equation (30) defines the ellipsoid at $t \neq t_0$ (Figure 2). Calculations by the MAPLE computer mathematics system give us the following limits:

$$\begin{gathered}
\lim_{t\to\infty}\sqrt{\frac{K}{A_1}} = +\infty, \quad \lim_{t\to\infty}\sqrt{K} = r, \quad \lim_{t\to\infty}\sqrt{\frac{K}{B_1}} = 0, \\
\lim_{t\to\infty}\left(-\frac{A_2}{2A_1}\right) = \operatorname{sign}(z_1)\infty, \quad \lim_{t\to\infty}\left(-\frac{B_2}{2B_1}\right) = 0, \\
\lim_{t\to\infty}\left(-\frac{A}{2}\right) = -\operatorname{sign}\left(\frac{\gamma}{\rho_0}\right)\infty.
\end{gathered} \tag{31}$$

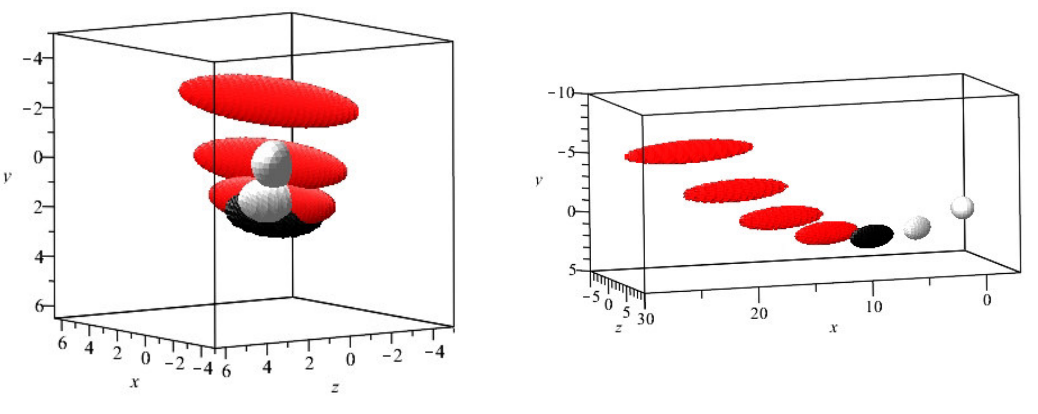

Figure 2. (**left**) The motion of the particles volume (26) with $x_1 = 0.1$, $y_1 = 0.1$, $z_1 = 0.1$, $r = 1$, $t_0 = -2$, $\rho_0 = 1$, $\gamma = 1$; $t = -2; -1$ (grey sphere and ellipsoid); $t = 0$ (black ellipsoid); and $t = 1; 2; 3$ (red ellipsoids). (**right**) The motion of the particles volume (26) with $x_1 = 0.1$, $y_1 = 0.1$, $z_1 = 4$, $r = 1$, $t_0 = -2$, $\rho_0 = 1$, $\gamma = 1$; $t = -2; -1$ (grey sphere and ellipsoid); $t = 0$ (black ellipsoid); and $t = 1; 2; 3; 4$ (red ellipsoids).

Thus, according to formulas (31), if $t \to \infty$, the ellipsoid (30) turns into the plane strip. The volume of the ellipsoid (30) is $V = \dfrac{4}{3}\pi r^3$ at $t \to \infty$.

4. Exact Solutions of the Gas Dynamics Equations with Monatomic Gas State Equation

We consider the basis generators of 3-dimensional subalgebra 3.4* from the optimal system of nonsimilar subalgebras of Lie algebra L_{14} for Equation (8) [42]:

$$\begin{gathered} -X_3+X_5=-\partial_z+t\partial_y+\partial_v, \\ a(X_2+X_6)+X_7+X_{10}+X_{12}= \\ =a(\partial_y+t\partial_z+\partial_w)+y\partial_z-z\partial_y+v\partial_w-w\partial_v+(1+t^2)\partial_t+ \\ +tx\partial_x+ty\partial_y+tz\partial_z+(x-tu)\partial_u+(y-tv)\partial_v+(z-tw)\partial_w-3t\rho\partial_\rho-5tp\partial_p, \\ X_2+X_6+bX_{14}=\partial_y+t\partial_z+\partial_w+b(\rho\partial_\rho+p\partial_p-\tfrac{2}{3}S\partial_S). \end{gathered} \tag{32}$$

Invariants have the following form:

$$\begin{gathered} x_1=\frac{x}{\sqrt{1+t^2}},\ u\sqrt{1+t^2}-\frac{tx}{\sqrt{1+t^2}},\ v+tw-\frac{2yt+z(t^2-1)}{1+t^2}, \\ w-tv-\frac{2zt-y(t^2-1)}{1+t^2},\ \rho(1+t^2)^{\frac{3}{2}}exp\left(b(a\tau-\frac{y+tz}{1+t^2})\right), \\ p(1+t^2)^{\frac{5}{2}}exp\left(b(a\tau-\frac{y+tz}{1+t^2})\right),\ \tau=\arctan t. \end{gathered}$$

The representation of the invariant solution is given as follows:

$$\begin{gathered} u=\frac{u_1(x_1)}{\sqrt{1+t^2}}+\frac{tx}{1+t^2}, v=\frac{v_1(x_1)-w_1(x_1)t+yt-z}{1+t^2}, \\ w=\frac{w_1(x_1)+v_1(x_1)t+y+zt}{1+t^2}, \\ \rho=\frac{\rho_1(x_1)}{(1+t^2)^{\frac{3}{2}}}exp\left(b(-a\tau+\frac{y+tz}{1+t^2})\right), p=\frac{p_1(x_1)}{(1+t^2)^{\frac{5}{2}}}exp\left(b(-a\tau+\frac{y+tz}{1+t^2})\right), \\ S=S_1(x_1)exp\left(-\frac{2}{3}b(-a\tau+\frac{y+tz}{1+t^2})\right), S_1(x_1)=\frac{p_1(x_1)}{\rho_1(x_1)^{\frac{5}{3}}}. \end{gathered} \tag{33}$$

Substituting (33) in (1) and (8), we obtain the invariant submodel:

$$\begin{gathered} u_1u_{1x_1}+\frac{p_{1x_1}}{\rho_1}=-x_1, \\ u_1v_{1x_1}=2w_1-b\frac{p_1}{\rho_1}, \\ u_1w_{1x_1}=-2v_1, \\ u_1\rho_{1x_1}+\rho_1u_{1x_1}=b\rho_1(a-v_1), \\ u_1S_{1x_1}=-\frac{2}{3}bS_1(a-v_1),\ S_1=p_1\rho_1^{-\frac{5}{3}}. \end{gathered} \tag{34}$$

The system (34) has an integral $\rho_1u_1S_1^{\frac{3}{2}}=\left(\frac{p_1}{\rho_1}\right)^{\frac{3}{2}}u_1=K_0$ and admits reflection $a\to-a, b\to-b, v_1\to-v_1, w_1\to-w_1$.

If the condition $u_1=0$ is satisfied, then there is a solution:

$$ab=0, u_1=0, v_1=0, w_1=\frac{bp_1}{2\rho_1}, \rho_1=-\frac{p_{1x_1}}{x_1},$$

where $p_1(x_1)$ is an arbitrary function.

Let us distinguish among them solutions with a linear velocity field $w_1=-\frac{bx_1p_1}{2p_{1x_1}}=a_3x_1+b_3$:

$$1)\ b=0, u_1=0, v_1=0, w_1=0, \rho_1=-\frac{p_{1x_1}}{x_1};\ \ p_1(x_1) \text{ is arbitrary function,} \tag{35}$$

$$\begin{aligned} &2)\ a=0, bb_3>0, u_1=0, v_1=0, w_1=b_3,\\ &\rho_1=\rho_0 e^{-\frac{b}{4b_3}x_1^2},\ \ p_1=\frac{2b_3}{b}\rho_0 e^{-\frac{b}{4b_3}x_1^2}; \end{aligned} \tag{36}$$

$$\begin{aligned} &3)\ a=0, a_3\neq 0, b(a_3x_1+b_3)>0, u_1=0, v_1=0, w_1=a_3x_1+b_3,\\ &\rho_1=\rho_0|a_3x_1+b_3|^{\frac{bb_3}{2a_3^2}-1}e^{-\frac{b}{2a_3}x_1},\ \ p_1=\frac{2}{|b|}\rho_0|a_3x_1+b_3|^{\frac{bb_3}{2a_3^2}}e^{-\frac{b}{2a_3}x_1}. \end{aligned} \tag{37}$$

If $u_1\neq 0$ and we assume $\vec{u}_1=(u_1,v_1,w_1)=(a_1x_1+b_1,a_2x_1+b_2,a_3x_1+b_3)$ with a linear velocity field, then there are no solutions.

Solution (35) produces solution to the gas dynamics Equations (1) and (8):

$$\begin{gathered} u=\frac{tx}{1+t^2},\ \ v=\frac{yt-z}{1+t^2},\ \ w=\frac{y+zt}{1+t^2},\\ \rho=-\frac{p_{1x_1}}{x_1(1+t^2)^{\frac{3}{2}}},\ \ p=\frac{p_1(x_1)}{(1+t^2)^{\frac{5}{2}}},\\ S=p\rho^{-\frac{5}{3}};\, p_1(x_1) \text{ is arbitrary function of } x_1=\frac{x}{\sqrt{1+t^2}}, \end{gathered} \tag{38}$$

which was investigated in the work [43].

Analysis of the Exact Solution

Invariant solution (36) produces exact solution to the gas dynamics Equations (1) and (8):

$$\begin{gathered} u=\frac{tx}{1+t^2},\ \ v=\frac{yt-z-b_3t}{1+t^2},\ \ w=\frac{y+zt+b_3}{1+t^2},\\ \rho=\frac{\rho_0}{(1+t^2)^{\frac{3}{2}}}exp\left(\frac{b}{1+t^2}(-\frac{x^2}{4b_3}+y+tz)\right),\\ p=\frac{2b_3\rho_0}{b(1+t^2)^{\frac{5}{2}}}exp\left(\frac{b}{1+t^2}(-\frac{x^2}{4b_3}+y+tz)\right),\\ S=\frac{2b_3}{b}\rho_0^{-\frac{2}{3}}exp\left(-\frac{2b}{3(1+t^2)}(-\frac{x^2}{4b_3}+y+tz)\right),\ \ bb_3>0. \end{gathered} \tag{39}$$

The vortex is $\vec{\omega}=(w_y-v_z,u_z-w_x,v_x-u_y)=\left(\frac{2}{1+t^2},0,0\right)$. The motion of the particles is vortex.

Equations (17) and (39) give the world lines:

$$\begin{gathered} x(t)=x_0\sqrt{1+t^2},\\ y(t)=-(b_3\tau+z_0)t+y_0,\\ z(t)=y_0t+b_3\tau+z_0, \end{gathered} \tag{40}$$

which are space curves for $x_0\neq 0$.

Here, x_0,y_0,z_0 are the particle coordinates at the moment $t=0$ (the global Lagrange coordinates). The Jacobian of transformation (40) is $J=(1+t^2)^{\frac{3}{2}}\neq 0$. The motion of the particles occurs without blow-up. The particles lying in the plane $x=0$ do not leave it (plane curves). The motion of particles occurs in half-spaces symmetrically relative to the plane $x=0$.

The velocity components (39) along world lines (40) are as follows:

$$u=\frac{tx_0}{\sqrt{1+t^2}},\ \ v=-z_0-b_3\tau-\frac{b_3t}{1+t^2},\ \ w=y_0+\frac{b_3}{1+t^2}.$$

At $t=0$, we have $\vec{u}(x_0;y_0;z_0)=(0;-z_0;y_0+b_3)$; at $t\to\pm\infty$, we have $\vec{u}(x_0;y_0;z_0)=(\pm x_0;-z_0\mp b_3\frac{\pi}{2};y_0)$.

We introduce new axes in the plane Oyz:

$$\xi=\frac{y+zt}{\sqrt{1+t^2}},\ \ \eta=\frac{z-ty}{\sqrt{1+t^2}},$$

which, at the time $t = 0$, coincide with the axes Oy and Oz ($\xi_0 = y_0$ and $\eta_0 = z_0$, respectively), and at any other time t, they are rotated by an angle $\tau = \arctan t$ counter-clockwise relative to the axes Oy and Oz. If $t \to \pm\infty$, then $\xi \to \pm z, \eta \to \mp y, \tau \to \pm\frac{\pi}{2}$.

Thus, the formulas (40) are given by:

$$\begin{aligned} x(t) &= x_0\sqrt{1+t^2}, \\ \xi(t) &= \xi_0\sqrt{1+t^2}, \\ \eta(t) &= (\eta_0 + b_3\tau)\sqrt{1+t^2}. \end{aligned} \tag{41}$$

In the space (x, ξ, η), trajectories (41) are plane curves. In polar coordinates $x = r\cos\phi$, $\xi = r\sin\phi$, Equations (41) have the following form:

$$\begin{aligned} r(t) &= r_0\sqrt{1+t^2}, \\ \phi(t) &= \phi_0, \\ \eta(t) &= (\eta_0 + b_3\tau)\sqrt{1+t^2}. \end{aligned} \tag{42}$$

In the plane (r, η), the trajectories (42) are given by the equation $\eta = \left(\eta_0 \pm b_3 \arccos\frac{r_0}{r}\right)\frac{r}{r_0}$, which define the curves with oblique asymptotes $\eta_a = \dfrac{\eta_0 + 0.5\text{sign}(t)b_3\pi}{r_0}r - b_3\text{sign}(t)$ at $t \to \pm\infty$. If we consider the curve as $r = r(\eta)$, then it has a derivative $r'_\eta = \dfrac{r_t}{\eta_t} = \dfrac{r_0 t}{t\eta_0 + b_3(1+t\tau)}$ and an extremum at $t = 0$ in the point (η_0, r_0). For any ϕ_0, its graph is shown in the Figure 3.

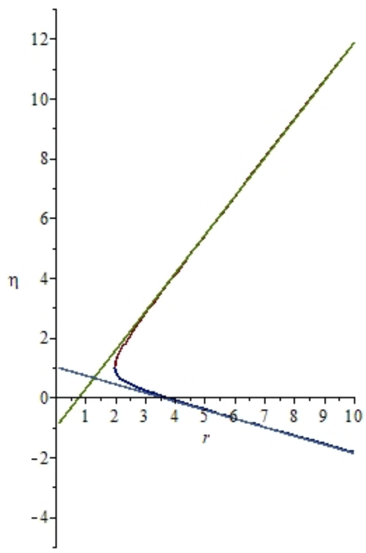

Figure 3. The trajectory (42) in the plane (r, η) at $r_0 = 2, \eta_0 = 1, b_3 = 1$.

Thus, the motion of particles can be represented as a complex motion consisting of the condensation of gas particles to the origin, followed by scattering and successive (uneven) rotation in the plane Oyz by an angle π. Figure 4 represents the trajectories of the same particles in spaces (x, ξ, η) (or (r, ϕ, η)) and (x, y, z), showing how the rotation of the coordinate system around the axis Ox changes the trajectories particles motion.

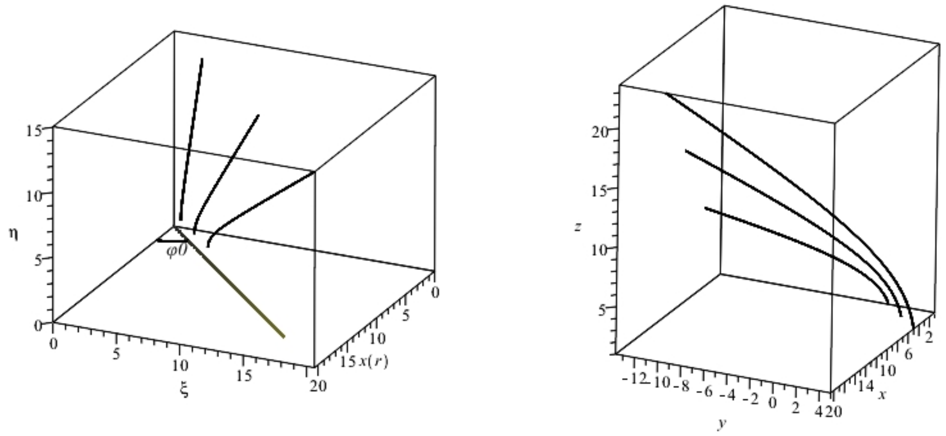

Figure 4. (**left**) The trajectories (41), (42) in the space (x, ξ, η) at $b_3 = 2, t = 0..4$ with coordinates $(x_0; y_0; z_0) = (1;1;1);\ (3;3;1);\ (5;5;1)$. (**right**) The trajectories (40) in the space (x, y, z) with the same parameters.

If the particle has coordinates $(x_{t_0}; y_{t_0}; z_{t_0})$ at the moment of time $t = t_0$, then the Equation (40) may be rewritten as follows:

$$\begin{aligned} x(t) &= \frac{x_{t_0}}{\sqrt{1+t_0^2}}\sqrt{1+t^2}, \\ y(t) &= -b_3 t(\tau-\tau_0) - \frac{z_{t_0} - y_{t_0}t_0}{1+t_0^2}t + \frac{y_{t_0} + z_{t_0}t_0}{1+t_0^2}, \\ z(t) &= b_3(\tau-\tau_0) + \frac{y_{t_0} + z_{t_0}t_0}{1+t_0^2}t + \frac{z_{t_0} - y_{t_0}t_0}{1+t_0^2}. \end{aligned} \tag{43}$$

If at $t = t_0$ the particle with coordinates $(x_{t_0}, y_{t_0}, z_{t_0})$ is on a sphere with a center (x_1, y_1, z_1) and radius $R_0 > 0$

$$(x_{t_0} - x_1)^2 + (y_{t_0} - y_1)^2 + (z_{t_0} - z_1)^2 = R_0^2,$$

then at time moment t, it will be on the sphere

$$\begin{aligned} &\left(x - \frac{x_1}{\sqrt{1+t_0^2}}\sqrt{1+t^2}\right)^2 + \left(y + b_3 t(\tau-\tau_0) + \frac{z_1 - y_1 t_0}{1+t_0^2}t - \frac{y_1 + z_1 t_0}{1+t_0^2}\right)^2 + \\ &+\left(z - b_3(\tau-\tau_0) - \frac{y_1 + z_1 t_0}{1+t_0^2}t - \frac{z_1 - y_1 t_0}{1+t_0^2}\right)^2 = \frac{R_0^2}{1+t_0^2}(1+t^2), \end{aligned} \tag{44}$$

which has a radius $\dfrac{R_0}{\sqrt{1+t_0^2}}\sqrt{1+t^2}$ and the center moves according to the law of motion (43).

The sound surface is given by the equation $u^2 + v^2 + w^2 = a_c^2$ [38] $\left(a_c^2 = \dfrac{5p}{3\rho} = \dfrac{10b_3}{3b(1+t^2)} \text{ from (3)}\right)$. It has the following form from (39):

$$\frac{t^2x^2}{1+t^2} + \left(y + b_3\frac{1-t^2}{1+t^2}\right)^2 + \left(z + b_3\frac{2t}{1+t^2}\right)^2 = \frac{10b_3}{3b}. \tag{45}$$

For $t \neq 0$, Equation (45) defines an ellipsoid with a center $\left(0; -b_3\frac{1-t^2}{1+t^2}; -b_3\frac{2t}{1+t^2}\right)$, one semiaxis is equal to $\sqrt{\frac{10b_3(1+t^2)}{3bt^2}}$, and the other two semiaxes are equal to $r_c = \sqrt{\frac{10b_3}{3b}}$.

For $t = 0$, Equation (45) defines a circular cylinder $(y+b_3)^2 + z^2 = \frac{10b_3}{3b}$ with radius r_c and an axis parallel to the Ox, passing through a point $(0; -b_3; 0)$.

For $t \to \pm\infty$, Equation (45) defines a sphere $x^2 + (y-b_3)^2 + z^2 = \frac{10b_3}{3b}$ with a center $(0, b_3, 0)$ and radius r_c.

The motion of the volume of particles (44) has been considered. The surface of the volume of particles (sphere) is the contact characteristic of the gas dynamics equations. For a normal gas, the gas dynamics equations also have two sound characteristics $C_\pm$.

We consider the equations of sound characteristics [38]:

$$C_\pm : h_t + uh_x + vh_y + wh_z \pm a\sqrt{h_x^2 + h_y^2 + h_z^2} = 0.$$

The equations of the bicharacteristics have the following form:

$$\frac{d\vec{x}}{dt} = \vec{u} \pm a\frac{\nabla h}{|\nabla h|}, \tag{46}$$

$$\frac{dh_j}{dt} = -\vec{u}_j \cdot \nabla h \mp a_j|\nabla h|, \ \ j = t, x, y, z. \tag{47}$$

Equations (39) and (47) arrive at the following formulas:

$$\begin{gathered} h_x = \frac{h_{x0}}{\sqrt{1+t^2}}, \ \ h_y = \frac{h_{y0} - th_{z0}}{1+t^2}, \ \ h_z = \frac{h_{z0} + th_{y0}}{1+t^2}, \\ |\nabla h| = \frac{|\nabla h_0|}{\sqrt{1+t^2}}, \ \ |\nabla h_0| = \sqrt{h_{x0}^2 + h_{y0}^2 + h_{z0}^2}. \end{gathered} \tag{48}$$

Equations (39), (46), and (48) give:

$$\begin{gathered} x = \left(\pm\sqrt{\frac{10b_3}{3b}}\frac{h_{x0}}{|\nabla h_0|}(\tau - \tau_0) + \frac{x_{t_0}}{\sqrt{1+t_0^2}}\right)\sqrt{1+t^2}, \\ y = \frac{y_{t_0} + t_0 z_{t_0}}{1+t_0^2} + \frac{t_0 y_{t_0} - z_{t_0}}{1+t_0^2}t - b_3 t(\tau - \tau_0) \pm \sqrt{\frac{10b_3}{3b}}\frac{h_{y0} - th_{z0}}{|\nabla h_0|}(\tau - \tau_0), \\ z = \frac{z_{t_0} - t_0 y_{t_0}}{1+t_0^2} + \frac{y_{t_0} + t_0 z_{t_0}}{1+t_0^2}t + b_3(\tau - \tau_0) \pm \sqrt{\frac{10b_3}{3b}}\frac{th_{y0} + h_{z0}}{|\nabla h_0|}(\tau - \tau_0). \end{gathered} \tag{49}$$

Let us construct a characteristic conoid $K(P)$ with a vertex P—a characteristic surface formed by all the bicharacteristics coming out of a given point $P(\vec{x}_{t_0}, t_0)$. In order to represent all such bicharacteristics, it is necessary to take into account the initial conditions to the system (46), (47) [38]:

$$\vec{x}(t_0) = \vec{x}_{t_0}, \ \ h_j(t_0) = h_{jo} \ \ (j = t, x, y, z). \tag{50}$$

In order to satisfy (50), it is necessary to make a replacement $h_{x0} \to h_{x0}\sqrt{1+t_0^2}; h_{y0} \to h_{y0} + t_0 h_{z0}; h_{z0} \to h_{z0} - t_0 h_{y0}; |\nabla h_0| \to |\nabla h_0|\sqrt{1+t_0^2}$ in (48), (49), but still, we exclude $h_{x0}, h_{y0}, h_{z0}, |\nabla h_0|$ from (49) and obtain the following equation:

$$\left(x-\frac{x_{t_0}}{\sqrt{1+t_0^2}}\sqrt{1+t^2}\right)^2+\left(y-\frac{y_{t_0}+t_0z_{t_0}}{1+t_0^2}-\frac{t_0y_{t_0}-z_{t_0}}{1+t_0^2}t+b_3t(\tau-\tau_0)\right)^2+$$
$$+\left(z-\frac{z_{t_0}-t_0y_{t_0}}{1+t_0^2}-\frac{y_{t_0}+t_0z_{t_0}}{1+t_0^2}t-b_3(\tau-\tau_0)\right)^2=\frac{10b_3}{3b}(\tau-\tau_0)^2(1+t^2). \tag{51}$$

The section of the conoid (51) by hyperplane $t = const$ is a sphere in $\mathbb{R}^3(\vec{x})$, the center of which moves according to the law of motion (43), and the radius is equal to $\sqrt{\frac{10b_3}{3b}(1+t^2)}|\tau-\tau_0|$.

At $t \to \infty$, the sound velocity $a_c \to 0$ and sound characteristic tends to the contact characteristic. Indeed, the conoid (51) tends to the sphere (44), which has a radius equal to $R_0 = \sqrt{\frac{10b_3}{3b}(1+t_0^2)}|\frac{\pi}{2}-\tau_0|$. Thus, for any moment of time $t > 0$, the characteristic conoid with a vertex $P(\vec{x}_{t_0}, t_0)$ beginning from the center of a spherical volume with a radius $R_0 = \sqrt{\frac{10b_3}{3b}(1+t_0^2)}|\frac{\pi}{2}-\tau_0|$ never reaches particles outside this volume. Figure 5 shows the propagation of a characteristic conoid (51) with a vertex $P(\vec{x}_{t_o}, t_0)$ and the motion of a spherical volume of gas (44) with the center at the same point at $t = t_0$ and radius R_0.

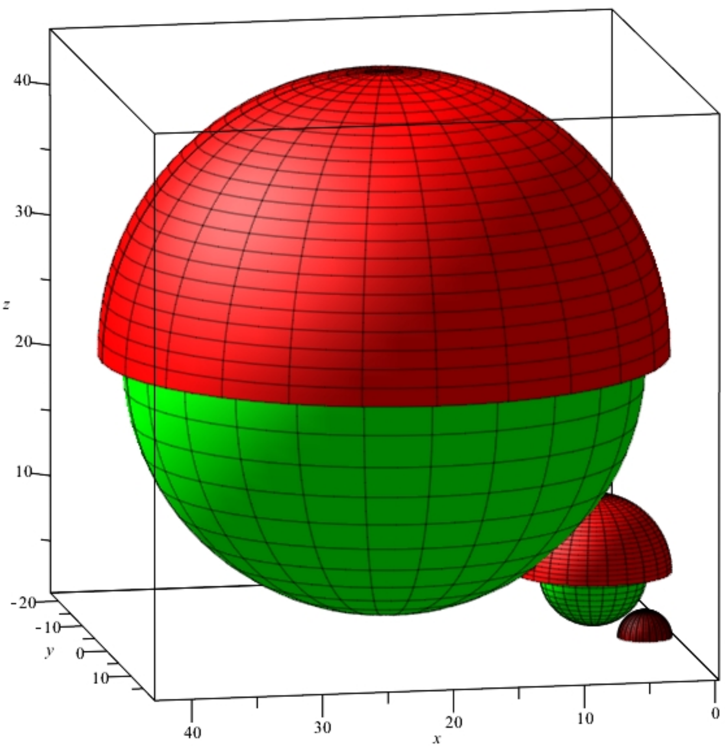

Figure 5. The characteristic conoid (51) with a vertex $P(2;2;1;1)$ is represented in green color, the motion of the sphere (44) with center $(x_1, y_1, z_1) = (2, 2, 1)$ and with the radius $R_0 = \frac{\sqrt{15}\pi}{6}$ at $t_0 = 1$ is red color; $b_3 = 1, b = 1$; $t = 1;4;15$.

Depending on where the point $P(\vec{x}_{t_0}, t_0)$ is located (in the subsonic region, on the sound surface, or in the supersonic region), the sections of the characteristic conoid by planes $t = const$ are nested into each other, touch at the point, or intersect each other, respectively (Figure 6).

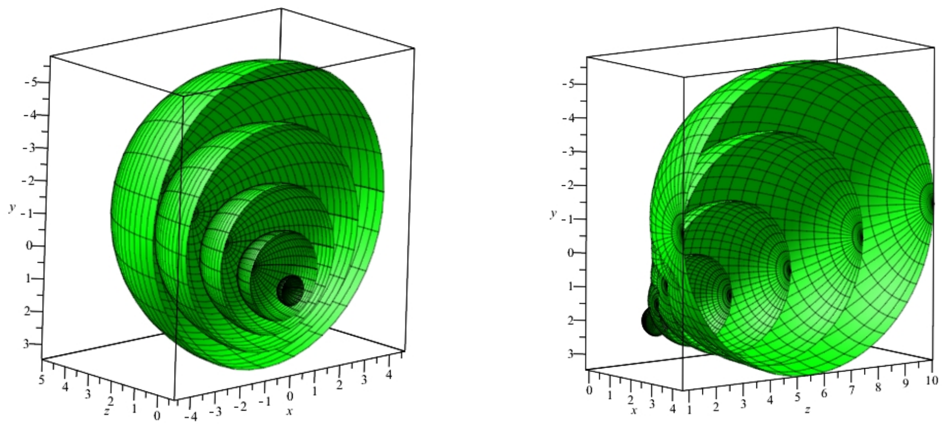

Figure 6. (**left**) The characteristic conoid with a vertex $P(0; 1; -1; 1)$ in the subsonic region is represented at $t = 1.2; 1.6; 2.0; 2.5; 3.0$; $b_3 = 3, b = 1$. (**right**) The characteristic conoid with a vertex $P(2; 2; 1; 1)$ in the supersonic region is represented at $t = 1.2; 1.6; 2.0; 2.5; 3.0$; $b_3 = 3, b = 1$.

At $t = const$, the pressure and density have the same values on parabolic cylinders.

5. Generalization of Exact Solutions of the Gas Dynamics Equations with a Linear Velocity Field

We consider the solution of the gas dynamics equations with a linear velocity field (11). Let us substitute the velocity representation (11) into the gas dynamics Equations (1) and (2) and express the time derivative and the gradient of the pressure function. On equating the shifted derivatives, we obtain a classifying relation for the matrix $B = A' + A^2$. Matrix B is decomposed into the sum of symmetric $\mathcal{S}$ and antisymmetric parts. We obtain one or another submodel, depending on the rank of the matrix $\mathcal{S}$. It is shown in [14] that there are 11 such submodels. Let us choose one of them so that the state Equations (6) and (8) satisfy the state equation of the selected submodel:

$$
\begin{gathered}
\mathcal{S}' + 2\mathcal{S}A = (1-\gamma)\mathcal{S}\mathrm{tr}A, \;\; A' + A^2 = \mathcal{S}, \;\; \mathcal{S} = \mathcal{S}^T, \\
\vec{v}' + A^T\vec{v} + \mathcal{S}\vec{u}_0 = (1-\gamma)\vec{v}\mathrm{tr}\,\mathrm{A}, \;\; \vec{u}_0' + A\vec{u}_0 = \vec{v}, \\
\rho = 2e^{-\int \mathrm{tr}A dt} R'(I), \\
I = (\vec{x}\cdot\mathcal{S}\vec{x} + 2\vec{v}\cdot\vec{x})e^{-(1-\gamma)\int \,\mathrm{tr}A dt} - 2\int \vec{u}_0\cdot\vec{v}e^{-(1-\gamma)\int \mathrm{tr}A dt} dt, \\
p = \rho^\gamma h_0(S) + P_0\frac{1-\rho^\gamma}{\gamma},
\end{gathered}
\tag{52}
$$

where γ, P_0 are arbitrary constants; R, h_0 are arbitrary function; tr$A$ is matrix trace A.

5.1. *Generalization of Exact Solution of the Gas Dynamics Equations with Special State Equation*

We consider an invariant solution (14) for the velocity components from subalgebra 3.36 (12), then matrix A and vector $\vec{u}_0$ from (11) have the form:

$$
A = \begin{pmatrix} 0 & -\dfrac{b}{a} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \;\; \vec{u}_0 = \begin{pmatrix} u_1(t) \\ v_1(t) \\ w_1(t) \end{pmatrix}.
$$

The state equation from (52) at $\gamma \to 0$ tends to:

$$
p = h_0(S) - P_0 \ln \rho. \tag{53}
$$

Since $\mathcal{S} = A' + A^2 \equiv 0$, substituting the matrix A into the differential equations of the submodel (52), the differential equation for the matrix $\mathcal{S}$ is fulfilled. After substituting the matrices A and $\mathcal{S}$ into the differential equation for vector $\vec{u}_0(t)$ from (52), we obtain a system of six-order differential equations for unknown functions $u_1(t)$, $v_1(t)$, $w_1(t)$:

$$u_1'' - \frac{b}{a}v_1' + w_1' = 0,$$

$$v_1'' - \frac{b}{a}v_1' + \frac{b^2}{a^2}v_1 - \frac{b}{a}w_1 = 0,$$

$$w_1'' + u_1' - \frac{b}{a}v_1 + w_1 = 0.$$

The solution of the last system has the form:

$$u_1 = -\frac{1}{a^3}\left(\widetilde{u}_{03}\frac{t^3}{6} + (\overline{u}_{04} - ab\overline{u}_{01})\frac{t^2}{2} + (\overline{u}_{05} - \overline{u}_{03}a^2)t\right) + \frac{\overline{u}_{06}}{a},$$

$$v_1 = -\frac{b}{a^2}\widetilde{u}_{03}\frac{t^2}{2} + \left(\frac{\overline{u}_{01}}{a} - \frac{b}{a^2}\overline{u}_{04}\right)t + \overline{u}_{02} - \frac{b}{a^2}\overline{u}_{05}, \quad w_1 = \frac{1}{a}\left(\overline{u}_{04}t + \widetilde{u}_{03}\frac{t^2}{2} + \overline{u}_{05}\right),$$

where $\overline{u}_{01}, \overline{u}_{02}, \overline{u}_{03}, \overline{u}_{04}, \overline{u}_{05}, \overline{u}_{06}$ are arbitrary constants; $\widetilde{u}_{03} = b\overline{u}_{02} - \overline{u}_{03}$.

Let us write the components of the velocity vector using the last formulas for u_1, v_1, and w_1 (14) and apply the Galilean translations with $\vec{b} = (\overline{u}_{06}a^{-1}, -\overline{u}_{05}ba^{-2}, \overline{u}_{05}a^{-1})$ and dilatation from (4) with $c = a$:

$$\begin{aligned} u &= az - by - \widetilde{u}_{03}\frac{t^3}{6} - \left(\frac{\overline{u}_{04}}{a} - b\overline{u}_{01}\right)\frac{t^2}{2} + \overline{u}_{03}t, \\ v &= -b\widetilde{u}_{03}\frac{t^2}{2} + \left(\overline{u}_{01} - \frac{b}{a}\overline{u}_{04}\right)t + \overline{u}_{02}, \\ w &= \overline{u}_{04}t + a\widetilde{u}_{03}\frac{t^2}{2}. \end{aligned} \tag{54}$$

The density function from the formulas (52) has the form:

$$\rho = 2R'(I), \tag{55}$$

where variable I has the form:

$$\begin{aligned} I = &-2\widetilde{u}_{03}x - 2\left(\frac{b}{a}(a\widetilde{u}_{03}t + \overline{u}_{04}) - \overline{u}_{01}\right)y + 2(a\widetilde{u}_{03}t + \overline{u}_{04})z - \frac{\widetilde{u}_{03}^2}{3}t^4 + \\ &+ \frac{4}{3}\frac{\widetilde{u}_{03}}{a}(ab\overline{u}_{01} - \overline{u}_{04})t^3 - \left(\overline{u}_{03}^2 - b^2\overline{u}_{02}^2 + \overline{u}_{04}^2 + (\overline{u}_{01} - \frac{b}{a}\overline{u}_{04})^2\right)t^2 - \\ &-2\overline{u}_{02}\left(\overline{u}_{01} - \frac{b}{a}\overline{u}_{04}\right)t. \end{aligned} \tag{56}$$

We need to solve the system of differential equations for the function p from the gas dynamics equations in order to define the function of pressure:

$$\begin{aligned} \nabla p &= -\rho(\mathcal{S}\vec{x} + \vec{v}), \\ p_t &= \rho(A\vec{x} + \vec{u}_0)\cdot(\mathcal{S}\vec{x} + \vec{v}) - \rho a_c^2(p,\rho)\mathrm{tr}A. \end{aligned} \tag{57}$$

Substituting the known function ρ, matrix S, and vector $\vec{v}$ into the first equation in (57), we obtain:

$$\nabla p = -2R'(I)\vec{v}$$

or

$$\nabla p = -R'(I)\nabla I.$$

The integration one of the two last differential equations gives us the form of the pressure:

$$p = -R(I) + \overline{p}_0(t), \tag{58}$$

where $\overline{p}_0$ is an arbitrary function. Without loss of generality, we can assume in (58) that $\overline{p}_0(t)$ is equal to zero because after substituting (58) in the second differential equation from (57), we find that $\overline{p}_0(t)$ is constant $\overline{p}_0$. Then, the pressure (58) has the form:

$$p = -R(I). \tag{59}$$

Thus, the solution of the gas dynamics Equations (1) with the state Equation (53) is given by the formulas (54), (55), (56), and (59).

Remark 2. *The solution* (54), (55), (56), *and* (59) *coincides with the solution* (16) *after using uniform dilatation from* (4) *with* $c = a$ *and if* $\overline{u}_{01} = -\gamma\rho_0^{-1}$, $\overline{u}_{02} = \overline{u}_{03} = \overline{u}_{04} = \widetilde{u}_{03} = 0$, $R(I) = 2^{-1}\rho_0 I$, *and* $P_0 = 0$.

The world lines of gas particles (17) are defined from solution (54):

$$\begin{aligned} x &= -\frac{\widetilde{u}_{03}t^2}{2} + (az_0 - by_0)t + x_0, \\ y &= -b\frac{\widetilde{u}_{03}}{6}t^3 + (\overline{u}_{01} - \frac{b}{a}\overline{u}_{04})\frac{t^2}{2} + \overline{u}_{02}t + y_0, \\ z &= \frac{a\widetilde{u}_{03}}{6}t^3 + \frac{\overline{u}_{04}}{2}t^2 + z_0, \end{aligned} \tag{60}$$

where x_0, y_0, and z_0 are global Lagrangian coordinates. The Jacobian $J = |\partial\vec{x}/\partial\vec{x}_0|$ [41] of transformations (60) is equal to 1, where $\vec{x}_0 = (x_0, y_0, z_0)$. This fact means that the motion of particles does not have singularities. Trajectories of the gas particles (60) are built in the Figure 7.

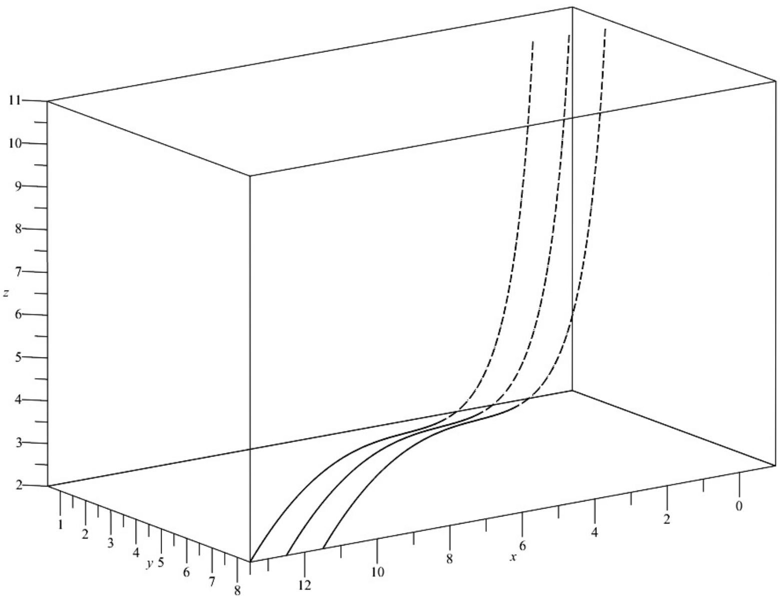

Figure 7. The trajectories (60) with parameters $a = 1$, $b = 0$, $\overline{u}_{0i} = 1$, $i = 1..4$, $x_0 = 1,2,3$; $y_0 = z_0 = 1$, and $t = -3..3$.

5.2. Generalization of Exact Solution of the Gas Dynamics Equations with State Equation of the Monatomic Gas

We consider the representation of invariant solution (33) for subalgebra 3.4* (32) when the functions u_1, v_1, and w_1 are linear

$$u_1 = \frac{a_1 x}{\sqrt{1+t^2}} + b_1, \quad v_1 = \frac{a_2 x}{\sqrt{1+t^2}} + b_2, \quad w_1 = \frac{a_3 x}{\sqrt{1+t^2}} + b_3,$$

where a_1, a_2, a_3, b_1, b_2, and b_3 are arbitrary constants. Then, matrix A and $\vec{u}_0$ from submodel (52) have the form:

$$A = \begin{pmatrix} \dfrac{a_1+t}{1+t^2} & 0 & 0 \\ \dfrac{a_2 - a_3 t}{(1+t^2)^{3/2}} & \dfrac{t}{1+t^2} & -\dfrac{1}{1+t^2} \\ \dfrac{a_3 + a_2 t}{(1+t^2)^{3/2}} & \dfrac{1}{1+t^2} & \dfrac{t}{1+t^2} \end{pmatrix}, \quad \vec{u}_0 = \begin{pmatrix} \dfrac{b_1}{\sqrt{1+t^2}} \\ \dfrac{b_2 - t b_3}{1+t^2} \\ \dfrac{b_3 + b_2 t}{1+t^2} \end{pmatrix}. \tag{61}$$

The state equation from (52) with $\gamma = \dfrac{5}{3}$, $P_0 = 0$ is equal to $p = \rho^{5/3} h_0(S)$. Matrix $A' + A^2$ has the form:

$$A' + A^2 = \begin{pmatrix} \dfrac{a_1^2 + 1}{(1+t^2)^2} & 0 & 0 \\ \dfrac{-2a_2 t + a_2 a_1 - 2a_3 - a_3 a_1 t}{(1+t^2)^{5/2}} & 0 & 0 \\ \dfrac{2a_2 - 2a_3 t + a_3 a_1 + a_2 a_1 t}{(1+t^2)^{5/2}} & 0 & 0 \end{pmatrix}$$

From the symmetry condition of the matrix $A' + A^2$, we obtain that $a_2 = a_3 = 0$. After substituting matrices $\mathcal{S}$ and A into the submodel Equation (52), we obtain a relation from which it follows that $a_1 = 0$.

Next, we substitute the known matrices into a vector differential equation for vector $\vec{u}_0$ from the submodel (52). The vector form $\vec{u}_0$ is taken from (61). From the three obtained relations, we find that $b_1 = 0$ and b_2 and b_3 are arbitrary constants. The components of the velocity vector have the form:

$$u = \frac{tx}{1+t^2}, \quad v = \frac{t(y - b_3) - z + b_2}{1+t^2}, \quad w = \frac{y + b_3 + t(z + b_2)}{1+t^2}. \tag{62}$$

Let us define the type of density function from (52). We receive:

$$\rho = \frac{2R'(I)}{(1+t^2)^{3/2}}, \tag{63}$$

where

$$I = \frac{x^2 - 4y(b_3 + t b_2) - 4z(b_3 t - b_2)}{1+t^2}. \tag{64}$$

From (57), we obtain the pressure

$$p = -\frac{R(I)}{(1+t^2)^{5/2}} + \overline{p}_0(t).$$

Function $\overline{p}_0(t)$ is defined from the second Equation (57), where $a_c^2(p, \rho) = \frac{5}{3} p \rho^{-1}$ from (3). As a result, the pressure function is:

$$p = -\frac{R(I)}{(1+t^2)^{5/2}}. \tag{65}$$

Thus, the solution of the gas dynamics Equations (1) with the state Equation (8) is given by the formulas (62), (63), (64), and (65).

Remark 3. *The solution* (62), (63), (64), *and* (65) *coincides with the solution* (38) *if* $b_2 = b_3 = 0$, $R(I) = -p(x_1)$, $I = x_1^2$.

Remark 4. *The solution* (62), (63), (64), *and* (65) *coincides with the solution* (39) *if* $b_2 = 0$, $R(I) = -\dfrac{2b_3\rho_0}{b}e^{-\frac{b}{4b_3}I}$.

Remark 5. *The solution* (62), (63), (64), *and* (65) *does not coincide with the solution* (33), (37) *because the matrix* $B = A' + A^2$ *has an antisymmetrical part which contradicts the formulas* (52).

The world lines of gas particles (17) are defined from solution (62):

$$x = x_0\sqrt{1+t^2},\ \ y = (b_2 - tb_3)\tau - z_0 t + y_0, \\ z = (tb_2 + b_3)\tau + y_0 t + z_0, \tag{66}$$

where x_0, y_0, and z_0 are global Lagrangian coordinates.

The Jacobian $J = |\partial\vec{x}/\partial\vec{x}_0|$ [41] of transformations (66) is equal to $(1+t^2)^{\frac{3}{2}} \neq 0$. At the initial moment of time $t = 0$, the gas particle is located at the point with coordinates (x_0, y_0, z_0). The trajectories of the gas particles (66) are built in the Figure 8.

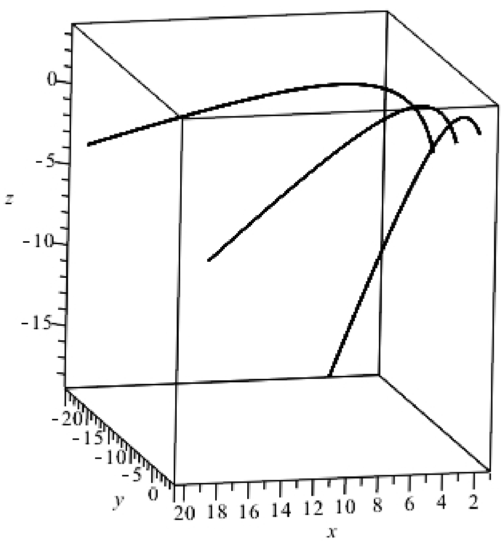

Figure 8. The trajectories (66) are represented with $b_2 = -5$, $b_3 = 2$, $(x_0, y_0, z_0) = (1,1,1); (3,3,1); (5,5,1)$, and t = 0..4.

6. Conclusions

In this work, the gas dynamics equations have been considered in the case of the state equation of a monatomic gas and the special state equation (a pressure equals to the sum of two arbitrary functions of density and entropy). One three-dimensional subalgebra has been selected for each specified state equation. Invariant submodels of rank one have been constructed for them, and exact solutions with a linear velocity field with inhomogeneous deformation have been obtained.

In the case of the special state equation, the submodel describes the isochoric motion of particles with constant pressure along each world line. The motion is vortex, limited by a moving plane. The motion of particles occurs along parabolas and along rays in parallel planes. The motion of the particles volume in the case of the subalgebra without one parameter is considered. The spherical volume of particles turns into an ellipsoid at finite moments of time, and as time tends to infinity, the particles end up on an infinite strip with a width equals to the diameter of the original sphere.

In the case of the state equation of a monatomic gas, the submodel describes a complex vortex condensation (compaction) to the origin and subsequent expansion of gas particles in half-spaces relative to the plane $x = 0$. The motion of any allocated spherical volume of gas retains a spherical shape and expands (narrows) for $t > 0$ (for $t < 0$). An equation defining a characteristic conoid with an arbitrary vertex in $\mathbb{R}_4$ is obtained. Its projection into $\mathbb{R}_3$ at each moment of time gives the equation of the sphere. It is shown that for any point $\mathbb{R}_3$ at time $t = t_0 > 0$, it is possible to obtain a radius of the sphere depending on t_0. The perturbation coming from the center does not reach the particles outside this sphere.

The obtained exact solutions with a linear velocity field are generalized if the representations for the velocities are chosen according to the considered invariant solutions. From the previously performed classification of submodels, a submodel with a state equation including the above is selected. As a result, exact solutions of a wider class have been obtained, since density and pressure may not be invariant with respect to the generators of the selected three-dimensional subalgebras.

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