mathematics

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Some Results on the q-Calculus and Fractional q-Differential Equations

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Abstract: In this paper, we first discuss some important properties of fractional q-calculus. Then, based on these properties and the q-Laplace transform, we translate a class of fractional q-differential equations into the equivalent q-differential equations with integer order. Thus, we propose a method for solving some linear fractional q-differential equations by means of solving the corresponding integer order equations. Several examples are provided to illustrate our solution method.

Keywords: fractional q-calculus; the fractional q-differential equations; equivalency theorem; solution method

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Citation: Sheng, Y.; Zhang, T. Some Results on the q-Calculus and Fractional q-Differential Equations. *Mathematics* **2022**, *10*, 64. https://doi.org/10.3390/math10010064

Academic Editor: Jian Cao

Received: 1 December 2021
Accepted: 23 December 2021
Published: 25 December 2021

1. Introduction

Recently, mathematicians and engineers have paid much attention to the q-calculus and fractional q-differential equations. The q-calculus (also called the quantum calculus) can be dated back to 1908, Jackson's work [1]. Based on the q-calculus, the q-differential equations were established which can describe some special physical processes appearing in quantum dynamics, discrete dynamical systems and discrete stochastic processes, and so forth. It should be pointed out that the q-differential equations are usually defined on a time scale set T_q, where q is the scale index. With the development of the q-calculus theory, some related concepts have also been introduced and studied such as the q-Laplace transform, q-Gamma and q-Beta functions, q-Mittag–Leffler functions, q-Taylor expansion, q-integral transforms theory and so forth. Please refer to articles [2–10] for more details on q-calculus and fractional q-differential equations. Up to now, compared with the classical fractional calculus, the study of the fractional q-calculus is still immature.

At present, there have been some studies on the existence and uniqueness of solutions for the fractional q-differential equations. In [11], Abdeljawad et al. proved the uniqueness of an initial value problem involving a nonlinear delay Caputo fractional q-difference system by using a new generalized version of discrete fractional q-Gronwall inequality. In [12], the authors proved that the Caputo q-fractional boundary value problem with the p-Laplacian operator has a unique solution by using the Banach's contraction mapping principle. In [13], Ren et al. considered the uniqueness of nontrivial solutions by virtue of the contraction mapping principle and also obtained the existence of multiple positive solutions under some appropriate conditions via standard fixed point theorems. In [14], Zhang et al. gave the existence and uniqueness of solution of the Caputo fractional q-differential equations by using the Ascoli–Arzela Theorem and a q-analogue Gronwall inequality. Furthermore, Zhang et al. in [15] discussed the existence of a unique solution in the q-integral space.

Although many research results have been given about the existence and uniqueness of the fractional q-differential equations of various types, there are very few studies about how to solve these problems analytically. The applications of the q-Mittag–Leffler function for presenting the solution of initial value problems of linear Caputo fractional q-differential equations were studied by Abdeljawad in [16,17]. Following the previous work, Abdeljawad et al. [11] provided a particular solution formula for the nonlinear delay Caputo fractional q-difference system by means of the q-Mittag–Leffler function. In [18], the solutions of some Caputo fractional q-difference equations are expressed by means of a new generalized type q-Mittag–Leffler function. In [15], Zhang et al. discussed the unique existence of solutions for the Caputo type nonlinear fractional q-differential equations and obtained the solutions expressed by means of the q-Mittag–Leffler function, which is defined as follows

$$E^q_{\alpha,\beta}(\lambda, z - z_0) = \sum_{k=0}^{\infty} \frac{\lambda^k (z - z_0)_q^{(k\alpha)}}{\Gamma_q(\beta + k\alpha)}.$$

We see that the q-Mittag–Leffler function is an infinite series. Although, in [15,18], the solutions were obtained by means of the q-Mittag–Leffler function, it is very difficult to express the solution in a finite analytic form, because the solution formula concerns the fractional q-integral of q-Mittag–Leffler function.

In this paper, we aim to find a method to obtain an analytic solution in a finite form for the initial value problem of the Caputo type fractional q-differential equation

$$\begin{cases} {}^cD_q^{\alpha} u(t) = \lambda u(t) + f(t),\ \alpha > 0, \\ D_q^k u(0) = d_k,\ k = 0, 1, \cdots, m,\ m = [\alpha]. \end{cases}$$

We first study some properties of the fractional q-calculus. Then we translate the above fractional q-differential equation into the equivalent q-differential equation with integer order by using the properties derived in this paper and the q-Laplace transform. Thus, we propose a method for solving the fractional q-differential equations by solving the corresponding integer order equations. At last, several examples are provided to illustrate our solution method.

This paper is organized as follows. Section 2 introduces some basic definitions and relevant results on q-calculus. In Section 3, we study some important properties about q-calculus. Section 4 is devoted to a new solution method to find out the analytic solutions of a class of fractional q-differential equations. In Section 5, we provide some examples to illustrate our solution method.

2. Preliminaries

Let $\mathbb{N} = \{1, 2, ...\}$ be the positive integer set and $0 < q < 1$. Introduce the q-shifted operations

$$(t - s)_q^{(0)} = 1,\ (t - s)_q^{(k)} = \prod_{i=0}^{k-1} (t - q^i s),\ k \in \mathbb{N}. \tag{1}$$

Let $\mathbb{C}$ be the set of complex numbers. If $\alpha \in \mathbb{C}$ and $\alpha \notin \mathbb{N}$, the q-shifted operation is defined by

$$(t - s)_q^{(\alpha)} = t^{\alpha} \prod_{i=0}^{\infty} \frac{t - q^i s}{t - q^{\alpha+i} s},\ 0 \le s \le t. \tag{2}$$

Introduce the notations

$$[\alpha]_q = \frac{1 - q^{\alpha}}{1 - q}, \quad [n]_q! = [n]_q [n-1]_q \cdots [1]_q.$$

For $\alpha \in \mathbb{C} \setminus \{-n, n \in \mathbb{N} \cup \{0\}\}$, define the q-analogue Gamma function $\Gamma_q(\alpha)$ as

$$\Gamma_q(\alpha) = (1 - q)_q^{(\alpha-1)} (1 - q)^{1-\alpha},\ 0 < q < 1. \tag{3}$$

Then, from the definition it is easy to see that

$$\Gamma_q(1) = 1, \quad \Gamma_q(n+1) = [n]_q!, \quad \Gamma_q(\alpha + 1) = [\alpha]_q \Gamma_q(\alpha).$$

For $\alpha, \beta \in \mathbb{C}$ and $Re(\alpha) > 0, Re(\beta) > 0$, define the q-Beta function

$$B_q(\alpha, \beta) = \int_0^1 s^{\alpha-1}(1-qs)_q^{(\beta-1)} d_q s. \tag{4}$$

The q-Gamma and q-Beta functions have the following relationship [2]

$$B_q(\alpha, \beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}.$$

For a given $q \in R$, a subset $A \subset R$ is called q-geometric if $qx \in A$ whenever $x \in A$. That is, $\forall x \in A$, set A includes all geometric sequences $\{xq^n\}_{n=0}^{\infty}$. A typical q-geometric set is $T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$, where $0 < q < 1, \mathbb{Z}$ is the set of integers.

Definition 1 ([1])**.** *Let $f(t)$ be a real-valued function on the set T_q and $0 < q < 1$. Define the q-derivative of $f(t)$ by*

$$D_q f(t) = \frac{d_q f(t)}{d_q t} = \frac{f(t) - f(qt)}{(1-q)t},\ t \in T_q \setminus \{0\}, \tag{5}$$

$$D_q f(0) = \frac{d_q f(t)}{d_q t}|_{t=0} = \lim_{n\to\infty} \frac{f(tq^n) - f(0)}{tq^n},\ t \neq 0.$$

From the definition, we see that the q-derivative is different from the classical derivative. It is a kind of discrete analogue of the classical derivative. On the basis of Definition 1, the higher order q-derivative $D_q^n f(t)$ is defined by $D_q^n f(t) = D_q(D_q^{n-1} f(t)), n \geq 2$.

Furthermore, for two real-valued functions $f(t)$ and $g(t)$, we have the following operation rules by a straightforward computation

$$\begin{aligned}
&D_q(af(t) \pm bg(t)) = aD_q f(t) \pm bD_q g(t),\ a, b \in R,\\
&D_q(f(t)g(t)) = g(t)D_q f(t) + f(qt)D_q g(t),\\
&D_q\Big(\frac{f(t)}{g(t)}\Big) = \frac{g(t)D_q f(t) - f(t)D_q g(t)}{g(t)g(qt)},\ g(t) \neq 0,\ g(qt) \neq 0.
\end{aligned}$$

Definition 2 ([19])**.** *Let $f(t)$ be a real-valued function defined on the set T_q and $0 < q < 1$. The q-integral of $f(t)$ is defined by*

$$\int_0^t f(s) d_q s = (1-q) \sum_{n=0}^{\infty} tq^n f(tq^n),\ t \in T_q, \tag{6}$$

and for $b > a$

$$\int_a^b f(s) d_q s = \int_0^b f(s) d_q s - \int_0^a f(s) d_q s,\ a, b \in T_q. \tag{7}$$

From Definition 2, it holds that

$$\Big|\int_0^b f(s) d_q s\Big| \leq \int_0^b |f(s)| d_q s,\ b > 0, \tag{8}$$

$$\int_a^b f(s) d_q s = \int_a^c f(s) d_q s + \int_c^b f(s) d_q s,\ a < c < b. \tag{9}$$

The operation of q-integration by parts is given in the lemma below.

Lemma 1 ([20]). *Suppose that $f(t)$ and $g(t)$ are real-valued functions defined on the set T_q, $0 < q < 1,\ a, b \in T_q,\ 0 \le a < b$. Then*

$$\int_a^b f(qt)D_q g(t)d_q t = (fg)(b) - (fg)(a) - \int_a^b g(t)D_q f(t)d_q t. \tag{10}$$

Now, we introduce the concept of fractional q-calculus. Let A be a q-geometric set.

Definition 3 ([21]). *Suppose that $t \in A$ and $\alpha \neq -1, -2, \cdots$. The α-order fractional q-integral of the Riemann–Liouville type is defined by $I_q^0 f(t) = f(t)$ and*

$$I_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)_q^{(\alpha-1)} f(s)d_q s. \tag{11}$$

Definition 4 ([22]). *Let $n = [\alpha]$ and $f(t)$ be a real-valued function defined on $(0, \infty)$. The α-order Caputo type fractional q-derivative of function $f(t)$ is defined by*

$$^cD_q^\alpha f(t) = \begin{cases} I_q^{-\alpha} f(t), & \alpha \le 0, \\ I_q^{n-\alpha} D_q^n f(t), & \alpha > 0, \end{cases} \tag{12}$$

where $[\alpha]$ represents the smallest integer which is greater than or equal to α.

For $\alpha > 0$, we have

$$^cD_q^\alpha f(t) = \frac{1}{\Gamma_q(n-\alpha)} \int_0^t (t - qs)^{(n-\alpha-1)} D_q^n f(s)d_q s. \tag{13}$$

Definition 5 ([16]). *Suppose that $\alpha \in R$ and $n = [\alpha]$, $f(t)$ is a real-valued function on $(0, \infty)$. The α-order Riemann–Liouville type fractional q-derivative of $f(t)$ is defined as follows*

$$D_q^\alpha f(t) = \begin{cases} I_q^{-\alpha} f(t), & \alpha \le 0, \\ D_q^n I_q^{n-\alpha} f(t), & \alpha > 0. \end{cases} \tag{14}$$

For $\alpha > 0$, we have

$$D_q^\alpha f(t) = \frac{1}{\Gamma_q(n-\alpha)} D_q^n \int_0^t (t - qs)_q^{(n-\alpha-1)} f(s)d_q s. \tag{15}$$

Under certain conditions, the Riemann–Liouville type fractional q-derivative and Caputo type fractional q-derivative have the following relationship [2,16]

$$^cD_q^\alpha f(t) = D_q^\alpha (f(t) - \sum_{k=0}^{n-1} \frac{D_q^k f(0)}{\Gamma_q(k+1)} t^k),\ t > 0,\ \alpha > 0,\ n = [\alpha]. \tag{16}$$

3. Some Properties on q-Calculus

Introduce the following q-integrable and q-continuous differentiable function spaces

$$_qL^p(0, b) = \{f(t) : \|f(t)\|_{_qL^p(0,b)} = \sup_{0<t\le b, (t,b)\in T_q} (\int_0^t |f(s)|^p d_q s)^{\frac{1}{p}} < \infty\},$$

$$C_q^{(m)}[0, b] = \{f(t) : D_q^k f(t) \in C[0, b], k = 0, 1, \cdots, m.\}.$$

Similarly, we can define $C_q^{(m)}(0,b]$. By the definition of q-derivative, we see that if $f(t) \in C[0,b]$, then $f(t) \in C_q^{(k)}(0,b]$, $k \geq 0$. Furthermore, if $f(t) \in C[0,b]$ and $f'(0)$ exists, then $f(t) \in C_q^{(1)}[0,b]$.

Define the integer order q-integral operator

$$I_q^0 f(t) = f(t), \ \ I_q f(t) = \int_0^t f(s) d_q s, \ \ I_q^n f(t) = I_q^{n-1} I_q f(t), n = 1, 2, \cdots.$$

Lemma 2. *Let $f(t) \in C[0,b]$, $0 < q < 1$, then we have*

$$D_q I_q f(t) = f(t), \ 0 < t \leq b, \tag{17}$$

$$I_q D_q f(t) = f(t) - f(0), \ 0 < t \leq b. \tag{18}$$

Proof. From (5) and (6), we have

$$\begin{aligned} D_q I_q f(t) &= (\int_0^t f(s) d_q s - \int_0^{qt} f(s) d_q s)/(1-q)t \\ &= [(1-q)\sum_{k=0}^{\infty} tq^k f(tq^k) - (1-q)\sum_{k=0}^{\infty} tq^{k+1} f(tq^{k+1})]/(1-q)t \\ &= \sum_{k=0}^{\infty} q^k f(tq^k) - \sum_{k=0}^{\infty} q^{k+1} f(tq^{k+1}) = f(t). \end{aligned}$$

Then (17) holds. For (18), from (6) and (5), we obtain

$$\begin{aligned} I_q D_q f(t) &= \int_0^t \frac{f(s) - f(qs)}{(1-q)s} d_q s \\ &= (1-q)\sum_{k=0}^{\infty} tq^k \frac{f(tq^k) - f(tq^{k+1})}{(1-q)tq^k} \\ &= \sum_{k=0}^{\infty} (f(tq^k) - f(tq^{k+1})) \\ &= f(t) - \lim_{k\to\infty} f(tq^k) = f(t) - f(0_+). \end{aligned}$$

This gives (18). □

Lemma 3. *Let $f(t) \in C[0,b]$, $0 < q < 1$, $\alpha \neq -1, -2, \cdots$. Then*

$$\lim_{\alpha \to 0_+} I_q^\alpha f(t) = f(t). \tag{19}$$

Proof. Since $D_{q,s}(t-s)_q^{(\alpha)} = -[\alpha]_q (t-qs)_q^{(\alpha-1)}$ (see (2.6) in [20]), we have from (10) that

$$\begin{aligned} I_q^\alpha f(t) &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)_q^{(\alpha-1)} f(s) d_q s \\ &= \frac{-1}{\Gamma_q(\alpha)[\alpha]_q} \int_0^t D_{q,s}(t-s)_q^{(\alpha)} f(s) d_q s \\ &= -\frac{1}{\Gamma_q(\alpha+1)} \{(t-s)_q^{(\alpha)} f(s)|_0^t - \int_0^t (t-qs)_q^{(\alpha)} D_q f(s) d_q s\} \\ &= \frac{1}{\Gamma_q(\alpha+1)} \{t^\alpha f(0) + \int_0^t (t-qs)_q^{(\alpha)} D_q f(s) d_q s\}. \end{aligned}$$

So by Lemma 2,

$$\lim_{\alpha\to 0_+} I_q^\alpha f(t) = f(0) + \int_0^t D_q f(s) d_q s = f(t).$$

□

Corollary 1. Let $f(t) \in C[0,b]$. Then we have

$$\lim_{\alpha\to n_-} {}^cD_q^\alpha f(t) = D_q^n f(t),\ 0 < q < 1,\ n-1 < \alpha < n,\ n \geq 1,\ t > 0. \tag{20}$$

Proof. From (12) and Lemma 3, we obtain

$$\lim_{\alpha\to n_-} {}^cD_q^\alpha f(t) = \lim_{\alpha\to n_-} I_q^{n-\alpha} D_q^n f(t) = D_q^n f(t).$$

□

Lemma 4 ([2])**.** *If $f(t) \in {}_qL^1[0,b]$, then the semigroup property holds*

$$I_q^\alpha I_q^\beta f(t) = I_q^\beta I_q^\alpha f(t) = I_q^{\alpha+\beta} f(t),\ \alpha \geq 0,\ \beta \geq 0. \tag{21}$$

Lemma 5. *Let $f(t) \in {}_qL^1[0,b]$, then*

$$\lim_{\alpha\to n_-} I_q^\alpha f(t) = I_q^n f(t),\ n-1 < \alpha < n. \tag{22}$$

Proof. When $n = 1$,

$$\begin{aligned} \lim_{\alpha\to 1_-} I_q^\alpha f(t) &= \lim_{\alpha\to 1_-} \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)_q^{(\alpha-1)} f(s) d_q s \\ &= \int_0^t f(s) d_q s = I_q f(t). \end{aligned}$$

When $n > 1$, let $\alpha = \alpha' + n - 1$ and $0 < \alpha' < 1$, by Lemma 4,

$$\lim_{\alpha\to n_-} I_q^\alpha f(t) = \lim_{\alpha'\to 1_-} I_q^{n-1+\alpha'} f(t) = \lim_{\alpha'\to 1_-} I_q^{\alpha'} I_q^{n-1} f(t) = I_q^n f(t).$$

□

Lemma 6 ([2])**.** *Let $0 < \alpha < 1$ and $f(t) \in C[0,b]$ such that $D_q f(t) \in C(0,b]$. Then*

$${}^cD_q^\alpha f(t) = D_q^\alpha f(t) - \frac{t^{-\alpha}}{\Gamma_q(1-\alpha)} f(0),\ t > 0. \tag{23}$$

Lemma 7. *The following properties hold*

(i) *If $|f(t)| \leq M$, then $I_q^\alpha f(0) = 0,\ \alpha > 0$,*

(ii) *If $|D_q f(t)| \leq M$, then ${}^cD_q^\alpha f(0) = 0,\ 0 < \alpha < 1$.*

Proof. For property (i), using (8) we can obtain

$$\begin{aligned}|I_q^\alpha f(t)| &\le \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)_q^{(\alpha-1)}|f(s)|d_qs \\ &\le \frac{M}{\Gamma_q(\alpha)}\int_0^t (t-qs)_q^{(\alpha-1)}d_qs \\ &= \frac{M}{\Gamma_q(\alpha)}t^\alpha\int_0^1 (1-q\tau)_q^{(\alpha-1)}d_q\tau \\ &= \frac{M}{\Gamma_q(\alpha)}t^\alpha B_q(\alpha,1) = \frac{t^\alpha M}{\Gamma_q(1+\alpha)} \to 0,\ t\to 0.\end{aligned}$$

Next, from (i) and ${}^cD_q^\alpha f(t) = I_q^{1-\alpha}D_qf(t)$, we see that the property (ii) holds. □

Lemma 8. *For $0<\alpha<1$ and $f(t)\in C[0,b]$, it holds*

$$D_q^\alpha I_q^\alpha f(t) = f(t),\ {}^cD_q^\alpha I_q^\alpha f(t) = f(t). \tag{24}$$

Proof. By Definition 5 and Lemma 2, we see that

$$D_q^\alpha I_q^\alpha f(t) = D_q I_q^{1-\alpha} I_q^\alpha f(t) = D_q I_q f(t) = f(t).$$

Then, the first equality holds. From Lemma 6 and noting $I_q^\alpha f(0)=0$,

$${}^cD_q^\alpha I_q^\alpha f(t) = D_q^\alpha I_q^\alpha f(t) = f(t).$$

The proof is completed. □

Lemma 9. *For $0<\alpha<1$, $n-1<\beta<n$, $n-1<\alpha+\beta\le n$ and $n\ge 1$, we have*

$${}^cD_q^\alpha\, {}^cD_q^\beta f(t) = {}^cD_q^{\alpha+\beta} f(t). \tag{25}$$

Proof. Applying Lemma 4 and Lemma 8, it yields that

$$\begin{aligned}{}^cD_q^\alpha\, {}^cD_q^\beta f(t) &= {}^cD_q^\alpha I_q^{n-\beta} D_q^n f(t) = {}^cD_q^\alpha I_q^\alpha I_q^{n-\alpha-\beta} D_q^n f(t) \\ &= {}^cD_q^\alpha I_q^\alpha\, {}^cD_q^{\alpha+\beta} f(t) = {}^cD_q^{\alpha+\beta} f(t).\end{aligned}$$

□

Remark 1. *In Lemma 9, condition $n-1<\alpha+\beta\le n$ is necessary. For $n=1$, a counterexample is as follows*

$${}^cD_q^{0.7}\,{}^cD_q^{0.5}t = \frac{1}{\Gamma_q(\frac{4}{5})}t^{-0.2},\ {}^cD_q^{1.2}t = 0.$$

4. The Fractional q-Differential Equations

In this section, we establish the solution method for a class of linear fractional q-differential equations.

Let us consider the fractional q-differential equations in the following form

$$\begin{cases}{}^cD_q^\alpha u(t) = \lambda u(t) + f(t),\ 0<t\le b, \\ D_q^k u(0) = d_k,\ k=0,1,\cdots,m-1,\end{cases} \tag{26}$$

where $\{d_k\}$ are constants, $m=[\alpha]$, λ is a constant and $\alpha = m-1+\frac{r}{p}$, $r,\ p\in N$, $1\le r<p$.

Introduce the q-Mittag–Leffler function

$$E^q_{\alpha,\beta}(\lambda, z - z_0) = \sum_{k=0}^{\infty} \frac{\lambda^k (z - z_0)_q^{(k\alpha)}}{\Gamma_q(\beta + k\alpha)},$$

where $\{\lambda, z, z_0, \alpha, \beta\} \in \mathbb{C}$ and $Re(\alpha) > 0$, $Re(\beta) \geq 0$.

Theorem 1. *Let $u(t)$ be the solution of problem (26) and assume that $|u(t)| \leq M$, $|\lambda b^\alpha| \leq 1$. Then $u(t)$ can be represented as follows*

$$u(t) = \sum_{k=0}^{m-1} E^q_{\alpha,k+1}(\lambda, t) d_k t^k + \int_0^t (t - qs)_q^{(\alpha-1)} E^q_{\alpha,\alpha}(\lambda, t - q^\alpha s) f(s) d_q s. \tag{27}$$

Proof. According to ([15], Theorem 1), the solution of problem (26) satisfies the equivalent integral equation

$$u(t) = \sum_{k=0}^{m-1} d_k \frac{t^k}{\Gamma_q(k+1)} + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)_q^{(\alpha-1)} [\lambda u(s) + f(s)] d_q s. \tag{28}$$

Let $d(t) = \sum\limits_{k=0}^{m-1} d_k \frac{t^k}{\Gamma_q(k+1)}$. Then by a circulative iteration and using Lemma 4, we obtain from (28) that

$$\begin{aligned}
u(t) &= d(t) + \lambda I_q^\alpha u(t) + I_q^\alpha f(t) \\
&= d(t) + \lambda I_q^\alpha [d(t) + \lambda I_q^\alpha u(t) + I_q^\alpha f(t)] + I_q^\alpha f(t) \\
&= d(t) + \lambda I_q^\alpha d(t) + \lambda^2 I_q^{2\alpha} u(t) + \lambda I_q^{2\alpha} f(t) + I_q^\alpha f(t) \\
&= d(t) + \lambda I_q^\alpha d(t) + \lambda^2 I_q^{2\alpha} [d(t) + \lambda I_q^\alpha u(t) + I_q^\alpha f(t)] + \lambda I_q^{2\alpha} f(t) + I_q^\alpha f(t) \\
&= d(t) + \lambda I_q^\alpha d(t) + \lambda^2 I_q^{2\alpha} d(t) + \lambda^3 I_q^{3\alpha} u(t) + \lambda^2 I_q^{3\alpha} f(t) + \lambda I_q^{2\alpha} f(t) + I_q^\alpha f(t) \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&= \sum_{n=0}^{\infty} \lambda^n I_q^{n\alpha} d(t) + \sum_{n=0}^{\infty} \lambda^n I_q^{(n+1)\alpha} f(t) + \lim_{n\to\infty} \lambda^n I_q^{n\alpha} u(t).
\end{aligned} \tag{29}$$

Since

$$\lambda^n I_q^{n\alpha} d(t) = \lambda^n I_q^{n\alpha} [\sum_{k=0}^{m-1} \frac{t^k}{\Gamma_q(k+1)} d_k] = \sum_{k=0}^{m-1} \frac{\lambda^n t^{n\alpha+k}}{\Gamma_q(n\alpha + k + 1)} d_k,$$

then

$$\begin{aligned}
\sum_{n=0}^{\infty} \lambda^n I_q^{n\alpha} d(t) &= \sum_{k=0}^{m-1} [\sum_{n=0}^{\infty} \frac{\lambda^n t^{n\alpha+k}}{\Gamma_q(n\alpha + k + 1)}] d_k \\
&= \sum_{k=0}^{m-1} [\sum_{n=0}^{\infty} \frac{\lambda^n t^{n\alpha}}{\Gamma_q(n\alpha + k + 1)}] t^k d_k \\
&= \sum_{k=0}^{m-1} E^q_{\alpha,k+1}(\lambda, t) t^k d_k.
\end{aligned} \tag{30}$$

Next, by the identity [21] $(t-s)_q^{(\beta+\gamma)} = (t-s)_q^{(\beta)}(t-q^\beta s)_q^{(\gamma)}$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\lambda^n I_q^{(n+1)\alpha} f(t) &= \sum_{n=0}^{\infty}\frac{\lambda^n}{\Gamma_q(n\alpha+\alpha)}\int_0^t (t-qs)_q^{(n\alpha+\alpha-1)} f(s)d_qs \\ &= \int_0^t (t-qs)_q^{(\alpha-1)}\sum_{n=0}^{\infty}\frac{\lambda^n(t-q^\alpha s)_q^{(n\alpha)}}{\Gamma_q(n\alpha+\alpha)} f(s)d_qs \\ &= \int_0^t (t-qs)_q^{(\alpha-1)} E_{\alpha,\alpha}^q(\lambda, t-q^\alpha s) f(s)d_qs. \end{aligned} \tag{31}$$

Furthermore, by the argument of Lemma 7

$$|\lambda^n I_q^{n\alpha} u(t)| \leq \frac{(\lambda b^\alpha)^n}{\Gamma_q(n\alpha+1)} M \leq \frac{M}{\Gamma_q(n\alpha+1)} \to 0, n\to\infty. \tag{32}$$

Then, combining (29)–(32), we complete the proof. □

Although formula (27) gives a solution representation of problem (26), we see that it is very difficult to obtain a finite form solution by formula (27), which concerns the complex fractional q-integral. Below, we will transform problem (26) into an equivalent integer order q-differential equation and then by using this integer order equation to find the solution of problem (26).

We first consider the case of $m=1$ and $r=1$ in problem (26). Corresponding to problem (26), introduce the auxiliary problem

$$\begin{cases} D_q z(t) = \lambda^p z(t) + w(t), \ 0 < t \leq b, \\ z(0) = u(0), \end{cases} \tag{33}$$

where

$$w(t) = \lambda^p \sum_{k=1}^{p} \theta_k t^{\frac{k}{p}} + \sum_{k=0}^{p-1} \lambda^{p-(k+1)}\, {}^cD_q^{\frac{k}{p}} f(t) - \theta_p, \tag{34}$$

$$\theta_k = \frac{\lambda^k u(0) + \lambda^{k-1} f(0)}{\Gamma_q(1+\frac{k}{p})}, \ k = 1, 2, \cdots, p. \tag{35}$$

Furthermore, we introduce the q-Laplace transform ${}_qL_s(\cdot)$ which is a linear transform and has the following properties [2]

$${}_qL_s\left(\frac{t^\gamma}{\Gamma_q(\gamma+1)}\right) = \frac{(1-q)^\gamma}{s^{\gamma+1}}, \ \gamma > -1, \ Re(s) > 0, \tag{36}$$

$${}_qL_s({}^cD_q^\alpha f(t)) = \frac{s^\alpha}{(1-q)^\alpha}\,({}_qL_s f)(s) - \frac{f(0)s^{\alpha-1}}{(1-q)^\alpha}, \ 0<\alpha<1, \tag{37}$$

$${}_qL_s(D_q f(t)) = \frac{s}{1-q}\,({}_qL_s f)(s) - \frac{f(0)}{1-q}. \tag{38}$$

Theorem 2. *Let functions $u(t)$ and $z(t)$ satisfy the relationship*

$$u(t) = z(t) + \sum_{k=1}^{p}\theta_k t^{\frac{k}{p}}. \tag{39}$$

Then, for $m=1$ and $r=1$, $u(t)$ is a solution of fractional q-differential problem (26) if and only if $z(t)$ is a solution of integer order problem (33).

Proof. We first assume that $u(t)$ is the solution of problem (26) and (39) holds. We need to prove that $z(t)$ is the solution of problem (33). Denote the q-Laplace transform of function $u(t)$ by $\overline{u}(s)$. Applying q-Laplace transform to equation (26) with $m=1, r=1$ and using (37), it yields

$$(S_q^{\frac{1}{p}}-\lambda)\overline{u}(s)=\frac{1}{1-q}S_q^{\frac{1}{p}-1}u(0)+\overline{f}(s), \tag{40}$$

where $S_q=\frac{s}{1-q}$. Multiplying (40) by $\sum\limits_{k=0}^{p-1}S_q^{\frac{p-1-k}{p}}\lambda^k$ and using the identity

$$a^n-b^n=(a-b)\sum_{k=0}^{n-1}a^{n-1-k}b^k,$$

we obtain

$$(S_q-\lambda^p)\overline{u}(s)=\frac{1}{1-q}u(0)\sum_{k=0}^{p-1}S_q^{\frac{1}{p}-1}S_q^{\frac{p-1-k}{p}}\lambda^k+\sum_{k=0}^{p-1}S_q^{\frac{p-1-k}{p}}\lambda^k\overline{f}(s). \tag{41}$$

Again applying q-Laplace transform to Equation (39) and using (36), we have

$$\begin{aligned}\overline{u}(s)&=\overline{z}(s)+\sum_{k=1}^{p}\theta_k\Gamma_q(\frac{k}{p}+1)(1-q)^{\frac{k}{p}}s^{-(\frac{k}{p}+1)}\\&=\overline{z}(s)+\sum_{k=1}^{p}\theta_k\Gamma_q(\frac{k}{p}+1)\frac{1}{1-q}S_q^{-(\frac{k}{p}+1)}.\end{aligned} \tag{42}$$

Combining equation (41) and (42) we obtain

$$\begin{aligned}(S_q-\lambda^p)\overline{z}(s)&=\frac{1}{1-q}u(0)\sum_{k=0}^{p-1}S_q^{-\frac{k}{p}}\lambda^k+\sum_{k=0}^{p-1}S_q^{\frac{p-1-k}{p}}\lambda^k\overline{f}(s)-\\&\quad-\sum_{k=1}^{p}\theta_k\Gamma_q(\frac{k}{p}+1)S_q^{-(\frac{k}{p}+1)}\frac{1}{1-q}(S_q-\lambda^p)\\&=\frac{\lambda^p}{1-q}\sum_{k=1}^{p}\theta_k\Gamma_q(\frac{k}{p}+1)S_q^{-(\frac{k}{p}+1)}-\frac{1}{1-q}\sum_{k=1}^{p}\theta_k\Gamma_q(\frac{k}{p}+1)S_q^{-\frac{k}{p}}+\\&\quad+\frac{1}{1-q}u(0)\sum_{k=0}^{p-1}S_q^{-\frac{k}{p}}\lambda^k+\sum_{k=0}^{p-1}S_q^{\frac{p-1-k}{p}}\lambda^k\overline{f}(s)\\&=\frac{\lambda^p}{1-q}\sum_{k=1}^{p}\theta_k\Gamma_q(\frac{k}{p}+1)S_q^{-(\frac{k}{p}+1)}-\frac{1}{1-q}\theta_p\Gamma_q(2)S_q^{-1}+\frac{u(0)}{1-q}+\\&\quad+\sum_{k=1}^{p-1}\frac{1}{(1-q)S_q^{k/p}}[\lambda^k u(0)-\theta_k\Gamma_q(\frac{k}{p}+1)]+\sum_{k=0}^{p-1}S_q^{\frac{p-1-k}{p}}\lambda^k\overline{f}(s).\end{aligned}$$

Then, using (35), we have

$$\begin{aligned}(S_q-\lambda^p)\overline{z}(s)&=\frac{\lambda^p}{1-q}\sum_{k=1}^{p}\theta_k\Gamma_q(\frac{k}{p}+1)S_q^{-(\frac{k}{p}+1)}-\frac{1}{1-q}\theta_pS_q^{-1}+\\&\quad+\frac{u(0)}{1-q}-\sum_{k=1}^{p-1}\frac{\lambda^{k-1}f(0)}{1-q}S_q^{-\frac{k}{p}}+\sum_{k=0}^{p-1}\lambda^kS_q^{\frac{p-1-k}{p}}\overline{f}(s).\end{aligned} \tag{43}$$

Since $u(0) = z(0)$ (see (39)), $S_q = \frac{s}{1-q}$, we have from (43) that

$$\frac{s}{1-q}\bar{z}(s) - \frac{1}{1-q}z(0) - \lambda^p \bar{z}(s) = \lambda^p \sum_{k=1}^{p} \theta_k \Gamma_q(\frac{k}{p}+1)s^{-\frac{k}{p}-1}(1-q)^{\frac{k}{p}}$$
$$+\sum_{k=0}^{p-2}[\lambda^k s^{1-\frac{k+1}{p}} \frac{1}{(1-q)^{\frac{p-k-1}{p}}}\bar{f}(s) - \lambda^k s^{-\frac{k+1}{p}} f(0)\frac{1}{(1-q)^{\frac{p-k-1}{p}}}] -$$
$$-\theta_p s^{-1} + \lambda^{p-1}\bar{f}(s). \tag{44}$$

Applying the inverse q-Laplace transform to Equation (44) and using (36)–(38), we arrive at

$$\begin{aligned} D_q z(t) - \lambda^p z(t) &= \lambda^p \sum_{k=1}^{p} \theta_k t^{\frac{k}{p}} - \theta_p + \sum_{k=0}^{p-2} {}^cD_q^{\frac{p-k-1}{p}} f(t)\lambda^k + \lambda^{p-1} f(t) \\ &= \lambda^p \sum_{k=1}^{p} \theta_k t^{\frac{k}{p}} - \theta_p + \sum_{k=0}^{p-1} {}^cD_q^{\frac{p-k-1}{p}} f(t)\lambda^k. \end{aligned}$$

So $z(t)$ satisfies Equation (33). Conversely, assume that $z(t)$ is the solution of problem (33) and relation (39) holds. Then, using again the above equalities and a backward deduction, we can prove that $u(t)$ is the solution of problem (26). □

Next, we consider the following fractional q-differential system of equations

$$\begin{cases} {}^cD_q^{\frac{1}{p}} U(t) = AU(t) + F(t), \\ U(0) = U_0, \end{cases} \tag{45}$$

where $U(t) = (u_1(t), \cdots, u_n(t))^T$, $F(t) = (f_1(t), \cdots, f_n(t))^T$, $U_0 = (u_1(0), \cdots, u_n(0))^T$, $A \in R^{n\times n}$ is a matrix. Similar to the argument of Theorem 2, we have the following result.

Theorem 3. *The system of equations (45) has the following solution*

$$U(t) = Z(t) + \sum_{k=1}^{p} \Theta_k t^{\frac{k}{p}},$$

where $Z(t)$ is the solution of problem

$$D_q Z(t) = A^p Z(t) + W(t),$$

and

$$W(t) = A^p \sum_{k=1}^{p} \Theta_k t^{\frac{k}{p}} + \sum_{k=0}^{p-1} A^{p-(k+1)}\, {}^cD_q^{\frac{k}{p}} F(t) - \Theta_p,$$

where

$$\Theta_k = \frac{A^k U(0) + A^{k-1}F(0)}{\Gamma_q(\frac{k+p}{p})}, \ k = 1, 2, \cdots, p.$$

In order to extend the conclusions of Theorem 2 and Theorem 3 to more general equations, we give the following results, which can be proved by using Lemma 9 and Lemma 7.

Theorem 4. *For $m = 1, 1 < r < p$, the initial value problem (26) is equivalent to the following fractional q-differential system*

$$\begin{cases} {}^cD_q^{\frac{1}{p}} u_1(t) = u_2(t), \\ {}^cD_q^{\frac{1}{p}} u_2(t) = u_3(t), \\ \qquad \cdots\cdots \\ {}^cD_q^{\frac{1}{p}} u_r(t) = \lambda u_1(t) + f(t), \\ u_1(0) = u(0),\ u_k(0) = 0,\ k = 2, \cdots, r. \end{cases}$$

Theorem 5. *For $\alpha = m - 1 + \frac{r}{p}, 1 \le r < p, m > 1$, the initial value problem (26) is equivalent to the following fractional q-differential system*

$$\begin{cases} {}^cD_q^{\frac{1}{p}} u_1(t) = u_2(t), \\ {}^cD_q^{\frac{1}{p}} u_2(t) = u_3(t), \\ \qquad \cdots\cdots \\ {}^cD_q^{\frac{1}{p}} u_{(m-1)p+r}(t) = \lambda u_1(t) + f(t), \end{cases}$$

with the initial value conditions

$$u_{(k-1)p+1}(0) = d_{k-1},\ k = 1, 2, \cdots, m,\ \ u_l(0) = 0,\ l \neq (k-1)p + 1.$$

5. Examples

In this section, we give some examples to illustrate the validity of the proposed method for solving the fractional q-differential equations.

Example 1. *Consider the fractional q-differential equation*

$$\begin{cases} {}^cD_q^{\frac{1}{2}} u(t) = u(t) + f(t), \\ u(0) = 1, \end{cases} \tag{46}$$

where $f(t) = 1 - t - \frac{1}{\Gamma_q(\frac{3}{2})} t^{\frac{1}{2}}$. According to Theorem 2, we only need to solve the integer order equation

$$\begin{cases} {}^cD_q z(t) = z(t) + w(t), \\ z(0) = 1, \end{cases}$$

where

$$w(t) = \sum_{k=1}^{2} \theta_k t^{\frac{k}{2}} + \sum_{k=0}^{1} {}^cD_q^{\frac{k}{2}} f(t) - \theta_2 = t - 2.$$

It is easy to see that the solution $z(t) = 1 - t$. So the solution of problem (46) is

$$u(t) = z(t) + \sum_{k=1}^{2} \theta_k t^{\frac{k}{p}} = 1 + t + \frac{2}{\Gamma_q(\frac{3}{2})} t^{\frac{1}{2}}. \tag{47}$$

On the other hand, by formula (27), the solution of problem (46) is

$$u(t) = E^q_{\frac{1}{2},1}(1, t) + \int_0^t (t - qs)^{(-\frac{1}{2})} E^q_{\frac{1}{2},\frac{1}{2}}(1, t - q^{\frac{1}{2}} s)(1 - s - \frac{1}{\Gamma_q(\frac{3}{2})} s^{\frac{1}{2}}) d_q s. \tag{48}$$

Compare these two solutions given by (47) and (48), respectively. Solution (47) is a finite analytic function, but solution (48) concerns the fractional q-integral of an infinite function series.

Example 2. *Consider a higher order problem*

$$\begin{cases} {}^cD_q^{\frac{3}{2}} u(t) = D_q u(t) + f(t), \\ u(0) = 1,\ D_q u(0) = 1, \end{cases} \tag{49}$$

where $f(t) = 1 - t - \frac{1}{\Gamma_q(\frac{3}{2})} t^{\frac{1}{2}}$. *Let* $u_1(t) = D_q u(t)$. *Since* ${}^cD_q^{\frac{3}{2}} u(t) = {}^cD_q^{\frac{1}{2}} (D_q u(t)) = {}^cD_q^{\frac{1}{2}} u_1(t)$, *then from (49), we see that* $u_1(t)$ *satisfies*

$$\begin{cases} {}^cD_q^{\frac{1}{2}} u_1(t) = u_1(t) + f(t), \\ u_1(0) = 1. \end{cases}$$

According to the result of Example 1, we obtain

$$u_1(t) = 1 + t + \frac{2}{\Gamma_q(\frac{3}{2})} t^{\frac{1}{2}}.$$

Then, by Lemma 2 we have

$$u(t) - u(0) = I_q D_q u(t) = I_q u_1(t).$$

That is,

$$\begin{aligned} u(t) &= u(0) + \int_0^t 1 + s + \frac{2}{\Gamma_q(\frac{3}{2})} s^{\frac{1}{2}} d_q s \\ &= 1 + t + \frac{1}{\Gamma_q(3)} t^2 + \frac{2}{\Gamma_q(\frac{5}{2})} t^{\frac{3}{2}}. \end{aligned}$$

Note that, for problem (49), the solution formula (27) is not applicable.

Example 3. *Consider the following problem*

$$\begin{cases} {}^cD_q^{\frac{2}{3}} u(t) = 2u(t) + f(t), \\ u(0) = 0. \end{cases} \tag{50}$$

where $f(t) = 1 - \frac{2t^{\frac{2}{3}}}{\Gamma_q(\frac{5}{3})}$. *Let* $u_1(t) = u(t), u_2(t) = D_q^{\frac{1}{3}} u_1(t)$. *Then, problem (50) is equivalent to the following system of equations*

$$\begin{cases} {}^cD_q^{\frac{1}{3}} u_1(t) = u_2(t),\ u_1(0) = 0, \\ {}^cD_q^{\frac{1}{3}} u_2(t) = 2u_1(t) + f(t),\ u_2(0) = 0, \end{cases}$$

or

$$\begin{cases} {}^cD_q^{\frac{1}{3}} U(t) = AU(t) + F(t), \\ U(0) = 0, \end{cases} \tag{51}$$

where

$$U(t) = (u_1, u_2)^T,\ F(t) = (0, f)^T,\ A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

According to Theorem 3, system (51) is equivalent to the following problem

$$\begin{cases} D_q Z(t) = A^3 Z(t) + W(t), \\ Z(0) = 0, \end{cases} \tag{52}$$

where $W(t) = (4t, -2)^T$ *is given by Theorem 3. It is easy to see that the solution of problem (52) is* $Z(t) = (0, -2t)^T$*. So the solution of problem (51) is*

$$U(t) = Z(t) + \sum_{k=1}^{3} \Theta_k t^{\frac{k}{3}} = (\frac{t^{\frac{2}{3}}}{\Gamma_q(\frac{5}{3})}, \frac{t^{\frac{1}{3}}}{\Gamma_q(\frac{4}{3})})^T.$$

Thus, we obtain the solution of problem (50)

$$u(t) = u_1(t) = \frac{t^{\frac{2}{3}}}{\Gamma_q(\frac{5}{3})}. \tag{53}$$

On the other hand, by formula (27), the solution of problem (50) is

$$u(t) = \int_0^t (t - qs)^{(-\frac{1}{3})} E^q_{\frac{2}{3},\frac{2}{3}}(2, t - q^{\frac{2}{3}} s)(1 - \frac{2s^{\frac{2}{3}}}{\Gamma_q(\frac{5}{3})}) d_q s. \tag{54}$$

Obviously, the expression of solution (53) is more concise than that of solution (54).

6. Conclusions

We consider how to solve a class of linear fractional q-differential equations. By transforming this class of equations into the equivalent integer order q-differential equations, we establish a solution method that can find out the analytical solutions under certain conditions. Our work further develops the solution theory of fractional q-differential equations.

Author Contributions: Conceptualization, Y.S. and T.Z.; Formal analysis, Y.S. and T.Z.; Methodology, Y.S. and T.Z.; Validation, T.Z.; Writing–original draft, Y.S.; Writing–review & editing, Y.S. and T.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the editor and the referees for their positive comments and useful suggestions which have improved this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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