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Design of ℓ_1 New Suboptimal Fractional Delays Controller for Discrete Non-Minimum Phase System under Unknown-but-Bounded Disturbance

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Abstract: ℓ_1 -regularization methodologies have appeared recently in many pattern recognition and image processing tasks frequently connected to ℓ_1 -optimization in the control theory. We consider the problem of optimal stabilizing controller synthesis for a discrete non-minimum phase dynamic plant described by a linear difference equation with an additive unknown-but-bounded noise. Under considering the “worst” case of noise, the solving of these optimization problem has a combinatorial complexity. The choosing of an appropriate sufficiently high sampling rate allows to achieve an arbitrarily small level of suboptimality using a noncombinatorial algorithm. In this paper, we suggest to use fractional delays to achieve a small level of suboptimality without increasing the sampling rate so much. We propose an approximation of the fractional lag with a combination of rounding and a first-order fractional lag filter. The suggested approximation of the fractional delay is illustrated via a simulation example with a non-minimum phase second-order plant. The proposed methodology appears to be suitable to be used in various pattern recognition approaches.

Keywords: non-minimum phase system; fractional delays; unknown-but-bounded noise; stabilizing controller



Citation: Ivanov, D.; Granichin, O.; Pankov, V.; Volkovich, Z. Design of ℓ_1 New Suboptimal Fractional Delays Controller for Discrete Non-Minimum Phase System under Unknown-but-Bounded Disturbance. *Mathematics* **2022**, *10*, 69. <https://doi.org/10.3390/math10010069>

Academic Editor: Michal Fečkan

Received: 27 November 2021

Accepted: 21 December 2021

Published: 26 December 2021

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1. Introduction

Many kinds of modeling problems can be represented as

$$\overline{A}\overline{\alpha} = \overline{B},$$

where \overline{A} is a linear operator acting in a Hilbert space, and $\overline{\alpha}$, and \overline{B} are appropriate vectors. After discretizing with sampling rate δ , this equation can be re-written as

$$A_\delta \alpha_\delta = B_\delta. \quad (1)$$

The case of A_δ with dimensionality $m \times N$ often arises in practice when unknown physical characteristics described by α_δ cannot be measured directly, but B_δ is observed. Matrix A_δ is an observation scheme matrix. The interesting case is $m \ll N$. It is a conventional ill-posed problem admitting perhaps an infinite number of solutions and is frequently resolved by a regularization method resting upon some preliminary information with the aim to discover a suitable α_δ .

One of the widely used regularization methods is ℓ_1 -minimization of the α_δ value, which is a typical situation in pattern recognition and signal processing operating with the compressive sensing (CS) methodology.

CS was introduced by Candens, Romberg and Tao [1] in 2006. This approach theoretically guarantees a precise recovery of s -sparse signal α_δ based on $m \ll N$ measurements sampled below the Nyquist rate. CS and its applications are reviewed in detail in [2]. CS is commonly used in such areas as super-resolution imaging [3–5], MRI fast image acquisition [6–8], and seismic data compression [9–11]. A signal possesses in these fields natural s -sparse representation corresponding to a specific domain, and the resultant regularized solutions of (1) with the minimal ℓ_1 norm leads to sufficiently correct outcomes.

As it can be seen, the task of finding a s -sparse solution α_δ is widely used in various applications. To formulate the overall problem, let us introduce $X_\delta = \text{col}(x_{1,\delta}, x_{2,\delta}, \dots, x_{s,\delta})$, the vector containing indices of the required nonzero elements of α_δ and $\alpha_\delta(X_\delta) = \text{col}(\alpha_{x_{1,\delta}}, \alpha_{x_{2,\delta}}, \dots, \alpha_{x_{s,\delta}})$ a vector containing the matching elements. Let $A_\delta(X_\delta)$ denote the restriction of operator A_δ to subspace X_δ . Then, (1) is rewritten as

$$A_\delta(X_\delta)\alpha_\delta(X_\delta) = B_\delta.$$

This much simpler relationship treats vectors of much smaller dimensionality because $s < m \ll N$, and it can be efficiently solved by standard techniques, such as LU or QR decomposition. Compared to the original problem, this system is not underdetermined. In the case of an invertible operator A_δ and $m = s$, $\alpha_\delta(X_\delta)$ is expressed as

$$\alpha_\delta(X_\delta) = A_\delta(X_\delta)^{-1}B_\delta. \tag{2}$$

So, the task of finding an s -sparse solution α_δ can be reformulated as an optimization problem: to find such X_δ corresponding to a minimal ℓ_1 norm of α_δ such that

$$\|\alpha_\delta(X_\delta)\|_1 = \sum_{i=1}^s |\alpha_{x_{i,\delta}}| \rightarrow \min_X. \tag{3}$$

Suppose that X_δ is not a vector containing integers, but a real-valued vector. In this case, a necessary condition for points X_δ where the gradient exists is

$$\nabla(\|A_\delta(X_\delta)^{-1}B_\delta\|_1, X_\delta) = 0, \tag{4}$$

where ∇ is the gradient operator. If the corresponding minimum point has all integer components, then it is a solution of original problem (3). Otherwise, a suboptimal one can be obtained by rounding the components to nearest integers. It can be shown that the nearest solution with integer components $\overline{\alpha_\delta(X_\delta)}$ has a suboptimality level of difference between values of (3) at real-valued and rounded solutions (see [12], which is based on [13] or [14]).

A more accurate outcome can be obtained by rounding the minimum to the nearest vector with rational components in the ϵ neighborhood and by choosing the sampling rate

$$\delta = \frac{1}{\text{lcd}(x_{1,\delta}, x_{2,\delta}, \dots, x_{s,\delta})},$$

where lcd is the least common denominator of X_δ coordinates. Further, the required solution is given just by multiplying by the lcd value. As was shown in [15], the fractional solution has a suboptimality level ϵ .

In this paper, the idea of a fractional solution is used for the synthesis of a suboptimal ℓ_1 stabilizing controller based on filters approximating fractional delay. This work is based on the method described in the paper [13], where the problem of synthesis of an optimal stabilized controller for a discrete non-minimal phase plant under an irregular bounded noise is considered. The result of [13] was applied in [16,17] for solving a SISO plant adaptive optimal control problem under unknown-but-bounded disturbances. The general solution to the problem of ℓ_1 stabilizing optimal controller was obtained in [18].

Non-minimal phase systems have zeros or poles of a transfer function with a positive real part. Such schemes are used to describe various processes, such as ship motion control [19], flexible manipulators [20], servomechanism [21,22], continuous stirred tank reactors [23], electronic circuits [24,25], cart inverted pendulum and coupled-tank systems [25], settle time applications [26], and so on. Moreover, similar structures can appear, for example, due to an approximation, such as a Pade approximation to unit delay, of processes described by partial differential equations [27].

The control of a non-minimal phase plant leads to many difficulties [28]. More precisely, these limitations can be concluded, for example, from the classical root-locus method [29], waterbed effect phenomena [30], and the LTR problem [31]. In linear time-invariant systems, the source of all of the limitations mentioned above is that the non-minimum phase zero of the given process cannot be annulled by the unstable controller pole since such a cancellation causes internal instability [32].

Another generalization of the minimal phase integer order systems consists of the fractional order systems used mainly to describe viscoelasticity/damping [33], electrochemical processes and flexible robots [34].

Automatic control algorithms, based on the concepts of fractional calculus, have been proposed both in the frequency [35] and in the time [36] domains. In [37], fractional order PID controller was proposed. The synthesis of linear quadratic regulator (LQR) based on fractional order PID controllers was suggested in [38–40]. A task of tuning of fractional order PID controllers was considered in [41]. Ref. [42] was based on a fractional order controller for compensating the unstable zeros. Various modifications of fractional-order controllers for controlling non-minimum phase systems was discussed in [43,44]. A comprehensive overview of fractional order control, fractional PID controllers, and fractional order calculus application for MIMO plants was described in [45–48]. The fundamental methods of fractional control were described in [49–52]. It is worth noting that fractional order controllers in published papers are proposed only for the deterministic continuous case.

This paper's main contribution consists of a new implementation of a suboptimal ℓ_1 controller based on filters approximating fractional delay. The problem of synthesizing an optimal stabilizing controller under conditions of arbitrary bounded noise is investigated. The results of [12] serve as a basis for creating an ℓ_1 suboptimal stabilizing control regulator with fractional delays for discrete non-minimal phase systems with bounded noise. But with an arbitrary fixed sampling rate for the dynamical system underlying the processes under study, it is impossible to achieve an arbitrarily small level of suboptimality. For a suboptimal solution, an algorithm for the synthesis of a fractional controller (with non-integer delay) is proposed and justified. We show that the use of fractional lag controllers allows to achieve the control aim that is closer to the optimal level, and with the help of fractional lag, it is not needed to greatly increase the sampling rate to achieve an arbitrarily small level of suboptimality. An advantage of the proposed approach is also the dimension reduction to the number of unstable zeros of a control plant, which is better compared to the infinite dimension of the original optimization problem.

The paper is organized as follows. Section 2 presents the problem statement and introduction of fractional delay and fractional delay filters for the implementation of fractional delay. Section 3 discusses the approximation errors of the controller. The simulation results are presented in Section 4. Finally, Section 5 concludes this paper.

2. Problem Statement and the Theoretical Framework

The results presented in this section are mostly based on the material stated in [12,15].

2.1. Controller for Continuous Non-Minimum Phase System under Unknown-But-Bounded Disturbance

Let us consider a continuous-time control plant with the input–output transfer function

$$G(z) = \frac{g_0(z - \lambda^{(1)}) \dots (z - \lambda^{(s)}) \dots (z - \lambda^{(n-1)})}{(z - \bar{\lambda}^{(1)}) \dots (z - \bar{\lambda}^{(2)}) \dots (z - \bar{\lambda}^{(n)})} \tag{5}$$

We assume that the excess of the poles and zeros of the system is equal to unity, the first s zeros of the transfer function $G(z)$ are unstable ($Re\lambda(i) > 0, i = 1, \dots, s$) and the remaining zeros are stable ($Re(\lambda(i)) < 0, i = s + 1, \dots, n - 1$). Here, $Re(\lambda)$ stands for the real part of a complex number λ . Let the poles $\lambda(1), \dots, \lambda(n)$ of the transfer function $G(z)$ not coincide with the first s unstable zeros.

We choose a discretization step $\delta > 0$ and study a family of piecewise constant functions defining the control actions varying at time instants $k\delta, k = 0, 1, 2, \dots$. By considering the discretization of the given continuous-time system in the zero approximation (see [21]), we obtain for a sufficiently small value of δ a discrete system with the transfer function

$$H_\delta(z) = \frac{h_\delta z(z - \lambda_\delta^{(1)}) \dots (z - \lambda_\delta^{(s)}) \dots (z - \lambda_\delta^{(n-1)})}{(z - \bar{\lambda}_\delta^{(1)}) \dots (z - \bar{\lambda}_\delta^{(2)}) \dots (z - \bar{\lambda}_\delta^{(n)})} \tag{6}$$

with the poles $\bar{\lambda}_\delta^{(1)}, \dots, \bar{\lambda}_\delta^{(s)}, \dots, \bar{\lambda}_\delta^{(n-1)}$ and zeros $\lambda_\delta^{(1)}, \lambda_\delta^{(2)}, \dots, \lambda_\delta^{(n)}$. It is well known that for $\delta \rightarrow 0$ (see [21]), the poles of the transfer function $H_\delta(z)$ are approximately related to the poles of $G(z)$ by $\bar{\lambda}_\delta^{(i)} = e^{-\delta\bar{\lambda}^{(i)}}, i = 1, \dots, n$; for zeros, these relations are

$$\lambda_\delta^{(i)} \approx e^{-\delta\lambda^{(i)}}, i = 1, \dots, n - 1.$$

2.2. Controller for Discrete Non-Minimum Phase Plant under Unknown-But-Bounded Disturbance

Let us consider a discrete dynamic control plant described by the following equation

$$a(q^{-1})y_t = b(q^{-1})x_t + v_t, \tag{7}$$

where

- y_t, x_t, v_t are the output, input, and disturbance signals at time instant t respectively;
- q^{-1} is the backward shift operator: $q^{-1}y_i = y_{i-1}$;
- $a(q^{-1})$ and $b(q^{-1})$ are polynomials of q^{-1} ;

$$a(\lambda) = 1 + \sum_{i=1}^{n_a} a_i \lambda^i, b(\lambda) = \sum_{i=r}^{n_b} b_i \lambda^i,$$

where $r \geq 1$ is a time lag.

It is additionally supposed that $\|v\|_\infty = \max_t |v_t| \leq C_v, C_v > 0$.

A linear stationary regulator carries out the control of a plant with known parameters

$$\alpha(q^{-1})u_t = \beta(q^{-1})y_t, \tag{8}$$

where $\alpha(\lambda)$ and $\beta(\lambda)$ are polynomials of q^{-1} :

$$\alpha(\lambda) = 1 + \sum_{i=1}^{n_c} \alpha_i \lambda^i, \beta(\lambda) = \sum_{i=1}^{k_c} \beta_i \lambda^i.$$

If the characteristic polynomial of the closed-loop system described by Equations (7) and (8)

$$\chi(\lambda) = \alpha(\lambda)a(\lambda) - \beta(\lambda)b(\lambda) \tag{9}$$

has no unstable roots (in unit disk), then regulator described by Equation (8) is stabilizing

$$\sup_t (|y_t| + |u_t|) < \infty, \tag{10}$$

i.e., output y_t and control u_t are bounded.

Denote by $\bar{y} \in l_\infty$ the desired output of the control plant described by Equation (7). The meaningful formulation of the problem is to build the polynomials $\alpha(\cdot)$ and $\beta(\cdot)$, which are guaranteeing Inequality (10)—stabilizing of output and input—and asymptotic bound

$$\overline{\lim}_{i \rightarrow \infty} |y_t - \bar{y}_t| \leq J(\alpha(\cdot), \beta(\cdot), C_v) \tag{11}$$

with performance index

$$J(\alpha(\cdot), \beta(\cdot), C_v) = \inf_{\alpha(\cdot), \beta(\cdot)} \sup_{\|v\|_\infty \leq C_v} \overline{\lim}_{i \rightarrow \infty} |y_t - \bar{y}_t|. \tag{12}$$

Let be $\psi_{y/v}(\lambda)$ transfer function of closed-loop system from disturbance v_t to input y_t

$$\psi_{y/v}(\lambda) = \frac{\alpha_{y/v}(\lambda)}{\chi(\lambda)} = \sum_{i=0}^{\infty} \psi_i \lambda^i.$$

So, we have

$$y_t = \sum_{i=0}^{\infty} \psi_i v_{t-i}.$$

The control problem described by Equations (11) and (12) has an infinite dimension. Equation (12) can be rewritten in the form

$$J(\alpha(\cdot), \beta(\cdot), C_v) = C_v \inf_{\psi_{y/v}(\cdot)} \sum_{i=0}^{\infty} |\psi_i|$$

due to the arbitrary nature of the disturbance performance index.

Denote $b_-(\lambda)$ and $b_+(\lambda)$ the unstable and stable part of polynomial $b(\lambda)$. So, we have $b(\lambda) = b_-(\lambda)b_+(\lambda)$.

Assumption 1. $a(\lambda)$ and $b_-(\lambda)$ are coprime polynomials, and polynomial $b_-(\lambda)$ has s unstable different not zero and not unit roots $\lambda_1, \dots, \lambda_s$ and does not have unit roots.

In [13] it was shown under Assumption 1 that optimal regulator characteristic polynomial $\chi(\lambda)$ is equal to $b_+(\lambda)$ and optimal polynomial $\alpha(\lambda)$ has following structure: $\alpha(\lambda) = f(\lambda)b_+(\lambda)$, where polynomial $f(\lambda)$ has not more s non-zero coefficient (s -sparse structure) and highest possible degree of $f(\lambda)$ is bounded and depended on highest of magnitude on non-stable zeros of polynomial $b(\lambda)$.

The following theorem from [12] gives the answer to the question about the achievable quality of control. It allows to reformulate the control problem (11)–(12) as a finite dimension problem, which is similar to (3).

Theorem 1. Under Assumption 1, the minimum value of the function $C_v \|F(X)\|_1$ is a lower bound estimate for minimum value of (12), i.e.

$$\min_{X \in \mathbb{R}_+^s} C_v \|F(X)\|_1 \leq J(\alpha(\cdot), \beta(\cdot), C_v) \tag{13}$$

where $F(X) = A^{-1}(X)B \in \mathbb{R}^{r+s}$, $X = (x_1, x_2, \dots, x_s)^T \in \mathbb{R}_+^s$, $x_i \geq 1$, $\|F\|_1 = \sum_{i=0}^{s+r-1} |f_i|$,

$$A(X) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & \ddots & 0 \\ 0 & 2a_{(1)} & 2 & 0 & \dots & 0 & \ddots & 0 \\ 0 & 6a_{(2)} & 6a_{(2)} & 3 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \lambda_1 & \dots & \dots & \lambda_1^{r-1} & \lambda_1^{r-1+x_1} & \dots & \lambda_1^{r-1+\sum_{j=1}^s x_j} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_s & \dots & \dots & \lambda_s^{r-1} & \lambda_s^{r-1+x_s} & \dots & \lambda_s^{r-1+\sum_{j=1}^s x_j} \end{pmatrix},$$

$$B = \left(1 \quad -a_1 \quad \dots \quad (r-1)a_{r-1} \quad \frac{1}{a(\lambda_1)} \quad \dots \quad \frac{1}{a(\lambda_s)} \right)^T.$$

If the minimum of the left side of Equation (13) is achieved in point X_0 with integer-value components, then the polynomial of the ℓ_1 -optimal stabilizing controller can be obtained by the formulas (see [12])

$$\alpha(\lambda) = f(\lambda)b_+(\lambda), \tag{14}$$

$$\beta(\lambda) = \frac{(a(\lambda)f(\lambda) - 1)}{b_-(\lambda)}, \tag{15}$$

where $f(\lambda) = \sum_{j=0}^{r-1} f_j \lambda^j + \sum_{j=r}^{s+r-1} f_j(X_0) \lambda^{D_j}$, $D_j = r - 1 + \sum_{i=1}^{j-r+1} x_i$.

For the general case, in order for the controller to be physically attainable, it is necessary to resolve Equation (13) in the positive integers.

$$X_z = \arg \min_{X \in \mathbb{Z}_+^s} J(X) = \arg \min_{X \in \mathbb{Z}_+^s} C_v \sum_{j=0}^{s+r} |f_j(X)|. \tag{16}$$

A NP-hard problem of finding this point admits a relaxation consisting of resolving in the real numbers field, followed by rounding to the nearest integer. However, the continuous optimization methods applied here can provide ill-treatment solutions. To avoid such a circumstance, the current paper proposes using fractional delay filters to implement a consistent suboptimal controller.

More precisely, in the case non-integer-value components X_0 , we can relax the considered problem to the fractional case when delays D_j may be fractional in controller Equations (14) and (15). Next, we consider more carefully the fractional delay for discrete systems.

2.3. Fractional Delay Filters

The ideal fractional delay element is a digital version of a continuous-time delay line

$$q^{-D_j} u_i = u_{i-D_j}. \tag{17}$$

The impulse response of an ideal fractional delay (FD) filter is a shifted and sampled sinc function

$$h_j(i) = \frac{\sin(\pi i - D_j)}{\pi i} \tag{18}$$

where D_j is the delay with an integral part $\tilde{D}_j = \text{floor}(D_j)$ is a rounding of D_j to the nearest integers less than or equal to D_j , and d_j is the fractional part $d_j = D_j - \tilde{D}_j$.

For case $d_j > 0$, the impulse response is an infinite length. The impulse response of the ideal fractional delay filter is presented in Figure 1.

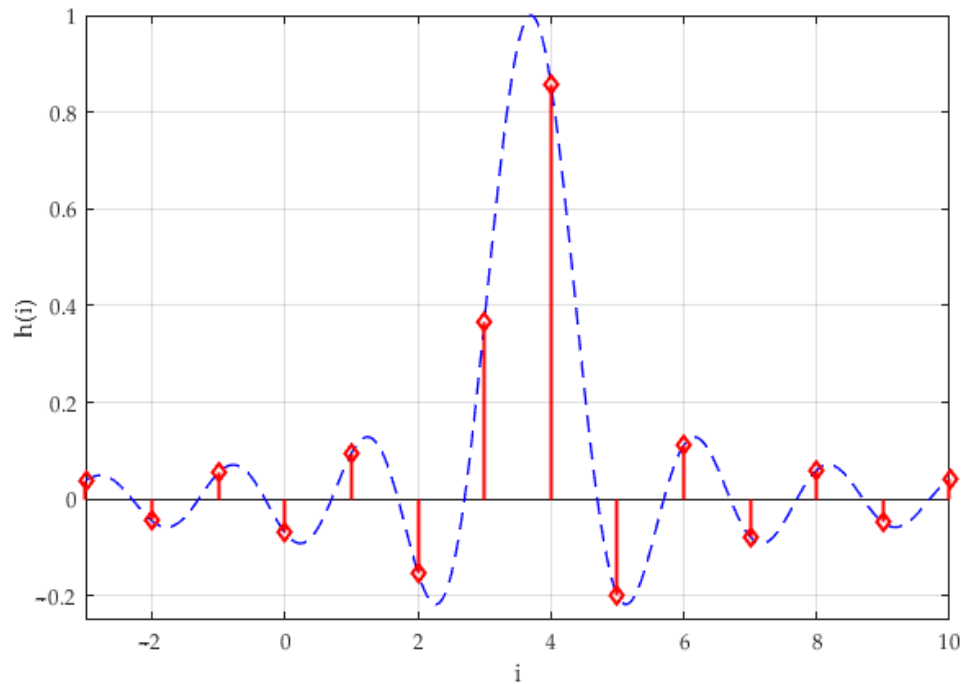


Figure 1. The impulse response of ideal fractional delay filter.

The infinite length of the impulse response leads to a noncausal filter that cannot be made causal by a finite shift in time. In addition, the filter is not stable since the impulse response is not absolutely summable. Thus, the ideal FD filter is nonrealizable.

The problem of a suboptimal controller is faced with the problem of design s fractional delay filters. To produce a realizable finite-length fractional delay filter, causal approximation for the sinc function can be used [53].

Generally, today, there are a large number of papers deal with the design of filters with fractional delays of both filters with finite impulse response (FIR) [54] and infinite impulse response IIR [55,56].

This article discusses the most straightforward non-recursive first-order filters. These filters are consistently stable [57,58] and somewhat increase the complexity of the synthesized controller. More complex filters should be considered on a case-by-case basis.

3. Approximation Errors of the Controller

The approximation of the fractional delay leads to an additional error in the plants. Let us define this error as ΔF , so

$$(A + \Delta A)(F + \Delta F) = B, \tag{19}$$

This is shown in [59]

$$\frac{\|\Delta F\|_1}{\|F\|_1} \leq \mu \frac{\|\Delta A\|_1}{\|A\|_1}, \tag{20}$$

where $\mu = \|A^{-1}\|_1 \|A\|_1$ is the condition number.

Substituting (20) into Equation (19), we have

$$\|\Delta F\|_1 \leq \|\Delta A\|_1 \|A^{-1}\|_1 \|F\|_1. \tag{21}$$

Equation [23] shows that the approximation error $\|\Delta F\|_1$ is naturally bounded by an estimator linearly depending on the error $\|\Delta A\|_1$. According to the definition of the ℓ_1 norm, a decrease in errors of all its elements will lead to a decrease in the total error.

Let us compare the approximation error of the rounding method [12] and the simplest non-recursive fractional delay filter and introduce

$$\hat{D}(q^{-1}) = (1 - d) + dq^{-1} \tag{22}$$

Consider the approximation by the example of an unstable zero $\lambda_1^{x_1}$ representing it in the irreducible fraction of an integer and a fractional part $\lambda_1^{x_1} = \lambda_1^{\hat{D}} \lambda_1^d$.

An approximation of the fractional part λ_1^d by the method of rounding to integer powers is [12]

$$\lambda_1^d \approx \begin{cases} 1, & d < 0.5 \\ \lambda_1, & d \geq 0.5 \end{cases} \tag{23}$$

An approximation of the fractional part λ_1^d for a non-recursive fractional delay filter is

$$\lambda_1^d \approx 1 - d + d\lambda_1. \tag{24}$$

The rounding approximation error is defined as

$$\epsilon_r(d, \lambda_1) = \begin{cases} 1 - \lambda_1^d, & d < 0.5 \\ \lambda_1^d - \lambda_1, & d \geq 0.5 \end{cases} \tag{25}$$

The error of approximation by the fractional lag filter is defined as

$$\epsilon_f(d, \lambda_1) = 1 - d + d\lambda_1 - \lambda_1^d. \tag{26}$$

Consider for which values of the parameters λ_1 and d the approximation error of (26) is less than (25).

Theorem 2. *The approximation error (26) is less than approximation error (25) if parameter $\lambda_1 \in (0, 1)$, delay $d \in (0, 1)$, and*

$$d < \frac{\lambda_1 + 1}{1 - \lambda_1} - \frac{\text{LambertW}\left(\frac{-2\lambda_1^{(\lambda_1+1)/(1-\lambda_1)} \ln(\lambda_1)}{\lambda_1 - 1}\right)}{\ln(\lambda_1)}, \tag{27}$$

where LambertW is Lambert W function, namely the branches of the converse relation of the function $f(w) = we^w$.

Proof. Let us consider two possible cases.

Case 1. $\lambda_1^d \approx \lambda_1^0 = 1$. Consider the inequality $1 - d + d\lambda_1 - \lambda_1^d < 1 - \lambda_1^d$. After simplification, the inequality takes the form

$$d(\lambda_1 - 1) < 0, \tag{28}$$

and it always holds true for the values $d \in [0, 0.5)$, $\lambda \in (0, 1)$.

Case 2. $\lambda_1^d \approx \lambda$. Consider the inequality $1 - d + d\lambda_1 - \lambda_1^d < \lambda_1^d - \lambda_1$. After simplification, the inequality takes the form

$$1 - d + \lambda_1(1 + d) < 2\lambda_1^d. \tag{29}$$

After finding a solution to the inequality (29) with respect to d , it is expressed as (27).

Figure 2 shows the region (marked in blue) in which inequality (29) is satisfied. As we can see from the figure, this is possible only for the values $d > 0.1$, $\lambda_1 < 0.285$. □

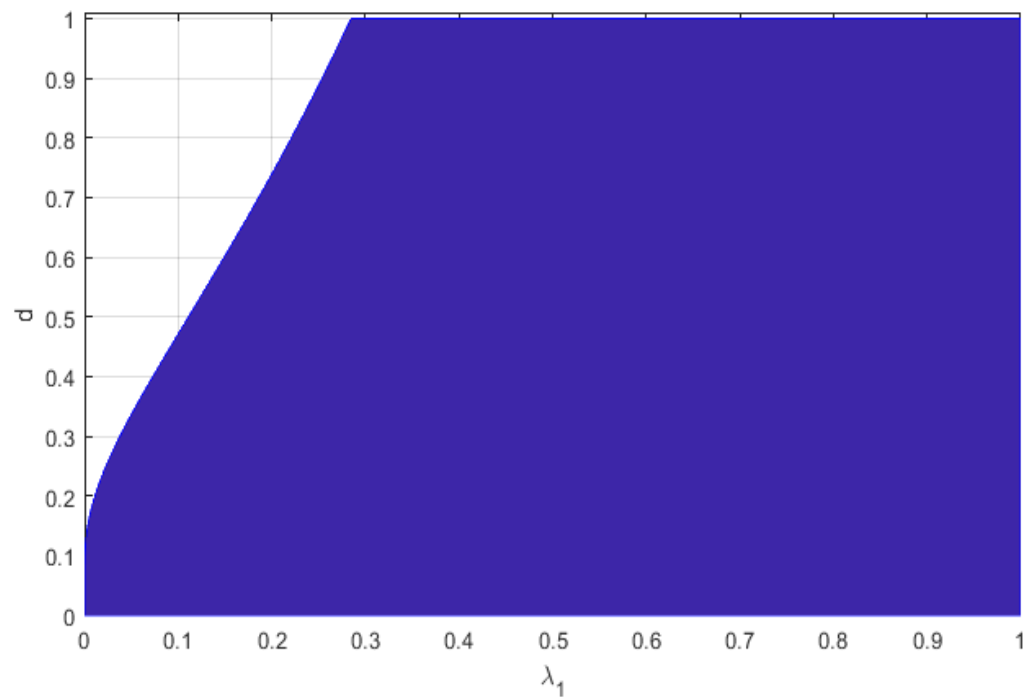


Figure 2. Region when Inequality (29) is satisfied.

When Inequality (27) is satisfied, an approximation based on a fractional delay filter should be chosen. If Inequality (27) is not satisfied, the approximation is based on rounding. The filter approximating the fractional delay according to this rule can be defined as

$$d(q^{-1}) = \begin{cases} 1 - d + dq^{-1}, & d < \frac{\lambda+1}{1-\lambda} - \frac{\text{LambertW}\left(\frac{-2\lambda^{(\lambda+1)/(1-\lambda)} \ln(\lambda)}{\lambda-1}\right)}{\ln(\lambda)}, \\ q^{-1}, & d \geq \frac{\lambda+1}{1-\lambda} - \frac{\text{LambertW}\left(\frac{-2\lambda^{(\lambda+1)/(1-\lambda)} \ln(\lambda)}{\lambda-1}\right)}{\ln(\lambda)}. \end{cases} \tag{30}$$

An approximation by fractional delay filters of other elements of the matrix is carried out similarly. The use of filters approximating the fractional delay allows to reduce the approximation error of most elements without resorting to increasing the sampling frequency in the controller.

Remark 1. This approach can be applied to design a controller for a continuous system. The sampling error can be considered as an unknown-but-bounded noise [44].

4. Simulation Results

The class of non-minimum phase plants is described by the equation

$$y_t - 1.91y_{t-1} + 5.2y_{t-2} = \lambda_1\lambda_2u_{t-1} - (\lambda_1 + \lambda_2)u_{t-2} + u_{t-3} + v_t, \tag{31}$$

where the appropriate vector and matrix are

$$A(X) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \lambda_1^{x_1} & \lambda_1^{x_1+x_2} \\ 1 & \lambda_2^{x_1} & \lambda_2^{x_1+x_2} \end{bmatrix}, \quad B = \begin{pmatrix} 1 \\ 1/a(\lambda_1) \\ 1/a(\lambda_2) \end{pmatrix}$$

and $r = 1, C_v = 1$.

The controller for the experiment was implemented according to the following steps:

1. The minimum of the function (12) is found using optimization algorithm;
2. Extraction of fractional parts d_j of delay D_j is performed;

3. Fractional parts d_i are approximated according to the condition (30);
4. The controller implementation is performed based on the approximation from the previous step and Formulas (14) and (15).

The minimum of function (9) is a function of the values of unstable zeros λ_1, λ_2 :

$$J_{\min}(\lambda_1, \lambda_2) = \arg \min_X J(X, \lambda_1, \lambda_2). \tag{32}$$

Figure 3 presents the surface of the function $J_{\min}(\lambda_1, \lambda_2)$ (32). It shows the minimum achievable error for different values of unstable zeros.

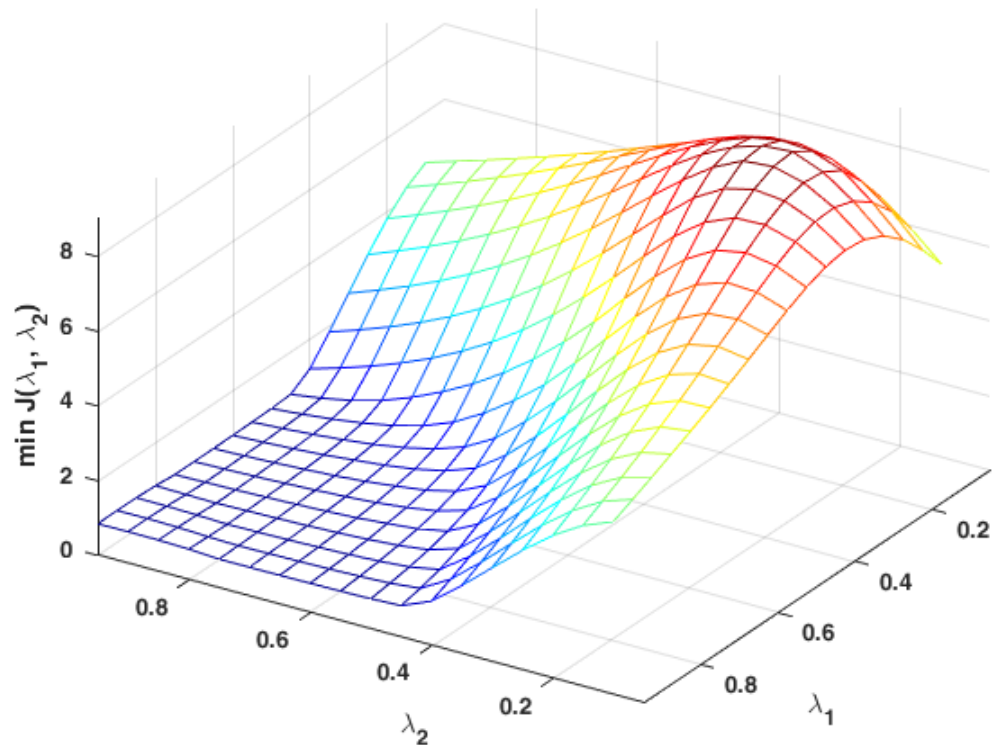


Figure 3. Function Surface (33), which depicts minimum achievable error for different values of unstable zeros.

Determine the absolute error in the assessment using a suboptimal controller (23)

$$\Delta J_1(\lambda_1, \lambda_2) = J_1(\lambda_1, \lambda_2) - J_{\min}(\lambda_1, \lambda_2), \tag{33}$$

and controller (30)

$$\Delta J_2(\lambda_1, \lambda_2) = J_2(\lambda_1, \lambda_2) - J_{\min}(\lambda_1, \lambda_2). \tag{34}$$

The function surface of error (33) is shown on the Figure 4, and the function surface of error (34) is shown on the Figure 5.

As it can be seen from Figure 5, the error caused by the fractional lag filter is less than the error caused by rounding (Figure 4) for all values of λ_1, λ_2 . The use of a filter approximating fractional delay allows to obtain a more accurate solution compared to a suboptimal controller [12]. The histogram presented on Figure 6 shows that errors of fractional lag filter have a lower variance with maximum value 0.5, compared to Figure 7, where the maximum additional error is greater than 2.

Table 1 shows the mean and standard deviation of additional errors for all λ_1, λ_2 . Compared to the rounding method, the mean value and standard deviation of errors caused by the proposed algorithm are smaller.

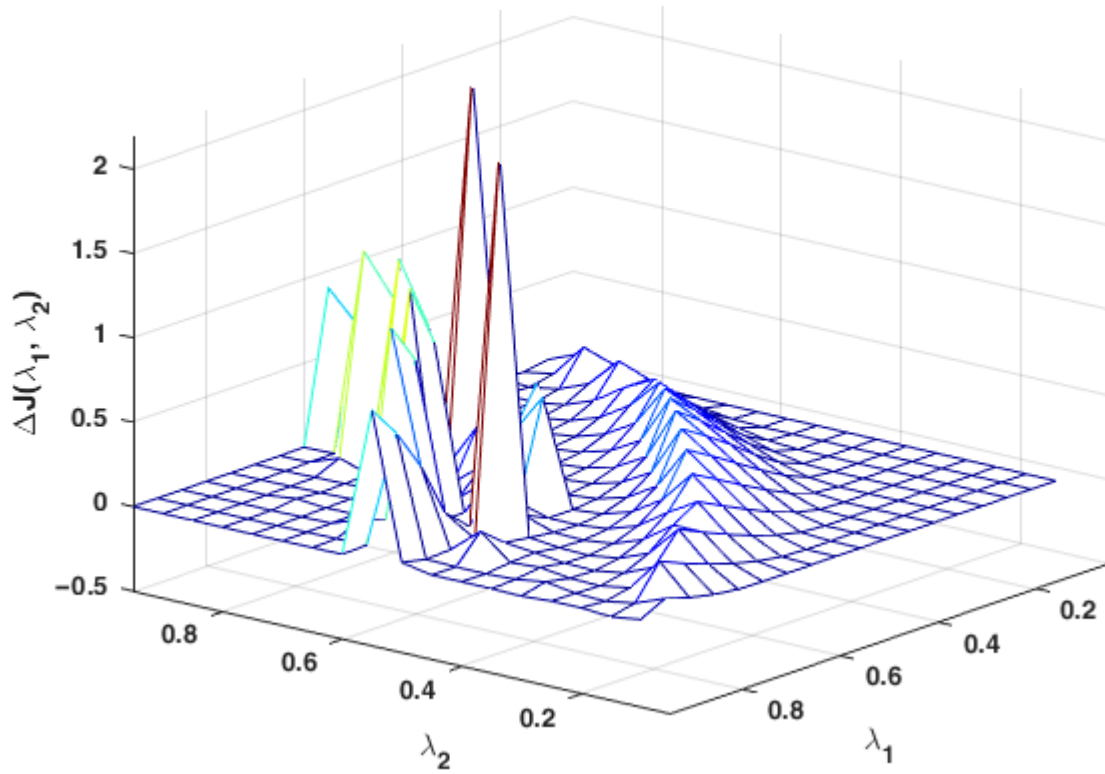


Figure 4. Function Surface (33), which depicts additional error caused by rounding.

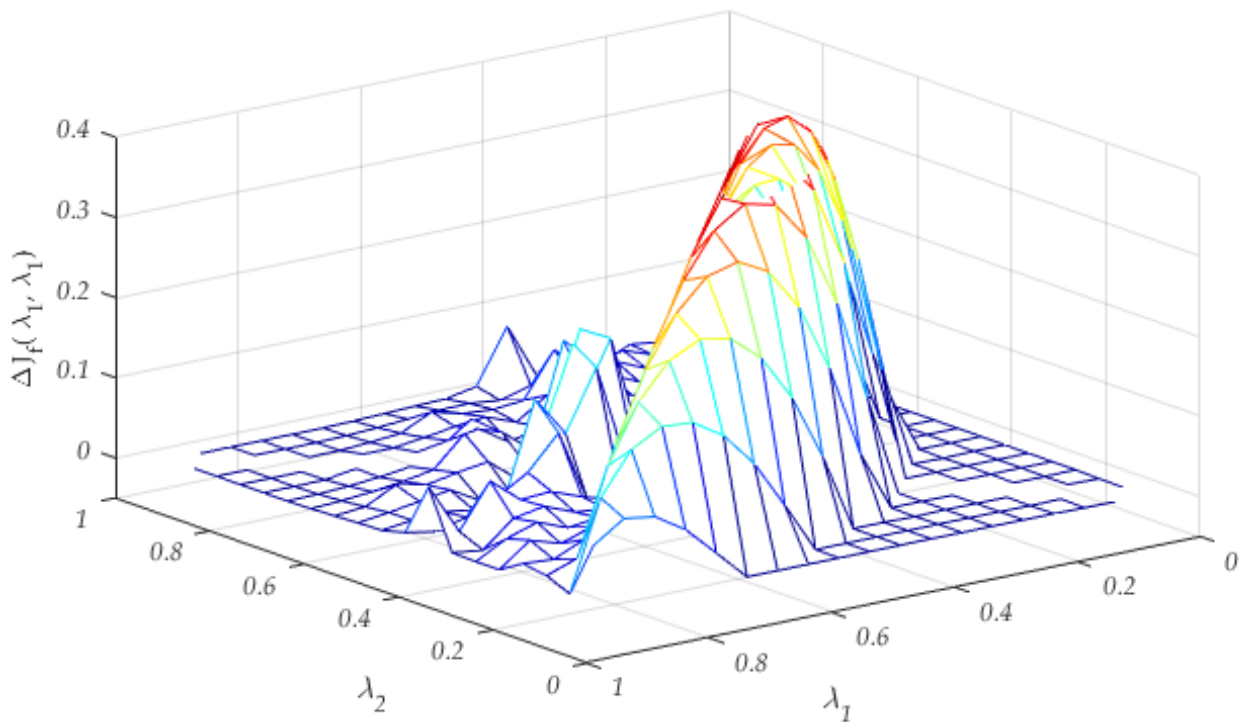


Figure 5. Function Surface (34), which depicts additional error caused by using of fractional lag filter.

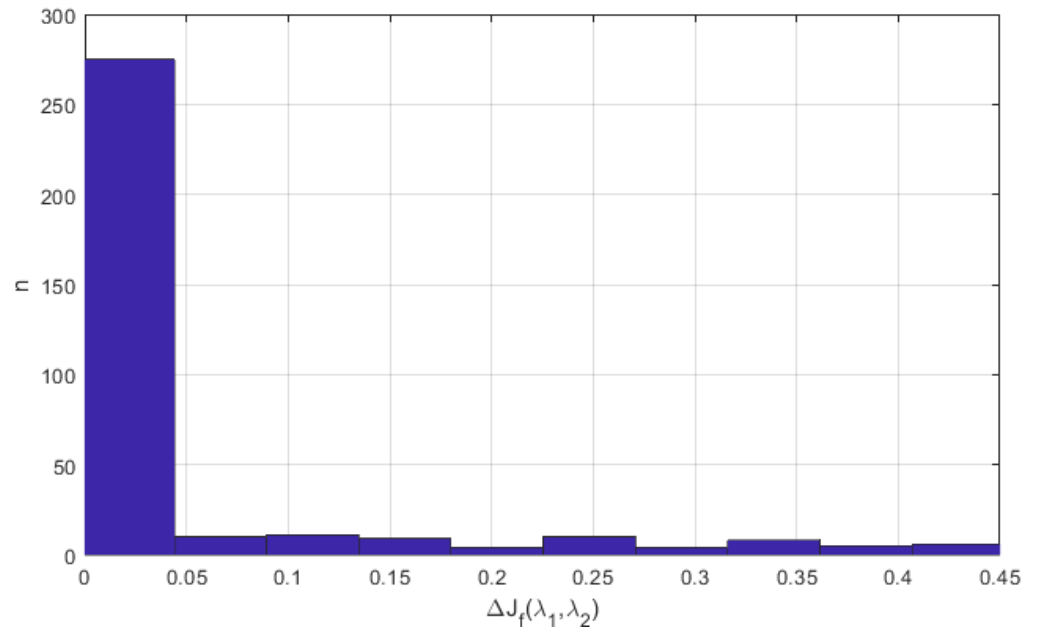


Figure 6. Histogram of additional errors caused by using of fractional lag filter.

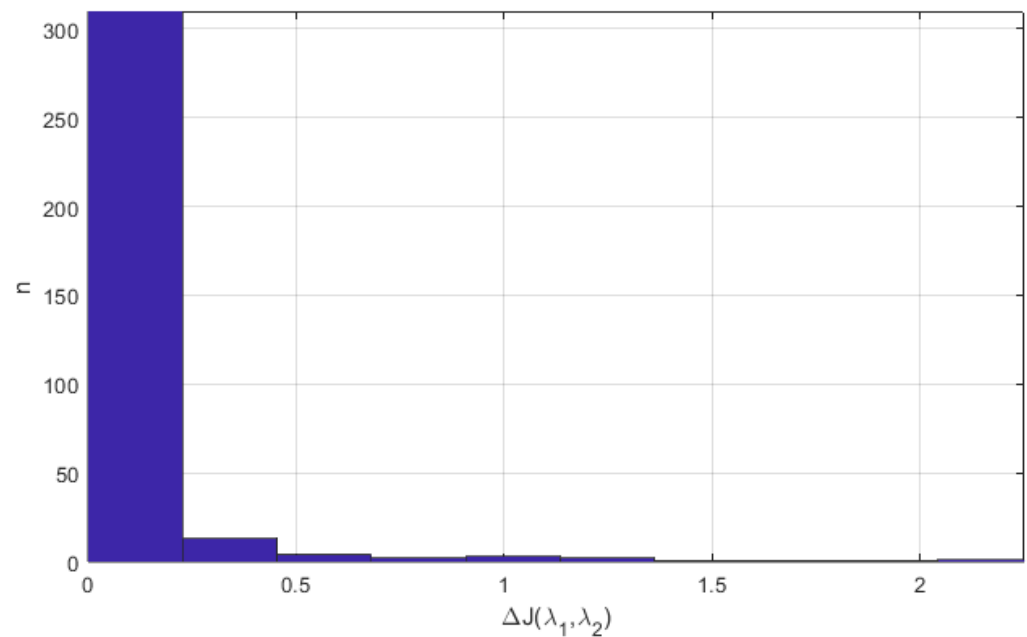


Figure 7. Histogram of additional errors caused by rounding.

Table 1. Mean and standard deviation of additional errors.

Method	Mean	Std
Rounding	0.0591	0.2050
Fractional filter	0.0468	0.1018

For instance, let us compare suboptimality levels for the case when $\lambda_1 = 0.5, \lambda_2 = 0.7$. The rounding method consists of the following steps:

1. $A(X) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.5^{x_1} & 0.5^{x_1+x_2} \\ 1 & 0.7^{x_1} & 0.7^{x_1+x_2} \end{bmatrix}$, $B = \begin{pmatrix} 1 \\ 0.743 \\ 0.452 \end{pmatrix}$ by definition in Theorem 1;
2. $F(X) = A^{-1}(X)B$;
3. $J_{min}(x_1, x_2) = 1 + |f_1(x_1, x_2)| + |f_2(x_1, x_2)|$;

4. The minimum value of $J_{min} = 2.224$ at the point $X^{opt} = (2.255, 2.409)$;
5. The coefficients $\alpha_1 \approx -0.973, \alpha_2 \approx -0.421$ of the polynomial α is found from the system of equations

$$\begin{aligned} 1 + \alpha_1 \lambda_1^{\lfloor x_1^{opt} \rfloor} + \alpha_2 \lambda_1^{\lfloor x_1^{opt} + x_2^{opt} \rfloor} &= \frac{1}{a(\lambda_1)}, \\ 1 + \alpha_1 \lambda_2^{\lfloor x_1^{opt} \rfloor} + \alpha_2 \lambda_2^{\lfloor x_1^{opt} + x_2^{opt} \rfloor} &= \frac{1}{a(\lambda_2)}, \end{aligned}$$

where $\lfloor x \rfloor$ denotes rounding of x to the nearest integer;

6. The suboptimality level is calculated as $1 + |\alpha_1| + |\alpha_2| - J_{min} = 0.170$.

The proposed method consists of the following steps:

1. Extraction of fractional parts $d_1 = 0.745 + 0.255q^{-1}, d_2 = 0.336 + 0.664q^{-1}$ of delay is performed using the (30);
2. The coefficients $\alpha = (\alpha_{11} + \alpha_{12}, \alpha_{21} + \alpha_{22})$ is found using the formula (14), where $\alpha_{11} = -0.855, \alpha_{21} = -0.049$ are coefficients of integer part, and $\alpha_{12} = -0.293, \alpha_{22} = -0.098$ are coefficients of fractional part.
3. The suboptimality level is calculated as $1 + |\alpha_{11}| + |\alpha_{12}| + |\alpha_{21}| + |\alpha_{22}| - J_{min} = 0.071$.

If a more accurate approximation of the fractional delay filter is necessary, multi-rate signal processing with a higher sampling frequency in the controller can be used, allowing to use a filter of a higher order.

5. Conclusions

The paper proposes a new method of synthesis of a ℓ_1 suboptimal stabilizing controller with fractional delays for a discrete non-minimum phase plant described by a linear difference equation with an additive unknown-but-bounded noise. The fractional delays approach is used to achieve a small level of suboptimality without significantly increasing the sampling rate. This approach consists of an approximation of the fractional lag with a combination of rounding and a first-order fractional lag filter. The suggested approximation of the fractional delay is illustrated via a simulation example with a non-minimum phase second-order plant. The results of the simulation shows that the proposed methodology allows to obtain a more accurate solution compared to approximation of the fractional delay by rounding.

The proposed methodology appears to be suitable to be used in various pattern recognition approaches, such as CS. A relationship between the fractional delay control problem and CS remains open and will be investigated in the future.

Author Contributions: Conceptualization, D.I.; investigation, D.I., O.G., Z.V. and V.P. The authors have equally contributed to the writing, editing, and style of the paper. All authors have read and agreed to the published version of the manuscript.

Funding: Sections 1 and 2 were supported by the St. Petersburg State University (project No. 73555239). Sections 3 and 4 were supported by the Ministry of Science and Higher Education of the Russian Federation (Project No. 075-15-2021-573, IPME RAS).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are grateful to the reviewers for constructive and useful comments, which made it possible to significantly improve the presentation of the results obtained.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Candes, E.; Romberg, J.; Tao, T. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. *IEEE Trans. Inf. Theory* **2006**, *52*, 489–509. [\[CrossRef\]](#)
2. Granichin, O.; Pavlenko, D. Randomization of data acquisition and ℓ_1 -optimization (recognition with compression). *Autom. Remote Control* **2010**, *71*, 2259–2282. [\[CrossRef\]](#)
3. Pan, Z.; Yu, J.; Huang, H.; Hu, S.; Zhang, A.; Ma, H.; Sun, W. Super-resolution based on compressive sensing and structural self-similarity for remote sensing images. *IEEE Trans. Geosci. Remote Sens.* **2013**, *51*, 4864–4876. [\[CrossRef\]](#)
4. Zhang, X.; Li, C.; Meng, Q.; Liu, S.; Zhang, Y.; Wang, J. Infrared image super resolution by combining compressive sensing and deep learning. *Sensors* **2018**, *18*, 2587. [\[CrossRef\]](#) [\[PubMed\]](#)
5. Yang, S.; Sun, F.; Wang, M.; Liu, Z.; Jiao, L. Novel super resolution restoration of remote sensing images based on compressive sensing and example patches-aided dictionary learning. In Proceedings of the 2011 International Workshop on Multi-Platform/Multi-Sensor Remote Sensing and Mapping, Xiamen, China, 10–12 January 2011; IEEE: Piscataway, NJ, USA, 2011 ; pp. 1–6.
6. Mardani, M.; Gong, E.; Cheng, J.Y.; Vasanawala, S.S.; Zaharchuk, G.; Xing, L.; Pauly, J.M. Deep generative adversarial neural networks for compressive sensing MRI. *IEEE Trans. Med. Imaging* **2018**, *38*, 167–179. [\[CrossRef\]](#)
7. Zhang, M.; Li, M.; Zhou, J.; Zhu, Y.; Wang, S.; Liang, D.; Chen, Y.; Liu, Q. High-dimensional embedding network derived prior for compressive sensing MRI reconstruction. *Med. Image Anal.* **2020**, *64*, 101717. [\[CrossRef\]](#)
8. Yang, G.; Yu, S.; Dong, H.; Slabaugh, G.; Dragotti, P.; Ye, X.; Liu, F.; Arridge, S.; Keegan, J.; Guo, Y.; et al. DAGAN: Deep De-Aliasing Generative Adversarial Networks for Fast Compressed Sensing MRI Reconstruction. *IEEE Trans. Med. Imaging* **2017**, *37*, 1310–1321. [\[CrossRef\]](#) [\[PubMed\]](#)
9. Zhang, X.; Zhang, S.; Lin, J.; Sun, F.; Zhu, X.; Yang, Y.; Tong, X.; Yang, H. An efficient seismic data acquisition based on compressed sensing architecture with generative adversarial networks. *IEEE Access* **2019**, *7*, 105948–105961. [\[CrossRef\]](#)
10. Siahkoochi, A.; Kumar, R.; Herrmann, F. Seismic data reconstruction with generative adversarial networks. In Proceedings of the 80th EAGE Conference and Exhibition 2018. European Association of Geoscientists & Engineers, Copenhagen, Denmark, 11–14 June 2018; Volume 2018; pp. 1–5.
11. Baraniuk, R.G.; Steeghs, P. Compressive sensing: A new approach to seismic data acquisition. *Lead. Edge* **2017**, *36*, 642–645. [\[CrossRef\]](#)
12. Granichin, O. Design of suboptimal controller of the linear object in bounded noise. *Autom. Remote Control* **1990**, *51*, 184–187.
13. Barabanov, A.; Granichin, O. Optimal controller for linear plant with bounded noise. *Autom. Remote Control* **1984**, *45*, 578–584.
14. Dahleh, M.; Pearson, J. L_1 -optimal feedback controllers for MIMO discrete-time systems. *IEEE Trans. Autom. Control* **1987**, *32*, 314–322. [\[CrossRef\]](#)
15. Granichin, O. Designing the discrete suboptimal controller of the continuous-time object in nonregular bounded noise. *Autom. Remote Control* **2001**, *62*, 422–429. [\[CrossRef\]](#)
16. Amelin, K.; Granichin, O. Randomized control strategies under arbitrary external noise. *IEEE Trans. Autom. Control* **2016**, *61*, 1328–1333.
17. Vakhitov, A.; Vlasov, V.; Granichin, O. Adaptive control of SISO plant with time-varying coefficients based on random test perturbation. In Proceedings of the 2010 American Control Conference, Baltimore, MD, USA, 30 June–2 July 2010; pp. 4004–4009. [\[CrossRef\]](#)
18. Khammash, M. A new approach to the solution of the ℓ_1 control problem: The scaled-Q method. *IEEE Trans. Autom. Control* **2000**, *45*, 180–187. [\[CrossRef\]](#)
19. Perez, T. *Ship Motion Control: Course Keeping and Roll Stabilisation Using Rudder and Fins*; Springer: Berlin/Heidelberg, Germany, 2005.
20. Kwon, D.S.; Book, W. A time-domain inverse dynamic tracking control of a single-link flexible manipulator. *J. Dyn. Syst. Meas. Control ASME* **1994**, *116*, 193–200. [\[CrossRef\]](#)
21. Qiu, L.; Davison, E. Performance limitations of non-minimum phase systems in the servomechanism problem. *Automatica* **1993**, *29*, 337–349. [\[CrossRef\]](#)
22. Jemaa, L.; Davison, E. Performance limitations in the robust servomechanism problem for discrete time periodic systems. *Automatica* **2003**, *39*, 1053–1059. [\[CrossRef\]](#)
23. Kravaris, C.; Daoutidis, P. Nonlinear state feedback control of second-order nonminimum-phase nonlinear systems. *Comput. Chem. Eng.* **1990**, *14*, 439–449. [\[CrossRef\]](#)
24. Gray, P.; Meyer, R. *Analysis and Design of Analog Integrated Circuits*; John Wiley & Sons, Inc.: Hoboken, NJ, USA, 1993.
25. Rigney, B.; Pao, L.; Lawrence, D. Nonminimum Phase Dynamic Inversion for Settle Time Applications. *Control Syst. Technol.* **2009**, *17*, 989–1005. [\[CrossRef\]](#)
26. Bose, S.; Hote, Y.; Kennedy, R. Nonminimum-phase equalization and its subjective importance in room acoustics. *IEEE Trans. Speech Audio Process.* **2000**, *8*, 728–737.
27. Curtain, R.; Morris, K. Transfer functions of distributed parameter systems: A tutorial. *Automatica* **2009**, *45*, 1101–1116. [\[CrossRef\]](#)
28. Seron, M.; Braslavsky, J.; Goodwin, G. *Fundamental Limitations in Filtering and Control*; Springer: Berlin/Heidelberg, Germany, 1997.
29. Hoagg, J.; Bernstein, D. Nonminimum-phase zeros. *IEEE Control Syst. Mag.* **2007**, *27*, 45–57. [\[CrossRef\]](#)
30. Doyle, J.; Francis, B.; Tannenbaum, A. *Feedback Control Theory*; Macmillan: New York, NY, USA, 1992.

31. Skogestad, S.; Postlethwaite, I. *Multivariable Feedback Control*; Wiley: New York, NY, USA, 1996.
32. Kailath, T. *Linear Systems. Englewood Cliffs*; Prentice-Hall: Hoboken, NJ, USA, 1980.
33. Stiasnie, M. On the application of fractional calculus for the formulation of viscoelastic models. *Appl. Math. Model.* **1979**, *3*, 300–302. [[CrossRef](#)]
34. Vinagre, B.; Feliu, V. Modeling and control of dynamic system using fractional calculus: Application to electro-chemical processes and flexible structures. In Proceedings of the 41st IEEE Conference on Decision and Control, Las Vegas, NV, USA, 10–13 December 2002; pp. 214–239.
35. Oustaloup, A. The CRONE control of resonant plants: Application to a flexible transmission. *Eur. J. Control* **1995**, *1*, 113–121. [[CrossRef](#)]
36. Machado, J. Analysis and design of fractional order digital control systems. *SAMS-J. Syst. Anal. Model. Simul.* **1997**, *27*, 107–122.
37. Podlubny, I. Fractional-order systems and PID-controllers. *IEEE Trans. Autom. Control* **1999**, *44*, 208–214. [[CrossRef](#)]
38. Das, S.; Pan, I.; Halder, K.; Das, S.; Gupta, A. Optimum weight selection based LQR formulation for the design of fractional order PIAD μ controllers to handle a class of fractional order systems. In Proceedings of the Computer Communication and Informatics (ICCCI), Coimbatore, India, 4–6 January 2013; pp. 1–6.
39. Das, S.; Pan, I.; Halder, K.; Gupta, A. LQR based improved discrete PID controller design via optimum selection of weighting matrices using fractional order integral performance index. *Appl. Math. Model.* **2013**, *37*, 4253–4268. [[CrossRef](#)]
40. Das, S.; Pan, I.; Das, S. Multi-objective LQR with optimum weight selection to design FOPID controllers for delayed fractional order processes. *ISA Trans.* **2015**, *58*, 35–49. [[CrossRef](#)]
41. Monje, C.A.; Vinagre, B.M.; Feliu, V.; Chen, Y. Tuning and auto-tuning of fractional order controllers for industry applications. *Control Eng. Pract.* **2008**, *16*, 798–812. [[CrossRef](#)]
42. Merrikh-Bayat, F. Fractional-order unstable pole-zero cancellation in linear feedback systems. *J. Process Control* **2013**, *23*, 817–825. [[CrossRef](#)]
43. Arya, P.; Chakrabarty, S. IMC based fractional order controller design for specific non-minimum phase systems. *IFAC-PapersOnLine* **2018**, *51*, 847–852. [[CrossRef](#)]
44. Muresan, C.; Dulf, E.; Copot, C.; Keyser, R.D.; Ionescu, C. Design and analysis of a multivariable fractional order controller for a non-minimum phase system. *J. Vib. Control* **2016**, *22*, 2187–2195. [[CrossRef](#)]
45. Chen, Y.; Petras, I.; Xue, D. Fractional order control—a tutorial. In Proceedings of the 2009 American Control Conference, St. Louis, MO, USA, 10–12 June 2009; IEEE: Piscataway, NJ, USA, 2009; pp. 1397–1411.
46. Shah, P.; Agashe, S. Review of fractional PID controller. *Mechatronics* **2016**, *38*, 29–41. [[CrossRef](#)]
47. Almeida, A.M.d.; Lenzi, M.K.; Lenzi, E.K. A survey of fractional order calculus applications of multiple-input, multiple-output (MIMO) process control. *Fractal Fract.* **2020**, *4*, 22. [[CrossRef](#)]
48. Birs, I.; Muresan, C.; Nascu, I.; Ionescu, C. A survey of recent advances in fractional order control for time delay systems. *IEEE Access* **2019**, *7*, 30951–30965. [[CrossRef](#)]
49. Monje, C.A.; Chen, Y.; Vinagre, B.M.; Xue, D.; Feliu-Battle, V. *Fractional-Order Systems and Controls: Fundamentals and Applications*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2010.
50. Baleanu, D.; Machado, J.A.T.; Luo, A.C. *Fractional Dynamics and Control*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2011.
51. Eckert, M.; Kupper, M.; Hohmann, S. Functional fractional calculus for system identification of battery cells. *Automatisierungstechnik* **2014**, *62*, 272–281. [[CrossRef](#)]
52. Valério, D.; da Costa, J. *An Introduction to Fractional Control*; Control, Robotics and Sensors Series; Institution of Engineering and Technology: London, UK, 2013.
53. Valimäki, V.; Laakso, T. Principles of fractional delay filters. In Proceedings of the 2000 IEEE International Conference on Acoustics, Speech, and Signal Processing, Istanbul, Turkey, 5–9 June 2000; pp. 3870–3873.
54. Farrow, W. A continuously variable digital delay element. In Proceedings of the IEEE International Symposium on Circuits and Systems, Espoo, Finland, 7–9 June 1988; Volume 3, pp. 2641–2645.
55. Thiran, P. Recursive digital filters with maximally flat group delay. *IEEE Trans. Circ. Theory* **1971**, *18*, 659–664. [[CrossRef](#)]
56. Diaz-Carmona, J.; Dolecek, G. Fractional delay digital filters. *Appl. MATLAB Sci. Eng.* **1996**, *1*, 247–272.
57. Williamson, G. *Digital Signal Processing Handbook*; Madisetti, V.K., Williams, D.B., Eds.; CRC Press LLC.: Boca Raton, FL, USA, 1999.
58. Mulgrew, B.; Grant, P.; Thompson, J. Finite impulse response digital filters. In *Digital Signal Processing*; Springer: Berlin/Heidelberg, Germany, 1999; pp. 150–175.
59. Laakso, T.; Välimäki, V.; Karjalainen, M.; Laine, U. Splitting the unit delay tools for fractional delay filter design. *IEEE Signal Process. Mag.* **1996**, *13*, 30–60. [[CrossRef](#)]