mathematics

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A Krasnoselskii–Ishikawa Iterative Algorithm for Monotone Reich Contractions in Partially Ordered Banach Spaces with an Application

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Abstract: Iterative algorithms have been utilized for the computation of approximate solutions of stationary and evolutionary problems associated with differential equations. The aim of this article is to introduce concepts of monotone Reich and Chatterjea nonexpansive mappings on partially ordered Banach spaces. We describe sufficient conditions for the existence of an approximate fixed-point sequence (AFPS) and prove certain fixed-point results using the Krasnoselskii–Ishikawa iterative algorithm. Moreover, we present some interesting examples to highlight the superiority of our results. Lastly, we provide both weak and strong convergence results for such mappings and consider an application of our results to prove the existence of a solution to an initial value problem.

Keywords: approximate fixed-point sequence; Krasnoselskii–Ishikawa iterative algorithm; ordered Reich and Chatterjea nonexpansive mappings; weak and strong convergence

MSC: 47H07; 47H09; 47H10; 54H24

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Citation: Hussain, N.; Alsulami, S.M.; Alamri, H. A Krasnoselskii–Ishikawa Iterative Algorithm for Monotone Reich Contractions in Partially Ordered Banach Spaces with an Application. *Mathematics* **2022**, *10*, 76. https://doi.org/10.3390/math10010076

Academic Editor: Christopher Goodrich

Received: 8 December 2021
Accepted: 22 December 2021
Published: 27 December 2021

1. Introduction

The study of the existence of fixed points for monotone nonexpansive and asymptotically nonexpansive mappings in the setting of a metric space have been widely investigated. Let U be a normed linear space. A self mapping $\mathcal{S} : U \to U$ is said to be nonexpansive if $\|\mathcal{S}\mu - \mathcal{S}\omega\| \leq \|\mu - \omega\|$ for all $\mu, \omega \in U$. The set of fixed points of $\mathcal{S}$ is denoted by $\mathcal{F}(\mathcal{S})$, that is, $\mathcal{F}(\mathcal{S}) = \{\mu \in U : \mathcal{S}\mu = \mu\}$. A mapping $\mathcal{S}$ is called quasi-nonexpansive if $\|\mathcal{S}\mu - \nu\| \leq \|\mu - \nu\|$ for all $\mu \in U$, and $\nu \in \mathcal{F}(\mathcal{S})$. A mapping $\mathcal{S} : U \to U$ is said to be asymptotically nonexpansive if there exists a sequence $\{a_n\}$ with $a_n \geq 1$ and $\lim_{n\to\infty} a_n = 1$ such that the condition $\|\mathcal{S}^n\mu - \mathcal{S}^n\omega\| \leq a_n\|\mu - \omega\|$ holds for all $n \in \mathbb{N}$. A sequence $\{\mu_n\}$ in U is asymptotically regular if $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ for all $n \in \mathbb{N}$. We know that every nonexpansive mapping or asymptotically nonexpansive mapping on a non-empty, closed, bounded, convex subset of a uniformly convex Banach space has at least one fixed point [1–3]. Debnath and La Sen [4] studied the fixed points of a broad category of set-valued maps that may include discontinuous maps extending the notions of orbitally continuous and asymptotically regular mappings. In [5], the authors established fixed-point results for some asymptotically regular multivalued mappings satisfying the Kannan type contractive condition without assuming compactness of the underlying metric space or continuity of underlying mapping.

Many authors have studied an approximate fixed-point sequence (AFPS) and approximate fixed-point property (AFPP) for different types of mappings [6–9]. A sequence $\{\mu_n\}$ in a normed space U is said to be an (AFPS) for a self mapping $\mathcal{S} : U \to U$ if $\lim_{n\to\infty} \|\mu_n - \mathcal{S}\mu_n\| = 0$. Matouškovà and Reich [10] found that every infinite dimensional Banach space contained an unbounded closed convex set with the approximate point fixed property for nonexpansive mappings. These results have become an important tool in solving a variety of problems, such as integral equations, partial differential equations, optimization problems and boundary value problems; see [10–15].

Recently, Som et al. in [16] introduced two types of mappings, Reic-type nonexpansive and Chatterjea-type nonexpansive mappings, and discussed that these classes of mappings possess an APFS under some conditions using the Krasnoselskii iteration method in Banach spaces. They explored some properties of the fixed-point sets of these mappings like closedness, convexity, and remotality, and obtained sufficient conditions under which a Reich-type nonexpansive mapping reduces to that of a nonexpansive one. For further considerations on Reich and Chatterjea contractions, see [17–20]. Let $\mu_1 \in U$ be arbitrary. Then, consider the Krasnoselskii-Ishikawa iteration defined by the sequence $\{\mu_n\} \subset U$ as follows:

$$\mu_{n+1} = \lambda \mathcal{S}\mu_n + (1-\lambda)\mu_n, \qquad (KIS)$$

where $\lambda \in (0,1)$, $n \in \mathbb{N}$ and $\mathcal{S}$ is a self map of U. Very recently, Popescu and Stan [21] have obtained certain interesting results related to Reich- and Chatterjea-type nonexpansive mappings. In ref. [22], Debnath et al. studied the existence and uniqueness of fixed points of Reich-type G-contraction on closed and bounded subsets of a metric space endowed with a graph. Khamsi and Khan [23] discussed the behavior of the Krasnoselskii–Ishikawa iteration process for monotone nonexpansive mappings in $L_1([0,1])$; see also [12,24].

The fixed-point theory of monotone contractions defined on partially ordered metric spaces has been researched by a number of scholars. The contractivity criterion on the nonlinear map is only considered to hold elements that are comparable in the partial order in such findings. The reader is directed to the ground-breaking work on the subject, which has applications in integral, differential, and nonlinear fractional evolution equations and equilibrium problems; see [2,25–34].

Iterative algorithms have been utilized for the computation of approximate solutions of stationary and evolutionary problems associated with differential equations. This paper is organized as follows: In Section 2, some basic definitions and propositions are stated. In Section 3, we develop sufficient conditions for the existence of an approximate fixed point sequence (AFPS) with certain fixed-point results using the Krasnoselskii–Ishikawa iteration algorithm. A number of examples are presented to illustrate the results. In Section 4, we provide some weak and strong convergence results for these mappings. In Section 5, we study the existence of a solution for a nonlinear differential equation. Section 6 is concerned with conclusion.

2. Preliminaries

Let $(U, \|.\|)$ be a Banach space endowed with a partial order $\preceq$. Throughout, it is assumed that order intervals are convex and closed. Recall that an order interval is any of the subsets

$$[\mu, \rightarrow) = \{\nu \in U; \mu \preceq \nu\} \ \ and \ \ (\leftarrow, \omega] = \{\nu \in U; \nu \preceq \omega\},$$

for any $\mu, \omega \in U$. Therefore, the order interval $[\mu, \omega]$ for all $\mu, \omega \in U$ is given by

$$[\mu, \omega] = \{\nu \in U; \mu \preceq \nu \preceq \omega\} = [\mu, \rightarrow) \cap (\leftarrow, \omega],$$

is also closed and convex for any $\mu, \omega \in U$.

Definition 1 ([24])**.** *Let E be a nonempty subset of the ordered Banach space U. A map $\mathcal{S} : E \rightarrow E$ is said to be*

(i) monotone if $\mathcal{S}\mu \preceq \mathcal{S}\omega$ for all $\mu, \omega \in E$ with $\mu \preceq \omega$;
(ii) monotone nonexpansive if $\mathcal{S}$ is monotone and $\|\mathcal{S}\mu - \mathcal{S}\omega\| \leq \|\mu - \omega\|$ for any $\mu, \omega \in E$ with $\mu \preceq \omega$.

The sequence $\{\mu_n\}$ is said to be bounded in a partially ordered set $(U, \preceq)$ if there exists a point $\nu \in U$ such that $\mu_n \preceq \nu$ for all $n \in \mathbb{N}$.

Now, we give the definition of ordered Reich and Chatterjea nonexpansive mappings.

Definition 2. *Let U be an ordered normed space, wherein E is a non-empty closed convex subset of U. A mapping $\mathcal{S} : E \to E$ is said to be an ordered Reich nonexpansive mapping if there exist non-negative real numbers α, β, γ with $\alpha + \beta + \gamma = 1$ such that*

$$\|\mathcal{S}\mu - \mathcal{S}\omega\| \leq \alpha\|\mu - \omega\| + \beta\|\mu - \mathcal{S}\mu\| + \gamma\|\omega - \mathcal{S}\omega\|, \tag{1}$$

for all $\mu, \omega \in E$ with $\mu \preceq \omega$.

The mapping $\mathcal{S}$ is said to be an ordered Chatterjea nonexpansive mapping if there exist non-negative real numbers α, β, γ with $\alpha + \beta + \gamma = 1$ such that the condition

$$\|\mathcal{S}\mu - \mathcal{S}\omega\| \leq \alpha\|\mu - \omega\| + \beta\|\mu - \mathcal{S}\omega\| + \gamma\|\omega - \mathcal{S}\mu\|, \tag{2}$$

holds, for all $\mu, \omega \in E$ such that $\mu \preceq \omega$.

The following proposition is proved in [35,36].

Proposition 1. *Let $(U, d, \preceq)$ be a partially ordered metric space. Assume that the sequences $\{\mu_n\}$ and $\{\omega_n\}$ in U with $t_n \in [0, 1)$ satisfy following conditions for all $n \in \mathbb{N}$:*

(i) $\mu_{n+1} \in [\mu_n, \omega_n]$ *with* $d(\mu_n, \omega_{n+1}) = t_n d(\mu_n, \omega_n)$*;*

(ii) $d(\omega_{n+1}, \omega_n) \leq d(\mu_{n+1}, \mu_n)$.

Then,

$$\left(1 + \sum_{s=i}^{i+n-1} t_s\right) d(\omega_i, \mu_i) \leq d(\omega_{i+n}, \mu_i) + \prod_{s=i}^{i+n-1} (1 - t_s)^{-1} [d(\omega_i, \mu_i) - d(\omega_{i+n}, \mu_{i+n})]$$

for all $n \geq 1$ and $i \geq 0$.

3. Main Results

The following technical lemma will be useful to develop further results.

Lemma 1. *Let $(U, d, \preceq)$ be a partially ordered Banach space and E be a nonempty convex subset of U. Let $\mathcal{S} : E \to E$ be a monotone mapping satisfying the following condition:*

$$\xi d(\mu, \mathcal{S}\mu) \leq d(\mu, \omega) \Rightarrow d(\mathcal{S}\mu, \mathcal{S}\omega) \leq d(\mu, \omega), \tag{3}$$

where $\xi \in (0, 1)$. Assume that there exists $\mu_1 \in E$ such that μ_1 and $\mathcal{S}\mu_1$ are comparable. Consider the sequence $\{\mu_n\}$ in E as defined by KIS, such that $\xi < \lambda$ and $\lambda \in (0, 1)$. Then, for any $n \geq 1$ and $i \geq 0$, we have

$$(1 + n\lambda) d(\mathcal{S}\mu_i, \mu_i) \leq d(\mathcal{S}\mu_{i+n}, \mu_i) + (1 - \lambda)^{-n} [d(\mathcal{S}\mu_i, \mu_i) - d(\mathcal{S}\mu_{i+n}, \mu_{i+n})].$$

Moreover, if $\{\mu_n\}$ has two subsequences which converge to ω and ν, then $\omega = \nu$.

Proof. The induction method is employed to demonstrate $\mu_n \preceq \mu_{n+1} \preceq \mathcal{S}\mu_n$ for any $n \geq 1$. By assumption, there is $\mu_1 \preceq \mathcal{S}\mu_1$. Using the convexity of the order interval $[\mu_1, \mathcal{S}\mu_1]$ and KIS, we obtain

$$\mu_1 \preceq \lambda\mathcal{S}\mu_1 + (1 - \lambda)\mu_1 = \mu_2 \preceq \mathcal{S}\mu_1.$$

As $\mathcal{S}$ is monotone, $\mathcal{S}\mu_1 \preceq \mathcal{S}\mu_2$, which implies

$$\mu_1 \preceq \mu_2 \preceq \mathcal{S}\mu_1 \preceq \mathcal{S}\mu_2.$$

Thus, by induction, we have

$$\mu_n \preceq \mu_{n+1} \preceq \mathcal{S}\mu_n \preceq \mathcal{S}\mu_{n+1},$$

for all $n \in \mathbb{N}$. This implies that

$$\mu_{n+1} \in [\mu_n, \mathcal{S}\mu_n]. \tag{4}$$

Thus, by KIS, we have

$$d(\mu_{n+1}, \mathcal{S}\mu_n) = (1-\lambda)d(\mu_n, \mathcal{S}\mu_n).$$

Furthermore,

$$d(\mu_{n+1}, \mu_n) = \lambda d(\mu_n, \mathcal{S}\mu_n). \tag{5}$$

Since $\xi < \lambda$, for all $\lambda \in (0,1)$, there is

$$\xi d(\mu_n, \mathcal{S}\mu_n) \leq \lambda d(\mu_n, \mathcal{S}\mu_n) = d(\mu_{n+1}, \mu_n).$$

Now, (3) this implies that

$$d(\mathcal{S}\mu_{n+1}, \mathcal{S}\mu_n) \leq d(\mu_{n+1}, \mu_n) \tag{6}$$

for all $n \in \mathbb{N}$. All of the assumptions of Proposition 1 hold, with $\omega_n = \mathcal{S}\mu_n$. On the other hand

$$\begin{cases} (1+n\lambda) \leq 1 + \sum_{s=i}^{i+n-1} t_s, \\ \prod_{s=i}^{i+n-1}(1-t_s)^{-1} \leq (1-\lambda)^{-n}, \\ d(\mu_i, \mathcal{S}\mu_{i+n}) \leq \delta(E) = sup\{d(\mu,\omega) : \mu, \omega \in E\}, \end{cases} \tag{7}$$

for all $n \in \mathbb{N}$ and $\lambda \in (0,1)$, which implies that

$$(1+n\lambda)d(\mathcal{S}\mu_i, \mu_i) \leq d(\mathcal{S}\mu_{i+n}, \mu_i) + (1-\lambda)^{-n}[d(\mathcal{S}\mu_i, \mu_i) - d(\mathcal{S}\mu_{i+n}, \mu_{i+n})]$$

for any $i \geq 0$ and $n \geq 1$ as required.

Next, let $\{\mu_{n_k}\}$ be a subsequence of $\{\mu_n\}$ that converges to ω. Fix $m \geq 1$. Since the order interval $[\mu_k, \rightarrow)$ is closed and convex and $\{\mu_n\}$ is monotone increasing, then $\omega \in [\mu_m, \rightarrow)$. Hence $\mu_m \preceq \omega$ for any $m \geq 1$. Consequently, if $\{\mu_n\}$ has another subsequence which converges to ν, it is necessary to have $\omega = \nu$. Indeed, since $\mu_n \preceq \omega$, for any $n \geq 1$, we get $\omega \preceq \nu$. Similarly, $\nu \preceq \omega$, which implies that $\omega = \nu$. □

Remark 1. *Under the assumptions of Lemma 1, if we assume $\mathcal{S}\mu_1 \preceq \mu_1$, then we have*

$$\mathcal{S}\mu_n \preceq \mu_{n+1} \preceq \mu_n$$

for any $n \geq 1$. Moreover, the conclusion on equality of limits of subsequences of μ_n holds.

Theorem 1. *Let $(U, \|.\|, \preceq)$ be a partially ordered Banach space and E be the nonempty closed convex and bounded subset of U. Let a self-mapping $\mathcal{S}$ of E be a monotone and an ordered Reich nonexpansive mapping with coefficients α, β, γ such that $\gamma < 1$. Assume that there exists $\mu_1 \in E$ such that μ_1 and $\mathcal{S}\mu_1$ are comparable. Furthermore, suppose that for any μ, ω in E with $\mu \preceq \omega$, we have*

$$\frac{(1-\gamma)}{6}\|\mu - \mathcal{S}\mu\| \leq \|\mu - \omega\| \Rightarrow \|\mathcal{S}\mu - \mathcal{S}\omega\| \leq \|\mu - \omega\|. \tag{8}$$

Then, $\mathcal{S}$ has an AFPS in E, which is asymptotically regular.

Proof. Let $\mu_1 \in E$ such that $\mu_1 \preceq \mathcal{S}\mu_1$. Considering the sequence $\{\mu_n\}$ in U defined by KIS, the following is obtained:

$$\|\mu_{n+1} - \mu_n\| = \lambda\|\mathcal{S}\mu_n - \mu_n\|. \tag{9}$$

In addition, we have

$$\|\mu_{n+2} - \mu_{n+1}\| = \lambda\|\mathcal{S}\mu_{n+1} - \mu_{n+1}\| \tag{10}$$

for all $n \geq 1$ where $\lambda \in [\frac{1}{2}, 1)$.

Now, for any $\mu \in U$, with $\mu = \mu_n$, and $\omega = \mu_{n+1}$, using (1) , we get

$$\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \leq \alpha\|\mu_n - \mu_{n+1}\| + \beta\|\mu_n - \mathcal{S}\mu_n\| + \gamma\|\mu_{n+1} - \mathcal{S}\mu_{n+1}\|, \tag{11}$$

where $\mu_n \preceq \mu_{n+1}$.

From (11), we get

$$\lambda\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \leq \lambda\alpha\|\mu_n - \mu_{n+1}\| + \beta\lambda\|\mu_n - \mathcal{S}\mu_n\| + \gamma\lambda\|\mu_{n+1} - \mathcal{S}\mu_{n+1}\|. \tag{12}$$

Using (9) and (10) in (12), we get

$$\lambda\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \leq \alpha\lambda\|\mu_n - \mu_{n+1}\| + \beta\|\mu_{n+1} - \mu_n\| + \gamma\|\mu_{n+2} - \mu_{n+1}\|. \tag{13}$$

Again, by KIS, we obtain,

$$\|\mu_{n+2} - \mu_{n+1}\| \leq \lambda\|\mathcal{S}\mu_{n+1} - \mathcal{S}\mu_n\| + (1-\lambda)\|\mu_{n+1} - \mu_n\|. \tag{14}$$

Then, using (14) in (13), we have

$$\begin{aligned}\lambda\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \leq{}& \alpha\lambda\|\mu_n - \mu_{n+1}\| + \beta\|\mu_{n+1} - \mu_n\| + \gamma\lambda\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \\ &+ \gamma(1-\lambda)\|\mu_n - \mu_{n+1}\|.\end{aligned}$$

Thus,

$$(\lambda - \gamma\lambda)\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \leq \alpha\lambda\|\mu_n - \mu_{n+1}\| + \beta\|\mu_{n+1} - \mu_n\| + \gamma(1-\lambda)\|\mu_n - \mu_{n+1}\|.$$

Since $\lambda < 1$, then $(1-\lambda) < 1$. We thus obtain

$$\lambda(1-\gamma)\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| < \alpha\|\mu_n - \mu_{n+1}\| + \beta\|\mu_{n+1} - \mu_n\| + \gamma\|\mu_n - \mu_{n+1}\|.$$

We get that

$$\lambda(1-\gamma)\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| < \|\mu_n - \mu_{n+1}\|,$$

where $\alpha + \beta + \gamma = 1$. On the other hand,

$$\|\mu_n - \mathcal{S}\mu_{n+1}\| = \|\mu_n - \mathcal{S}\mu_n + \mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \leq \|\mu_n - \mathcal{S}\mu_n\| + \|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\|.$$

Then, using (9), we have

$$\|\mu_n - \mathcal{S}\mu_{n+1}\| \leq \frac{1}{\lambda}\|\mu_n - \mu_{n+1}\| + \|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\|. \tag{15}$$

Hence,

$$\lambda(1-\gamma)\|\mu_n - \mathcal{S}\mu_{n+1}\| \leq \frac{\lambda(1-\gamma)}{\lambda}\|\mu_n - \mu_{n+1}\| + \lambda(1-\gamma)\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\|. \tag{16}$$

Using (15) into (16), we obtain

$$\begin{aligned}\lambda(1-\gamma)\|\mu_n - \mathcal{S}\mu_{n+1}\| &< (1-\gamma)\|\mu_n - \mu_{n+1}\| + \frac{\lambda(1-\gamma)}{\lambda(1-\gamma)}\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \\ &= (2-\gamma)\|\mu_n - \mu_{n+1}\|.\end{aligned}$$

Which implies

$$\frac{\lambda(1-\gamma)}{(2-\gamma)}\|\mu_n - \mathcal{S}\mu_{n+1}\| < \|\mu_n - \mu_{n+1}\|. \tag{17}$$

Now, since $1/2 \leq \lambda < 1$ and $0 \leq \gamma < 1$, we get

$$\frac{1-\gamma}{4} < \frac{\lambda(1-\gamma)}{2-\gamma}.$$

Therefore, (17) implies

$$\frac{(1-\gamma)}{4}\|\mu_n - \mathcal{S}\mu_{n+1}\| < \|\mu_n - \mu_{n+1}\|.$$

The following is obtained:

$$\begin{aligned}\frac{(1-\gamma)}{4}\|\mu_n - \mathcal{S}\mu_n\| &\leq \frac{1-\gamma}{4}\|\mu_n - \mathcal{S}\mu_{n+1}\| + \frac{1-\gamma}{4}\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \\ &< \|\mu_n - \mu_{n+1}\| + \frac{1-\gamma}{4\lambda(1-\gamma)}\|\mu_n - \mu_{n+1}\| \\ &\leq \|\mu_n - \mu_{n+1}\| + \frac{1}{4\lambda}\|\mu_n - \mu_{n+1}\| \\ &\leq \frac{3}{2}\|\mu_n - \mu_{n+1}\|.\end{aligned}$$

For any $1/2 \leq \lambda < 1$, we have that

$$\frac{(1-\gamma)}{6}\|\mu_n - \mathcal{S}\mu_n\| < \|\mu_n - \mu_{n+1}\|,$$

which implies that

$$\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \leq \|\mu_n - \mu_{n+1}\|. \tag{18}$$

Note that $\|\mu_n - \mathcal{S}\mu_n\|$ is decreasing if, and only if, $\|\mathcal{S}\mu_{n+1} - \mu_{n+1}\|$ is decreasing, which follows from the triangle inequality, (18) and (9),

$$\begin{aligned}\|\mathcal{S}\mu_{n+1} - \mu_{n+1}\| &\leq \|\mathcal{S}\mu_{n+1} - \mathcal{S}\mu_n\| + \|\mathcal{S}\mu_n - \mu_{n+1}\| \\ &\leq \|\mu_n - \mu_{n+1}\| + \|\mathcal{S}\mu_n - \mu_{n+1}\| \\ &\leq \lambda\|\mu_n - \mathcal{S}\mu_n\| + \|\mathcal{S}\mu_n - \mu_{n+1}\|.\end{aligned} \tag{19}$$

From the KIS definition, we obtain

$$\|\mathcal{S}\mu_n - \mu_{n+1}\| = (1-\lambda)\|\mu_n - \mathcal{S}\mu_n\|. \tag{20}$$

From (19) and (20), the implication is that

$$\begin{aligned}\|\mathcal{S}\mu_{n+1} - \mu_{n+1}\| &\leq \lambda\|\mu_n - \mathcal{S}\mu_n\| + (1-\lambda)\|\mu_n - \mathcal{S}\mu_n\| \\ &\leq \|\mu_n - \mathcal{S}\mu_n\|\end{aligned}$$

for all $n \in \mathbb{N}$. Consequently, $\|\mu_{n+1} - \mathcal{S}\mu_{n+1}\|$ is a decreasing and bounded sequence. Thus, there exists $R \geq 0$ such that

$$\lim_{n\to\infty}\|\mu_n - \mathcal{S}\mu_n\| = R \geq 0.$$

Since $\xi = (1-\gamma)/6 < \lambda < 1$, the inequality obtained in Lemma 1 implies that

$$(1+n\xi)\|\mu_i - \mathcal{S}\mu_i\| \leq (1+n\lambda)\|\mu_i - \mathcal{S}\mu_i\| \leq \delta(E) + (1-\lambda)^{-n}[\|\mu_i - \mathcal{S}\mu_i\| - \|\mu_{i+n} - \mathcal{S}\mu_{i+n}\|],$$

for all $i \geq 0$, and $n \geq 1$, with $\delta(E) = \sup\{\|\mu - \omega\|; \mu, \omega \in E\} < +\infty$. If we let $i \to +\infty$, we get

$$(1+n\lambda)R \leq \delta(E) + (1-\lambda)^{-n}[R - R].$$

Then,

$$R \leq \frac{\delta(E)}{1+n\lambda}$$

holds for all $n \geq 1$. Clearly this is possible only if $R = 0$, which implies $\lim_{n\to\infty} \|\mu_n - \mathcal{S}\mu_n\| = 0$. Thus, the sequence $\{\mu_n\}$ is an AFPS for $\mathcal{S}$. Further, we have

$$\|\mu_n - \mu_{n+1}\| = \lambda\|\mu_n - \mathcal{S}\mu_n\| \to 0, \quad \text{as } n \to \infty.$$

Therefore, the AFPS $\{\mu_n\}$ is asymptotically regular. □

Theorem 2. *Under the assumptions of Theorem 1, $\mathcal{S}$ has a fixed point, provided $\alpha < 1$.*

Proof. By Theorem 1, $\mathcal{S}$ has an AFPS (μ_n) which is an asymptotically regular sequence. By triangle inequality and asymptotic regularity, for $m \geq n \geq 1$, we obtain

$$\begin{aligned}\|\mu_m - \mu_n\| &\leq \|\mu_m - \mathcal{S}\mu_m\| + \|\mathcal{S}\mu_m - \mathcal{S}\mu_n\| + \|\mu_n - \mathcal{S}\mu_n\| \\ &\leq \|\mu_m - \mathcal{S}\mu_m\| + \alpha\|\mu_m - \mu_n\| + \beta\|\mu_m - \mathcal{S}\mu_m\| + \gamma\|\mu_n - \mathcal{S}\mu_n\| \\ &+ \|\mu_n - \mathcal{S}\mu_n\|,\end{aligned}$$

which implies that

$$(1-\alpha)\|\mu_m - \mu_n\| \leq \|\mu_m - \mathcal{S}\mu_m\| + \beta\|\mu_m - \mathcal{S}\mu_m\| + \gamma\|\mu_n - \mathcal{S}\mu_n\| + \|\mu_n - \mathcal{S}\mu_n\| \to 0,$$

as $m \geq n \to \infty$, $\alpha < 1$. Therefore,$\{\mu_n\}$ is Cauchy sequence in E and there exists μ in E such that $\mu_n \to \mu$. Then,

$$\|\mathcal{S}\mu_m - \mathcal{S}\mu_n\| \leq \alpha\|\mu_m - \mu_n\| + \beta\|\mu_m - \mathcal{S}\mu_m\| + \gamma\|\mu_n - \mathcal{S}\mu_n\| \to 0,$$

as $n, m \to \infty$. This implies that $\mathcal{S}\mu_n$ is a Cauchy sequence in E. Therefore, the following is obtained:

$$\|\mathcal{S}\mu_n - \mu\| \leq \|\mathcal{S}\mu_n - \mu_n\| + \|\mu_n - \mu\| \to 0$$

as $n \to \infty$. Again, there is

$$\|\mathcal{S}\mu_n - \mathcal{S}\mu\| \leq \alpha\|\mu_n - \mu\| + \beta\|\mu_n - \mathcal{S}\mu_n\| + \gamma\|\mu - \mathcal{S}\mu\|.$$

As $n \to \infty$, we get

$$\|\mu - \mathcal{S}\mu\| \leq \gamma\|\mu - \mathcal{S}\mu\| < \|\mu - \mathcal{S}\mu\|,$$

which implies that $\mu = \mathcal{S}\mu$, i.e., μ is a fixed point of $\mathcal{S}$. □

The next theorem provides sufficient conditions for the existence of AFPS in case S is an ordered Chatterjea nonexpansive mapping.

Theorem 3. *Let $(U, \|.\|, \preceq)$ be a partially ordered Banach space and E be a nonempty closed convex and bounded subset of U. Let $\mathcal{S} : E \to E$ be a monotone and an ordered Chatterjea nonexpansive mapping with coefficients α, β, γ such that $\alpha + \beta + \gamma = 1$ and $\beta < 1$. Assume that there exists $\mu_1 \in E$ such that μ_1 and $\mathcal{S}\mu_1$ are comparable. Assume that for any μ, ω in E with $\mu \preceq \omega$, we have*

$$\frac{(1-\beta)}{7}\|\mu - \mathcal{S}\omega\| \leq \|\mu - \omega\| \Rightarrow \|\mathcal{S}\mu - \mathcal{S}\omega\| \leq \|\mu - \omega\|.$$

Then, $\mathcal{S}$ has an AFPS in E, which is asymptotically regular.

Proof. Fix $\mu_1 \in E$. It is assumed that $\mu_1 \preceq \mathcal{S}\mu_1$. Consider the sequence $\{\mu_n\}$, defined by KIS in E, for all $n \geq 1$ where $\lambda \in [\frac{1}{2}, 1)$.
By our hypothesis, we obtain

$$\|\mathcal{S}\mu - \mathcal{S}\omega\| \leq \alpha\|\mu - \omega\| + \beta\|\mu - \mathcal{S}\omega\| + \gamma\|\omega - \mathcal{S}\mu\|. \tag{21}$$

Putting $\mu = \mu_n$, and $\omega = \mu_{n+1}$ in (21), we get

$$\begin{aligned} \|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| &\leq \alpha\|\mu_n - \mu_{n+1}\| + \beta\|\mu_n - \mathcal{S}\mu_{n+1}\| + \gamma\|\mu_{n+1} + \mathcal{S}\mu_n\| \\ &\leq \alpha\|\mu_n - \mu_{n+1}\| + \beta\{\|\mu_n - \mu_{n+1}\| + \|\mu_{n+1} - \mathcal{S}\mu_{n+1}\|\} \\ &+ \gamma\{\|\mu_{n+1} - \mu_n\| + \|\mu_n + \mathcal{S}\mu_n\|\} \\ &= (\alpha + \beta + \gamma)\|\mu_n - \mu_{n+1}\| + \beta\|\mu_{n+1} - \mathcal{S}\mu_{n+1}\| + \gamma\|\mu_n - \mathcal{S}\mu_n\| \\ &= \|\mu_n - \mu_{n+1}\| + \beta\|\mu_{n+1} - \mathcal{S}\mu_{n+1}\| + \gamma\|\mu_n - \mathcal{S}\mu_n\| \end{aligned} \tag{22}$$

for all $n \geq 1$. Using (9) and (10) in (22), we obtain

$$\lambda\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \leq \lambda\|\mu_n - \mu_{n+1}\| + \beta\|\mu_{n+1} - \mu_{n+2}\| + \gamma\|\mu_n - \mu_{n+1}\|, \tag{23}$$

where $\mu_n \preceq \mu_{n+1}$.

The definition of KIS and (23) imply that

$$\begin{aligned} \|\mu_{n+1} - \mu_{n+2}\| &\leq \lambda\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| + (1-\lambda)\|\mu_n - \mu_{n+1}\| \\ &\leq \lambda\|x_n - x_{n+1}\| + b\|x_{n+1} - x_{n+2}\| + c\|x_n - x_{n+1}\| \\ &+ (1-\lambda)\|\mu_n - \mu_{n+1}\|, \end{aligned}$$

and

$$\|\mu_{n+1} - \mu_{n+2}\| \leq \frac{(1+\gamma)}{(1-\beta)}\|\mu_n - \mu_{n+1}\|, \tag{24}$$

then

$$\lambda\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \leq \|\mu_{n+1} - \mu_{n+2}\| + (1-\lambda)\|\mu_n - \mu_{n+1}\|. \tag{25}$$

Now using (9), (24) and (25) we get

$$\begin{aligned} \|\mu_n - \mathcal{S}\mu_{n+1}\| &\leq \|\mu_n - \mathcal{S}\mu_n\| + \|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \\ &\leq \frac{1}{\lambda}\|\mu_n - \mu_{n+1}\| + \|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \\ \Rightarrow \lambda\|\mu_n - \mathcal{S}\mu_{n+1}\| &\leq \|\mu_n - \mu_{n+1}\| + \lambda\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| \\ &\leq \|\mu_n - \mu_{n+1}\| + \|\mu_{n+1} - \mu_{n+2}\| + (1-\lambda)\|\mu_n - \mu_{n+1}\| \\ &\leq (2-\lambda)\|\mu_n - \mu_{n+1}\| + \frac{(1+\gamma)}{(1-\beta)}\|\mu_n - \mu_{n+1}\| \\ &< \left(\frac{3}{2} + \frac{2}{1-\beta}\right)\|\mu_n - \mu_{n+1}\|. \end{aligned}$$

This implies that

$$\begin{aligned} \frac{1}{2}\|\mu_n - \mathcal{S}\mu_{n+1}\| &\leq \lambda\|\mu_n - \mathcal{S}\mu_{n+1}\| \\ &< \left(\frac{3}{2} + \frac{2}{1-\beta}\right)\|\mu_n - \mu_{n+1}\|. \end{aligned}$$

Therefore,

$$\frac{(1-\beta)}{7}\|\mu_n - \mathcal{S}\mu_{n+1}\| < \|\mu_n - \mu_{n+1}\|.$$

By our hypothesis, we obtain

$$\|\mathcal{S}\mu_n - \mathcal{S}\mu_{n+1}\| < \|\mu_n - \mu_{n+1}\|.$$

According to Theorem 1, the sequence $\|\mu_n - \mathcal{S}\mu_n\|$ is decreasing for any $n \geq 1$. Set $\lim_{n\to\infty} \|\mu_n - \mathcal{S}\mu_n\| = R$. Since $\xi = (1-\beta)/7 < \lambda < 1$, the inequality is obtained in Lemma 1, which in turn implies that

$$(1+n\xi)\|\mu_i - \mathcal{S}\mu_i\| \leq (1+n\lambda)\|\mu_i - \mathcal{S}\mu_i\| \leq \delta(E) + (1-\lambda)^{-n}[\|\mu_i - \mathcal{S}\mu_i\| - \|\mu_{i+n} - \mathcal{S}\mu_{i+n}\|]$$

for all $i \geq 0$ and $n \geq 1$. If we let $i \to +\infty$, we get

$$(1+n\lambda)R \leq \delta(E).$$

Then,

$$R \leq \frac{\delta(E)}{1+n\lambda}$$

holds for all $n \geq 1$. Clearly this is possible only if $R = 0$, and hence $\lim_{n\to\infty} \|\mu_n - \mathcal{S}\mu_n\| = 0$. Thus the sequence $\{\mu_n\}$ is an AFPS for $\mathcal{S}$ which is asymptotically regular. □

Theorem 4. *Suppose that all the conditions of Theorem 3 are satisfied. Further, assume that, for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\|\mu - \omega\| + \|\mu - \mathcal{S}\omega\| + \|\omega - \mathcal{S}\mu\| < 3\epsilon + \delta \Rightarrow \|\mathcal{S}\mu - \mathcal{S}\omega\| \leq \frac{\epsilon}{2}. \tag{26}$$

Then, $\mathcal{S}$ has a fixed point in E.

Proof. According to Theorem 3, $\mathcal{S}$ has an AFPS $\{\mu_n\}$, such that $\mu_n \preceq \mu_{n+1}$ for all $n \geq 1$, $1/2 \leq \lambda < 1$ and satisfies the following relation:

$$\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = \lambda \lim_{n\to\infty} \|\mathcal{S}\mu_n - \mu_n\| = 0. \tag{27}$$

Suppose that $\{\mu_n\}$ is not a Cauchy sequence. Then, there exists $\epsilon > 0$ such that

$$\limsup_{n,m\to\infty} \|\mu_n - \mu_m\| \geq 2\epsilon \tag{28}$$

for any $n \geq m$ and $n \geq 1$. By hypothesis, there exists $\delta > 0$ such that (26) holds. Without the loss of generality, we take $\delta < \epsilon$. Since $\{\mu_n\}$ is asymptotically regular from (27), there exists $N \in \mathbb{N}$ such that

$$\|\mu_{n+1} - \mu_n\| \leq \|\mathcal{S}\mu_n - \mu_n\| < \frac{\delta}{12},$$

for all $n \geq N$. Pick $m, n > N$ and $m \leq n$, so that $\|\mu_n - \mu_m\| > 2\epsilon$. For i in $[m, n]$, there is

$$|\|\mu_m - \mu_i\| - \|\mu_m - \mu_{i+1}\|| \leq \|\mu_i - \mu_{i+1}\| < \frac{\delta}{12}.$$

Since $\|\mu_m - \mu_{m+1}\| < \epsilon$, $\|\mu_n - \mu_m\| > \epsilon + \delta$, there exists $i \in [m, n]$. This implies that

$$\epsilon + \frac{\delta}{6} \leq \|\mu_m - \mu_i\| \leq \epsilon + \frac{\delta}{4}. \tag{29}$$

We obtain

$$\begin{aligned}
&\|\mu_m - \mu_i\| + \|\mu_m - \mathcal{S}\mu_i\| + \|\mu_i - \mathcal{S}\mu_m\| \\
&\leq \|\mu_m - \mu_i\| + \|\mu_m - \mu_i\| + \|\mu_i - \mathcal{S}\mu_i\| + \|\mu_i - \mu_m\| + \|\mu_m - \mathcal{S}\mu_m\| \\
&< 3\left(\epsilon + \frac{\delta}{4}\right) + 2\frac{\delta}{12} = 3\epsilon + \frac{11\delta}{12} < 3\epsilon + \delta.
\end{aligned}$$

Hence, we get

$$\|\mathcal{S}\mu_m - \mathcal{S}\mu_i\| \leq \epsilon.$$

It follows that

$$\begin{aligned}\|\mu_m - \mu_i\| &\leq \|\mu_m - \mathcal{S}\mu_m\| + \|\mathcal{S}\mu_m - \mathcal{S}\mu_i\| + \|\mathcal{S}\mu_i - \mu_i\| \\ &< \epsilon + 2\frac{\delta}{12} = \epsilon + \frac{\delta}{6},\end{aligned}$$

which is a contradiction. Thus $\{\mu_n\}$ is a Cauchy sequence and hence is convergent to some $\mu \in E$. Clearly, $\mathcal{S}\mu_n \to \mu$ as $n \to \infty$.

Since $\{\mu_n\}$ is increasing in bounded set E, then $\mu_n \preceq \mu$ for all $n \geq 1$, where $\mu \in E$. Consider

$$\begin{aligned}\|\mu - \mathcal{S}\mu\| &\leq \|\mu - \mu_n\| + \|\mu_n - \mathcal{S}\mu_n\| + \|\mathcal{S}\mu_n - \mathcal{S}\mu\| \\ &\leq \|\mu - \mu_n\| + \|\mu_n - \mathcal{S}\mu_n\| + \alpha\|\mu_n - \mu\| + \beta\|\mu_n - \mathcal{S}\mu\| + \gamma\|\mu - \mathcal{S}\mu_n\|.\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$\|\mu - \mathcal{S}\mu\| \leq \beta\|\mu - \mathcal{S}\mu\|,$$

which gives $\|\mu - \mathcal{S}\mu\| = 0$; thus, μ is a fixed point of $\mathcal{S}$. □

Example 1. *Consider the real sequence space $U = \ell^p$; let $1 < p < \infty$ be equipped with the usual ordering and standard norm*

$$\|\mu\|_p = (\sum_{i=1}^{\infty} |\mu_i|^p)^{\frac{1}{p}}.$$

Define $A = \{\mu = (\mu_1, \mu_2, \mu_3, ...) \in \ell^p : \mu_1 \leq 1, \mu_j = 0 \,\forall\, j \neq 1\}$. A is a closed, bounded and convex subset of U. Consider that the usual order $\preceq$ on U and $\mathcal{S} : A \to A$ is defined by

$$\mathcal{S}(\mu) = \begin{cases} (1 - \frac{\mu_1}{2}, 0, 0, ...) & \text{if } 0 \leq \mu_1 < \frac{1}{4}, \\ (\frac{\mu_1}{4} + \frac{1}{2}, 0, 0, ...) & \text{if } \frac{1}{4} \leq \mu_1 \leq 1. \end{cases}$$

Consider $\mu = (1/8, 0, 0, ...)$, $\omega = (1/4, 0, 0, ...)$; then, $\mathcal{S}(\mu) = (15/16, 0, ...)$ and $\mathcal{S}(\omega) = (9/16, 0, 0, ...)$, and we obtain

$$\begin{aligned}||\mathcal{S}(\mu) - \mathcal{S}(\omega)||_p &= \left|\frac{15}{16} - \frac{9}{16}\right| = \frac{6}{8} \\ &\geq \frac{1}{8} = \left|\frac{1}{8} - \frac{1}{4}\right| = \|\mu - \omega\|_p,\end{aligned}$$

which implies that $\mathcal{S}$ is not a nonexpansive mapping.

Now, we will show that $\mathcal{S}$ is monotone and an ordered Reich nonexpansive mapping. Choosing $\alpha = \beta = \gamma = \frac{1}{3}$, we consider different cases as follows:

(i) Let $0 \leq \mu_1 \leq \omega_1 < \frac{1}{4}$; we have

$$\begin{aligned}\alpha\|\mu-\omega\|_p+\beta\|\mu-\mathcal{S}(\mu)\|_p+\gamma\|\omega-\mathcal{S}(\omega)\|_p &= \frac{1}{3}|\mu_1-\omega_1|+\frac{1}{3}\left(\left|\mu_1-1+\frac{\mu_1}{2}\right|+\left|\omega_1-1+\frac{\omega_1}{2}\right|\right)\\ &= \frac{1}{3}|\mu_1-\omega_1|+\frac{1}{3}\left(\left|\frac{3\mu_1}{2}-1\right|+\left|\frac{3\omega_1}{2}-1\right|\right)\\ &\geq \frac{1}{3}|\mu_1-\omega_1|+\frac{1}{3}\left|\left(\frac{3\mu_1}{2}-1\right)-\left(\frac{3\omega_1}{2}-1\right)\right|\\ &= \frac{1}{3}|\mu_1-\omega_1|+\frac{1}{2}|\mu_1-\omega_1|\\ &\geq \frac{1}{2}|\mu_1-\omega_1| = \|\mathcal{S}\mu-\mathcal{S}\omega\|_p.\end{aligned}$$

(ii) Let $\frac{1}{4} \leq \mu_1 \leq \omega_1 \leq 1$; we get

$$\begin{aligned}\alpha\|\mu-\omega\|_p+\beta\|\mu-\mathcal{S}(\mu)\|_p+\gamma\|\omega-\mathcal{S}(\omega)\|_p &= \frac{1}{3}|\mu_1-\omega_1|+\frac{1}{3}\left(\left|\mu_1-\frac{\mu_1}{4}-\frac{1}{2}\right|+\left|\omega_1-\frac{\omega_1}{4}-\frac{1}{2}\right|\right)\\ &= \frac{1}{3}|\mu_1-\omega_1|+\frac{1}{3}\left(\left|\frac{3\mu_1}{4}-\frac{1}{2}\right|+\left|\frac{3\omega_1}{4}-\frac{1}{2}\right|\right)\\ &\geq \frac{1}{3}|\mu_1-\omega_1|+\frac{1}{3}\left|\left(\frac{3\mu_1}{4}-\frac{1}{2}\right)-\left(\frac{3\omega_1}{4}-\frac{1}{2}\right)\right|\\ &\geq \frac{7}{12}|\mu_1-\omega_1| \geq \frac{3}{12}|\mu_1-\omega_1| = \|\mathcal{S}\mu-\mathcal{S}\omega\|_p.\end{aligned}$$

(iii) Let $0 \leq \mu_1 < \frac{1}{4}$, and $\frac{1}{4} \leq \omega_1 \leq 1$; we obtain that

$$\begin{aligned}\alpha\|\mu-\omega\|_p+\beta\|\mu-\mathcal{S}(\mu)\|_p+\gamma\|\omega-\mathcal{S}(\omega)\|_p &= \frac{1}{3}|\mu_1-\omega_1|+\frac{1}{3}\left(\left|\frac{3\mu_1}{2}-1\right|+\left|\frac{3\omega_1}{4}-\frac{1}{2}\right|\right)\\ &\geq \frac{1}{3}|\mu_1-y_1|+\frac{1}{3}\left|\left(\frac{3\mu_1}{2}-1\right)+\left(\frac{3\omega_1}{4}-\frac{1}{2}\right)\right|\\ &= \frac{1}{3}|\mu_1-\omega_1|+\frac{1}{4}|2-2\mu_1-\omega_1|\\ &\geq \frac{1}{4}|2-2\mu_1-\omega_1| = \|\mathcal{S}\mu-\mathcal{S}\omega\|_p.\end{aligned}$$

Hence $\mathcal{S}$ is an ordered Rich nonexpansive mapping with coefficients $(1/3, 1/3, 1/3)$. The required AFPS is $\mu_n = (2n/(3n+1), 0, 0, ...)$ for all $n \in \mathbb{N}$ with $\mu = (2/3, 0, 0, ...)$ as a fixed point of $\mathcal{S}$. The mapping $\mathcal{S}$ satisfies the condition (8) of Theorem 1 for all $\mu, \omega \in A_1 \cup A_2$ such that $A_1 = \{\mu = (\mu_1, 0, 0, ...), \omega = (\omega_1, 0, 0, ...) : \mu_1 \in [0, 1/5], \omega_1 \in [1/4, 1]\}$ and $A_2 = \{\mu = (\mu_1, 0, 0, ...), \omega = (\omega_1, 0, 0, ...) : \mu_1 \in [0, 1/4), \omega_1 \in [3/10, 1]\}$.

Example 2. *Let $U = \mathbb{R}^2$, and $A = \{\mu = (\mu_1, \mu_2) : (\mu_1, \mu_2) \in [0, 1] \times [0, 1]\}$ be a subset of U with the norm $\|\mu\| = \|(\mu_1, \mu_2)\| = |\mu_1| + |\mu_2|$. Define the mapping $\mathcal{S} : A \to A$ by*

$$\mathcal{S}(\mu_1,\mu_2)=\begin{cases}(1-\mu_1, 1-\mu_2), & \text{if } (\mu_1,\mu_2)\in[0,\frac{1}{2}]\times[0,1],\\ \left(\frac{1+\mu_1}{3},\frac{1+\mu_2}{3}\right), & \text{if } (\mu_1,\mu_2)\in(\frac{1}{2},1]\times[0,1].\end{cases}$$

For $\mu = (0, 0)$ and $\omega = (0.51, 0.25)$, we have $\mathcal{S}(\mu) = (1, 1)$, $\mathcal{S}(\omega) = (151/300, 5/12)$,

$$\|\mathcal{S}(\mu)-\mathcal{S}(\omega)\| = 1.08 > 0.76 = \|\mu-\omega\|.$$

Therefore, $\mathcal{S}$ is not a nonexpansive mapping.

Now, we will show that $\mathcal{S}$ is an ordered Reich nonexpansive mapping. Choosing $\alpha = \beta = \gamma = 1/3$, we consider different cases as follows:

(i) Let $\mu = (\mu_1, \mu_2), \omega = (\omega_1, \omega_2) \in [0, \frac{1}{2}] \times [0, 1]$; we then have

$$\begin{aligned}\|\mathcal{S}(\mu) - \mathcal{S}(\omega)\| &= \|(1-\mu_1, 1-\mu_2) - (1-\omega_1, 1-\omega_2)\| \\ &= \|(\omega_1 - \mu_1, \omega_2 - \mu_2)\| = |\omega_1 - \mu_1| + |\omega_2 - \mu_2| \\ &= |\mu_1 - \omega_1| + |\mu_2 - \omega_2| = \|\mu - \omega\|\end{aligned}$$

and

$$\begin{aligned}\alpha\|\mu - \omega\| + \beta\|\mu - \mathcal{S}(\mu)\| + \gamma\|\omega - \mathcal{S}(\omega)\| &= \frac{1}{3}\|\mu - \omega\| + \frac{1}{3}[(|2\mu_1 - 1| \\ &+ |2\mu_2 - 1|)] + \frac{1}{3}[(|2\omega_1 - 1| + |2\omega_2 - 1|)] \\ &\geq \frac{1}{3}\|\mu - \omega\| + \frac{1}{3}[(|2\mu_1 - 1| + |1 - 2\omega_1|)] \\ &+ \frac{1}{3}[(|2\mu_2 - 1| + |1 - 2\omega_2|)] \\ &\geq \frac{1}{3}\|\mu - \omega\| + \frac{2}{3}(|\mu_1 - \omega_1| + |\mu_2 - \omega_2|) \\ &= \|\mu - \omega\| = \|\mathcal{S}\mu - \mathcal{S}\omega\|.\end{aligned}$$

(ii) Let $\mu = (\mu_1, \mu_2), \omega = (\omega_1, \omega_2) \in (\frac{1}{2}, 1] \times [0, 1]$; we then get

$$\begin{aligned}\|\mathcal{S}(\mu) - \mathcal{S}(\omega)\| &= \|(\frac{1+\mu_1}{3}, \frac{1+\mu_2}{3}) - (\frac{1+\omega_1}{3}, \frac{1+\omega_2}{3})\| \\ &= \frac{1}{3}\|(\mu_1 - \omega_1, \mu_2 - \omega_2)\| \\ &= \frac{1}{3}(|\mu_1 - \omega_1| + |\mu_2 - \omega_2|) = \frac{1}{3}\|\mu - \omega\|.\end{aligned}$$

In addition,

$$\begin{aligned}\alpha\|\mu - \omega\| + \beta\|\mu - \mathcal{S}(\mu)\| + \gamma\|\omega - \mathcal{S}(\omega)\| &= \frac{1}{3}\|\mu - \omega\| + \frac{1}{3}(|2\mu_1 - 1| + |2\mu_2 - 1|) \\ &+ \frac{1}{3}(|2\omega_1 - 1| + |2\omega_2 - 1|) \\ &\geq \frac{1}{3}\|\mu - \omega\| + \frac{1}{3}(|2\mu_1 - 1| + |1 - 2\omega_1|) \\ &+ \frac{1}{3}(|2\mu_2 - 1| + |1 - 2\omega_2|) \\ &\geq \frac{1}{3}\|\mu - \omega\| + \frac{4}{3}(|\mu_1 - \omega_1| + |\mu_2 - \omega_2|) \\ &= \frac{5}{3}\|\mu - \omega\| = 5\|\mathcal{S}\mu - \mathcal{S}\omega\| \geq \|\mathcal{S}\mu - \mathcal{S}\omega\|.\end{aligned}$$

(iii) Let $\mu = (\mu_1, \mu_2) \in [0, \frac{1}{2}] \times [0, 1]$, and $\omega = (\omega_1, \omega_2) \in (\frac{1}{2}, 1] \times [0, 1]$; then, we have

$$\begin{aligned}\|\mathcal{S}(\mu)-\mathcal{S}(\omega)\| &= \|(1-\mu_1,1-\mu_2)-(\frac{1+\omega_1}{3},\frac{1+\omega_2}{3})\| \\ &= \frac{1}{3}\|(2-3\mu_1-\omega_1,2-3\mu_1-\omega_2)\| \\ &= \frac{1}{3}[|\mu_1-\omega_1|+|\mu_2-\omega_2|+2|1-2\mu_1|+2|1-2\mu_2|] \\ &= \frac{1}{3}\|\mu-\omega\|+\frac{2}{3}[|1-2\mu_1|+|1-2\mu_2|] \leq \|\mu-\omega\|+2,\end{aligned}$$

and

$$\begin{aligned}\alpha\|\mu-\omega\|+\beta\|\mu-\mathcal{S}(\mu)\|+\gamma\|\omega-\mathcal{S}(\omega)\| &= \frac{1}{3}\|\mu-\omega\|+\frac{1}{3}(|2\mu_1-1|+|2\mu_2-1|) \\ &+\frac{1}{3}(|2\omega_1-1|+|2\omega_2-1|) \\ &\geq \frac{1}{3}\|\mu-\omega\|+\frac{2}{3}(|\mu_1-\omega_1|+|\mu_2-\omega_2|) \\ &\geq \|\mu-\omega\|+2 \geq \|\mathcal{S}\mu-\mathcal{S}\omega\|.\end{aligned}$$

Then, $\mathcal{S}$ is monotone and an ordered Reich nonexpansive mapping with coefficients $(1/3,1/3,1/3)$. The AFPS is (μ_n) where $\mu_n = ((n+1)/2n,(n+1)/2n)$ for all $n \in \mathbb{N}$. Furthermore, $\mathcal{S}$ has a fixed point $\mu = (0.5, 0.5)$. Since $\alpha < 1$, the mapping $\mathcal{S}$ satisfies the condition (8) of Theorem 1 for all $\mu \prec \omega$ in A_1, where $A_1 = \{\mu = (\mu_1,\mu_2), \omega = (\omega_1,\omega_2) : (\mu_1,\mu_2) \in [0,\frac{1}{2}] \times \{1\}; (\omega_1,\omega_2) \in (\frac{1}{2},1] \times [0,1]\}$.

4. Convergence Results

In this section we establish certain strong and weak convergence results in a partially ordered Banach space.

Definition 3. *Let $(U,\|.\|,\preceq)$ be a partially ordered Banach space.*

(i) According to [37], a space U satisfies the weak-Opial property if for any sequence $\{\mu_n\}$ in U which converges weakly to ν; thus, we have

$$\liminf_{n\to\infty}\|\mu_n-\nu\| < \liminf_{n\to\infty}\|\mu_n-\omega\|$$

for all $\omega \neq \nu$ in U;

(ii) According to [17], a space U satisfies the monotone weak-Opial property if for any monotone sequence $\{\mu_n\}$ in U which converges weakly to ν; thus, we have

$$\liminf_{n\to\infty}\|\mu_n-\nu\| < \liminf_{n\to\infty}\|\mu_n-\omega\|$$

for all $\omega \neq \nu$ in U, and ω is greater or less than all the elements of the sequence $\{\mu_n\}$.

Proposition 2. *Let $(U,\|.\|,\preceq)$ be a partially ordered Banach space and E be a nonempty closed and convex subset of U. Let $\mathcal{S}: E \to E$ be a monotone and an ordered Reich nonexpansive mapping satisfying the conditions of Theorem 2. Then, $\mathcal{S}$ is quasi-nonexpansive, provided $\gamma \leq \beta < 1$.*

Proof. By Theorem 2, $\mathcal{F}(\mathcal{S})$ is nonempty. Suppose $\nu \in \mathcal{F}(\mathcal{S})$ and $\mu \in E$ are such that $\mu \preceq \nu$:

$$\begin{aligned}\|\mathcal{S}\nu-\mathcal{S}\mu\| = \|\nu-\mathcal{S}\mu\| &\leq \alpha\|\nu-\mu\|+\beta\|\nu-\mathcal{S}\nu\|+\gamma\|\mu-\mathcal{S}\mu\| \\ &\leq \alpha\|\nu-\mu\|+\gamma\|\mu-\nu\|+\gamma\|\nu-\mathcal{S}\mu\| \\ \Rightarrow (1-\gamma)\|\nu-\mathcal{S}\mu\| &\leq (\alpha+\gamma)\|\mu-\nu\|,\end{aligned}$$

which implies that

$$\|\nu - \mathcal{S}\mu\| \leq \frac{\alpha + \gamma}{1 - \gamma}\|\nu - \mu\|.$$

Since $1 - \gamma > 0$, and $\frac{\alpha+\gamma}{1-\gamma} = \frac{1-\beta}{1-\gamma} \leq 1$, we obtain

$$\|\nu - \mathcal{S}\mu\| \leq \|\nu - \mu\|$$

as required. □

Lemma 2. *Let E be a nonempty subset of an ordered Banach space $(U, \|.\|, \preceq)$ and $\mathcal{S} : E \to E$ be a monotone and an ordered Reich nonexpansive mapping satisfying the conditions of Theorem 1. If for each $\mu, \omega \in E$ with $\mu \preceq \omega$, and $\gamma \leq \beta < 1$, then the following estimates hold true:*

(i) $\|\mathcal{S}\mu - \mathcal{S}^2\mu\| \leq \|\mu - \mathcal{S}\mu\|$;

(ii) *At least one of the following ((a) and (b)) holds:*

- *(a)* $\frac{1-\gamma}{6}\|\mu - \mathcal{S}\mu\| \leq \|\mu - \omega\|$;
- *(b)* $\frac{1-\gamma}{6}\|\mathcal{S}^2\mu - \mathcal{S}\mu\| \leq \|\mathcal{S}\mu - \omega\|$;

The condition (a) implies $\|\mathcal{S}\mu - \mathcal{S}\omega\| \leq \|\mu - \omega\|$,
and the condition (b) implies $\|\mathcal{S}^2\mu - \mathcal{S}\omega\| \leq \|\mathcal{S}\mu - \omega\|$;

(iii) $\|\mu - \mathcal{S}\omega\| \leq \frac{2+\alpha+\beta}{1-\gamma}\|\mu - \mathcal{S}\mu\| + \frac{1-\beta}{1-\gamma}\|\mu - \omega\|$.

Proof. (i) By the definition of an ordered Reich nonexpansive mapping, we obtain

$$\begin{aligned}
\|\mathcal{S}\mu - \mathcal{S}^2\mu\| &\leq \alpha\|\mu - \mathcal{S}\mu\| + \beta\|\mu - \mathcal{S}\mu\| + \gamma\|\mathcal{S}\mu - \mathcal{S}^2\mu\| \\
\Rightarrow (1 - \gamma)\|\mathcal{S}\mu - \mathcal{S}^2\mu\| &\leq (\alpha + \beta)\|\mu - S\mu\|,
\end{aligned}$$

which implies

$$\|\mathcal{S}\mu - \mathcal{S}^2\mu\| \leq \frac{1 - \gamma}{1 - \gamma}\|\mu - \mathcal{S}\mu\| = \|\mu - \mathcal{S}\mu\|;$$

(ii) Assume on the contrary that $(1 - \gamma)/6\|\mu - \mathcal{S}\mu\| > \|\mu - \omega\|$ and $(1 - \gamma)/6\|\mathcal{S}\mu - \mathcal{S}^2\mu\| > \|\mathcal{S}\mu - \omega\|$. Then, by triangle inequality with the assumption (i), we find that

$$\begin{aligned}
\|\mu - \mathcal{S}\mu\| &\leq \|\mu - \omega\| + \|\omega - \mathcal{S}\mu\| \\
&< \frac{1 - \gamma}{6}\|\mu - \mathcal{S}\mu\| + \frac{1 - \gamma}{6}\|\mathcal{S}\mu - \mathcal{S}^2\mu\|, \\
\Rightarrow \|\mu - \mathcal{S}\mu\| &< \frac{1 - \gamma}{3}\|\mu - \mathcal{S}\mu\|.
\end{aligned}$$

Since $(1 - \gamma)/3 < 1$, we obtain $\|\mu - \mathcal{S}\mu\| < \|\mu - \mathcal{S}\mu\|$, which is a contradiction. Therefore, at least one of (a) and (b) holds. Hence, (ii) holds;

(iii) For the first case, by our hypotheses, we have

$$\begin{aligned}
\|\mu - \mathcal{S}\omega\| &\leq \|\mu - \mathcal{S}\mu\| + \|\mathcal{S}\mu - \mathcal{S}\omega\| \\
&\leq \|\mu - \mathcal{S}\mu\| + \alpha\|\mu - \omega\| + \beta\|\mu - \mathcal{S}\mu\| + \gamma\|\omega - \mathcal{S}\omega\| \\
&\leq (1 + \beta)\|\mu - \mathcal{S}\mu\| + (\alpha + \gamma)\|\mu - \omega\| + \gamma\|\mu - \mathcal{S}\omega\| \\
\Rightarrow (1 - \gamma)\|\mu - \mathcal{S}\omega\| &\leq (1 + \beta)\|\mu - \mathcal{S}\mu\| + (\alpha + \gamma)\|\mu - \omega\|.
\end{aligned}$$

Since $\alpha + \beta + \gamma = 1$, and $1 - \gamma > 0$, then

$$\|\mu - \mathcal{S}\omega\| \leq \frac{1 + \beta}{1 - \gamma}\|\mu - \mathcal{S}\mu\| + \frac{1 - \beta}{1 - \gamma}\|\mu - \omega\|.$$

For the second case, using (i), we get

$$\begin{aligned}
\|\mu - \mathcal{S}\omega\| &\leq \|\mu - \mathcal{S}\mu\| + \|\mathcal{S}\mu - \mathcal{S}^2\mu\| + \|\mathcal{S}^2\mu - \mathcal{S}\omega\| \\
&\leq \|\mu - \mathcal{S}\mu\| + \|\mu - \mathcal{S}\mu\| + \alpha\|\mathcal{S}\mu - \omega\| + \beta\|\mathcal{S}\mu - \mathcal{S}^2\mu\| + \gamma\|\omega - \mathcal{S}\omega\| \\
&\leq (2+\alpha)\|\mu - \mathcal{S}\mu\| + \alpha\|\mu - \omega\| + \beta\|\mu - \mathcal{S}\mu\| + \gamma\|\mu - \omega\| + \gamma\|\mu - \mathcal{S}\omega\| \\
\Rightarrow (1-\gamma)\|\mu - \mathcal{S}\omega\| &\leq (2+\alpha+\beta)\|\mu - \mathcal{S}\mu\| + (\alpha+\gamma)\|\mu - \omega\|,
\end{aligned}$$

where $1-\gamma > 0$; thus, we have

$$\|\mu - \mathcal{S}\omega\| \leq \frac{2+\alpha+\beta}{1-\gamma}\|\mu - \mathcal{S}\mu\| + \frac{1-\beta}{1-\gamma}\|\mu - \omega\|,$$

where $\gamma \leq \beta < 1$, and $\alpha + \beta + \gamma = 1$. □

As an application of the above Lemma, we obtain the following result:

Theorem 5. *Let E be a nonempty convex and compact subset of an ordered Banach space $(\mathcal{U}, \|.\|, \preceq)$ and $\mathcal{S} : E \to E$ be a monotone and an ordered Reich nonexpansive mapping satisfying the conditions of Theorem 2. Suppose that $\{\mu_n\}$ is the sequence defined by KIS. Then, $\{\mu_n\}$ converges strongly to a fixed point of $\mathcal{S}$, provided $\gamma \leq \beta < 1$.*

Proof. From Theorem 1, we have

$$\lim_{n\to\infty} \|\mu_n - \mathcal{S}\mu_n\| = 0.$$

Since E is compact, there exists a subsequence $\{\mu_{n_j}\}$ of $\{\mu_n\}$ such that $\{\mu_{n_j}\} \to \nu$ as $j \to \infty$ for $\nu \in E$. and $\nu \in E$. By Lemma 2, we obtain

$$\|\mu_{n_j} - \mathcal{S}\nu\| \leq \frac{2+\alpha+\beta}{1-\gamma}\|\mu_{n_j} - \mathcal{S}\mu_{n_j}\| + \frac{1-\beta}{1-\gamma}\|\mu_{n_j} - \nu\|$$

for all $j \in \mathbb{N}$. Taking the limit as $j \to \infty$, we obtain that $\{\mu_{n_j}\}$ converges to $\mathcal{S}\nu$, which implies $\nu = \mathcal{S}\nu$. By Proposition 2, and the sequence $\{\mu_n\}$, which is defined by KIS, we infer that

$$\|\mu_{n+1} - \nu\| \leq \lambda\|\mathcal{S}\mu_n - \nu\| + (1-\lambda)\|\mu_n - \nu\| \leq \|\mu_n - \nu\|$$

for all $n \geq 1$. $\lim_{n\to\infty} \|x_n - \nu\|$ exists for every $\nu \in \mathcal{F}(\mathcal{S})$, so $\{\mu_n\}$ converges strongly to a fixed point of $\mathcal{S}$. □

Proposition 3. *Let $(\mathcal{U}, \|.\|, \preceq)$ be a partially ordered Banach space and E a nonempty convex, closed and bounded subset of $\mathcal{U}$ with a monotone weak-Opial property. Let $\mathcal{S} : E \to E$ be a monotone Reich nonexpansive mapping satisfying the conditions of Theorem 2. If $\{\mu_n\}$ converges weakly to ν, then, $\mathcal{S}\nu = \nu$, provided $\gamma \leq \beta < 1$.*

Proof. From Theorem 1, we obtain

$$\lim_{n\to\infty} \|\mu_n - \mathcal{S}\mu_n\| = 0. \tag{30}$$

Lemma 2 verifies that

$$\|\mu_n - \mathcal{S}\nu\| \leq \frac{2+\alpha+\beta}{1-\gamma}\|\mu_n - \mathcal{S}\mu_n\| + \frac{1-\beta}{1-\gamma}\|\mu_n - \nu\|$$

for all $n \in \mathbb{N}$. As $\frac{1-\beta}{1-\gamma} < 1$, so

$$\liminf_{n\to\infty} \|\mu_n - \mathcal{S}\nu\| \leq \liminf_{n\to\infty} \|\mu_n - \nu\|.$$

We show that $\mathcal{S}\nu = \nu$. Suppose that $\mathcal{S}\nu \neq \nu$; by the monotone weak-Opial property, we obtain

$$\liminf_{n\to\infty} \|\mu_n - \nu\| < \liminf_{n\to\infty} \|\mu_n - \mathcal{S}\nu\|,$$

which is a contradiction. Thus, $\mathcal{S}\nu = \nu$. □

Theorem 6. *Let E be a nonempty convex and weakly compact subset of a partially ordered Banach space $(U, \|.\|, \preceq)$ with a monotone weak-Opial property. Let $\mathcal{S} : E \to E$ be a monotone and an ordered Reich nonexpansive mapping satisfying the condition of Theorem 2. Assume that there exists $\mu_1 \in E$ such that $\mu_1 \preceq \mathcal{S}\mu_1$ and $\mathcal{F}(\mathcal{S})$ is nonempty. Then, the sequence $\{\mu_n\}$ defined by KIS converges weakly to a fixed point of $\mathcal{S}$.*

Proof. By Theorem 1, we have

$$\lim_{n\to\infty} \|\mu_n - \mathcal{S}\mu_n\| = 0.$$

Since E is weakly compact, there exists a subsequence $\{\mu_{n_j}\}$ of $\{\mu_n\}$ and $\nu \in E$ such that $\{\mu_{n_j}\}$ converges weakly to ν. From Proposition 3, we infer that ν is a fixed point of $\mathcal{S}$. As in the proof of Theorem 1, we prove that $\{\|\mu_n - \nu\|\}$ is a decreasing sequence. Now, suppose that the sequence $\{\mu_n\}$ does not converge to ν. Therefore, there exists a subsequence $\{\mu_{n_i}\}$ of $\{\mu_n\}$ which converges weakly to κ such that $\nu \neq \kappa$ and $\mathcal{S}\kappa = \kappa$. Now from the monotone weak-Opial property,

$$\begin{aligned}\liminf_{n\to\infty} \|\mu_n - \nu\| &= \liminf_{j\to\infty} \|\mu_{n_j} - \nu\| < \liminf_{j\to\infty} \|\mu_{n_j} - \kappa\| = \liminf_{n\to\infty} \|\mu_n - \kappa\| \\ &= \liminf_{i\to\infty} \|\mu_{n_i} - \kappa\| < \liminf_{i\to\infty} \|\mu_{n_i} - \nu\| = \liminf_{n\to\infty} \|\mu_n - \nu\|,\end{aligned}$$

which is a contradiction. Thus, $\nu = \kappa$. Therefore, $\{\mu_n\}$ converges weakly to a fixed point of $\mathcal{S}$. This completes the proof. □

5. Application to Initial-Value Problems

Fixed-point theorems for monotone nonexpansive mappings in ordered metric spaces have been widely investigated and have found various applications in matrix equations and differential and integral equations [14,26,29,30,38–40]. In this section we discuss the existence of solutions of initial-value problems for nonlinear first-order ordinary differential equations.

Consider the differential equation

$$\begin{cases} \mu'(t) = g(t, \mu(t)), & t \in I, \\ \mu(0) = \nu_0. \end{cases} \tag{31}$$

Theorem 7. *Let E be a compact subset of $U = C[0,1]$, with the space of continuous functions on $I = [0,1]$ with partial order $\preceq$ defined by $\mu \preceq \omega$ if, and only if, $\mu(t) \leq \omega(t)$ for all $\mu, \omega \in U$ and $t \in I$. Suppose U is equipped with a supremum norm defined by $\|\mu - \omega\| = \sup_{t\in I} |\mu(t) - \omega(t)|$. Assume $\nu_0 \in \mathbb{R}$, and the following conditions hold:*

(a) $g : I \times E \to E$ *is continuous;*
(b) *For any $t \in I$, the function $g(t, .)$ is nondecreasing;*
(c) *There exists a continuous function $f : I \to \mathbb{R}$ satisfying*

$$\sup_{t\in I}\{f(t) - \int_0^t g(s, f(s))ds\} \leq \nu_0;$$

(d) *There exist non-negative real numbers α, β, γ such that $\alpha + \beta + \gamma = 1$ with $\alpha \leq 1$, and for any continuous functions $\mu(t), \omega(t) : I \to \mathbb{R}$ such that $\mu(t) \preceq \omega(t)$ for all $t \in I$, we have*

$$\int_0^t |g(s,\mu(s)) - g(s,\omega(s))|ds \leq \alpha|\mu(t)-\omega(t)| + \beta|\mu(t) - \nu_0 - \int_0^t g(s,\mu(s))ds| + \gamma|\omega(t) - \nu_0 - \int_0^t g(s,\omega(s))ds|$$

for all $t \in I$;

(e) $\mathcal{S} : E \to E$ *is a mapping defined by*

$$\mathcal{S}\mu(t) = \nu_0 + \int_0^1 g(s,\mu(s))ds, \quad t \in I, \ \lambda \geq 0.$$

Then, the nonlinear differential Equation (31) has a solution in $C[0,1]$, *provided* $\mathcal{S}$ *admits an AFPS.*

Proof. For $\mu, \omega \in E$, such that $\mu \preceq \omega$, we have

$$\begin{aligned}|\mathcal{S}\mu(t) - \mathcal{S}\omega(t)| &= \left|\nu_0 + \int_0^t g(s,\mu(s))ds - \nu_0 - \int_0^t g(s,\omega(s))ds\right| \\ &\leq \int_0^t |g(s,\mu(s)) - g(s,\omega(s))|ds \\ &\leq \alpha|\mu(t)-\omega(t)| + \beta|\mu(t) - \nu_0 - \int_0^t g(s,\mu(s))| \\ &+ \gamma|\omega(t) - \nu_0 - \int_0^t g(s,\omega(s))| \\ &\leq \alpha|\mu(t)-\omega(t)| + \beta|\mu(t) - \mathcal{S}\mu(t)| + \gamma|\omega(t) - \mathcal{S}\omega(t)|.\end{aligned}$$

Taking the supremum norm on both sides, we get

$$\|\mathcal{S}\mu(t) - \mathcal{S}\omega(t)\| \leq \alpha\|\mu - \omega\| + \beta\|\mu - \mathcal{S}\mu\| + \gamma\|\omega - \mathcal{S}\omega\|.$$

Moreover, we show the mapping $\mathcal{S}$ is monotone. Let $\mu, \omega \in E$ with $\mu \preceq \omega$. Then, $\mu(t) \preceq \omega(t)$ for all $t \in I$. By (b), we obtain

$$\begin{aligned}|\mathcal{S}\mu(t) - \mathcal{S}\omega(t)| &= \left|\nu_0 + \int_0^t g(s,\mu(s))ds - \nu_0 - \int_0^t g(s,\omega(s))ds\right| \\ &\leq \int_0^t |g(s,\mu(s)) - g(s,\omega(s))|ds \geq 0, \quad \text{for all } t \in I,\end{aligned}$$

which implies $\mathcal{S}\mu \leq \mathcal{S}\omega$.

Hence, $\mathcal{S}$ is a monotone Reich nonexpansive map. Thus, all of the assumptions of Theorem 1 are satisfied; therefore, (31) has a solution in $E \subseteq C[0,1]$. □

Example 3. *Consider the following functional differential equation:*

$$\begin{cases}\mu'(t) = 3e^{3t} \ \ for \quad t \in [0,1], \\ \mu(0) = 3.\end{cases} \tag{32}$$

The problem has a solution as follows:

$$\mu(t) = e^{3t} + 1, \quad t \in [0,1].$$

Therefore, the continuous function $f : [0,1] \to \mathbb{R}$ *defined by*

$$f(t) = e^{3t}, \quad t \in [0,1]$$

satisfies the condition $\sup_{t\in I}\{f(t) - \int_0^t g(s, f(s))ds\} \leq 3$, *where* $g(t, f(t)) = 3e^{3t}$ *for* $t \in [0, 1]$. *Let* $v_0 = 3$. *It is obvious that the function* $g(t, \mu) = 3e^{3t}$ *satisfies hypotheses (a), (b) and (d) in Theorem 7. Hence, Theorem 7 guarantees the existence of solution of the problem* (32).

6. Conclusions

The concepts of monotone Reich and Chatterjea nonexpansive mappings on partially ordered Banach spaces are introduced and studied. Certain sufficient conditions for the existence of approximate fixed-point sequences have been established to derive some fixed-point results by employing the Krasnoselskii–Ishikawa iteration method. Furthermore, we provided some weak and strong convergence results for such mappings together with an application to nonlinear differential equations. As a future research plan, we suggest to researchers to prove these results for partially ordered metric linear spaces, partially ordered convex metric spaces and partially ordered hyperbolic spaces.

Author Contributions: Conceptualization, N.H. and H.A.; methodology, N.H. and H.A.; validation, N.H. and S.M.A.; investigation, N.H. and H.A.; writing—original draft preparation, N.H. and H.A.; writing—review and editing, N.H. and H.A.; supervision, N.H. and S.M.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are thankful to the referees and the editor for their constructive comments to improve the presentation of the paper.

Conflicts of Interest: The authors declare no conflict of interests.

References

1. Browder, F.E. Nonexpansive nonlinear operators in a Banach space. *Proc. Natl. Acad. Sci. USA* **1965**, *54*, 1041. [CrossRef]
2. Debnath, P.; Konwar, N.; Radenovic, S. *Metric Fixed Point Theory: Applications in Science, Engineering and Behavioural Sciences*, 1st ed.; Springer: Berlin/Heidelberg, Germany, 2021.
3. Goebel, K.; Kirk, W.A. A fixed point theorem for asymptotically nonexpansive mappings. *Proc. Am. Math. Soc.* **1972**, *35*, 171–174. [CrossRef]
4. Debnath, P.; de La Sen, M. Contractive inequalities for some asymptotically regular set-valued mappings and their fixed points. *Symmetry* **2020**, *12*, 411. [CrossRef]
5. Debnath, P.; Mitrovic, Z.D.; Srivastava, H.M. Fixed points of some asymptotically regular multivalued mappings satisfying a Kannan-Type condition. *Axioms* **2021**, *10*, 24. [CrossRef]
6. Chidume, C.E.; Zegeye, H. Approximate fixed point sequences and convergence theorems for asymptotically pseudocontractive mappings. *J. Math. Anal. Appl.* **2003**, *278*, 354–366. [CrossRef]
7. Hussain, N.; Khan, A.R.; Agarwal, R.P. Krasnosel'skii and Ky Fan type fixed point theorems in ordered Banach spaces. *J. Nonlinear Convex Anal.* **2010**, *11*, 475–489.
8. Lim, T.; Xu, H. Fixed point theorems for asymptotically nonexpansive mappings. *Nonlinear Anal. Theory Methods Appl.* **1994**, *22*, 1345–1355. [CrossRef]
9. Schu, J. Approximation of fixed points of asymptotically nonexpansive mappings. *Proc. Am. Math. Soc.* **1991**, *112*, 143–151. [CrossRef]
10. Matoušková, E.; Reich, S. Reflexivity and approximate fixed points. *Stud. Math.* **2003**, *159*, 403–415. [CrossRef]
11. Berinde, V. *Iterative Approximation of Fixed Points*; Springer: Berlin/Heidelberg, Germany, 2007; Volume 1912.
12. Dehaish, B.A.B.; Khamsi, M.A. Browder and Göhde fixed point theorem for monotone nonexpansive mappings. *Fixed Point Theory Appl.* **2016**, *2016*, 20. [CrossRef]
13. Goebel, K.; Simeon, R. *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*; Marcel Dekker Inc.: New York, NY, USA, 1984.
14. Hussain, N.; Taoudi, M.A. Krasnosel'skii-type fixed point theorems with applications to Volterra integral equations. *Fixed Point Theory Appl.* **2013**, *2013*, 196. [CrossRef]
15. Yang, H.; Agarwal, R.P.; Nashine, H.K.; Liang, Y. Fixed point theorems in partially ordered Banach spaces with applications to nonlinear fractional evolution equations. *J. Fixed Point Theory Appl.* **2017**, *19*, 1661–1678. [CrossRef]

16. Som, S.; Petrusel, A.; Garai, H.; Dey, L.K. Some characterizations of Reich and Chatterjea type nonexpansive mappings. *J. Fixed Point Theory Appl.* **2019**, *21*, 1–21. [CrossRef]
17. Alfuraidan, M.; Khamsi, M.A. A fixed point theorem for monotone asymptotically nonexpansive mappings. *Proc. Am. Math. Soc.* **2018**, *146*, 2451–2456. [CrossRef]
18. Chatterjea, S.K. Fixed-point theorems. *Dokl. Bolg. Akad. Nauk.* **1972,** *25,* 727. [CrossRef]
19. Reich, S. Some remarks concerning contraction mappings. *Can. Math. Bull.* **1971**, *14*, 121–124 [CrossRef]
20. Reich, S. Fixed points of contractive functions. *Boll. Unione Mat. Ital.* **1972**, *5*, 26–42.
21. Popescu, O.; Stan, G. Some remarks on Reich and Chatterjea type nonexpansive mappings. *Mathematics* **2020**, *8*, 1270. [CrossRef]
22. Debnath, P.; Neog, M.; Radenović, S. Set valued Reich type G-contractions in a complete metric space with graph. *Rend. Circ. Mat. Palermo Ser. 2* **2020**, *69*, 917–924. [CrossRef]
23. Khamsi, M.A.; Khan, A.R. On monotone nonexpansive mappings in $L_1([0,1])$. *Fixed Point Theory Appl.* **2015**, *2015*, 94. [CrossRef]
24. Dehaish, B.A.B.; Khamsi, M.A. Mann iteration process for monotone nonexpansive mappings. *Fixed Point Theory Appl.* **2015**, *2015*, 177. [CrossRef]
25. Beg, I.; Butt, A.R. Fixed point for set-valued mappings satisfying an implicit relation in partially ordered metric spaces. *Nonlinear Anal. Theory Methods Appl.* **2009**, *71*, 3699–3704. [CrossRef]
26. Guran, L.; Mitrović, Z.D.; Reddy, G.; Belhenniche, A.; Radenović, S. Applications of a fixed point result for solving nonlinear fractional and integral differential equations. *Fractal Fract.* **2021**, *5*, 211. [CrossRef]
27. Harjani, J.; Sadarangani, K. Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations. *Nonlinear Anal. Theory Methods Appl.* **2010**, *72*, 1188–1197. [CrossRef]
28. Li, J. Several fixed point theorems on partially ordered Banach spaces and their applications to integral equations. *Fixed Point Theory* **2020**, *21*, 259–270. [CrossRef]
29. Nieto, J.J.; RodrÍguez-López, R. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order* **2005**, *22*, 223–239. [CrossRef]
30. Ran, A.C.; Reurings, M.C. A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Am. Math. Soc.* **2004**, *132*, 1435–1443. [CrossRef]
31. Samet, B.; Vetro, C. Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces. *Nonlinear Anal. Theory Methods Appl.* **2011**, *74*, 4260–4268. [CrossRef]
32. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorems for α-ψ-contractive type mappings. *Nonlinear Anal.* **2012**, *75*, 2154–2165. [CrossRef]
33. Younis, M.; Singh, D.; Abdou, A.A. A fixed point approach for tuning circuit problem in dislocated b-metric spaces. *Math. Methods Appl. Sci.* **2021**, 11–21. [CrossRef]
34. Zhang, C.; Wang, Y. Applications of order-theoretic fixed point theorems to discontinuous quasi-equilibrium problems. *Fixed Point Theory Appl.* **2015**, *2015*, 54. [CrossRef]
35. Goebel, K.; Kirk, W.A. Iteration processes for nonexpansive mappings. *Contemp. Math* **1983**, *21*, 115–123.
36. Goebel, K.; Kirk, W.A. *Topics in Metric Fixed Point Theory*; Cambridge University Press: Cambridge, CA, USA, 1990.
37. Opial, Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Am. Math. Soc.* **1967**, *73*, 591–597. [CrossRef]
38. Rao, N.S.; Kalyani, K.; Khatri, K. Contractive mapping theorems in Partially ordered metric spaces. *Cubo (Temuco)* **2020**, *22*, 203–214. [CrossRef]
39. Agarwal, R.P.; Hussain, N.; Taoudi, M.A. Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations. *Abstract Appl. Anal.* **2012**, *2012*, 245872. [CrossRef]
40. Al-Sulami, H.H.; Ahmad, J.; Hussain, N.; Latif, A. Relation Theoretic (Θ, R) Contraction results with applications to nonlinear matrix equations. *Symmetry* **2018**, *10*, 767. [CrossRef]