mathematics

MDPI

Article

Rota–Baxter Operators on Cocommutative Weak Hopf Algebras

Zhongwei Wang [1], Zhen Guan [2], Yi Zhang [3] and Liangyun Zhang [2,*]

[1] School of Mathematics, Southeast University, Nanjing 211189, China; wangzhongwei@jit.edu.cn
[2] Department of Mathematics, Nanjing Agricultural University, Nanjing 210095, China; 23115207@njau.edu.cn
[3] School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China; zhangy2016@nuist.edu.cn
* Correspondence: zlyun@njau.edu.cn

Abstract: In this paper, we first introduce the concept of a Rota–Baxter operator on a cocommutative weak Hopf algebra H and give some examples. We then construct Rota–Baxter operators from the normalized integral, antipode, and target map of H. Moreover, we construct a new multiplication "$*$" and an antipode S_B from a Rota–Baxter operator B on H such that $H_B = (H, *, \eta, \Delta, \varepsilon, S_B)$ becomes a new weak Hopf algebra. Finally, all Rota–Baxter operators on a weak Hopf algebra of a matrix algebra are given.

Keywords: weak Hopf algebra; Rota–Baxter operator; normalized integral; matrix algebra

Citation: Wang, Z.; Guan, Z.; Zhang, Y.; Zhang, L. Rota–Baxter Operators on Cocommutative Weak Hopf Algebras. *Mathematics* **2022**, *10*, 95. https://doi.org/10.3390/math10010095

Academic Editor: Askar Tuganbaev

Received: 30 November 2021
Accepted: 22 December 2021
Published: 28 December 2021

1. Introduction and Preliminaries

Weak bialgebras and weak Hopf algebras were introduced by Böhm, Nill, and Szlachányi in 1999 [1]. They are one kind of the generalizations of ordinary bialgebras and Hopf algebras by weakening some conditions. In general, a weak Hopf algebra H is both an (associative and unital) algebra and a (coassociative and counital) coalgebra, but its comultiplication of the unit is allowed to be non-unital, namely $\Delta(1_H) = \sum 1_1 \otimes 1_2 \neq 1_H \otimes 1_H$. In addition, the multiplicativity of the counit does not hold any longer but is replaced by a weaker condition: for any $h, g \in H$, $\varepsilon(hg) = \sum \varepsilon(h1_1)\varepsilon(1_2 g)$. Weak Hopf algebras have a wide range of applications, such as quantum field theory and operator algebras.

Let A be an arbitrary algebra over a field K, $\lambda \in K$. A linear map $R : A \to A$ is called a *Rota–Baxter operator* [2] of weight λ on A if for all $x, y \in A$:

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy).$$

It is obvious that the Rota–Baxter operator on an associative algebra satisfies an identity abstracted from the integration by part formula in calculus. The Rota–Baxter algebra is widely used in pure mathematics and more recently mathematical physics. It originated from the paper by Baxter in 1960, which helped him understand Spitzer's identity in fluctuation [3]. Soon afterwards, Rota began a study of Rota–Baxter algebras from an algebraic and combinatorial perspective in connection with hypergeometric functions, incidence algebras, and symmetric functions [2]. In recent years, many mathematicians, such as Guo, Bai, Sheng, and Brzeziński et al., studied Rota–Baxter algebras and obtained some interesting results. They succeeded in constructing connections between Rota–Baxter algebras and pre-Lie algebras and dendriform algebras (see [4–8] for examples).

Recently, the relationships between Rota–Baxter operators and Hopf algebras have attracted many researchers, such as Jian [9], Yu, Guo and Thibon [10], and Zheng, Guo, and Zhang [11]. In 2021, Guo, Lang, and Sheng gave the notion of a Rota–Baxter operator on a group [12], moreover, based on the above notion, Goncharov introduced the definition of a Rota–Baxter operator on a cocommutative Hopf algebra and proved that the Rota–Baxter operator on the universal enveloping algebra $U(L)$ of a Lie algebra L is one to one corresponding to the Rota–Baxter operator on L [13]. As we know, a weak Hopf algebra is a generalization of a Hopf algebra. For this purpose, we introduce and study Rota–Baxter operators on cocommutative weak Hopf algebras, which is the motivation of this paper.

This paper is organized as follows. In Section 1, we shall recall some definitions and useful properties on weak Hopf algebras. In Section 2, we introduce the definition of a Rota–Baxter operator on a cocommutative weak Hopf algebra and investigate its properties. Meanwhile, we present some examples of Rota–Baxter weak Hopf algebras. In particular, we construct Rota–Baxter operators by using the normalized integral, antipode, and target map of weak Hopf algebras, respectively. In Section 3, for a given arbitrary Rota–Baxter weak Hopf algebra (H, B), we define a new multiplication " $*$ " and a new antipode S_B of (H, B) such that $H_B = (H, *, \eta, \Delta, \varepsilon, S_B)$ becomes a new weak Hopf algebra, which will be called a descent weak Hopf algebra. In Section 4, we give all Rota–Baxter operators on a weak Hopf algebra of an n-dimensional matrix algebra.

Throughout the paper, we always work on a fixed field K and use the Sweedler's notations [14]. If A is a vector space over K and $\Delta : A \to A \otimes A$ is a comultiplication on A, then we shall use the following sumless Sweedler notation for the image of $a \in A$:

$$\Delta(a) = a_1 \otimes a_2.$$

Definition 1. *[1] A weak bialgebra $H = (H, m, \eta, \Delta, \varepsilon)$ is both an algebra and a coalgebra such that for any $x, y, z \in H$, the following identities hold:*

$$\begin{aligned} \Delta(xy) &= \Delta(x)\Delta(y), \\ \varepsilon(xyz) &= \varepsilon(xy_1)\varepsilon(y_2 z) = \varepsilon(xy_2)\varepsilon(y_1 z), \\ \Delta^2(1_H) &= (\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H)) \\ &= (1_H \otimes \Delta(1_H))(\Delta(1_H) \otimes 1_H), \end{aligned}$$

where $\Delta(1_H) = 1_1 \otimes 1_2$ and $\Delta^2 = (\Delta \otimes id_H) \circ \Delta$.

Further, if there exists a linear map $S : H \to H$ such that for all $h \in H$,

$$h_1 S(h_2) = \varepsilon(1_1 h)1_2, \quad S(h_1)h_2 = 1_1 \varepsilon(h 1_2), \quad S(h_1)h_2 S(h_3) = S(h),$$

then we call $(H, m, \mu, \Delta, \varepsilon, S)$ a weak Hopf algebra and S an antipode of H. We say that the weak Hopf algebra H is cocommutative if $\Delta = \tau \circ \Delta$, where τ is the flipping map.

The antipode S of a weak Hopf algebra H is both anti-multiplicative and anti-comultiplicative. Meanwhile, the unit and counit are S-invariant. That means for any $h, g \in H$,

$$S(hg) = S(g)S(h), \quad \Delta(S(h)) = S(h_2) \otimes S(h_1), \quad S(1_H) = 1_H, \quad \varepsilon \circ S = \varepsilon.$$

For H, a weak bialgebra, we define the maps Π^L and $\Pi^R : H \to H$ by the formulas:

$$\Pi^L(h) = \varepsilon(1_1 h)1_2, \quad \Pi^R(h) = \varepsilon(h 1_2)1_1,$$

which are called the *target map* and *source map*, respectively. Their images H^L =ImΠ^L and H^R =ImΠ^R are both separable subalgebras of H and commute with each other. Furthermore, by [1,15–17], we have the following Equations (1)–(4) for all $x \in H^L, y \in H^R$ and $h, g \in H$:

$$\Delta(x) = 1_1 x \otimes 1_2, \quad \Delta(y) = 1_1 \otimes y 1_2, \tag{1}$$

$$h_1 \otimes \Pi^L(h_2) = 1_1 h \otimes 1_2, \quad \Pi^R(h_1) \otimes h_2 = 1_1 \otimes h 1_2, \tag{2}$$

$$\varepsilon(h\Pi^L(g)) = \varepsilon(hg) = \varepsilon(\Pi^R(h)g), \tag{3}$$

$$h\Pi^L(g) = \varepsilon(h_1 g)h_2, \quad g_1 \varepsilon(h g_2) = \Pi^R(h)g. \tag{4}$$

For H, a weak Hopf algebra with an antipode S and $h \in H$, we have:

$$\Pi^L \circ S = \Pi^L \circ \Pi^R = S \circ \Pi^R, \quad \Pi^R \circ S = \Pi^R \circ \Pi^L = S \circ \Pi^L, \tag{5}$$

$$\Pi^L(h_1) \otimes h_2 = S(1_1) \otimes 1_2 h, \quad h_1 \otimes \Pi^R(h_2) = h1_1 \otimes S(1_2). \tag{6}$$

For a cocommutative weak Hopf algebra H, it is known that the antipode S is an involution (i.e., $S^2 = id_H$) and the elements in H^L and H^R are all S-invariant. Hence,

$$H^L = H^R = H^L H^R,$$

is a weak Hopf subalgebra of H by [18].

Example 1. *Let R be the real field. We set*

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in R \right\}.$$

Then, A is an algebra under matrix multiplication with the basis

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Define its comultiplication and counit:

$$\Delta(e_1) = e_1 \otimes e_1, \quad \Delta(e_2) = e_2 \otimes e_2, \quad \varepsilon(e_1) = 1 = \varepsilon(e_2).$$

Then, by [19], A is a cocommutative weak Hopf algebra with the antipode $S = id$.

Example 2. *Let G be a finite groupoid (a category with finite many morphisms, such that each morphism is invertible). Then, the groupoid algebra KG (generated by morphisms in G with the product of two morphisms being equal to their composition if the latter is defined and* 0 *otherwise) is a quantum groupoid (weak Hopf algebra) in [18] via*

$$\Delta(g) = g \otimes g, \;\; \varepsilon(g) = 1, \;\; S(g) = g^{-1}, \;\; g \in G.$$

2. Rota–Baxter Operators on Cocommutative Weak Hopf Algebras

In this section, we first mainly introduce the definition of a Rota–Baxter operator on a cocommutative weak Hopf algebra. Then, we investigate its properties and present some examples.

Definition 2. *Let H be a cocommutative weak Hopf algebra. A map $B : H \to H$ that is both a left H^L-module map and a coalgebra map is called a Rota–Baxter operator on H, if for all $h, g \in H$,*

$$B(h)B(g) = B(h_1 B(h_2) g S(B(h_3))). \tag{7}$$

Here, H is a left H^L-module via its multiplication.

Lemma 1. *Let H be a cocommutative weak Hopf algebra and $B : H \to H$ a Rota–Baxter operator on H. Then for all $h \in H$,*

$$\Pi^L(B(h)) = \Pi^L(h), \;\; \Pi^R(B(h)) = \Pi^L \Pi^R(B(h)). \tag{8}$$

Moveover, if $H^L \subseteq C(H)$ (the center of H), then B is H^L-invariant.

Proof. Indeed, we have

$$\begin{aligned}\Pi^L(B(h)) &= \varepsilon(1_1 B(h))1_2 = \varepsilon(B(1_1 h))1_2\\ &= \varepsilon(1_1 h)1_2 = \Pi^L(h),\\ \Pi^R(B(h)) &= \Pi^R(\Pi^R(B(h))) = 1_1\varepsilon(\Pi^R(B(h))1_2)\\ &= 1_1\varepsilon(1_2\Pi^R(B(h))) = \varepsilon(1_1\Pi^R(B(h)))1_2\\ &= \Pi^L(\Pi^R(B(h))),\end{aligned}$$

as required.

Now suppose that $H^L \subseteq C(H)$. Since B is a Rota–Baxter operator on H, on one hand, we obtain that $\Delta(B(1_H)) = \Delta(1_H)(B(1_H) \otimes B(1_H)) = (B(1_H) \otimes B(1_H))\Delta(1_H)$ and $\Pi^L(B(1_H)) = 1_H = \Pi^R(B(1_H))$. Hence, $B(1_H)$ is a group-like element and invertible by [20] (Corollary 5.2). On the other hand, by Equation (7), we obtain

$$B(1_H)B(1_H) = B(1_1 B(1_2)S(B(1_3))) = B(1_1\Pi^L(B(1_2))) = B(1_1 1_2) = B(1_H).$$

Then, we have $B(1_H) = 1_H$. Hence, for all $x \in H^L$, we have $B(x) = xB(1_H) = x$. □

A weak Hopf algebra H is called *quantum commutative* if $h_1 g\Pi^R(h_2) = hg$ for any $h, g \in H$. By [11,21], H is quantum commutative if and only if $H^R \subseteq C(H)$. Note that the two weak Hopf algebras in Examples 1 and 2 are both quantum commutative.

Example 3. *Let H be a quantum commutative cocommutative weak Hopf algebra. Then Π^L is a Rota–Baxter operator on H.*

In fact, for all $h, g \in H$, we have

$$\begin{aligned}\Pi^L(h_1\Pi^L(h_2)gS(\Pi^L(h_3))) &= \Pi^L(h_1\Pi^R(h_2)gS(\Pi^L(h_3)))\\ &= \Pi^L(h_1 g\Pi^L(h_2)) = \Pi^L(\Pi^L(h_1)h_2 g)\\ &= \Pi^L(hg) = \Pi^L(h)\Pi^L(g),\end{aligned}$$

where the last equality holds by [1] (Equation (2.5a)).

Moreover, it is known from [1] that Π^L is left H^L-linear. Meanwhile, for all $h \in H$,

$$\begin{aligned}\Delta(\Pi^L(h)) &\overset{(1)}{=} 1_1\Pi^L(h) \otimes 1_2 = \Pi^L(1_1 h) \otimes 1_2\\ &\overset{(2)}{=} \Pi^L(h_1) \otimes \Pi^L(h_2),\end{aligned}$$

which implies that Π^L is a Rota–Baxter operator on H.

Example 4. *Let H be a cocommutative weak Hopf algebra. If H is also a quantum commutative, then $S : H \to H$ is a Rota–Baxter operator on H.*

Indeed, since H is cocommutative, S is a coalgebra map. Furthermore, S is a left H^L-module map since $S(xh) = S(hx) = S(x)S(h) = xS(h)$ for all $x \in H^L$ and $h \in H$. Moreover, S satisfies Equation (7):

$$\begin{aligned}S(h_1 S(h_2)gS(S(h_3))) &= S(\Pi^L(h_1)gh_2) = S(g\Pi^L(h_1)h_2)\\ &= S(gh) = S(h)S(g),\end{aligned}$$

thus, S is a Rota–Baxter operator on H.

Definition 3. *Let H be a weak Hopf algebra, and a left integral in H is an element $\ell \in H$ satisfying*

$$h\ell = \Pi^L(h)\ell, \quad \textit{for all } h \in H.$$

A left integral ℓ is normalized if $\Pi^L(\ell) = 1_H$.

Example 5. *Let H be a quantum commutative cocommutative weak Hopf algebra and ℓ a normalized left integral. Moreover, if ℓ satisfies*

$$\Delta(\ell) = \Delta(1_H)(\ell \otimes \ell) = (\ell \otimes \ell)\Delta(1_H),$$

then the map

$$B : H \to H, \quad B(h) = h\ell,$$

is a Rota–Baxter operator on H.

As a matter of fact, it is easy to see that B is a left H^L-module map and a coalgebra map. Meanwhile, for all $h, g \in H$, we have

$$\begin{aligned} B(h_1 B(h_2) g S(B(h_3))) &= B(h_1 h_2 \ell g S(h_3 \ell)) = B(h_1 (h_2 \ell)_1 g S((h_2 \ell)_2)) \\ &= B(h_1 (\Pi^L(h_2)\ell)_1 g S((\Pi^L(h_2)\ell)_2)) \overset{(2)}{=} B(1_1 h 1_2 \ell_1 g S(\ell_2)) \\ &= B(1_1 h \Pi^L(1_2) \ell_1 g S(\ell_2)) = h\ell_1 g S(\ell_2)\ell \\ &\overset{(5)}{=} h\ell_1 g \Pi^L(\Pi^R(\ell_2))\ell \overset{(6)}{=} h\ell 1_1 g S(1_2)\ell \\ &= h\ell g \ell = B(h)B(g). \end{aligned}$$

Therefore, B satisfies Equation (7); that is, it is a Rota–Baxter operator on H.

Remark 1. *Let H be a commutative and cocommutative weak Hopf algebra. Then, a coalgebra map $B : H \to H$ is a Rota–Baxter operator on H if and only if B is an algebra map. In this case, B is a weak bialgebra map. Indeed, for all $h, g \in H$, we have*

$$\begin{aligned} B(h)B(g) &= B(h_1 B(h_2) g S(B(h_3))) = B(h_1 g B(h_2) S(B(h_3))) \\ &= B(h_1 g \Pi^L(B(h_2))) \overset{(8)}{=} B(\Pi^L(h_1) h_2 g) = B(hg). \end{aligned}$$

Proposition 1. *Let H be a quantum commutative cocommutative weak Hopf algebra and B a Rota–Baxter operator on H. Then*

$$\widetilde{B}(h) = S(h_1)B(S(h_2)), \; for\ all\ h \in H,$$

is also a Rota–Baxter operator on H.

Proof. Clearly, $\widetilde{B}$ is a linear map. We prove that B is a coalgebra map. For all $h \in H$,

$$\begin{aligned} \Delta(\widetilde{B}(h)) &= \Delta(S(h_1)B(S(h_2))) = \Delta(S(h_1))\Delta(B(S(h_2))) \\ &= (S(h_2) \otimes S(h_1))(B(S(h_4)) \otimes B(S(h_3))) \\ &= S(h_1)B(S(h_2)) \otimes S(h_3)B(S(h_4)) \\ &= \widetilde{B}(h_1) \otimes \widetilde{B}(h_2), \\ \varepsilon(\widetilde{B}(h)) &= \varepsilon(S(h_1)B(S(h_2))) = \varepsilon(\Pi^R(S(h_1))B(S(h_2))) \\ &\overset{(6)}{=} \varepsilon(1_2 B(S(h)1_1)) = \varepsilon(B(1_2 S(h) 1_1)) \\ &= \varepsilon(1_2 S(h) 1_1) = \varepsilon(h). \end{aligned}$$

Next, we show that $\widetilde{B}$ is a left H^L-module map. As a matter of fact, for all $x \in H^L$ and $h \in H$, we obtain that

$$\begin{aligned} \widetilde{B}(xh) &= S(x_1 h_1)B(S(x_2 h_2)) \overset{(1)}{=} S(xh_1)B(S(h_2)) \\ &= xS(h_1)B(S(h_2)) = x\widetilde{B}(h). \end{aligned}$$

As a consequence, we obtain the following equation that will be used next,

$$1_1 S(h_1)B(S(h_2)) \otimes 1_2 h_3 = S(h_1)B(S(h_2)) \otimes h_3. \tag{9}$$

Finally, we notice again that B is left H^L-linear and that S is an involution. For all $h, g \in H$, we consider

$$\begin{aligned}
&\widetilde{B}\Big(h_1 \widetilde{B}(h_2) g S(\widetilde{B}(h_3))\Big) \\
&= \widetilde{B}\Big[h_1 S(h_2)B(S(h_3)) g S\Big(S(h_{(4)})B(S(h_5))\Big)\Big] \\
&= \widetilde{B}\Big[\Pi^L(h_1)B(S(h_2)) g S\Big(B(S(h_3))\Big)h_4\Big] \overset{(6)}{=} \widetilde{B}\Big[S(1_1)B(S(1_2 h_1)) g S\Big(B(S(h_2))\Big)h_3\Big] \\
&= \widetilde{B}\Big[B\Big(S(1_2 h_1 1_1)\Big) g S\Big(B(S(h_2))\Big)h_3\Big] = \widetilde{B}\Big[B\Big(S(h_1)\Big) g S\Big(B(S(h_2))\Big)h_3\Big] \\
&= S(h_{31})B(S(h_{21}))S(g_1)S\Big(B(S(h_{12}))\Big)B\Big[S(h_{32})B(S(h_{22}))S(g_2)S\Big(B(S(h_{11}))\Big)\Big] \\
&= S(h_1)B(S(h_2))S(g_1)S\Big(B(S(h_3))\Big)B\Big[S(h_4)B(S(h_5))S(g_2)S\Big(B(S(h_6))\Big)\Big] \\
&\overset{(7)}{=} S(h_1)B(S(h_2))S(g_1)S(B(S(h_3)))B(S(h_4))B(S(g_2)) \\
&= S(h_1)B(S(h_2))S(g_1)\Pi^R(B(S(h_3)))B(S(g_2)) \\
&\overset{(2)}{=} S(h_1)B(S(h_2))S(g_1)B(S(g_2)) \\
&= \widetilde{B}(h)\widetilde{B}(g).
\end{aligned}$$

Therefore, $\widetilde{B}$ is a Rota–Baxter operator on H. □

Remark 2. *Let H be a quantum commutative cocommutative weak Hopf algebra and ℓ a normalized left integral. Moreover, if ℓ satisfies*

$$\Delta(\ell) = \Delta(1_H)(\ell \otimes \ell) = (\ell \otimes \ell)\Delta(1_H),$$

then, according to Example 5 and Proposition 1, it is not difficult to prove that $\widetilde{B_\ell}(h)$ is also a Rota–Baxter operator on H.

Proposition 2. *Let H be a quantum commutative cocommutative weak Hopf algebra. Suppose that H^1 and H^2 are two weak Hopf subalgebras of H such that $H = H^1H^2$ as a weak Hopf algebra. Define a map B on H by*

$$B(\sum_i h^i g^i) = \sum_i \Pi^L(h^i)S(g^i), \quad \textit{for all } h^i \in H^1, g^i \in H^2.$$

Then B is a Rota–Baxter operator on H.

Proof. It is easy to see that B is left H^L-linearity by [1] (Lemma 2.5). Taking $t = \sum_i h^i g^i$, where $h^i \in H^1$, $g^i \in H^2$, we compute that

$$\begin{aligned}
B(t_1) \otimes B(t_2) &= \sum_i \Pi^L(h^i_1)S(g^i_2) \otimes \Pi^L(h^i_2)S(g^i_1) \overset{(2)}{=} \sum_i \Pi^L(1_1 h^i)S(g^i_2) \otimes 1_2 S(g^i_1) \\
&= \sum_i 1_1 \Pi^L(h^i)S(g^i_1) \otimes 1_2 S(g^i_2) = \sum_i \Delta(\Pi^L(h^i)S(g^i)) = \Delta(B(t)), \\
\varepsilon(B(t)) &= \sum_i \varepsilon(\Pi^L(h^i)S(g^i)) = \sum_i \varepsilon(1_1 h^i)\varepsilon(1_2 S(g^i)) \\
&= \sum_i \varepsilon(h^i 1_1)\varepsilon(1_2 g^i) = \sum_i \varepsilon(h^i g^i) = \varepsilon(t).
\end{aligned}$$

Thus, B is a coalgebra map.

For $u = hg$ and $v = h'g'$, where $h, h' \in H^1$ and $g, g' \in H^2$, we have that

$$\begin{aligned} B\Big(u_1 B(u_2) v S(B(u_3))\Big) &= B\Big(h_1 g_1 B(h_2 g_2) v S(B(h_3 g_3))\Big) \\ &= B\Big(h_1 g_1 \Pi^L(h_2) S(g_2) v S(\Pi^L(h_3) S(g_3))\Big) \\ &= B\Big(h_1 \Pi^R(h_2) \Pi^R(h_3) g_1 S(g_2) v g_3)\Big) \\ &= B\Big(h_1 \Pi^R(h_2) v \Pi^L(g_1) g_2)\Big) \\ &= B(hvg) = \Pi^L(hh') S(g'g) \\ &= \Pi^L(h) S(g) \Pi^L(h') S(g') \\ &= B(hg) B(h'g'), \end{aligned}$$

and since $H_1 H_2$ is spanned by elements of the form hg, Equation (7) holds for all $u, v \in H$. Hence, B is a Rota–Baxter operator on H. □

At the end of this section, we construct a Rota–Baxter weak Hopf algebra from a Rota–Baxter Hopf algebra. To do this, we need the following lemma (see [22] (Theorem 2.12)), which provides us a way to extend an n-dimensional Hopf algebra to an $(n+1)$-dimensional weak Hopf algebra.

Lemma 2. *Let $\mathcal{H}$ be a Hopf algebra and e its unit. We consider the set H as a result of adjoining a unit 1_H to $\mathcal{H}$ with respect to the multiplication. We extend the comultiplication Δ, the counit ε and the antipode S in the following way*

$$\Delta(1_H) = (1_H - e) \otimes (1_H - e) + e \otimes e, \quad \varepsilon(1_H) = 2, \quad S(1_H) = 1_H.$$

Then H becomes a weak Hopf algebra.

Proposition 3. *With the notions as in Lemma 2, a Rota–Baxter Hopf algebra $(\mathcal{H}, \mathcal{B})$ (see [13] for the definition) gives rise to a Rota–Baxter weak Hopf algebra (H, B), where*

$$B(h) = \begin{cases} \mathcal{B}(h), & h \neq 1_H; \\ 1_H, & h = 1_H. \end{cases}$$

Proof. Followed by Lemma 2, the cocommutativity of $\mathcal{H}$ yields to that of H. Meanwhile, we can easily obtain that

$$\Pi^L(h) = \begin{cases} \varepsilon(h)e, & h \in \mathcal{H}; \\ 1_H, & h = 1_H. \end{cases}$$

Then, obviously, $H^L \subseteq C(H)$, and the left H^L-linearity of B follows from a direct verification. Consequently, $B : H \to H$ is a coalgebra map since $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is a coalgebra map satisfying $\mathcal{B}(e) = e$. Furthermore, it is easy to see that B satisfies Equation (7). Hence, B is a Rota–Baxter operator on H. □

By [13] (Theorem 2), the following corollary is straightforward.

Corollary 1. *A Rota–Baxter operator of weight 1 on a Lie algebra gives rise to a Rota–Baxter operator on a weak Hopf algebra.*

3. A New Weak Hopf Algebra Constructed by Rota–Baxter Operators

In this section, we construct a new Rota–Baxter weak Hopf algebra by a given Rota–Baxter weak Hopf algebra, which is called a descent weak Hopf algebra.

Lemma 3. *Let H be a quantum commutative cocommutative weak Hopf algebra and B a Rota–Baxter operator on H. Define a new operation on H as follows:*

$$h * g = h_1 B(h_2) g S(B(h_3)), \quad for\ all\ h, g \in H.$$

Then $(H, *, \eta, \Delta, \varepsilon)$ is a weak bialgebra.

Proof. Taking $x, y, z \in H$, we have

$$\begin{aligned}(x*y)*z &= (x_1*y_1)B(x_2*y_2)zS(B(x_3*y_3))\\ &= x_1B(x_2)y_1S(B(x_3))B(x_4)B(y_2)zS(B(y_3))S(B(x_5))\\ &= x_1B(x_2)y_1\Pi^R(B(x_3))B(y_2)zS(B(y_3))S(B(x_4))\\ &= x_1B(x_2)\Pi^R(B(x_3))y_1B(y_2)zS(B(y_3))S(B(x_4))\\ &= x_1B(x_2)\underbrace{y_1B(y_2)zS(B(y_3))}S(B(x_3))\\ &= x_1B(x_2)(y*z)S(B(x_3)) = x*(y*z).\end{aligned}$$

Thus, the operation " $*$ " is associative. Furthermore,

$$\begin{aligned}1_H * x &= 1_1B(1_2)xS(B(1_3)) = 1_1xB(1_2)S(B(1_3)) = 1_1x\Pi^L(B(1_2))\\ &= 1_1x\Pi^L(1_2) = 1_1\Pi^L(1_2)x = 1_1\Pi^R(1_2)x = x,\\ x*1_H &= x_1B(x_2)1S(B(x_3)) = x_1\Pi^L(B(x_2)) = x_1\Pi^L(x_2) = x_1\Pi^R(x_2) = x.\end{aligned}$$

Therefore, 1_H is the unit respect to " $*$ ". Thus, $(H, *)$ is an associative algebra. Moreover,

$$\begin{aligned}\Delta(x*y) &= \Delta(x_1B(x_2)yS(B(x_3))) = \Delta(x_1)\Delta(B(x_2))\Delta(y)\Delta(S(B(x_3)))\\ &= (x_1\otimes x_2)(B(x_3)\otimes B(x_4))(y_1\otimes y_2)(S(B(x_6))\otimes S(B(x_5)))\\ &= (x_1B(x_2)y_1S(B(x_3)))\otimes(x_4B(x_5)y_2S(B(x_6)))\\ &= (x_1*y_1)\otimes(x_2*y_2).\end{aligned}$$

Hence, the comultiplication is multiplicative with respect to " $*$ ".

Next, we prove the weak multiplicativity of the counit ε. Since S and B are Rota–Baxter operators on H, we have

$$\begin{aligned}\varepsilon(x*y*z) &= \varepsilon(x_1B(x_2)(y*z)S(B(x_3))) = \varepsilon(x_1B(x_2)y_1B(y_2)zS(B(x_3)B(y_3)))\\ &\overset{(3)}{=} \varepsilon(\Pi^R(x_1)B(x_2)y_1B(y_2)zS(B(y_3))\Pi^L(S(B(y_3))))\\ &\overset{(2)}{=} \varepsilon(1_1B(x_11_2)y_1B(y_2)zS(B(y_3))\Pi^L(x_2))\\ &\overset{(2)}{=} \varepsilon(B(1_1'x)y_1B(y_2)zS(B(y_3))1_2'))\\ &= \varepsilon(xy_1B(y_2)z\Pi^L(S(B(y_3))))\\ &= \varepsilon(xy_1B(y_2)z\Pi^L(y_3))\\ &= \varepsilon(xy_1B(y_2)z) = \varepsilon(xy_11_1)\varepsilon(1_2B(y_2)z) = \varepsilon(xyz)\end{aligned}$$

and

$$\begin{aligned}\varepsilon(x*y_1)\varepsilon(y_2*z) &= \varepsilon(x_1B(x_2)y1S(B(x_3)))\varepsilon(y_2B(y_3)zS(B(y_4)))\\ &= \varepsilon(\Pi^R(x_1)B(x_2)y1\Pi^L(S(B(x_3))))\varepsilon(\Pi^R(y_2)B(y_3)z\Pi^L(S(B(y_4))))\\ &= \varepsilon(B(x_1)y1\Pi^L(x_2))\varepsilon(B(y_2)z\Pi^L(y_3))\\ &= \varepsilon(B(x)y1)\varepsilon(y_2z)\\ &= \varepsilon(xyz).\end{aligned}$$

Similarly, we can also prove

$$\varepsilon(x * y * z) = \varepsilon(x * y_2)\varepsilon(y_1 * z).$$

Last, we prove the weak comultiplicativity of the unit according to

$$\begin{aligned}\Delta^2(1_H) &= (\Delta \otimes id) \circ \Delta(1_H) = (\Delta \otimes id)(1_1 \otimes 1_2) = 1_1 \otimes 1_2 \otimes 1_3 \\ &= 1_1 \otimes 1_2 1_1^{'} \otimes 1_2^{'} = 1_1^{'} \otimes 1_1 1_2^{'} \otimes 1_2,\end{aligned}$$

we have

$$\begin{aligned}(\Delta(1_H) \otimes 1_H) * (1_H \otimes \Delta(1_H)) &= (1_1 \otimes 1_2 \otimes 1_H) * (1_H \otimes 1_1^{'} \otimes 1_2^{'}) \\ &= 1_1 \otimes 1_2 * 1_1^{'} \otimes 1_2^{'} \\ &= 1_1 \otimes 1_{21} B(1_{22}) 1_1^{'} S(B(1_{23})) \otimes 1_2^{'} \\ &= 1_1 \otimes 1_{21} B(1_{22}) S(B(1_{23})) 1_1^{'} \otimes 1_2^{'} \\ &= 1_1 \otimes 1_{21} \Pi^L(B(1_{22})) 1_1^{'} \otimes 1_2^{'} \\ &= 1_1 \otimes 1_{21} \Pi^R(1_{22}) 1_1^{'} \otimes 1_2^{'} \\ &= 1_1 \otimes 1_2 1_1^{'} \otimes 1_2^{'}.\end{aligned}$$

Thus, we have

$$(\Delta(1_H) \otimes 1_H) * (1_H \otimes \Delta(1_H)) = 1_1 \otimes 1_2 1_1^{'} \otimes 1_2^{'} = 1_1 \otimes 1_2 \otimes 1_3.$$

Similarly, we can prove that

$$(1_H \otimes \Delta(1_H)) * (\Delta(1_H) \otimes 1_H) = 1_1^{'} \otimes 1_1 1_2^{'} \otimes 1_2 = 1_1 \otimes 1_2 \otimes 1_3.$$

This completes the proof. □

Theorem 1. *Let H be a quantum commutative cocommutative weak Hopf algebra and B a Rota–Baxter operator on H. Then $H_B = (H, *, \eta, \Delta, \varepsilon, S_B)$ is also a cocommutative weak Hopf algebra called the descendent weak Hopf algebra of the Rota–Baxter weak Hopf algebra (H, B). Here, the antipode is given by*

$$S_B : H \to H, \quad S_B(h) = S(B(h_1))S(h_2)B(h_3), \ \textit{for all} \ h \in H.$$

Proof. Note that $\Pi^L = \Pi^R$ by assumption. To prove H_B is a cocommutative weak Hopf algebra, we are only left to show that S_B is indeed an antipode of H by Lemma 3. Before doing this, we shall give the following useful equations.

$$\Pi^L(h) = B(h_1)B(S_B(h_2)), \quad \Pi^R(h) = B(S_B(h_1))B(h_2), \tag{10}$$

for all $h \in H$.

As a matter of fact, by Lemma 1, we have

$$\begin{aligned}
B(h_1)B(S_B(h_2)) &= B\Big(h_1B(h_2)S_B(h_3)S(B(h_4))\Big)\\
&= B\Big(h_1B(h_2)S(B(h_3))S(h_4)B(h_5)S(B(h_6))\Big)\\
&= B\Big(h_1\Pi^L(B(h_2))S(h_3)\Pi^L(B(h_4))\Big)\\
&= B\Big(h_1\Pi^L(h_2)S(h_3)\Pi^L(h_4)\Big)\\
&= B\Big(h_1\Pi^R(h_2)S(h_3)\Big)\\
&= B(h_1S(h_2)) = B(\Pi^L(h))\\
&= \Pi^L(h),
\end{aligned}$$

and

$$\begin{aligned}
B(S_B(h_1))B(h_2) &= B(S_B(h_1))B(h_2)\Pi^R(B(h_3))\\
&= \Pi^R(B(h_1))B(S_B(h_2))B(h_3)\\
&= S(B(h_1))\underbrace{B(h_2)B(S_B(h_3))}B(h_4)\\
&= S(B(h_1))\Pi^L(h_2)B(h_3)\\
&= S(B(h_1))B(h_2\Pi^R(h_3))\\
&= S(B(h_1))B(h_2) = \Pi^R B(h)\\
&= \Pi^R(h),
\end{aligned}$$

as required.

For all $x \in H^L$ and $h \in H$, we can easily obtain that $x * h = xh$ from the definition of the operation "$*$". Meanwhile, S_B is left H^L-linear since both S and B are H^L-linear. Then, according to Lemma 1, we have

$$\begin{aligned}
h_1 * S_B(h_2) &= h_1B(h_2)S_B(h_3)S(B(h_4))\\
&= h_1B(h_2)S(B(h_3))S(h_4)B(h_5)S(B(h_6))\\
&= h_1\Pi^L(h_2)S(h_3)\Pi^L(h_4)\\
&= h_1S(h_2) = \Pi^L(h),
\end{aligned}$$

$$\begin{aligned}
S_B(h_1) * h_2 &= S_B(h_1)B(S_B(h_2))h_3S(B(S_B(h_4)))\\
&= S(B(h_1))S(h_2)B(h_3)B(S_B(h_4))h_5S(B(S_B(h_6)))\\
&\overset{(10)}{=} S(B(h_1))S(h_2)\Pi^L(h_3)h_4S(B(S_B(h_5)))\\
&= S(B(h_1))S(h_2)h_3S(B(S_B(h_4)))\\
&= S(B(h_1))\Pi^R(h_2)S(B(S_B(h_3)))\\
&= S(B(h_1\Pi^R(h_2)))S(B(S_B(h_3)))\\
&= S(B(S_B(h_1))B(h_2))\\
&= S(\Pi^R(h)) = \Pi^R(h),
\end{aligned}$$

and so we have

$$\begin{aligned}
S_B(h_1) * h_2 * S_B(h_3) &= \Pi^R(h_1) * S_B(h_2) = \Pi^R(h_1)S_B(h_2)\\
&= S_B(\Pi^L(h_1)h_2) = S_B(h).
\end{aligned}$$

Hence, H_B is a cocommutative weak Hopf algebra. □

For an algebra A with a multiplication m_A and a coalgebra C, we have the convolution algebra $Conv(C, A) = Hom(C, A)$ as spaces, with the multiplication given by

$$(\alpha \star \beta)(c) = \alpha(c_1)\beta(c_2),$$

for all $\alpha, \beta \in Hom(C, A)$ and $c \in C$.

Definition 4. *Let H be a weak bialgebra and A an algebra. We define the following set:*

$$WC(H, A) = \{\alpha \in Conv(H, A) \mid (\alpha \star \beta)(h) = \alpha(\Pi^L(h)), \quad (\beta \star \alpha)(h) = \alpha(\Pi^R(h)), \\ \exists \beta \in Conv(H, A), \quad \forall h \in H\}.$$

In this case, we say that β is a weak convolution invertible element of α in $WC(H, A)$.

It is known from [23] that if a weak convolution inverse of an element in $WC(H, A)$ exists, then it is unique.

Proposition 4. *Let H be a quantum commutative cocommutative weak Hopf algebra and B a Rota–Baxter operator on H. Then, B is a weak Hopf algebra homomorphism from H_B to H and a Rota–Baxter operator on the weak Hopf algebra H_B.*

Proof. Since $B : H_B \to H$ is an algebra map by Lemma 3 and a coalgebra map as well, B is a weak bialgebra map. Thus, in order to prove that B is a weak Hopf algebra map, it is left to prove that $B \circ S_B = S \circ B$.

In fact, for all $h \in H$, on one hand, we have

$$S(B(h_1))B(h_2) = \Pi^R(B(h)) = \Pi^L(B(h)) = \Pi^L(h) = SB(\Pi^L(h)),$$
$$B(h_1)S(B(h_2)) = \Pi^L(B(h)) = \Pi^R(h) = SB(\Pi^R(h)).$$

This means that the map $S \circ B$ is the weak convolution inverse for the map B in $WC(H, H)$. On the other hand, we obtain

$$B(S_B(h_1))B(h_2) \overset{(10)}{=} B(\Pi^R(h)) = B(S_B(\Pi^L(h))),$$
$$B(h_1)B(S_B(h_2)) \overset{(10)}{=} B(\Pi^L(h)) = B(\Pi^R(h)).$$

Thus, $B \circ S_B$ is also the weak convolution inverse for B in $WC(H, H)$. So $B \circ S_B = S \circ B$.

Clearly, B is both a coalgebra map and a left H^L-module map. Meanwhile, B satisfies Equation (7) since for all $h, g \in H$,

$$B(h_1 * B(h_2) * g * S_B(B(h_3))) = B(h_1)B(B(h_2))B(g)B(S_B(B(h_3)) = B(h) * B(g).$$

Thus, B is a Rota–Baxter operator on the weak Hopf algebra H_B. □

4. Rota–Baxter Operators on a Weak Hopf Algebra of a Matrix Algebra

In this section, we will give all Rota–Baxter operators on a weak Hopf algebra of an n-dimensional matrix algebra.

Proposition 5. *Let A be a weak Hopf algebra in Example 1 and $B \in End_K(A)$. Then, B is a Rota–Baxter operator on A if and only if B is an identity map, that is to say,*

$$B(e_1) = e_1, B(e_2) = e_2.$$

Proof. As $\{e_1, e_2\}$ is a basis of the weak Hopf algebra A, we only need to prove for e_1 and e_2 that B is a Rota–Baxter operator.

Let $B(e_1) = ke_1 + le_2 = \begin{pmatrix} k & 0 \\ 0 & l \end{pmatrix}$, $B(e_2) = me_1 + ne_2 = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$, $k, l, m, n \in R$. Then

$$B(e_1)B(e_1) = (ke_1 + le_2)(ke_1 + le_2) = k^2e_1 + l^2e_2 = \begin{pmatrix} k^2 & 0 \\ 0 & l^2 \end{pmatrix},$$
$$B(e_1B(e_1)e_1S(B(e_1))) = B(e_1B(e_1)e_1B(e_1)) = k^2B(e_1) = \begin{pmatrix} k^3 & 0 \\ 0 & k^2l \end{pmatrix}.$$

In order to let the map B satisfy Equation (7), we must make $k^3 = k^2$, $k^2l = l^2$. Similarly,

$$B(e_1)B(e_2) = (ke_1 + le_2)(me_1 + ne_2) = kme_1 + lne_2 = \begin{pmatrix} km & 0 \\ 0 & ln \end{pmatrix},$$
$$B(e_1B(e_1)e_2S(B(e_1))) = B(e_1B(e_1)e_2B(e_1)) = B(ke_1e_2B(e_1)) = B(0) = 0,$$

so $km = ln = 0$.

$$B(e_2)B(e_1) = (me_1 + ne_2)(ke_1 + le_2) = mke_1 + nle_2 = \begin{pmatrix} mk & 0 \\ 0 & nl \end{pmatrix},$$
$$B(e_2B(e_2)e_1S(B(e_2))) = B(ne_2e_1B(e_2)) = B(0) = 0,$$

so $mk = nl = 0$.

$$B(e_2)B(e_2) = (me_1 + ne_2)(me_1 + ne_2) = m^2e_1 + n^2e_2 = \begin{pmatrix} m^2 & 0 \\ 0 & n^2 \end{pmatrix},$$
$$B(e_2B(e_2)e_2S(B(e_2))) = B(ne_2e_2B(e_2)) = n^2B(e_2) = \begin{pmatrix} n^2m & 0 \\ 0 & n^3 \end{pmatrix},$$

so $n^2m = m^2$, $n^3 = n^2$.

By calculation, we can obtain the following classification:

(1) $k = l = 0; m = n = 0.$
(2) $k = l = 0; m = 0, n = 1.$
(3) $k = l = 0; m = n = 1.$
(4) $k = 1, l = 0; m = n = 0.$
(5) $k = 1, l = 0; m = 0, n = 1.$
(6) $k = l = 1; m = n = 0.$

The corresponding Rota–Baxter operator forms are

(1) $B(e_1) = B(e_2) = 0.$
(2) $B(e_1) = 0$, $B(e_2) = e_2.$
(3) $B(e_1) = 0$,$B(e_2) = e_1 + e_2.$
(4) $B(e_1) = e_1$, $B(e_2) = 0.$
(5) $B(e_1) = e_1$, $B(e_2) = e_2.$
(6) $B(e_1) = e_1 + e_2$, $B(e_2) = 0.$

Because B needs to be a coalgebra map, we have the following cases .

- For (1), $\varepsilon(B(e_1)) = 0 \neq 1 = \varepsilon(e_1)$.
- For (2), $\varepsilon(B(e_1)) = 0 \neq 1 = \varepsilon(e_1)$.
- For (3), $\varepsilon(B(e_1)) = 0 \neq 1 = \varepsilon(e_1)$.
- For (4), $\varepsilon(B(e_2)) = 0 \neq 1 = \varepsilon(e_2)$.
- For (5), $\varepsilon(B(e_1)) = \varepsilon(e_1)$, $\varepsilon(B(e_2)) = \varepsilon(e_2)$.
- For (6), $\varepsilon(B(e_2)) = 0 \neq 1 = \varepsilon(e_2)$.

Only (5) satisfies the condition. Obviously, the map B in (5) is a A^L-module map. At the same time,

$$\Delta(B(e_1)) = \Delta(e_1) = e_1 \otimes e_1 = (B \otimes B)\Delta(e_1),$$

$$\Delta(B(e_2)) = \Delta(e_2) = e_2 \otimes e_2 = (B \otimes B)\Delta(e_2).$$

Thus, B is a Rota–Baxter operator on A if and only if $B = id$. □

Remark 3. *Let R be the real field. We set*

$$A = \left\{ \begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 \\ 0 & a_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix} \mid a_{ii} \in R, i = 1, 2, \cdots, n \right\}.$$

Then A is an algebra under matrix multiplication with the basis

$$e_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \cdots, e_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Define its comultiplication and counit:

$$\Delta(e_1) = e_1 \otimes e_1, \Delta(e_2) = e_2 \otimes e_2, \cdots, \Delta(e_n) = e_n \otimes e_n, \quad \varepsilon(e_1) = \varepsilon(e_2) = \cdots = \varepsilon(e_n) = 1.$$

Then, A is a weak Hopf algebra with the antipode $S = id$. Using a similar method as in Proposition 5 , we can prove that a map $B : A \to A$ is a Rota–Baxter operator on A if and only if B is an identity map.

5. Conclusions

In this paper, we proposed the notion of a Rota–Baxter weak Hopf algebra and studied Rota–Baxter operators on weak Hopf algebras. A number of examples of Rota–Baxter weak Hopf algebras were presented. From the algebraic perspective, we believe that Rota–Baxter operators on weak Hopf algebras deserve to be studied further, and we think that it may be possible to remove the condition of "cocommutative" on the title of this article .

Author Contributions: Formal analysis, Z.G.; Methodology, Z.W. and Y.Z.; Writing—original draft, L.Z. All authors have read and agreed to the published version of the manuscript.

Funding: The China Postdoctoral Science Foundation (Grant No.2018M642128) and the Qing Lan Project of Jiangsu Province.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Böhm, G.; Nill, F.; Szlachányi, K. Weak Hopf algebras (I): Integral theory and C^*-structure. *J. Algebra* **1999**, *221*, 385–438. [CrossRef]
2. Rota, G.C. Baxter algebras and combinatorial identities I, II. *Bull. Amer. Math. Soc.* **1969**, *75*, 325–329. [CrossRef]
3. Baxter, G. An analytic problem whose solution follows from a simple algebraic identity. *Pac. J. Math.* **1960**, *10*, 731–742. [CrossRef]
4. An, H.H.; Bai, C.M. From Rota-Baxter algebras to pre-Lie algebras. *J. Phys. A Math. Theor.* **2008**, *41*, 145–150. [CrossRef]
5. Tang, R.; Bai, C.M.; Guo, L.; Sheng, Y.H. Deformations and their controlling cohomologies of $\mathcal{O}$-operators. *Commun. Math. Phys.* **2019**, *368*, 665–700. [CrossRef]
6. Brzeziński, T. Rota-Baxter systems, dendriform algebras and covariant bialgebras. *J. Algebra* **2016**, *460*, 1–25.
7. Zhang, T.J.; Gao, X.; Guo, L. Hopf algebras of rooted forests, cocyles, and free Rota-Baxter algebras. *J. Math. Phys.* **2016**, *57*, 492–532. [CrossRef]

8. Guo, L. *An Introduction to Rota-Baxter Algebras, Surveys of Modern Mathematics 4*; International Press: Somerville, MA, USA; Higher Education Press: Beijing, China, 2012.
9. Jian, R.Q. Construction of Rota-Baxter algebras via Hopf module algebras. *Sci. China Math.* **2014**, *57*, 2321–2328. [CrossRef]
10. Yu, H.Y.; Guo, L.; Thibon, J.Y. Weak composition quasi-symmetric functions, Rota-Baxter algebras and Hopf algebras. *Adv. Math.* **2019**, *344*, 1–34. [CrossRef]
11. Zheng, H.H.; Guo, L.; Zhang, L.Y. Rota-Baxter paired modules and their constructions from Hopf algebras. *J. Algebra* **2020**, *559*, 601–624. [CrossRef]
12. Guo, L.; Lang, H.L.; Sheng, Y.H. Integration and geometrization of Rota-Baxter Lie algebras. *Adv. Math.* **2021**, *387*, 107834. [CrossRef]
13. Goncharov, M. Rota-Baxter operators on cocommutative Hopf algebras. *J. Algebra* **2021**, *582*, 39–56. [CrossRef]
14. Sweedler, M.E. *Hopf Algebras*; Benjamin: New York, NY, USA, 1969.
15. Caenepeel, S.; De Groot, E. Modules over weak entwining structures. *Contemp. Math.* **2000**, *267*, 31–54.
16. Wang, Z.W.; Chen, C.; Zhang, L.Y. Morita equivalence for weak Hopf-Galois extensions. *Comm. Algebra* **2017**, *45*, 162–182. [CrossRef]
17. Wang, Z.W.; Chen, Y.Y.; Zhang, L.Y. Total integrals for weak Doi-Koppinen data. *Alg. Represent. Theory* **2013**, *16*, 931–953. [CrossRef]
18. Nikshych, D. On the structure of weak Hopf algebras. *Adv. Math.* **2002**, *170* , 257–286. [CrossRef]
19. Zhang, L.Y.; Chen, H.X.; Li, J.Q. Twisted smash products over weak Hopf algebras. *Acta Math. Sci.* **2004**, *221*, 385–438. [CrossRef]
20. Vecsernyés, P. Larson-Sweedler theorem and the role of grouplike elements in weak Hopf algebras. *J. Algebra* **2003**, *270*, 471–520. [CrossRef]
21. Bagio, D.; Flores, D.; Antana, A. Inner actions of weak Hopf algebras. *J. Algebra Appl.* **2015**, *12*, 393–403. [CrossRef]
22. Chebel, Z.; Makhlouf, A. Kaplansky's construction type and classification of weak bialgebras and weak Hopf algebras. *Mathematics* **2010**, *38*, 291–293.
23. Böhm, G.; Brzeziński, T. Cleft extensions of Hopf algebroids. *Appl. Categ. Struct.* **2006**, *14*, 431–469. [CrossRef]