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Generalized Wintgen Inequality for Statistical Submanifolds in Hessian Manifolds of Constant Hessian Curvature

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Abstract: The geometry of Hessian manifolds is a fruitful branch of physics, statistics, Kaehlerian and affine differential geometry. The study of inequalities for statistical submanifolds in Hessian manifolds of constant Hessian curvature was truly initiated in 2018 by Mihai, A. and Mihai, I. who dealt with Chen-Ricci and Euler inequalities. Later on, Siddiqui, A.N., Ahmad K. and Ozel C. came with the study of Casorati inequality for statistical submanifolds in the same ambient space by using algebraic technique. Also, Chen, B.-Y., Mihai, A. and Mihai, I. obtained a Chen first inequality for such submanifolds. In 2020, Mihai, A. and Mihai, I. studied the Chen inequality for $\delta(2,2)$-invariant. In the development of this topic, we establish the generalized Wintgen inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature. Some examples are also discussed at the end.

Keywords: statistical manifold; hessian manifold; hessian sectional curvature; generalized Wintgen inequality

MSC: 53C21; 53C25

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1. Introduction

Differential geometry is a traditional yet currently very active branch of pure mathematics, with notable applications in a number of areas of physics. Until recently, applications in the theory of statistics were fairly limited, but within the last few years, there has been intensive interest in the subject. For this reason, the geometric study of statistical submanifolds is new and has many research problems. For geometers, it would be natural to try to build the classical submanifold theory in statistical manifolds.

A statistical manifold becomes a Riemannian manifold by considering each point as a probability distribution. This manifold gives a setting for the field of information geometry and the Fisher information metric gives a metric on this manifold. The aim of information geometry is to use tools from Riemannian geometry to take out information from the underlying statistical model. For instance, if we take the family of all normal distributions as a parametric space of dimension 2 with two parameters (the expected value and the variance), which is also equipped with the Riemannian metric given by the Fisher information matrix. Thus, it can be treated as a statistical manifold with a geometry modeled on hyperbolic 2-space $\mathbb{H}^2$. The idea has fortunately been applied in numerous areas, including statistical inference and manifold learning. The first and second authors have discussed physical motivation of the concept of statistical manifolds in [1].

In 1985, Amari, S. [2] has developed and defined statistical manifolds as follows: a Riemannian manifold $(\overline{N}, \overline{g})$ with a Riemannian metric $\overline{g}$ is a statistical manifold if it has a pair of torsion-free affine connections $\overline{\nabla}, \overline{\nabla}^*$ (called dual connections on $\overline{N}$) and satisfying the following condition:

$$G\overline{g}(E,F) = \overline{g}(\overline{\nabla}_G E, F) + \overline{g}(E, \overline{\nabla}^*_G F),$$

for any $E, F, G \in \Gamma(T\overline{N})$. It is usually denoted by $(\overline{N}, \overline{\nabla}, \overline{\nabla}^*, \overline{g})$. In addition, $(\overline{\nabla}^*)^* = \overline{\nabla}$ holds. It is noted that $(\overline{N}, \overline{\nabla}, \overline{g})$ is a statistical structure, and so is $(\overline{N}, \overline{\nabla}^*, \overline{g})$.

An affine manifold $(\overline{N}, \overline{\nabla}, \overline{g})$ with a flat affine connection $\overline{\nabla}$ and a Hessian metric $\overline{g}$ is called a Hessian manifold if a pair $(\overline{\nabla}, \overline{g})$ on $\overline{N}$ satisfies the Codazzi equation given below [3–5]:

$$(\overline{\nabla}_E \overline{g})(F, G) = (\overline{\nabla}_F \overline{g})(E, G),$$

for any $E, F, G \in \Gamma(T\overline{N})$.

Next, suppose a difference in the tensor field $\overline{K} = \overline{\nabla} - \overline{\nabla}^0$ on a Hessian manifold $(\overline{N}, \overline{\nabla})$. Then, the Hessian curvature tensor $\overline{Q}$ with respect to $\overline{\nabla}$ is the tensor field of type $(1,3)$ and is defined by [3–5]:

$$\overline{Q}(E,F) = [\overline{K}_E, \overline{K}_F],$$

for any $E, F \in \Gamma(T\overline{N})$. Here, the Lie bracket $[,]$ of two vector fields is an operator that is assigned to any two vector fields on a smooth manifold and gives a third vector field.

Moreover, we denote the Riemannian curvature tensor with respect to the Levi–Civita connection $\overline{\nabla}^0$ and the dual affine connections $\overline{\nabla}$ and $\overline{\nabla}^*$ on $\overline{N}$ by $\overline{S}^0$, $\overline{S}$, and $\overline{S}^*$, respectively. Therefore, we have the following relation:

$$\overline{S}(E,F) + \overline{S}^*(E,F) = 2\overline{S}^0(E,F) + 2\overline{Q}(E,F). \tag{1}$$

A Hessian sectional curvature $\overline{\mathcal{K}}$ with respect to the Hessian curvature tensor $\overline{Q}$ on a Hessian manifold is defined as follows: Let $\mathcal{P}$ be a plane in $T_x\overline{N}$, $x \in \overline{N}$. We suppose that $\{E, F\}$ is an orthonormal basis of $\mathcal{P}$ and expresses the following:

$$\begin{aligned} \overline{\mathcal{K}}(E \wedge F) &= \overline{g}(\overline{Q}(E,F)F, E) \\ &= \frac{1}{2}\left[\overline{g}(\overline{S}(E,F)F,E) + \overline{g}(\overline{S}^*(E,F)F,E)\right] - \overline{g}(\overline{S}^0(E,F)F,E). \end{aligned}$$

It is important to note that the number $\overline{\mathcal{K}}(E \wedge F)$ is free from the choice of an orthonormal basis.

A Hessian manifold $(\overline{N}, \overline{\nabla}, \overline{g})$ is said to be of constant Hessian sectional curvature c if and only if [3–5]:

$$\overline{Q}(E,F,G,H) = \frac{c}{2}\Big[\overline{g}(E,F)\overline{g}(G,H) + \overline{g}(E,H)\overline{g}(F,G)\Big], \tag{2}$$

for any $E, F, G, H \in \Gamma(T\overline{N})$.

A Euclidean space $(\mathbb{R}^n, \overline{\nabla}, \overline{g})$ with a standard connection is a Hessian manifold of constant Hessian sectional curvature $c = 0$.

On a statistical manifold, Opozda, B. [6,7] gave the definition of a sectional curvature with respect to the dual affine connections as they are not metric in general, and thus it is not possible to define a sectional curvature with respect to them by using the standard definition from Riemannian geometry.

P. Wintgen [8] proposed a nice relationship between the Gauss curvature, the normal curvature, and the squared mean curvature of any surface N in a four-dimensional Euclidean space $\mathbb{E}^4$, and also discussed the necessary and sufficient conditions under which the equality case holds. I.V. Guadalupe and L. Rodriguez extended Wintgen's inequality to a surface of arbitrary codimension in a real space form $\mathbb{R}^{m+2}(c)$, $m \geq 2$. After that, B.-Y. Chen extended this inequality to surfaces in a four-dimensional pseudo-Euclidean space $\mathbb{E}^4_2$ with a neutral metric.

In [9], P.J. DeSmet, F. Dillen, L. Verstraelen, and L. Vrancken found the following DDVV conjecture (called the generalized Wintgen inequality) for the isometric immersion of an m-dimensional Riemannian manifold N in a real space form $\mathbb{R}(c)$ of constant curvature c, at each point $x \in N$:

$$\rho_{nor} \leq ||\mathcal{H}||^2 - \rho^{\perp} + c,$$

where ρ_{nor} denotes the normalized scalar curvature of N, $\rho^{\perp}$ denotes the normalized normal scalar curvature of N, and $\mathcal{H}$ is the mean curvature vector of N. Additionally, in [9], they conjectured this inequality for a submanifold with a codimension of 2 in a real space form $\mathbb{R}^{m+2}(c)$. The solution of this conjecture was independently proven by Z. Lu [10] and by J. Ge and Z. Tang [11] for a general case. Since then, many remarkable articles have been published, and several inequalities of this type have been obtained for other classes of submanifolds in several ambient spaces (see [12–23]).

The purpose of this article consists in an attempt to clarify the use of Riemannian manifolds in statistical theory. Should the readers have great interest in submanifold theory, they can reexamine the theorems of their choice in the statistical submanifolds setting. However, we would like to point out here that the study on the properties of statistical submanifolds should not be derived directly by performing simple modifications to the classical setting. Thus, in this sense, the present paper would be helpful for beginners in the study of statistical submanifold theory. In this paper, by using scalar and squared mean curvatures, and through the generalization of Riemannian connections, that is, via dual connections, we would like to perform research in the statistical immersion of statistical manifolds in a Hessain manifold of constant hessian curvature c. For this reason, we would extend the generalized Wintgen inequality to statistical submanifolds in the same ambient space. Therefore, we believe that current theories are very attractive and interesting for researchers in both Mathematics and Statistics.

2. Preliminaries

Let N be a submanifold in a statistical manifold $(\overline{N}, \overline{\nabla}, \overline{g})$. Then, (N, ∇, g) is also a statistical manifold with the induced statistical structure (∇, g) on N from $(\overline{\nabla}, \overline{g})$, and (N, ∇, g) becomes a statistical submanifold in $(\overline{N}, \overline{\nabla}, \overline{g})$. Furthermore, we call (N, ∇^*, g) a statistical submanifold in $(\overline{N}, \overline{\nabla}^*, \overline{g})$.

In the statistical setting, the Gauss and Weingarten formulae are respectively given below [24]:

$$\left.\begin{aligned} \overline{\nabla}_E F &= \nabla_E F + h(E,F), & \overline{\nabla}^*_E F &= \nabla^*_E F + h^*(E,F), \\ \overline{\nabla}_E \xi &= -A_\xi(E) + \nabla^{\perp}_E \xi, & \overline{\nabla}^*_E \xi &= -A^*_\xi(E) + \nabla^{\perp *}_E \xi, \end{aligned}\right\} \tag{3}$$

for any $E, F \in \Gamma(TN)$ and $\xi \in \Gamma(T^{\perp}N)$. The symmetric and bi-linear imbedding curvature tensor of N in $\overline{N}$ in relation to $\overline{\nabla}$ and $\overline{\nabla}^*$ are respectively indicated by h and h^*. The relation between h (respectively, h^*) and A^*_ξ (respectively, A_ξ) is given by [24]:

$$\left.\begin{aligned} \overline{g}(h(E,F),\xi) &= g(A^*_\xi E, F), \\ \overline{g}(h^*(E,F),\xi) &= g(A_\xi E, F), \end{aligned}\right\} \tag{4}$$

for any $E, F \in \Gamma(TN)$ and $\xi \in \Gamma(T^{\perp}N)$.

Next, we assume that S and S^* are the Riemannian curvature tensors with respect to ∇ and ∇^*, respectively. Then, the corresponding Gauss equations in relation to $\overline{\nabla}$ and $\overline{\nabla}^*$ are respectively given by the following:

$$\begin{aligned}\overline{g}(\overline{S}(E,F)G,H) &= g(S(E,F)G,H) + \overline{g}(h(E,G),h^*(F,H)) \\ &\quad -\overline{g}(h^*(E,H),h(F,G)),\end{aligned} \tag{5}$$

$$\begin{aligned}\overline{g}(\overline{S}^*(E,F)G,H) &= g(S^*(E,F)G,H) + \overline{g}(h^*(E,G),h(F,H)) \\ &\quad -\overline{g}(h(E,H),h^*(F,G)),\end{aligned} \tag{6}$$

for any $E,F,G,H \in \Gamma(TN)$.

The Ricci equations for $\nabla^{\perp}$ and $\nabla^{\perp *}$ are respectively given below [24]:

$$\overline{g}(S^{\perp}(E,F)\xi,\eta) = \overline{g}(\overline{S}(E,F)\xi,\eta) + g([A^*_{\xi},A_{\eta}]E,F), \tag{7}$$

$$\overline{g}(S^{\perp *}(E,F)\xi,\eta) = \overline{g}(\overline{S}^*(E,F)\xi,\eta) + g([A_{\xi},A^*_{\eta}]E,F), \tag{8}$$

for any $E,F \in \Gamma(TN)$ and $\xi,\eta \in \Gamma(T^{\perp}N)$. Here, $S^{\perp}$ and $S^{\perp *}$ are the normal curvature tensors for $\nabla^{\perp}$ and $\nabla^{\perp *}$ on $T^{\perp}N$, respectively.

3. Generalized Wintgen Inequality

In this section, we obtain the generalized Wintgen inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature c. Before proceeding, we first give the following propositions and lemmas, which are useful in obtaining the said inequality.

Consider a statistical submanifold N of dimension m in an n-dimensional Hessian manifold $\overline{N}$. We take a local orthonormal tangent frame $\{e_1,\ldots,e_m\}$ of T_xN and a local orthonormal normal frame $\{\xi_1,\ldots,\xi_n\}$ of $T^{\perp}_xN$ in $\overline{N}$, $x \in N$. Then, the mean curvature vectors $\mathcal{H}$, $\mathcal{H}^*$ and $\mathcal{H}^0$ of N in $\overline{N}$ for $\overline{\nabla}$, $\overline{\nabla}^*$, and $\overline{\nabla}^0$ are respectively defined as follows:

$$\begin{aligned}\mathcal{H} &= \frac{1}{m}\sum_{i=1}^{r} h(e_i,e_i) = \frac{1}{m}\sum_{i=1}^{m} h_{ii}, \\ \mathcal{H}^* &= \frac{1}{m}\sum_{i=1}^{r} h^*(e_i,e_i) = \frac{1}{m}\sum_{i=1}^{m} h^*_{ii}, \\ \mathcal{H}^0 &= \frac{1}{m}\sum_{i=1}^{r} h^0(e_i,e_i) = \frac{1}{m}\sum_{i=1}^{m} h^0_{ii},\end{aligned}$$

In addition, we set $h^a_{ij} = g(h(e_i,e_j),e_a)$, $i,j \in \{1,\ldots,m\}$, $a \in \{1,\ldots,n\}$, and $||h||^2 = \sum_{i,j=1}^{m} g(h(e_i,e_j),h(e_i,e_j))$. Similarly, we have the same notations for h^* and the second fundamental form h^0 with respect to the Levi–Civita connection.

The Casorati curvature C^0 for the Levi–Civita connection of N is defined by the following equation:

$$C^0 = \frac{1}{m}||h^0||^2. \tag{9}$$

The normalized scalar curvature ρ_{nor} of N with respect to the Hessian curvature tensor Q is given by the following:

$$\begin{aligned}\rho_{nor} &= \frac{2}{m(m-1)}\sum_{1\leq i<j\leq m}\mathcal{K}(e_i\wedge e_j)\\ &= \frac{1}{m(m-1)}\sum_{1\leq i<j\leq m}\Big[g(S(e_i,e_j)e_j,e_i)+g(S^*(e_i,e_j)e_j,e_i)\\ &\quad -2g(S^0(e_j,e_i)e_j,e_i)\Big].\end{aligned} \tag{10}$$

The normalized normal scalar curvature of N with respect to the Hessian curvature tensor Q is given by the following:

$$\begin{aligned}\rho^{\perp} &= \frac{1}{m(m-1)}\left\{\sum_{1\leq a<b\leq n}\sum_{1\leq i<j\leq m}\Big[g(S^{\perp}(e_i,e_j)\xi_a,\xi_b)+g(S^{\perp *}(e_i,e_j)\xi_a,\xi_b)\Big]^2\right\}^{1/2}\\ &= \frac{1}{m(m-1)}\left\{\sum_{1\leq a<b\leq n}\sum_{1\leq i<j\leq m}\Big[\overline{g}(\overline{S}(e_i,e_j)\xi_a,\xi_b)+g([A^*_{\xi_a},A_{\xi_b}]e_i,e_j)\right.\\ &\quad \left.+\overline{g}(\overline{S}^*(e_i,e_j)\xi_a,\xi_b)+g([A_{\xi_a},A^*_{\xi_b}]e_i,e_j)\Big]^2\right\}^{1/2}.\end{aligned}$$

It is easy to find the following equation:

$$\begin{aligned}\rho^{\perp} &= \frac{1}{m(m-1)}\left\{\sum_{1\leq a<b\leq n}\sum_{1\leq i<j\leq m}\Big[g([A^*_{\xi_a},A_{\xi_b}]e_i,e_j)\right.\\ &\quad \left.+g([A_{\xi_a},A^*_{\xi_b}]e_i,e_j)\Big]^2\right\}^{1/2}.\end{aligned} \tag{11}$$

In order to prove the main theorem of this section, we need the following useful results:

Lemma 1. *Let $(\overline{N}(c),\overline{\nabla},\overline{g})$ be an n-dimensional Hessian manifold of constant Hessian curvature c, and (N,∇,g) be an m-dimensional statistical submanifold in $\overline{N}(c)$. Then, the normalized scalar curvature of N with respect to the Hessian curvature tensor Q is given by the following equation:*

$$\begin{aligned}\rho_{nor} &= \frac{1}{m(m-1)}\sum_{a=1}^{m}\sum_{1\leq i<j\leq m}\Big[2h^a_{ij}h^{*a}_{ij}-h^{*a}_{ii}h^a_{jj}-h^a_{ii}h^{*a}_{jj}\Big]\\ &\quad -\frac{m}{m-1}||\mathcal{H}^0||^2+\frac{1}{m(m-1)}||h^0||^2+\frac{c}{4},\end{aligned} \tag{12}$$

Proof. From (5), (6), and (10), we derive the following:

$$\begin{aligned}\rho_{nor} &= \frac{1}{m(m-1)}\sum_{1\leq i<j\leq m}\Big[g(S(e_i,e_j)e_j,e_i)+g(S^*(e_i,e_j)e_j,e_i)-2g(S^0(e_i,e_j)e_j,e_i)\Big]\\ &= \frac{1}{m(m-1)}\sum_{1\leq i<j\leq m}\Big[g(h(e_i,e_i),h^*(e_j,e_j))-g(h^*(e_i,e_j),h(e_i,e_j))\\ &\quad +g(h^*(e_i,e_i),h(e_j,e_j))-g(h(e_i,e_j),h^*(e_i,e_j))\Big]-\rho^0_{nor},\end{aligned}$$

which can be rewritten as follows:

$$\rho_{nor}+\rho^0_{nor}=\frac{1}{m(m-1)}\sum_{a=1}^{m}\sum_{1\leq i<j\leq m}\Big[2h^a_{ij}h^{*a}_{ij}-h^{*a}_{ii}h^a_{jj}-h^a_{ii}h^{*a}_{jj}\Big] \tag{13}$$

We denote by ρ^0_{nor} the normalized scalar curvature of the Levi–Civita connection ∇^0 on N.

$$\rho^0_{nor} = \frac{1}{m(m-1)}\left[m^2||\mathcal{H}^0||^2 - ||h^0||^2 - m(m-1)\frac{c}{4}\right]. \quad (14)$$

Thus, by combining (13) and (14), we arrive at our desired result (12). □

In light of (11), it is easy to prove the following result:

Lemma 2. *Let $(\overline{N}(c), \overline{\nabla}, \overline{g})$ be an n-dimensional Hessian manifold of constant Hessian curvature c, and (N, ∇, g) be an m-dimensional statistical submanifold in $\overline{N}(c)$. Then, the normalized normal scalar curvature of N with respect to the Hessian curvature tensor Q is given by the following:*

$$\begin{aligned}\rho^{\perp} &= \frac{1}{m(m-1)}\left\{\sum_{1\leq a<b\leq n}\sum_{1\leq i<j\leq m}\left[\sum_{k=1}^{m}(h^b_{ik}h^{*a}_{jk} - h^{*a}_{ik}h^b_{jk}\right.\right. \\ &\quad \left.\left. + h^{*b}_{ik}h^a_{jk} - h^a_{ik}h^{*b}_{jk})\right]^2\right\}^{1/2}.\end{aligned}$$

Furthermore, we use the algebraic inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2),\ a,b,c \in \mathbb{R}$ and Lemma 2 to prove the following proposition:

Proposition 1. *Let $(\overline{N}(c), \overline{\nabla}, \overline{g})$ be an n-dimensional Hessian manifold of constant Hessian curvature c, and (N, ∇, g) be an m-dimensional statistical submanifold in $\overline{N}(c)$. Then we have the following:*

$$\begin{aligned}\rho^{\perp} &\leq \frac{3}{2}[||\mathcal{H}||^2 + ||\mathcal{H}^*||^2] + 24||\mathcal{H}^0||^2 - \frac{3}{m(m-1)}\sum_{a=1}^{n}\sum_{1\leq i<j\leq m}\left[20h^{0a}_{ii}h^{0a}_{jj}\right. \\ &\quad \left. - h^{*a}_{ii}h^a_{jj} - h^a_{ii}h^{*a}_{jj} - 20(h^{0a}_{ij})^2 + 2h^a_{ij}h^{*a}_{ij}\right].\end{aligned}$$

Proof. By following the same steps adopted in [25], one can easily obtain the desired inequality. □

In [26], M.E. Aydin and I. Mihai discussed the Wintgen inequality for a statistical surface in a four-dimensional Hessian statistical manifold of Hessian curvature 0, given by:

$$G^c + |G^{\perp}| + 2G^0 \leq \frac{1}{2}(||H||^2 + ||H^*||^2),$$

where G^c, $G^{\perp}$, and G^0 are the Gauss curvature, the normal curvature, and the Gauss curvature with respect to the Levi–Civita connection, respectively. Now, we give the following conjecture for the Wintgen inequality on a statistical submanifold in a Hessian manifold of constant Hessian curvature c:

Theorem 1. *Let $(\overline{N}(c), \overline{\nabla}, \overline{g})$ be an n-dimensional Hessian manifold of constant Hessian curvature c, and (N, ∇, g) be an m-dimensional statistical submanifold in $\overline{N}(c)$. Then we have the following equation:*

$$\begin{aligned}\rho^{\perp} &\leq 3\rho_{nor} - 30[\overline{\rho}^0_{nor} + \rho^0_{nor}] - \frac{3c}{4} \\ &\quad + \frac{3}{2}[||\mathcal{H}||^2 + ||\mathcal{H}^*||^2] + \left(\frac{27m-24}{m-1}\right)||\mathcal{H}^0||^2. \quad (15)\end{aligned}$$

Proof. By using Lemma 1 and Proposition 1, we derive the following:

$$\begin{aligned}\rho^{\perp} \leq\ & \frac{3}{2}[||\mathcal{H}||^2 + ||\mathcal{H}^*||^2] + \left(\frac{27m-24}{m-1}\right)||\mathcal{H}^0||^2 + 3\rho_{nor} \\ & -\frac{3}{m(m-1)}||h^0||^2 - \frac{3c}{4} - \frac{60}{m(m-1)}\sum_{a=1}^{n}\sum_{1\leq i<j\leq m}\Big[h_{ii}^{0a}h_{jj}^{0a} \\ & -(h_{ij}^{0a})^2\Big].\end{aligned} \tag{16}$$

The normalized scalar curvature $\overline{\rho}^0_{nor}$ of the Levi–Civita connection $\overline{\nabla}^0$ on $\overline{N}$ is as follows:

$$\overline{\rho}^0_{nor} = \frac{2}{m(m-1)}\sum_{1\leq i<j\leq m}\overline{g}(\overline{S}^0(e_i,e_j)e_j,e_i),$$

then, in using the Gauss equation for the Levi–Civita connection, we get the following equation:

$$\overline{\rho}^0_{nor} + \rho^0_{nor} = \frac{2}{m(m-1)}\sum_{a=1}^{n}\sum_{1\leq i<j\leq m}\Big[h_{ii}^{0a}h_{jj}^{0a} - (h_{ij}^{0a})^2\Big]. \tag{17}$$

In view of (16) and (17), we have the following:

$$\begin{aligned}\rho^{\perp} \leq\ & \frac{3}{2}[||\mathcal{H}||^2 + ||\mathcal{H}^*||^2] + \left(\frac{27m-24}{m-1}\right)||\mathcal{H}^0||^2 + 3\rho_{nor} \\ & -\frac{3}{m(m-1)}||h^0||^2 - \frac{3c}{4} - 30[\overline{\rho}^0_{nor} + \rho^0_{nor}].\end{aligned} \tag{18}$$

Thus, we get our desired inequality (15). □

Remark 1. *Shima, H. [5] noticed that a Hessian manifold of constant Hessian sectional curvature c is a statistical manifold of constant curvature zero and also a Riemannian space form of constant sectional curvature $-c/4$ with respect to the Levi–Civita connection. Thus, our main Theorem 1 can also be stated for such ambient spaces.*

Let us take a minimal submanifold N for the Levi–Civita connection, which gives $\mathcal{H} + \mathcal{H}^* = 0$ because of $\mathcal{H}^0 = 0$; thus, we have the relation $\frac{1}{2}[||\mathcal{H}||^2 + ||\mathcal{H}^*||^2] = -\overline{g}(\mathcal{H},\mathcal{H}^*)$. Then, Theorem 1 implies the following:

Corollary 1. *Let $(\overline{N}(c),\overline{\nabla},\overline{g})$ be an n-dimensional Hessian manifold of constant Hessian curvature c, and (N,∇,g) be a minimal m-dimensional statistical submanifold in $\overline{N}(c)$. Then we have the following:*

$$\rho^{\perp} \leq 3[\rho_{nor} - 10(\overline{\rho}^0_{nor} + \rho^0_{nor}) - \frac{c}{4} - \overline{g}(\mathcal{H},\mathcal{H}^*)].$$

The following is the generalized Wintgen inequality for a statistical submanifold in a Hessian manifold of constant Hessian curvature c by taking into account the Casorati curvature C^0 for the Levi–Civita connection on N:

Theorem 2. *Let $(\overline{N}(c),\overline{\nabla},\overline{g})$ be an n-dimensional Hessian manifold of constant Hessian curvature c, and (N,∇,g) be an m-dimensional statistical submanifold in $\overline{N}(c)$. Then we have the following:*

$$\rho^{\perp} \leq 3\rho_{nor} - 30[\overline{\rho}^{0}_{nor} + \rho^{0}_{nor}] - \frac{3c}{4} - \frac{3}{m-1}C^{0} + \frac{3}{2}[||\mathcal{H}||^2 + ||\mathcal{H}^*||^2] + \left(\frac{27m-24}{m-1}\right)||\mathcal{H}^0||^2. \tag{19}$$

Proof. On using (9) and (18), one can easily get (19). □

4. Some Related Examples

A statistical submanifold N is said to be minimal if $\mathcal{H} = \mathcal{H}^* = 0$. Hence, we give a new example for a minimal statistical submanifold in a Hessian manifold of constant Hessian curvature c, which can be written as follows:

Example 1. *Let* $(\mathbb{H}^3 = \{(x,y,z) \in \mathbb{R}^3 | z > 0\}, \overline{g} = \frac{(dx)^2+(dy)^2+(dz)^2}{z^2})$ *be the upper half space of constant sectional curvature* -1 *[26]. We set orthonormal basis on* $\mathbb{H}^3$ *as*

$$e_1 = z\partial x, \;\; e_2 = z\partial y, \;\; e_3 = z\partial z$$

with respect to $\overline{g}$*. Then, an affine connection* $\overline{\nabla}$ *on* $\mathbb{H}^3$ *is defined as [26]:*

$$\frac{1}{2}\overline{\nabla}_{e_1}e_1 = \frac{1}{2}\overline{\nabla}_{e_2}e_2 = \overline{\nabla}_{e_3}e_3 = e_3, \quad \overline{\nabla}_{e_i}e_j = 0,$$

for $i,j = 1,2,3$*. Hence,* $(\mathbb{H}^3, \overline{\nabla}, \overline{g})$ *is a Hessian manifold of constant Hessian curvature* 4*. Now, we consider the statistical immersion* $F : N^2 \to \mathbb{H}^3$ *as follows:*

$$F(s,t) = (as, bs, st), \quad a,b \in \mathbb{R}.$$

Then, we have the following:

$$F_s = \frac{a}{st}e_1 + \frac{b}{st}e_2 + \frac{1}{s}e_3, \quad F_t = \frac{1}{t}e_3.$$

Thus, we compute the unit normal vector ξ *of* N^2 *in* $\mathbb{H}^3$ *as follows:*

$$\xi = \frac{1}{||F_s \times F_t||}\left(\frac{b}{st^2}e_1 - \frac{a}{st^2}e_2\right), \tag{20}$$

and we also calculate as follows:

$$\left.\begin{aligned} \overline{\nabla}_{F_s}F_s &= \left(\frac{2(a^2+b^2)}{s^2t^2} + \frac{1}{s^2}\right)e_3, \\ \overline{\nabla}_{F_s}F_t &= \frac{1}{st}e_3, \\ \overline{\nabla}_{F_t}F_t &= \frac{1}{t^2}e_3. \end{aligned}\right\} \tag{21}$$

By using (20) and (21), we observe that N^2 *is a minimal* ($\mathcal{H} = \mathcal{H}^* = 0$) *statistical surface in* $\mathbb{H}^3$*. Hence,* N^2 *is minimal with respect to the Levi–Civita connection as well.*

Example 2. *We recall the upper-half space of the constant sectional curvature* $-k$ $(\mathbb{H}^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | x_n > 0\}, \overline{g})$*, where* $\overline{g} = \frac{(dx_1)^2+(dx_2)^2+\cdots+(dx_n)^2}{kx_n^2}$ *and* $k > 0$*. Define* $\overline{K} \in \Gamma(T\mathbb{H}^n)$ *by [27]:*

$$\overline{K}(\partial_i, \partial_j) = \frac{\delta_{ij}}{x_n}\partial_n, \;\; \overline{K}(\partial_i, \partial_n) = \frac{1}{x_n}\partial_i, \;\; \overline{K}(\partial_n, \partial_n) = \frac{2}{x_n}\partial_n,$$

where $\partial_i = \frac{\partial}{\partial x_i}$ *for* $i,j = \{1, \ldots, n-1\}$*. Then,* $(\mathbb{H}^n, \overline{\nabla} = \overline{K} + \overline{\nabla}^0, \overline{g})$ *is a Hessian manifold of constant Hessian curvature* $4k$*.*

Next, we suppose that $N^m = \mathbb{H}^m$ is a statistical submanifold of $\mathbb{H}^n$, with $0 < m < n$. Then, N^m is a totally geodesic submanifold of $\mathbb{H}^n$ with respect to the Levi–Civita connection.

5. Discussion

The curvature invariants are the main Riemannian invariants and the most natural ones. They also play an important role in physics. The most studied curvature invariants are the sectional, scalar, and Ricci curvatures. In submanifold theory, one fascinating problem is to find the relation between intrinsic invariants and extrinsic invariants (in particular, the squared mean curvature) of any submanifold. In this article, we used the normalized scalar, the normalized normal scalar, and the squared mean curvatures to establish the DDVV conjecture for the statistical immersion of an m-dimensional statistical manifold in a Hessian manifold of constant Hessian curvature c. The main Theorems 1 and 2 can be easily derived by the algebraic result of the following for the symmetric and trace-free operators in [10]: Let N be an n-dimensional Riemannian submanifold of an $(n+m)$-dimensional Riemannian space form $\overline{N}(c)$. For every set $\{B_1, \ldots, B_m\}$ of symmetric $(m \times m)$-matrices with a trace of zero, the following inequality holds:

$$\sum_{r,s=1}^{m} ||B_r, B_s||^2 \leq \left(\sum_{r=1}^{m} ||B_r||^2 \right)^2.$$

Finally, we provided new examples for a minimal statistical submanifold in a Hessian manifold of constant Hessian curvature c. Therefore, in the development of this topic, a lot of similar relationships for different kinds of invariants (of similar nature) for statistical submanifolds in Hessian manifolds of constant Hessian curvature can be discussed.

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