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Third Order Melnikov Functions of a Cubic Center under Cubic Perturbations

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Abstract: In this paper, cubic perturbations of the integral system $(1+x)^2dH$ where $H=(x^2+y^2)/2$ are considered. Some useful formulae are deduced that can be used to compute the first three Melnikov functions associated with the perturbed system. By employing the properties of the ETC system and the expressions of the Melnikov functions, the existence of exactly six limit cycles is given. Note that there are many cases for the existence of third-order Melnikov functions, and some existence conditions are very complicated—the corresponding Melnikov functions are not presented.

Keywords: Melnikov functions; bifurcation; limit cycles; generators

MSC: 34A99; 34C37; 37G15

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1. Introduction

In the study of planar systems of differential equations, one of the most challenging problems is the second part of Hilbert's 16th problem. Consider a planar system of differential equations in the following form

$$\dot{x} = P(x,y),\ \dot{y} = Q(x,y), \tag{1}$$

where P and Q are polynomials of degree n. Then, the second part of Hilbert's 16th problem asks for the upper bound for the number of limit cycles in planar polynomial systems of degree n and their relative positions (see [1,2], for example). Due to its difficulty, a weakened version, now known as the weakened 16th Hilbert problem is posed by Arnold [3] that asks for the upper bound for the number of limit cycles of the perturbed system

$$\dot{x} = \frac{\partial H}{\partial y} + \epsilon P(x,y),\ \dot{y} = -\frac{\partial H}{\partial y} + \epsilon Q(x,y), \tag{2}$$

where $|\epsilon| \ll 1$ and $H(x,y)$ is a Hamiltonian function. The problem has been researched for $n=2$ by several research groups independently [4–6], and to answer this weakened version, various methods have been developed, among them a popular method is based on the Melnikov functions. By computing the higher-order Melnikov functions based on the algorithm of [7,8], researchers have studied the number of limit cycles bifurcated from the above perturbed system (see [9–20]). The results in [12] showed that quasi-homogeneous polynomial Hamiltonian systems have a bound on the number of limit cycle bifurcations from the period annulus at any order of Melnikov functions. Asheghi and Nabavi [16] discussed the following perturbed system

$$\begin{cases} \dot{x} = -y(1-x^2) + \epsilon \sum\limits_{i+j=0}^{2} a_{ij} x^i y^j, \\ \dot{y} = x(1-x^2) - \epsilon \sum\limits_{i+j=0}^{2} b_{ij} x^i y^j. \end{cases} \tag{3}$$

They studied the limit cycles by using Melnikov functions up to order three and proved that there are six or seven limit cycles of system (3).

Yang and Yu [17] considered the following system with $n = 2$

$$\begin{cases} \dot{x} = y(1+x)^2 - \epsilon \sum\limits_{i+j=0}^{n} a_{ij} x^i y^j, \\ \dot{y} = -x(1+x)^2 + \epsilon \sum\limits_{i+j=0}^{n} b_{ij} x^i y^j, \end{cases} \tag{4}$$

using the Melnikov function of any order and proved that the upper bound for the number of limit cycles is three.

Following the work of [17], Liu [18] studied the number of limit cycles bifurcated from the origin of the following perturbed system

$$\begin{cases} \dot{x} = -y(1+x)^3 + \epsilon \sum\limits_{i+j=0}^{3} a_{ij} x^i y^j, \\ \dot{y} = x(1+x)^3 - \epsilon \sum\limits_{i+j=0}^{3} b_{ij} x^i y^j. \end{cases} \tag{5}$$

The main results showed that the first four Melnikov functions associated with the perturbed system can yield five limit cycles.

For system (4), it is easy to see when $\epsilon = 0$; there exists a family of periodic orbits $\Gamma_h : H(x,y) = \frac{1}{2}(x^2+y^2) = h$ for $0 < h < 1/2$ surrounding the origin. Then, the higher order Melnikov functions can be defined in the following way: fixing a transversal segment to the flow in (4) and using the energy level h to parameterize it, the corresponding displacement function is

$$d(h,\epsilon) = \epsilon M_1(h) + \epsilon^2 M_2(h) + \epsilon^3 M_3(h) + \cdots, h \in \left(0, \frac{1}{2}\right), \tag{6}$$

where $M_k(h)$ is called the kth-order Melnikov function of system (4). For example, the first-order Melnikov function of system (4) takes the form

$$M_1(h) = \oint_{H=h} \frac{Q(x,y)}{(1+x)^2} dx + \frac{P(x,y)}{(1+x)^2} dy. \tag{7}$$

If $M_1(h) \not\equiv 0$, every simple zero $h_0 \in \left(0, \frac{1}{2}\right)$ of $M_1(h)$ corresponds to a limit cycle of the perturbed system (4) near Γ_{h_0}. If $M_1(h) \equiv 0$, then the simple zero of $M_2(h)$ will become very important in investigating the number of limit cycles of the perturbed system. Similarly, we may use $M_3(h)$, $M_4(h)$,$\cdots$ to study the number of limit cycles [9,10,13,15,17].

In this paper, we also consider system (4), but with $n = 3$. Thus, the system becomes a planar cubic system. We will study the upper bound of limit cycles of the system by using the higher-order Melnikov functions. Although, our system seems similar to the system in [17,18], it needs more-difficult 1-form decompositions than theirs.

The organization of the paper is as follows: in Section 2, some Lemmas and formulae are presented as preliminaries; in Section 3, the first three Melnikov functions are computed, and the corresponding bifurcated limit cycles are given; in Section 4, a brief discussion is presented.

2. Preliminaries

Let

$$\omega = \frac{Q(x,y)}{(1+x)^2}dx + \frac{P(x,y)}{(1+x)^2}dy,$$

then the Pfaffian form of system (4) is $dH = \epsilon\omega$.

The following Lemmas and Remarks shall be used to prove the main Theorems. The algorithm of calculating $M_k(h)$ is shown by Francoise [7] and Iliev [8].

Lemma 1 ([8,17])**.** *Assume Γ_h is the period annulus defined by $H(x,y) = h$, the polynomial function H(x,y) and the 1-form ω satisfy $\oint_{H=h} \omega = 0$ if and only if there are two analytic functions $q(x,y)$ and $S(x,y)$ in a neighborhood of Γ_h such that $\omega = qdH + dS$.*

Lemma 2 ([7,17])**.** *Let $\omega = \bar{q}_0 dH + d\bar{Q}_0 + N_0$, then $M_1(h) = \oint_{H=h} \omega = \oint_{H=h} N_0$;*

(1) *If $M_1(h) \equiv 0$, then $N_0 = \tilde{q}_0 dH + d\tilde{Q}_0$. Let $q_0 = \bar{q}_0 + \tilde{q}_0$, $Q_0 = \bar{Q}_0 + \tilde{Q}_0$; we have $\omega = q_0 dH + dQ_0$, and $q_0\omega$ can be decomposed into $q_0\omega = \bar{q}_1 dH + d\bar{Q}_1 + N_1$; then $M_2(h) = \oint_{H=h} q_0\omega = \oint_{H=h} N_1$;*

(2) *If $M_2(h) \equiv 0$, then $N_1 = \tilde{q}_1 dH + d\tilde{Q}_1$. Let $q_1 = \bar{q}_1 + \tilde{q}_1$, $Q_1 = \bar{Q}_1 + \tilde{Q}_1$; we have $q_0\omega = q_1 dH + dQ_1$, and $q_1\omega$ can be decomposed into $q_1\omega = \bar{q}_2 dH + d\bar{Q}_2 + N_2$; then $M_3(h) = \oint_{H=h} q_1\omega = \oint_{H=h} N_2$;*
By this way, if $M_1(h) = M_2(h) = \cdots = M_{i-1}(h) \equiv 0$, then $q_j\omega = q_{j+1}dH + dQ_{j+1}, j \leq i-2$ and we assume $q_{-1} = 1$. We have $M_i(h) = \oint_{H=h} q_{i-2}\omega = \oint_{H=h} N_{i-1}$, where $q_{i-2}\omega = \bar{q}_{i-1}dH + d\bar{Q}_{i-1} + N_{i-1}$.

Remark 1. *The authors in [17] have noted that $q_i\omega = q_{i+1}dH + dQ_{i+1}$, dQ_i is not used in the subsequent calculative process when $i > 0$. Thus, the specific form of dQ_i for $i \geq 1$ will not be shown in the following sections.*

In the next lemma, an algorithm to decompose $q_i\omega$ is given to simplify the expression of $M_k(h)$.

Define

$$\begin{aligned} \omega_{ij}^k &= \frac{x^i y^j}{(1+x)^k}dx, \quad \delta_{ij}^k = \frac{x^i y^j}{(1+x)^k}dy, \quad \Omega_{ij}^k = \frac{x^i y^j \ln(1+x)}{(1+x)^k}dx, \\ \Delta_{ij}^k &= \frac{x^i y^j \ln(1+x)}{(1+x)^k}dy, \quad \Omega_{ij2}^k = \frac{x^i y^j \ln^2(1+x)}{(1+x)^k}dx, \\ \Delta_{ij2}^k &= \frac{x^i y^j \ln^2(1+x)}{(1+x)^k}dy, \quad J_k(h) = \oint_{H=h} \delta_{00}^k, \quad \bar{J}_k(h) = \oint_{H=h} \Delta_{00}^k, \quad k \geq 0, \end{aligned} \tag{8}$$

where $J_k(h)$ and $\bar{J}_k(h)$ are called the generators of $M_i(h)$.

Therefore, ω also can be rewritten as

$$\omega = \sum_{i+j=0}^{3} (a_{ij}\delta_{ij}^2 + b_{ij}\omega_{ij}^2). \tag{9}$$

Lemma 3. (i) *For $k = 2$, there exist 1-forms ω_{ij} and δ_{ij} as follows:*

$$
\begin{aligned}
\delta_{01}^2 =& \frac{dH}{(1+x)^2} - \frac{xdx}{(1+x)^2}, \quad \delta_{10}^2 = \delta_{00}^1 - \delta_{00}^2, \quad \delta_{02}^2 = 2H\delta_{00}^2 - dy + 2\delta_{00}^1 - \delta_{00}^2, \\
\delta_{11}^2 =& \frac{xdH}{(1+x)^2} - \frac{x^2dx}{(1+x)^2}, \quad \delta_{20}^2 = \delta_{00}^2 - 2\delta_{00}^1 + dy, \quad \omega_{00}^2 = \frac{dx}{(1+x)^2}, \\
\omega_{10}^2 =& \frac{xdx}{(1+x)^2}, \quad \omega_{20}^2 = \frac{x^2dx}{(1+x)^2}, \quad \omega_{01}^2 = \frac{ydx}{(1+x)^2} - \frac{dy}{1+x} + \delta_{00}^1, \\
\omega_{02}^2 =& 2\frac{dH}{1+x} - 2d\left(\frac{H}{1+x}\right) - \frac{x^2dx}{(1+x)^2}, \\
\omega_{11}^2 =& \frac{ydH}{(1+x)^2} - 2H\delta_{00}^2 + \delta_{00}^2 - 2\delta_{00}^1 + dy, \\
\omega_{11}^1 =& d(-y\ln(1+x)) + \omega_{01}^0 + \Delta_{00}^0, \quad \omega_{03}^2 = d\left(-\frac{y^3}{1+x}\right) + \frac{3ydH}{1+x} - 3\omega_{11}^1, \\
\omega_{12}^2 =& d\left(2H\left(\frac{1}{1+x} + \ln(1+x)\right)\right) - 2\left(\frac{1}{1+x} + \ln(1+x)\right)dH - \frac{x^3dx}{(1+x)^2}, \\
\delta_{03}^2 =& \frac{y^2dH}{(1+x)^2} - \omega_{12}^2, \quad \omega_{21}^2 = \omega_{11}^1 - \omega_{11}^2, \quad \delta_{12}^2 = \frac{xydH}{(1+x)^2} - \omega_{11}^1 + \omega_{11}^2, \\
\delta_{30}^2 =& 2H(\delta_{00}^1 - \delta_{00}^2) - \delta_{12}^2, \quad \omega_{30}^2 = \frac{x^3dx}{(1+x)^2}, \quad \delta_{21}^2 = \frac{x^2dH}{(1+x)^2} - \frac{x^3dx}{(1+x)^2}.
\end{aligned}
\tag{10}
$$

(ii) *If j is even,*

$$
\begin{aligned}
\omega_{04}^k =& \frac{(2H-x^2)^2}{(1+x)^k}dx = \frac{4H^2 - 4Hx^2 + x^4}{(1+x)^k}dx \\
=& 4H^2 d\left(\frac{-1}{(k-1)(1+x)^{k-1}}\right) - 4Hd\left(\int \frac{x^2}{(1+x)^k}dx\right) + \frac{x^4}{(1+x)^k}dx
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
=& d\left(-\frac{4H^2}{(k-1)(1+x)^{k-1}}\right) + \frac{8H}{(k-1)(1+x)^{k-1}}dH - d\left(4H\int \frac{x^2}{(1+x)^k}dx\right) \\
& + 4\int \frac{x^2}{(1+x)^k}dH + \frac{x^4}{(1+x)^k}dx \\
=& \left(\frac{8H}{(k-1)(1+x)^{k-1}} + 4\int \frac{x^2}{(1+x)^k}dx\right)dH + dQ_{04}^k(x,H), \\
\omega_{06}^k =& \left(\frac{24H^2}{(k-1)(1+x)^{k-1}} + 24H\int \frac{x^2}{(1+x)^k}dx - 6\int \frac{x^4}{(1+x)^k}dx\right)dH + dQ_{06}^k(x,H), \\
\omega_{02}^k =& \frac{2}{(k-1)(1+x)^{k-1}}dH + dQ_{02}^k(x,H), \\
\omega_{08}^k =& \left(\frac{64H^3}{(k-1)(1+x)^{k-1}} + 96H^2\int \frac{x^2}{(1+x)^k}dx - 48H\int \frac{x^4}{(1+x)^k}dx \right. \\
& \left. + 8\int \frac{x^6}{(1+x)^k}dx\right)dH + dQ_{08}^k(x,H),
\end{aligned}
$$

$$
\begin{aligned}
\delta_{04}^k =& \frac{(2H-x^2)^2}{(1+x)^k}dy = (2H-1)^2\delta_{00}^k + (8H-4)\delta_{00}^{k-1} + (6-4H)\delta_{00}^{k-2} - 4\delta_{00}^{k-3} + \delta_{00}^{k-4},\\
\delta_{06}^k =& 8H^3\delta_{00}^k - 12H^2\left(\delta_{00}^k - 2\delta_{00}^{k-1} + \delta_{00}^{k-2}\right) + 6H\left(\delta_{00}^k - 4\delta_{00}^{k-1} + 6\delta_{00}^{k-2} - 4\delta_{00}^{k-3} + \delta_{00}^{k-4}\right)\\
& - \delta_{00}^k + 6\delta_{00}^{k-1} - 15\delta_{00}^{k-2} + 20\delta_{00}^{k-3} - 15\delta_{00}^{k-4} + 6\delta_{00}^{k-5} - \delta_{00}^{k-6},\\
\delta_{02}^k =& 2H\delta_{00}^k - \delta_{00}^k - \delta_{00}^{k-2} + 2\delta_{00}^{k-1},\\
\Omega_{02}^k =& \frac{(2H-x^2)\ln(1+x)}{(1+x)^k}dx = 2Hd\int\frac{\ln(1+x)}{(1+x)^k}dx - \frac{x^2\ln(1+x)}{(1+x)^k}dx\\
&= -2\int\frac{\ln(1+x)}{(1+x)^k}dxdH + d\bar{Q}_{02}^k,\\
\Omega_{04}^k =& \left(-8H\int\frac{\ln(1+x)}{(1+x)^k}dx + 4\int\frac{x^2\ln(1+x)}{(1+x)^k}dx\right)dH + d\bar{Q}_{04}^k,\\
\Delta_{02}^k =& 2H\Delta_{00}^k - \Delta_{00}^k - \Delta_{00}^{k-2} + 2\Delta_{00}^{k-1},\\
\Delta_{04}^k =& (2H-1)^2\Delta_{00}^k + (8H-4)\Delta_{00}^{k-1} + (6-4H)\Delta_{00}^{k-2} - 4\Delta_{00}^{k-3} + \Delta_{00}^{k-4},
\end{aligned}
$$

If j is odd,

$$
\begin{aligned}
\omega_{0j}^k =& d\left(\frac{-y^j}{(k-1)(1+x)^{k-1}}\right) + \frac{j}{k-1}\delta_{0,j-1}^{k-1} = dS_{0j}^k(x,y) + \frac{j}{k-1}\delta_{0,j-1}^{k-1},\\
\delta_{0j}^k =& \frac{y^{j-1}}{(1+x)^k}dH - \omega_{0,j-1}^{k-1} + \omega_{0,j-1}^k,\\
\Omega_{0j}^k =& d\left(y^j\int\frac{\ln(1+x)}{(1+x)^k}dx\right) + \frac{j}{k-1}\Delta_{0,j-1}^{k-1} + \frac{j}{(k-1)^2}\delta_{0,j-1}^{k-1}\\
&= d\bar{S}_{0j}^k(x,y) + \frac{j}{k-1}\Delta_{0,j-1}^{k-1} + \frac{j}{(k-1)^2}\delta_{0,j-1}^{k-1},\\
\Delta_{0j}^k =& \frac{y^{j-1}\ln(1+x)}{(1+x)^k}dH - \Omega_{0,j-1}^{k-1} + \Omega_{0,j-1}^k.
\end{aligned}
$$

Proof. (i) Here, we use partial integration and the Laurent expansion method to decompose ω_{ij}^2 and δ_{ij}^2, and some decompositions have been proved in [17].

$$
\begin{aligned}
\omega_{11}^1 =& \frac{xy}{1+x}dx = yd(x-\ln(1+x)) = d(y(x-\ln(1+x))) - (x-\ln(1+x))dy\\
&= d(-y\ln(1+x)) + \omega_{01}^0 + \Delta_{00}^0,\\
\omega_{03}^2 =& y^3d\left(-\frac{1}{1+x}\right) = d\left(-\frac{y^3}{1+x}\right) + \frac{3y^2}{1+x}dy = d\left(-\frac{y^3}{1+x}\right) + \frac{3y}{2(1+x)}dy^2\\
&= d\left(-\frac{y^3}{1+x}\right) + \frac{3y}{1+x}dH - 3\omega_{11}^1,\\
\omega_{12}^2 =& \frac{x(2H-x^2)}{(1+x)^2}dx = 2Hd\left(\frac{1}{1+x} + \ln(1+x)\right) - \frac{x^3}{(1+x)^2}dx\\
=& d\left(2H\left(\frac{1}{1+x} + \ln(1+x)\right)\right) - 2\left(\frac{1}{1+x} + \ln(1+x)\right)dH - \frac{x^3dx}{(1+x)^2},\\
\delta_{03}^2 =& \frac{y^2}{2(1+x)^2}dy^2 = \frac{y^2}{(1+x)^2}dH - \omega_{12}^2,\\
\delta_{30}^2 =& \frac{x(2H-y^2)}{(1+x)^2}dy = 2H(\delta_{00}^1 - \delta_{00}^2) - \delta_{12}^2.
\end{aligned}
\tag{12}
$$

The Formula (10) is obtained. The proof of (ii) is omitted. □

Lemma 4. *The generators $J_k(h)$ shown in (8) can be obtained through Maple software*

$$
\begin{aligned}
J_1(h) =& -2\pi\left(1-\frac{1}{\sqrt{1-2h}}\right), \quad J_2(h) = \frac{4h\pi}{(1-2h)^{3/2}}, \quad J_3(h) = \frac{6h\pi}{(1-2h)^{5/2}},\\
J_4(h) =& \frac{4h\pi(h+2)}{(1-2h)^{7/2}}, \quad J_5(h) = \frac{5h\pi(3h+2)}{(1-2h)^{9/2}}, \quad J_6(h) = \frac{6h\pi(h^2+6h+2)}{(1-2h)^{11/2}},\\
J_7(h) =& \frac{7h\pi(5h^2+10h+2)}{(1-2h)^{13/2}}, \quad J_8(h) = \frac{2h\pi(5h^3+60h^2+60h+8)}{(1-2h)^{15/2}},\\
J_{01}(h) =& \oint\limits_{H=h} \omega_{01}^0 = 2\pi h.
\end{aligned} \tag{13}
$$

Define

$$
F_k(r) = \int_0^{2\pi} \frac{\cos^k\theta \ln(1+r\cos\theta)}{1+r\cos\theta} d\theta, \quad G_k(r) = \int_0^{2\pi} \cos^k\theta \ln(1+r\cos\theta) d\theta, \tag{14}
$$

then

$$
\begin{aligned}
F_0(r) =& \frac{2\pi}{\sqrt{1-r^2}} \ln\left(\frac{2(1-r^2)}{1+\sqrt{1-r^2}}\right),\\
F_2(r) =& \frac{2\pi\left(1-\sqrt{1-r^2}\right)}{r^2} - \frac{2\pi}{r^2}\ln\frac{1+\sqrt{1-r^2}}{2} + \frac{2\pi}{r^2\sqrt{1-r^2}}\ln\left(\frac{2(1-r^2)}{1+\sqrt{1-r^2}}\right),\\
G_0(r) =& 2\pi\ln\frac{1+\sqrt{1-r^2}}{2}, \quad G_1(r) = \frac{2\pi\left(1-\sqrt{1-r^2}\right)}{r}.
\end{aligned} \tag{15}
$$

The proof of the expressions $F_0(r)$, $F_2(r)$, $G_0(r)$ and $G_1(r)$ can be seen in [10].

Lemma 5. *The following expressions can be deduced by using the above formulae*

$$
\begin{aligned}
\oint\limits_{H=h} \Delta_{002}^0 =& -2r^2(F_0(r)-F_2(r)), \quad \bar{J}_0(r) = -rG_1(r), \quad \bar{J}_1(r) = -rF_1(r) = F_0(r) - G_0(r),\\
\bar{J}_2(r) =& rF_0'(r) + J_2(r), \quad \bar{J}_3(r) = \frac{1}{2}J_3(r) + \frac{r}{2}(F_0(r)+\bar{J}_2(r))', \quad \bar{J}_{10}(r) = -r^2G_2(r),
\end{aligned} \tag{16}
$$

where

$$
G_2(r) = \pi\left(\ln\frac{1+\sqrt{1-r^2}}{2} + \frac{\sqrt{1-r^2}-1}{r^2} + \frac{1}{2}\right), \quad r = \sqrt{2h}. \tag{17}
$$

Proof. Since

$$
\begin{aligned}
\Omega_{01}^1 =& \frac{y\ln(1+x)}{1+x}dx = yd\left(\frac{1}{2}\ln^2(1+x)\right) = d\left(\frac{1}{2}y\ln^2(1+x)\right) - \frac{1}{2}\ln^2(1+x)dy\\
=& d\left(\frac{1}{2}y\ln^2(1+x)\right) - \frac{1}{2}\Delta_{002}^0,
\end{aligned} \tag{18}
$$

therefore

$$
\oint\limits_{H=h} \Omega_{01}^1 = -\frac{1}{2}\oint\limits_{H=h} \Delta_{002}^0. \tag{19}
$$

Let $r = \sqrt{2h}$, $x = r\cos\theta$ and $y = r\sin\theta$, then

$$\oint\limits_{H=h}\Omega_{01}^{1} = r^2\int\limits_0^{2\pi}\frac{\sin^2\theta\ln(1+r\cos\theta)}{1+r\cos\theta}d\theta = r^2\int\limits_0^{2\pi}\frac{(1-\cos^2\theta)\ln(1+r\cos\theta)}{1+r\cos\theta}d\theta \\ = r^2(F_0(r)-F_2(r)), \tag{20}$$

which implies $\oint\limits_{H=h}\Delta_{002}^{0} = -2r^2(F_0(r)-F_2(r))$. We also have

$$\begin{aligned} \bar{J}_0(r) &= \oint\limits_{H=h}\ln(1+x)dy = -r\int_0^{2\pi}\cos\theta\ln(1+r\cos\theta)d\theta = -rG_1(r). \\ \bar{J}_1(r) &= \oint\limits_{H=h}\frac{\ln(1+x)}{1+x}dy = -r\int_0^{2\pi}\frac{\cos\theta\ln(1+r\cos\theta)}{1+r\cos\theta}d\theta = -rF_1(r) \\ &= -r\int_0^{2\pi}\ln(1+r\cos\theta)\left(\frac{1}{r}-\frac{1}{r(1+r\cos\theta)}\right)d\theta = F_0(r)-G_0(r). \end{aligned} \tag{21}$$

By

$$F_0'(r) = \int_0^{2\pi}-\frac{\cos\theta(\ln(1+r\cos\theta)-1)}{(1+r\cos\theta)^2}d\theta \tag{22}$$

we have

$$\begin{aligned} \bar{J}_2(r) &= \oint\limits_{H=h}\frac{\ln(1+x)}{(1+x)^2}dy = -r\int_0^{2\pi}\frac{\cos\theta\ln(1+r\cos\theta)}{(1+r\cos\theta)^2}d\theta \\ &= r\left(F_0'(r)-\int_0^{2\pi}\frac{\cos\theta}{(1+r\cos\theta)^2}d\theta\right) = rF_0'(r)+J_2(r). \end{aligned} \tag{23}$$

Because

$$\begin{aligned} F_{02}(r) :&= \int_0^{2\pi}\frac{\ln(1+r\cos\theta)}{(1+r\cos\theta)^2}d\theta \\ &= \int_0^{2\pi}\ln(1+r\cos\theta)\left(\frac{1}{1+r\cos\theta}-\frac{r\cos\theta}{(1+r\cos\theta)^2}\right)d\theta = F_0(r)+\bar{J}_2(r), \end{aligned} \tag{24}$$

on the other hand, we have

$$F_{02}'(r) = \int_0^{2\pi}\frac{\cos\theta(1-2\ln(1+r\cos\theta))}{(1+r\cos\theta)^3}d\theta, \tag{25}$$

therefore,

$$\begin{aligned} \bar{J}_3(r) &= -r\int_0^{2\pi}\frac{\cos\theta\ln(1+r\cos\theta)}{(1+r\cos\theta)^3}d\theta = -r\left(\frac{1}{2}\int_0^{2\pi}\frac{\cos\theta}{(1+r\cos\theta)^3}-\frac{1}{2}F_{02}'(r)\right) \\ &= \frac{1}{2}J_3(r)+\frac{r}{2}(F_0(r)+\bar{J}_2(r))'. \\ \oint\limits_{H=h}\Delta_{10}^{0} &= \oint\limits_{H=h}x\ln(1+x)dy = -r^2\int_0^{2\pi}\cos^2\theta\ln(1+r\cos\theta)d\theta = -r^2G_2(r), \\ G_2'(r) &= \int_0^{2\pi}\frac{\cos^3\theta}{1+r\cos\theta}d\theta = -\frac{1}{r^3}\oint\limits_{H=h}\frac{x^2}{1+x}dy \\ &= \frac{2}{r^3}\int_{-r}^{r}\frac{x^3}{(1+x)\sqrt{r^2-x^2}}dx = \frac{\pi\left(r^4+r^2+2\sqrt{1-r^2}-2\right)}{r^3(r^2-1)}, \\ G_2(r) &= \pi\left(\ln\frac{1+\sqrt{1-r^2}}{2}+\frac{\sqrt{1-r^2}-1}{r^2}+\frac{1}{2}\right) \quad \text{for } G_2(0)=0. \end{aligned} \tag{26}$$

□

The Chebyshev criterion will be used in the following (see [16,21,22] for example).

Let f_0, f_1,...,f_{n-1} be analytic functions on an open interval L of R. An ordered set $(f_0, f_1, ..., f_{n-1})$ is an extended complete Chebyshev system (in short, ECT system) on L if, for all $i = 1, 2, ..., n-1$, any nontrivial linear

$$\lambda_0 f_0(x) + \lambda_1 f_1(x) + ... + \lambda_i f_i(x) \tag{27}$$

has at most $i-1$ isolated zeros on L counted with multiplicities.

Lemma 6 ([21])**.** *$(f_0, f_1, ..., f_{n-1})$ is an ECT system on L if, and only if, for each $i = 1, 2, ..., n$,*

$$W_{i-1}(x) = W[f_0(x), f_1(x), \cdots, f_{i-1}(x)] \neq 0 \tag{28}$$

for all $x \in L$.

Remark 2. *If $(f_0, f_1, ..., f_{n-1})$ is an ECT system on L, then for each $i = 1, 2, ..., n-1$ there exists a linear combination (27) with exactly $i-1$ simple zeros on L (see, for instance, Remark 3.7 in [22]).*

3. The Calculation of $M_k(h)$

3.1. The Melnikov Function of First-Order $M_1(h)$

Lemma 7. *The 1-form ω of (9) can be expressed*

$$\omega = \bar{q}_0 dH + d\bar{Q}_0 + N_0, \tag{29}$$

where

$$\begin{aligned} \bar{q}_0 =& \bar{q}_0(x,y) = \frac{a_{03}y^2 + (b_{11} - b_{21})y}{(1+x)^2} + \frac{(a_{12} - a_{30} + 3b_{03})y}{1+x} + \frac{a_{01} - a_{11} + a_{21}}{(1+x)^2} \\ &+ \frac{a_{11} - 2a_{21} + 2a_{03} + 2b_{02} - 2b_{12}}{1+x} + a_{21} + 2(a_{03} - b_{12})\ln(1+x), \end{aligned} \tag{30}$$

$$\begin{aligned} d\bar{Q}_0 =& d\bar{Q}_0(x,y,H) = -2\left(\frac{a_{03} + b_{02} - b_{12}}{1+x} + (a_{03} - b_{12})\ln(1+x)\right)dH + \\ &\left(\frac{b_{03}y^3 - (b_{12} - a_{03} - b_{02})y^2 + b_{01}y}{(1+x)^2} + \frac{(b_{12} - a_{03})y^2 + (a_{12} - a_{30} + 3b_{03} - b_{21})y}{1+x}\right)dx \\ &- \frac{((a_{21} - b_{30})x^3 + (a_{11} - b_{20})x^2 + (a_{01} - b_{10})x - b_{00})dx}{(1+x)^2} \\ &+ \Big((a_{12} - a_{30} + 3b_{03} - b_{21})\ln(1+x) + a_{20} - a_{02} - a_{30} + a_{12} \\ &- \frac{b_{01}}{1+x} + b_{11} - b_{21} - \frac{3b_{03}y^2}{1+x}\Big)dy, \\ N_0 =& A_0\omega_{01}^0 + (HA_{21} + A_{20})\delta_{00}^2 + (HA_{11} + A_{10})\delta_{00}^1 + A_0\Delta_{00}^0, \text{ with} \\ A_0 =& a_{30} - a_{12} + b_{21} - 3b_{03}, \ A_{21} = 2a_{02} - 2a_{12} - 2b_{11} + 2b_{21}, \\ A_{20} =& a_{00} - a_{10} + a_{20} - a_{02} - a_{30} + a_{12} + b_{11} - b_{21}, \\ A_{11} =& 2a_{30}, \ A_{10} = 2a_{02} + a_{10} - 2a_{12} - 2a_{20} + 2a_{30} + b_{01} - 2b_{11} + 2b_{21}. \end{aligned} \tag{31}$$

Theorem 1. *$M_1(h)$ has at most three zeros, i.e., system (4) has at most three limit cycles by the first-order Melnikov function, and the upper bound for the number of limit cycles is reached.*

Proof. Let $z = \sqrt{1-2h}$, $z \in (0,1)$; by Lemmas 2, 4 and 5,

$$
\begin{aligned}
M_1(h) =& \oint_{H=h} N_0 = A_0\bar{J}_1(h) + (hA_{21}+A_{20})J_2(h) + (hA_{11}+A_{10})J_1(h) + A_0J_0(h)\\
&= M_1\left(\frac{1-z^2}{2}\right) = \frac{\pi(1-z)}{z^3}\Big(n_4z^4+n_3z^3+n_2z^2+n_0(z+1)\Big),\\
n_0 =& 2(a_0-a_{10}+a_{20}-a_{30}),\\
n_2 =& 2(a_2+a_{10}-a_{12}-2a_{20}+3a_{30}+b_1-b_{11}+b_{21}),\\
n_3 =& (-2a_2+3a_{12}-a_{30}+3b_3+2b_{11}-3b_{21}),\quad n_4 = (-a_{12}-a_{30}-3b_3+b_{21}).
\end{aligned} \tag{32}
$$

It is easy to verify that $(z+1, z^2, z^3, z^4)$ is an ECT system for $z>0$ by Lemma 6 and since $\frac{\partial(n_0,n_2,n_3,n_4)}{\partial(a_{10},a_{30},b_{11},b_{21})} = -8 \neq 0$. Therefore, $M_1(h)$ has exactly three zeros when taking appropriate coefficients by Remark 2. □

3.2. The Melnikov Function of Second-Order $M_2(h)$

$M_1(h) \equiv 0$ implies that $n_0 = n_2 = n_3 = n_4 = 0$; solving them, we have

$$
\begin{aligned}
a_{00} =& -a_{30}+a_{20}-b_{01},\quad a_{10} = -2a_{30}+2a_{20}-b_{01},\quad a_{12} = -a_{30}+b_{21}-3b_{03},\\
b_{11} =& 2a_{30}+3b_{03}+a_{02}.
\end{aligned} \tag{33}
$$

Let (33) hold; we compute $M_2(h)$. First, it has

$$
\begin{aligned}
N_0 =& a_{30}\Big(2H\delta^1_{00} - 2H\delta^2_{00} + 2\Delta^0_{00} - 2\delta^1_{00} + \delta^2_{00} + 2\omega_{01}\Big)\\
=& a_{30}\left(\frac{\Big(2(1+x)^2\ln(1+x) + x^3 + xy^2 - 2x - 1\Big)dy}{(1+x)^2} + 2ydx\right)\\
=& a_{30}\left(\frac{xy^2dy}{(1+x)^2} + 2\ln(1+x) + \frac{(x^2-x-1)dy}{1+x} + 2ydx\right)\\
=& a_{30}\left(\frac{xydH}{(1+x)^2} + d\left(2\ln(1+x) + \frac{x^2-x-1}{1+x}\right)y\right),
\end{aligned} \tag{34}
$$

which implies that

$$
\tilde{q}_0 = \frac{a_{30}yx}{(1+x)^2},\quad d\tilde{Q}_0 = d\left(2\ln(1+x) + \frac{x^2-x-1}{1+x}\right)y \tag{35}
$$

by Lemma 2. Then

$$
\begin{aligned}
q_0 =& \bar{q}_0 + \tilde{q}_0 = f_0(x) + g_0(x,y),\\
f_0(x) =& \frac{a_{01}-a_{11}+a_{21}}{(1+x)^2} + \frac{a_{11}-2a_{21}+2a_{03}+2b_{02}-2b_{12}}{1+x} + a_{21} +\\
& 2\ln(1+x)(a_{03}-b_{12}),\\
g_0(x,y) =& \frac{a_{03}y^2 + (2a_{30}+3b_{03}+a_{02}-b_{21})y}{(1+x)^2} + \frac{(b_{21}-2a_{30})y}{1+x} + \frac{a_{30}yx}{(1+x)^2},\\
dQ_0 =& d\bar{Q}_0 + d\tilde{Q}_0 = \Omega_1(x) + \Omega_2(x,y) + h(x)dH,\\
\Omega_1(x) =& -\frac{((a_{21}-b_{30})x^3 + (a_{11}-b_{20})x^2 + (a_{01}-b_{10})x - b_{00})dx}{(1+x)^2},\\
\Omega_2(x,y) =& \left(\frac{b_{03}y^3}{(1+x)^2} - \frac{((a_{03}-b_{12})x - b_{02})y^2}{(1+x)^2} + \frac{(x^2a_{30}+2xa_{30}+b_{01})y}{(1+x)^2}\right)dx\\
& + \left(\frac{x^2a_{30} + (a_{20}-a_{30})x - 3b_{03}y^2 + a_{20} - a_{30} - b_{01}}{1+x}\right)dy,\\
h(x) =& -\frac{2(a_{03}+b_{02}-b_{12})}{1+x} - 2(a_{03}-b_{12})\ln(1+x),\ \text{and}\ \omega = q_0dH + dQ_0.
\end{aligned} \tag{36}
$$

Lemma 8. *The 1-form $q_0\omega$ can be decomposed into*

$$q_0\omega = \bar{q}_1 dH + d\bar{Q}_1 + N_1, \tag{37}$$

where

$$\begin{aligned}
\bar{q}_1 =& q_0 h(x) + q_0^2 + q_{10},\\
q_{10} =& 2(a_{03}-b_{12})^2\ln^2(1+x) + \left(2\beta_{03}^2 - 2\alpha_{02}^1 + 4\alpha_{04}^3 + \frac{4(a_{03}-b_{12})(a_{03}+b_{02}-b_{12})}{1+x}\right)\\
&\times\ln(1+x) + \frac{2\alpha_{02}^4+3\beta_{01}^3}{3(1+x)^3} + \frac{\alpha_{02}^3+\beta_{01}^2}{(1+x)^2} + \frac{2\alpha_{02}^2+\beta_{01}^1}{1+x} + \frac{4\alpha_{04}^4(-3x^2+2H-3x-1)}{3(1+x)^3}\\
&- \frac{\beta_{03}^3(2x^2-y^2+3x+1)}{(1+x)^3} + \frac{y(3\alpha_{03}^2+\beta_{02}^1)}{1+x} + \frac{2\alpha_{04}^3(2H+3+4x)}{(1+x)^2}\\
&+ \frac{\beta_{03}^2(y^2+2x+2)}{(1+x)^2} + \frac{2(a_{03}-b_{12})(3yb_{03}+2a_{03}+2b_{02}-2b_{12})}{1+x},\\
N_1 =& \left(4\left(\frac{5\alpha_{05}^4}{3}+\beta_{04}^3\right)H^2 + 2\left(\beta_{02}^3 - \frac{10\alpha_{05}^4}{3} - 2\beta_{04}^3 + \alpha_{03}^4\right)H - \beta_{02}^3 + \frac{5}{3}\alpha_{05}^4 + \beta_{04}^3 - \alpha_{03}^4\right.\\
&\left.+\frac{1}{3}\alpha_{01}^4 + \beta_{00}^3\right)\delta_{00}^3 + \left(\left(\left(2\beta_{02}^2 + \frac{40\alpha_{05}^4}{3} + 8\beta_{04}^3 + 3\alpha_{03}^3\right)H + 2\beta_{02}^3 - \beta_{02}^2 - \frac{20\alpha_{05}^4}{3}\right.\right.\\
&\left.-4\beta_{04}^3 + 2\alpha_{03}^4 - \frac{3}{2}\alpha_{03}^3 + \frac{1}{2}\alpha_{01}^3 + \beta_{00}^2\right)\delta_{00}^2 + (2(a_{03}-b_{12})(b_{01}-a_{30})\\
&\left.-4\left(\frac{5\alpha_{05}^4}{3}+\beta_{04}^3\right)H - \beta_{02}^3 + 2\beta_{02}^2 + 10\alpha_{05}^4 + 6\beta_{04}^3 + \alpha_{01}^2 - \alpha_{03}^4 + 3\alpha_{03}^3 + \beta_{00}^1\right)\delta_{00}^1\\
&- \left(2(a_{03}-b_{12})(a_{30}+3b_{03}) + \frac{5}{3}\alpha_{05}^4 + \beta_{04}^3 + \beta_{10}^0 - \alpha_{01}^0 + \beta_{02}^1 + 3\alpha_{03}^2\right)\omega_{01}^0\\
&+ \left(2(a_{03}-b_{12})(a_{20}-3b_{03}-3a_{30}) - \alpha_{01}^1 - \beta_{02}^1 - 3\alpha_{03}^2\right)\Delta_{00}^0.
\end{aligned} \tag{38}$$

Proof. Following (36), we have

$$\begin{aligned}
q_0\omega =& q_0(q_0 dH + dQ_0) = q_0^2 dH + q_0 dQ_0 = q_0^2 dH + (f_0(x) + g_0(x,y))(\Omega_1(x)\\
&+ \Omega_2(x,y) + h(x)dH)\\
=& (q_0^2 + q_0 h(x))dH + f_0(x)\Omega_1(x) + g_0(x,y)\Omega_1(x) + q_0(x,y)\Omega_2(x,y)\\
=& (q_0^2 + q_0 h(x))dH + f_0(x)\Omega_1(x) + \sum_{k=0}^{4}\sum_{j=1}^{k+1}\alpha_{0j}^k\omega_{0j}^k + \sum_{k=0}^{3}\sum_{j=0}^{k+1}\beta_{0j}^k\delta_{0j}^k\\
&+ \sum_{j=1}^{3}\bar{\alpha}_{0j}^2\Omega_{0j}^2 + \bar{\alpha}_{02}^1\Omega_{02}^1 + \bar{\alpha}_{01}^0\Omega_{01}^0\\
&+ \beta_{10}^0\delta_{10}^0 + \bar{\beta}_{02}^1\Delta_{02}^1 + \bar{\beta}_{00}^1\Delta_{00}^1 + \bar{\beta}_{00}^0\Delta_{00}^0 + \bar{\beta}_{10}^0\Delta_{10}^0,
\end{aligned} \tag{39}$$

and the expressions of α_{ij}^k and β_{ij}^k are omitted.

From Lemma 3, we can calculate the decomposed expressions of ω_{0j}^k and δ_{0j}^k for $k \geq 2$. Together with

$$
\begin{aligned}
\omega_{01}^1 =& d(y\ln(1+x)) - \Delta_{00}^0, \quad \omega_{02}^1 = d(2H\ln(1+x)) - 2\ln(1+x)dH - \frac{x^2}{1+x}dx,\\
\Omega_{03}^2 =& d\left(-\frac{1+\ln(1+x)}{1+x}y^3\right) + 3y^2\frac{1+\ln(1+x)}{1+x}dy\\
&= d\left(-\frac{1+\ln(1+x)}{1+x}y^3\right) + 3(\delta_{02}^1 + \Delta_{02}^1),\\
\Omega_{02}^2 =& \frac{2(1+\ln(1+x))}{1+x}dH + d\bar{Q}_{02}^2(x,H),\\
\Omega_{02}^1 =& -(\ln(1+x))^2 dH + d\bar{Q}_{02}^1(x,H),\ \Omega_{01}^1 = d\left(\frac{1}{2}y\ln^2(1+x)\right) - \frac{1}{2}\Delta_{002}^0,\\
\Omega_{01}^2 =& d\left(-\frac{1+\ln(1+x)}{1+x}y\right) + \frac{1+\ln(1+x)}{1+x}dy = \delta_{00}^1 + \Delta_{00}^1 + d\bar{S}_{01}^2(x,y),\\
\Omega_{01}^0 =& dy(1+x)(\ln(1+x)-1) - (1+x)(\ln(1+x)-1)dy\\
&= -\Delta_{00}^0 - \Delta_{10}^0 - \omega_{01}^0 + d\bar{S}_{01}^0(x,y),\\
\Delta_{02}^1 =& \frac{y^2\ln(1+x)}{1+x}dy = \frac{y\ln(1+x)}{2(1+x)}dy^2 = \frac{y\ln(1+x)}{1+x}dH - \frac{xy\ln(1+x)}{1+x}dx\\
=& \frac{y\ln(1+x)}{1+x}dH - yd((1+x)(\ln(1+x)-1) - \frac{1}{2}\ln^2(1+x))\\
=& \frac{y\ln(1+x)}{1+x}dH + d\bar{S}_{02}^1(x,y) + \Delta_{00}^0 + \Delta_{10}^0 + \omega_{01}^0 - \frac{1}{2}\Delta_{002}^0,\\
\delta_{02}^1 =& \frac{y}{2(1+x)}dy^2 = \frac{y}{1+x}dH - \omega_{11}^1 = \frac{y}{1+x}dH + d(y\ln(1+x)) - \omega_{01}^0 - \Delta_{00}^0.
\end{aligned} \tag{40}
$$

By (39) and (40) and the expressions of ω_{0j}^k and δ_{0j}^k, (37) is given. □

Theorem 2. *Let $M_1(h) \equiv 0$; $M_2(h)$ has at most four zeros, then system (4) has at most four limit cycles by the second-order Melnikov function, and the maximum number can be attained.*

Proof. Let $z = \sqrt{1-2h}$, $z \in (0,1)$. Together with Lemmas 2, 4 and 5,

$$
M_2\left(\frac{1-z^2}{2}\right) = \frac{\pi(z-1)}{z^5}(l_6 z^6 + l_5 z^5 + l_4 z^4 + l_3 z^2(z+1) + l_1(z+1)). \tag{41}
$$

The function $(z+1, z^2(z+1), z^4, z^5, z^6)$ is an ECT system for $z > 0$ by Lemma 6. Furthermore, when we take $b_{03} = a_{21} = b_{30} = b_{10} = a_{03} = a_{01} = b_{02} = b_{00} = b_{20} = a_{20} = 0$, then $\frac{\partial(l_1,l_3,l_4,l_5,l_6)}{\partial(a_{02},a_{30},a_{11},b_{12},b_{01})} = 4a_{11}b_{21}((5a_{30}-3b_{21})a_{11}^2 + b_{12}(4a_{02}+21a_{30}-2b_{01}-7b_{21})a_{11} + 3b_{12}^2(-2b_{01}+3a_{30}+a_{02}-b_{21})) \neq 0$; it follows from Remark 2 that $M_2(h)$ has exactly four zeros by choosing appropriate coefficients $h \in (0, \frac{1}{2})$. □

3.3. The Third-Order Melnikov Function

Let $M_2(h) \equiv 0$, i.e., $l_1 = l_3 = l_4 = l_5 = l_6 = 0$ of Theorem 2. By using the Maple software, there are 28 cases. Some cases are short; however, some cases are very long, and we cannot continue to compute the higher-order Melnikov functions. Here, we only show five cases.

$$
\begin{aligned}
&\textit{Case}\,(1)\quad a_{02}=3b_{21},\, a_{20}=2b_{21},\, a_{30}=b_{01}=b_{21},\, b_{02}=-a_{03}-\frac{1}{2}a_{11}+a_{21},\\
&b_{03}=-\,b_{21},\, b_{12}=a_{03};\\
&\textit{Case}\,(2)\quad a_{01}=a_{21}+\frac{4}{3}a_{03}-2\,b_{02},\, a_{02}=-3b_{03},\, a_{11}=2\,a_{21}+\frac{4}{3}a_{03}-2\,b_{02},\\
&a_{20}=-\,6b_{03},\, a_{30}=b_{21}=0,\, b_{01}=-6b_{03},\, b_{12}=\frac{5}{3}a_{03};\\
&\textit{Case}\,(3)\quad a_{01}=a_{21}+\frac{2}{3}b_{12}-2\,b_{02},\, a_{03}=\frac{2}{3}b_{12},\, a_{11}=2a_{21}+\frac{2}{3}b_{12}-2b_{02},\\
&a_{20}=b_{01},\, a_{02}=a_{30}=b_{03}=b_{21}=0;\\
&\textit{Case}\,(4)\quad a_{01}=a_{21}-2b_{02},\, a_{02}=3b_{21},\, a_{03}=b_{12},\, a_{11}=2(a_{21}-b_{02}),\\
&a_{20}=a_{30}=b_{21},\, b_{03}=-b_{21};\\
&\textit{Case}\,(5)\quad a_{02}=a_{20},\, a_{03}=b_{12}=0,\, b_{21}=a_{30}=-b_{03},\\
&b_{00}=-\,\frac{(a_{20}-b_{01}+b_{03})(2a_{01}-a_{11}+2b_{02})}{a_{20}+3b_{03}},\\
&b_{10}=-\,\frac{2a_{1}a_{20}+2a_{01}b_{03}-2a_{11}b_{01}-a_{11}b_{03}+4a_{20}b_{02}-2a_{20}b_{20}-2b_{01}b_{2}+2b_{01}b_{20}}{2(a_{20}+3b_{03})}\\
&\qquad-\frac{5b_{02}b_{03}-3b_{03}b_{20}}{2(a_{20}+3b_{03})},\\
&b_{30}=-\,\frac{a_{11}b_{03}-a_{20}b_{02}-a_{20}b_{20}-2b_{02}b_{03}-4b_{03}b_{20}}{2(a_{20}+3b_{03})}.
\end{aligned}
\tag{42}
$$

Assume that $M_1(h)=M_2(h)\equiv 0$, then

$$q_1\omega=\bar{q}_2 dH+d\bar{Q}_2+N_2. \tag{43}$$

Lemma 9. *In Case (1), there exists* (43)*, where*

$$
\begin{aligned}
q_1=&\bar{q}_1+\tilde{q}_1=f_1(x)+g_1(x,y),\\
f_1(x)=&\frac{(a_{01}-a_{11}+a_{21})^2}{(1+x)^4}+\frac{2a_{03}(a_{01}-a_{11}+a_{21}+b_0-b_{10}+b_{20}-b_{30})}{3(1+x)^3}\\
&+\frac{-2(a_{01}-a_{11}+a_{21})(4a_{03}-a_{11}+2a_{21})}{3(1+x)^3}\\
&-\frac{-2a_{03}^2+(3a_{01}-a_{11}-a_{21}-3b_{10}+6b_{20}-9b_{30})a_{03}-6(a_{01}-a_{11}+a_{21})a_{21}}{3(1+x)^2}\\
&+\frac{2a_{03}(-a_{21}+4a_{03}-a_{11}+3b_{20}-9b_{30})}{3(1+x)}+a_{21}^2+2a_{03}(a_{21}-b_{30})M,
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
g_1(x,y) =& \frac{y}{(1+x)^4}\Big(a_{03}^2y^3+\frac{2}{3}a_{03}b_{21}(5x+2)y^2+\Big(2(a_{01}-a_{11}+a_{21})a_{03}+b_{21}^2\Big)y\\
&-2(a_{01}-a_{11}+a_{21})b_{21}\Big)\\
&-\frac{y\big(\big(10a_{03}^2-a_{03}a_{11}+2a_{03}a_{21}+6b_{21}^2\big)y-3b_{21}(4a_{01}+2a_{03}-5a_{11}+6a_{21})\big)}{3(1+x)^3}\\
&+\frac{y\big(2b_{21}(a_{03}+a_{21})x+\big(2a_{03}a_{21}+b_{21}^2\big)y-b_{21}(2a_{03}-a_{11}+2a_{21})\big)}{(1+x)^2},\\
\bar{q}_2 =& q_1q_0+q_1h(x)+q_{20},\\
N_2 =& \Big(4b_{21}a_{03}(a_{21}-b_{30})+\xi_{01}^0-\eta_{02}^1-3\xi_{03}^2-\frac{5}{3}\xi_{05}^4-\eta_{04}^3\Big)\omega_{01}^0+(4(a_{21}-b_{30})a_{03}b_{21}\\
&-\xi_{01}^1-\eta_{02}^1-3\xi_{03}^2\Big)\Delta_{00}^0+\Big(\eta_{10}^0-\frac{7}{5}\xi_{07}^6-\eta_{06}^5\Big)\delta_{10}^0+\Big(\frac{1}{5}\xi_{01}^6\\
&+\frac{1}{5}(2H-1)^3\Big(7\xi_{07}^6+5\eta_{06}^5\Big)+(2H-1)^2\Big(\xi_{05}^6+\eta_{04}^5\Big)\\
&+\frac{1}{5}\Big(5\eta_{02}^5+3\xi_{03}^6\Big)(2H-1)\Big)\delta_{00}^5+\Big(\frac{1}{4}\xi_{01}^5+\eta_{00}^4+\frac{1}{20}(2H-1)^2\Big(20\eta_{04}^4\\
&+120\eta_{06}^5+25\xi_{05}^5+168\xi_{07}^6\Big)+\frac{1}{4}\Big(4\eta_{02}^4+16\eta_{04}^5+3\xi_{03}^5+16\xi_{05}^6\Big)(2H-1)\\
&+\frac{6}{5}\xi_{03}^6+2\eta_{02}^5\Big)\delta_{00}^4+\Big(3\xi_{03}^3-\xi_{03}^4-5\xi_{05}^5+\xi_{01}^2+\xi_{05}^6+\eta_{04}^5\\
&-4\eta_{04}^4+2\eta_{02}^2-\eta_{02}^3+\eta_{00}^1+\frac{3}{5}\Big(7\xi_{07}^6+5\eta_{06}^5\Big)(2H-5)\\
&-\frac{2}{3}\Big(5\xi_{05}^4+3\eta_{04}^3\Big)(2H-3)\Big)\delta_{00}^1+\Big(\frac{1}{2}\xi_{01}^3-\frac{3}{4}\xi_{03}^5-\eta_{02}^4+\eta_{00}^2+2\xi_{03}^4-4\xi_{05}^6\\
&-4\eta_{04}^5+2\eta_{02}^3+\frac{1}{6}\Big(6\eta_{02}^2+24\eta_{04}^3+9\xi_{03}^3+40\xi_{05}^4\Big)(2H-1)\\
&-\frac{1}{2}\Big(4\eta_{04}^4+5\xi_{05}^5\Big)(2H-3)-\frac{4}{5}\Big(7\xi_{07}^6+5\eta_{06}^5\Big)(6H-5)\Big)\delta_{00}^2\\
&+\Big(\frac{1}{3}\xi_{01}^4-\frac{3}{5}\xi_{03}^6-\eta_{02}^5+\eta_{00}^3+\frac{3}{2}\xi_{03}^5+2\eta_{02}^4\\
&+\Big(\eta_{02}^3+4\eta_{04}^4+\xi_{03}^4+5\xi_{05}^5\Big)(2H-1)+\frac{1}{3}(2H-1)^2\Big(5\xi_{05}^4+3\eta_{04}^3\Big)\\
&-2(2H-3)\Big(\xi_{05}^6+\eta_{04}^5\Big)-\frac{3}{5}(2H-1)(2H-5)\Big(7\xi_{07}^6+5\eta_{06}^5\Big)\Big)\delta_{00}^3.
\end{aligned} \tag{45}
$$

Proof. Under *Case* (1), there exist $M_1(h)=M_2(h)\equiv 0$, Together with Lemma 2, N_1 can be written as $N_1=\tilde{q}_1dH+d\tilde{Q}_1$ by $\oint_{H=h} N_1=0$. By *Case* (1), we have

$$
\begin{aligned}
N_1 =& \frac{1}{3}b_{21}\Big(2(2H-1)(4Ha_{03}+3a_{01}-2a_{03}-3a_{11}+3a_{21})\delta_{00}^3+(8Ha_{03}\\
&+9a_{01}-4a_{03}-9a_{11}+9a_{21})\delta_{00}^2-2a_{03}\Big(8H\delta_{00}^1+3\Delta_{00}^0-3\delta_{00}^1+8\omega_{01}^0\Big)\Big)\\
=&\Big(\frac{4a_{03}b_{21}y^4}{3(1+x)^3}-\frac{b_{21}\left(4a_{03}x^2(x+2)-2(3a_{01}+a_{03}-3a_{11}+3a_{21})x\right)}{3(1+x)^2}\\
&-\frac{b_{21}(-3a_{01}-6a_{03}+3a_{11}-3a_{21})}{3(1+x)^2}-\frac{2b_{21}(2xa_{03}-a_{01}+2a_{03}+a_{11}-a_{21})y^2}{(1+x)^3}\Big)dy\\
&-\frac{2}{3}a_{03}b_{21}(3\ln(1+x)dy+8ydx)
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
&=\tilde{q}_1 dH + d\tilde{Q}_1, \text{where} \\
\tilde{q}_1 &= -\frac{2b_{21}y\left(3a_{03}x^2 - 2y^2a_{03} + 9xa_{03} - 3a_{01} + 6a_{03} + 3a_{11} - 3a_{21}\right)}{3(1+x)^3}, \\
d\tilde{Q}_1 &= d\left(\frac{2b_{21}a_{03}y^3(1+2x)}{3(1+x)^2} + b_{21}\left(\frac{2(a_{03}+a_{01}-a_{11}+a_{21})}{1+x} - \frac{(a_{01}-a_{11}+a_{21})}{(1+x)^2}\right.\right. \\
&\quad \left.\left. -\frac{2}{3}b_{21}a_{03}(2x+3M)\right)y\right).
\end{aligned} \tag{47}
$$

By Lemma 2, $q_1 = \bar{q}_1 + \tilde{q}_1 := f_1(x) + g_1(x,y)$. Then, we have $q_1\omega = q_1q_0dH + q_1dQ_0$, where

$$
\begin{aligned}
q_1dQ_0 =& q_1(\Omega_1(x) + \Omega_1(x,y) + h(x)dH) = q_1h(x)dH + f_1(x)\Omega_1(x) \\
&+ g_1(x,y)\Omega_1(x) + q_1\Omega_2(x,y) \\
=& q_1h(x)dH + f_1(x)\Omega_1(x) + \sum_{k=j-1}^{6}\sum_{j=1}^{7}\xi_{0j}^k\omega_{0j}^k + \sum_{k=j-1}^{5}\sum_{j=2}^{6}\eta_{0j}^k\delta_{0j}^k \\
&+ \sum_{k=0}^{4}(\eta_{00}^k\delta_{00}^k + \eta_{01}^k\delta_{01}^k) + \eta_{10}^0\delta_{10}^0 + \Xi_{01}^0\Omega_{01}^0 + \Xi_{02}^2\Omega_{02}^2 + \Xi_{32}^2\Omega_{03}^2 \\
&+ \bar{\eta}_{02}^1\Delta_{02}^1 + \bar{\eta}_{10}^0\Delta_{10}^0.
\end{aligned} \tag{48}
$$

□

Theorem 3. *Let $M_1(h) = M_2(h) \equiv 0$ and Case (1) hold. $M_3(h)$ has exactly six simple zeros; moreover, system (4) has exactly six limit cycles by the third-order Melnikov function.*

Proof. Let $z = \sqrt{1-2h}$, $z \in (0,1)$. By Lemmas 2–5, it yields

$$
\begin{aligned}
M_3(h) =& \oint_{H=h} N_2 = \frac{b_{21}\pi}{24z^9}(z-1)(k_{10}z^{10} + k_9z^9 + k_8z^8 + k_7z^6(z+1) + k_5z^4(z+1) \\
&+ k_3z^2(z+1) + k_1(z+1)),
\end{aligned} \tag{49}
$$

It is easy to verify that $(z+1, z^2(z+1), z^4(z+1), z^6(z+1), z^8, z^9, z^{10})$ is an ECT system according to Lemma 6. Let $a_{11} = a_{21} = 0$; one can obtain that

$$
\begin{aligned}
&\frac{\partial(k_1,k_3,k_5,k_7,k_8,k_9,k_{10})}{\partial(a_{01},a_{03},b_{00},b_{10},b_{20},b_{21},b_{30})} = -513684799488b_{21}a_{01}\Big(160(a_{03}-3b_{30})a_{01}^4 \\
&-4a_{03}(28a_{03} + 40b_{10} + 64b_{20} - 247b_{30})a_{01}^3 + a_{03}^2(218a_{03} - 16b_{10} + 328b_{20} - 777b_{30})a_{01}^2 \\
&-2a_{03}^3(14a_{03} + 16b_0 - 27b_{10} + 97b_{20} - 167b_{30})a_{01} + 56a_{03}^4(b_{00} - b_{10} + b_{20} - b_{30})\Big) \neq 0
\end{aligned} \tag{50}
$$

by Remark 2, one can choose enough coefficients such that $M_3(h)$ has exactly six simple zeros. □

By taking similar steps, we consider *Cases* (2)–(5).

Theorem 4. *Let $M_1(h) = M_2(h) \equiv 0$, then*

(i) *in Case (2), $M_3(h)$ has at most six zeros;*
(ii) *in Case (3), $M_3(h)$ has exactly four simple zeros;*
(iii) *in Case (4), $M_3(h)$ has exactly five simple zeros;*
(iv) *in Case (5), $M_3(h)$ has exactly six simple zeros.*

Proof. Let $z = \sqrt{1-2h}$. (i) In *Case* (2),

$$\begin{aligned} M_3(h) &= M_3\left(\frac{1-z^2}{2}\right) = \oint_{H=h} N_2 \\ &= -\frac{a_{03}b_{03}\pi}{18z^7}\Big[32a_{03}z^9 - 4(46a_{03}+45a_{21}-45b_{30})z^8 + 3(262a_{03}+360a_{21} \\ &+15b_2+60b_{20}-480b_{30})z^7 - 3(386a_{03}+444a_{21}+105b_{02}-40b_{10} \\ &+380b_{20}-1164b_{30})z^6 - 3(2a_{03}-72a_{21}-15b_{00}-99b_{02}+155b_{10} \\ &-367b_{20}+651b_{30})z^4(z+1) + 27(4a_{03}-17b_{00}-6b_{02}+25b_{10}-33b_{20} \\ &+41b_{30})z^2(z+1) + 270(b_{00}-b_{10}+b_{20}-b_{30})(z+1)\Big]. \end{aligned} \tag{51}$$

Here, we can prove that the function $(z+1, z^2(z+1), z^4(z+1), z^6, z^7, z^8, z^9)$ is an ECT system. However, the corresponding seven coefficients are not independent, and $M_3(h)$ has at most six zeros.

(ii) In *Case* (3), by Lemmas 2–5

$$\begin{aligned} M_3(h) &= M_3\left(\frac{1-z^2}{2}\right) = \oint_{H=h} N_2 \\ &= \frac{b_{12}b_{01}\pi(z-1)}{27z^7}\Big[(108a_{21}+54b_{12}-108b_{30})z^7 + (-180a_{21}-36b_{02} \\ &-74b_{12}-108b_{20}+396b_{30})z^6 + (36a_{21}+45b_{02}-54b_{10}+b_{12}+144b_{20} \\ &-270b_{30})z^4(z+1) - 9(7b_{00}+3b_{02}-11b_{10}-b_{12}+15b_{20}-19b_{30}) \\ &z^2(z+1) + 45(z+1)(b_0-b_{10}+b_{20}-b_{30})\Big]. \end{aligned} \tag{52}$$

Since the function $(z+1, z^2(z+1), z^4(z+1), z^6, z^7)$ is an ECT system and the corresponding five coefficients are free parameters, by Lemma 6 and Remark 2, $M_3(h)$ has exactly four simple zeros.

(iii) In *Case* (4),

$$\begin{aligned} M_3(h) =& M_3\left(\frac{1-z^2}{2}\right) = \frac{\pi(z-1)}{12z^7}\Big[K_8z^8 + K_7z^7 + K_6z^6 + K_5z^4(z+1) \\ &+ K_3z^2(z+1) + K_1(z+1)\Big]. \end{aligned} \tag{53}$$

One can verify that the function $(z+1, z^2(z+1), z^4(z+1), z^6, z^7, z^8)$ is an ECT system, and when $b_{02} = 0$, then we have

$$\begin{aligned} &\frac{\partial(K_1,K_3,K_5,K_6,K_7,K_8)}{\partial(a_{21},b_{30},b_{00},b_{10},b_{20},b_{12})} = -29859840b_{01}b_{12}^3b_{21}^2(5b_{01}-2b_{21})\Big(36a_{21}^2b_{21}^2(b_{00} \\ &-b_{10}+b_{20}-b_{30}) - b_{12}((8b_{00}-11b_{10}-21b_{12}+20b_{20}-29b_{30})a_{21} \\ &-21b_{12}(4b_{00}-b_{10}))b_{21}^2 + 3b_{01}b_{12}((32b_{00}-36b_{10}-29b_{12}+48b_{20}-60b_{30})a_{21} \\ &-b_{12}(96b_{00}-12b_{10}-8b_{20}-b_{30}))b_{21} + 12b_{01}^2b_{12}^2(a_{21}+8b_{20}-17b_{30})\Big) \neq 0. \end{aligned} \tag{54}$$

By Lemma 6 and Remark 2, $M_3(h)$ has exactly five simple zeros.

(iv) In *Case* (5),

$$
\begin{aligned}
M_3(h) =& M_3\left(\frac{1-z^2}{2}\right) \\
=& -\frac{\pi(z-1)}{4(a_{20}+3b_{03})z^9}\Big[c_{10}z^{10}+c_9z^9+c_8z^8+c_7z^6(z+1)+c_5z^4(z+1)+ \\
& c_3z^2(z+1)+c_1(z+1)\Big],
\end{aligned} \tag{55}
$$

By Lemma 6, we know that $(z+1, z^2(z+1), z^4(z+1), z^6(z+1), z^8, z^9, z^{10})$ is an ECT system. Furthermore, we have $\frac{\partial(c_1,c_3,c_5,c_7,c_8,c_9,c_{10})}{\partial(a_{11},a_{01},a_{20},b_{01},b_{20},b_{02},b_{03})} \neq 0$; then $M_3(h)$ has exactly six simple zeros by Remark 2. □

4. Conclusions

Based on the previous results, by deducing some new formulae in Lemmas 3 and 5, the first three Melnikov functions of system (4) with $n = 3$ are considered. There are many third-order Melnikov functions; since some cases are difficult to compute, we just list five cases. We obtain exactly six limit cycles. It is difficult to obtain the conditions of the existence of the fourth-order Melnikov functions; we do not continue to consider the fourth-order Melnikov function.

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