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On the Commutators of Marcinkiewicz Integral with a Function in Generalized Campanato Spaces on Generalized Morrey Spaces

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Abstract: This paper is devoted to exploring the mapping properties for the commutator $\mu_{\Omega,b}$ generated by Marcinkiewicz integral μ_Ω with a locally integrable function b in the generalized Campanato spaces on the generalized Morrey spaces. Under the assumption that the integral kernel Ω satisfies certain log-type regularity, it is shown that $\mu_{\Omega,b}$ is bounded on the generalized Morrey spaces with variable growth condition, provided that b is a function in generalized Campanato spaces, which contain the $BMO(\mathbb{R}^n)$ and the Lipschitz spaces $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ $(0 < \alpha \le 1)$ as special examples. Some previous results are essentially improved and generalized.

Keywords: Marcinkiewicz integrals; commutators; generalized Campanato spaces; generalized Morrey spaces

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1. Introduction

Let $\mathbb{R}^n$, $n \ge 2$, be the n-dimensional Euclidean spaces and S^{n-1} the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega \in L^1(S^{n-1})$ and the following property

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1}$$

where $x' = x/|x|$ for any $x \neq 0$.

The Marcinkiewicz integral operator μ_Ω is defined by

$$\mu_\Omega(f)(x) = \Big(\int_0^\infty |F_{\Omega,t}(f)(x)| \frac{dt}{t^3} \Big)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y| \le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

As is well known, Marcinkiewicz integral is one of the classical operators in harmonic analysis, which belongs to the broad class of the Littlewood-Paley g-functions and plays important roles in harmonic analysis and partial differential equations. The research on the mapping properties of Marcinkiewicz integral and its commutators in various function spaces has been an active topic. In 1958, Stein [1] first introduced the operator μ_Ω, which is the higher dimensional generalization of Marcinkiewicz integral in one-dimension, and showed that μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \le 2$ and weak type $(1,1)$, provided $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \le 1$. Subsequently, the boundedness of μ_Ω was studied extensively, see [2–8], etc. and therein references. In particular, Al-Salman et al. [2] obtained the L^p-boundedness of μ_Ω for $1 < p < \infty$, provided that $\Omega \in L(\log L)^{1/2}(S^{n-1})$. In addition, the boundedness of μ_Ω on generalized Morrey spaces and generalized weighted Morrey spaces was also established; see [9–11], etc.

In this paper, we will focus on the commutators $\mu_{\Omega,b}$ generated by μ_Ω with $b \in L_{\text{loc}}(\mathbb{R}^n)$ by

$$\mu_{\Omega,b}(f)(x) = \left(\int_0^\infty |[b, F_{\Omega,t}](f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$[b, F_{\Omega,t}](f)(x) = b(x)F_{\Omega,t}(f)(x) - F_{\Omega,t}(bf)(x) = \int_{|x-y|\le t} [b(x) - b(y)] \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

In 1990, Torchinsky and Wang [8] first studied the commutators $\mu_{\Omega,b}$ and showed that $\mu_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, provided that $\Omega \in \text{Lip}_\alpha(S^{n-1})$, $0 < \alpha \le 1$, $b \in BMO(\mathbb{R}^n)$. Subsequently, this result was improved and extended to the cases of rough kernels in [12–14], etc. Chen and Ding [15] also showed that $b \in BMO(\mathbb{R}^n)$ is necessary for the boundedness of $\mu_{\Omega,b}$ on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, under the assumption that Ω satisfies the following logarithm type regularity:

$$|\Omega(x') - \Omega(y')| \lesssim \left(\log \frac{2}{|x'-y'|} \right)^{-\gamma} \quad \text{for any } x', y' \in S^{n-1}, \text{ and some } \gamma > 1. \tag{2}$$

In addition, see [16] for the cases of the weighted versions with rough kernels. Furthermore, Aliev and Guliyev [9] obtained that, for $b \in BMO(\mathbb{R}^n)$ and $\Omega \in \text{Lip}_\alpha(S^{n-1})$, $\mu_{\Omega,b}$ is bounded from the generalized Morrey spaces $L^{p,\varphi_1}(\mathbb{R}^n)$ to $L^{p,\varphi_2}(\mathbb{R}^n)$ with certain appropriate positive functions. The boundedness of $\mu_{\Omega,b}$, for $b \in BMO(\mathbb{R}^n)$ and $\Omega \in \text{Lip}_\alpha(S^{n-1})$, on the generalized weighted Morrey spaces, Orlicz–Morrey spaces and the mixed Morrey spaces were also found in [4,11,17,18], etc.

On the other hand, Arai and Nakai [19] recently studied the commutators $[b, T]$ of the Calderón–Zygmund operator T on the generalized Morrey spaces and showed that, if b is a function of generalized Campanato spaces $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, which contain the BMO spaces and the Lipschitz spaces as special examples, then $[b, T]$ is bounded on the generalized Morrey spaces. The corresponding result for the commutators of general fractional integrals was also obtained.

Based on the results above, it is natural to ask the following question:

Question: What is the mapping properties of $\mu_{\Omega,b}$ on the generalized Morrey spaces when b is a function in the generalized Campanato spaces?

The main purpose of this paper is to address this question. To state our main results, we first recall some relevant definitions and notations.

Let $B(x,r)$ be the open ball centered at $x \in \mathbb{R}^n$ and of radius r, that is,

$$B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}.$$

For a measurable set $E \subset \mathbb{R}^n$, we denote by $|E|$ and χ_E the Lebesgue measure of E and the characteristic function of E, respectively. For a function $f \in L^1_{loc}(\mathbb{R}^n)$ and a ball B, let

$$f_B = \fint_B f(y)dy = \frac{1}{|B|} \int_B f(y)dy.$$

To introduce the generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with $p \in [1,\infty)$ and variable growth function $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, for a ball $B = B(x,r)$, we denote by $\varphi(B) = \varphi(x,r)$.

Definition 1 ([19]). *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $p \in [1,\infty)$, the generalized Morrey space $L^{(p,\varphi)}(\mathbb{R}^n)$ is defined as the set of all functions f such that*

$$\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)} = \sup_B \left(\frac{1}{\varphi(B)} ⨍_B |f(y)|^p dy \right)^{1/p} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

We know that $\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)}$ is a norm and $L^{(p,\varphi)}(\mathbb{R}^n)$ is a Banach space. If $\varphi_\lambda(x,r) = r^\lambda$ for $\lambda \in [-n,0]$, then $L^{(p,\varphi)}(\mathbb{R}^n)$ is the classical Morrey space, that is,

$$\|f\|_{L^{(p,\varphi_\lambda)}(\mathbb{R}^n)} = \sup_B \left(\frac{1}{\varphi_\lambda(B)} ⨍_B |f(y)|^p dy \right)^{1/p} = \sup_{B=B(x,r)} \left(\frac{1}{r^\lambda} ⨍_B |f(y)|^p dy \right)^{1/p}.$$

In particular, $L^{(p,\varphi_{-n})}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, and $L^{(p,\varphi_0)}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

Recall that a locally integrable function b is said to be in $BMO(\mathbb{R}^n)$ if

$$\|b\|_{BMO(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} ⨍ \mid b(x) - b_B \mid dx < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

We also consider the generalized Campanato spaces with variable growth condition, which are defined as follows.

Definition 2 ([19]). *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $p \in [1,\infty)$, the generalized Campanato space $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$ is the set of all functions f such that*

$$\|f\|_{\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)} = \sup_B \left(\frac{1}{\varphi(B)} ⨍_B |f(y) - f_B|^p dy \right)^{1/p} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

It is easy to check that $\|f\|_{\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)}$ is a norm modulo constant functions and thereby $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$ is a Banach space. If $p = 1$ and $\varphi \equiv 1$, then $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n) = BMO(\mathbb{R}^n)$. If $p = 1$ and $\varphi(x,r) = r^\alpha$ $(0 < \alpha \leq 1)$, then $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$ coincides with $\mathrm{Lip}_\alpha(\mathbb{R}^n)$.

We say that a function $\theta : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ satisfies the doubling condition if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and r, $s \in (0,\infty)$,

$$\frac{1}{C} \leq \frac{\theta(x,r)}{\theta(x,s)} \leq C, \text{ if } \frac{1}{2} \leq \frac{r}{s} \leq 2. \tag{3}$$

We also consider the following condition that there exists a positive constant C such that, for all x, $y \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\frac{1}{C} \leq \frac{\theta(x,r)}{\theta(y,r)} \leq C, \text{ if } |x-y| \leq r. \tag{4}$$

For two functions θ, $\kappa : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, we write $\theta \sim \kappa$ if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\frac{1}{C} \leq \frac{\theta(x,r)}{\kappa(x,r)} \leq C. \tag{5}$$

Definition 3. *(i) Let $\mathcal{G}^{dec}$ be the set of all functions $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that φ is almost decreasing and that $r \mapsto \varphi(x,r)r^n$ is almost increasing. That is, there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and r, $s \in (0,\infty)$,*

$$C\varphi(x,r) \geq \varphi(x,s), \;\; \varphi(x,r)r^n \leq C\varphi(x,s)s^n, \;\; \text{if} \;\; r < s.$$

(ii) Let $\mathcal{G}^{inc}$ be the set of all functions $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that φ is almost increasing and that $r \mapsto \varphi(x,r)/r$ is almost decreasing. That is, there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and r, $s \in (0,\infty)$,

$$\varphi(x,r) \leq C\varphi(x,s), \;\; C\varphi(x,r)/r \geq \varphi(x,s)/s, \;\; \text{if} \;\; r < s.$$

If $\varphi \in \mathcal{G}^{dec}$ or $\varphi \in \mathcal{G}^{inc}$, then φ satisfies the doubling condition (3).

It follows from [19] that, for $\varphi \in \mathcal{G}^{dec}$, if φ satisfies

$$\lim_{r\to 0} \varphi(x,r) = \infty, \quad \lim_{r\to\infty} \varphi(x,r) = 0, \tag{6}$$

then there exists $\widetilde{\varphi} \in \mathcal{G}^{dec}$ such that $\varphi \sim \widetilde{\varphi}$ and that $\widetilde{\varphi}(x,\cdot)$ is continuous, strictly decreasing and bijective from $(0,\infty)$ to itself for each x.

For $f \in L^{(p,\varphi)}(\mathbb{R}^n)$, $1 < p < \infty$, we define $\mu_\Omega(f)$ on each ball B by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left|F_{\Omega,t}(f\chi_{2B})(x) + F_{\Omega,t}(f\chi_{(2B)^\complement})(x)\right|^2 \frac{dt}{t^3}\right)^{1/2}, \quad x \in B. \tag{7}$$

Here, and in what follows, $E^\complement = \mathbb{R}^n \backslash E$ denotes the complementary set of any measurable subset E of $\mathbb{R}^n$. Then,

$$\mu_\Omega(f)(x) \leq \mu_\Omega(f\chi_{2B})(x) + \mu_\Omega(f\chi_{(2B)^\complement})(x).$$

Note that $\mu_\Omega(f\chi_{2B})$ is well defined since $f\chi_{2B} \in L^p(\mathbb{R}^n)$, and it easy to check that

$$\mu_\Omega(f\chi_{(2B)^\complement})(x) \leq \int_{(2B)^\complement} \frac{\Omega(x-y)}{|x-y|^n}|f(y)|dy,$$

which converges absolutely. Moreover, $\mu_\Omega(f)(x)$ defined in (7) is independent of the choice of the ball containing x. Furthermore, we can show that μ_Ω is bounded on $L^{(p,\varphi)}(\mathbb{R}^n)$. See Proposition 1 for the details.

For $f \in L^{(p,\varphi)}(\mathbb{R}^n)$, $1 < p < \infty$, we define $\mu_{\Omega,b}(f)$ on each ball B by

$$\mu_{\Omega,b}(f)(x) = \left(\int_0^\infty \left|[b,F_{\Omega,t}](f\chi_{2B})(x) + [b,F_{\Omega,t}](f\chi_{(2B)^\complement})(x)\right|^2 \frac{dt}{t^3}\right)^{1/2}, \quad x \in B. \tag{8}$$

Now, we can formulate our main result as follows.

Theorem 1. *Let $1 < p \leq q < \infty$ and φ, $\psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $\psi \in \mathcal{G}^{inc}$ satisfies (4), $\varphi \in \mathcal{G}^{dec}$ satisfies (6) and for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$\int_r^\infty \frac{\varphi(x,t)}{t} dt \leq C\varphi(x,r) \tag{9}$$

and

$$\psi(x,r)\varphi(x,r)^{1/p} \leq C_0\varphi(x,r)^{1/q}. \tag{10}$$

If $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, then $\mu_{\Omega,b}(f)$ in (8) is well defined for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$, and there exists a positive constant C, independent of b and f, such that

$$\|\mu_{\Omega,b}(f)\|_{L^{(q,\varphi)}} \leq C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

Remark 1. *For $b \in L^1_{loc}(\mathbb{R}^n)$, Chen and Ding [15] showed that, if $\mu_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, then $b \in BMO(\mathbb{R}^n)$, under the assumption of that Ω satisfies the logarithm type regularity condition (2). It is not clear that, for $b \in L^1_{loc}(\mathbb{R}^n)$, under the same assumptions of Theorem 1, if $\mu_{\Omega,b}$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$. This is an interesting open problem. Moreover, it is also interesting whether or not the corresponding conclusions are still true if the regularity of Ω is weakened or removed. In addition, for $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, it is also worth exploring the mapping properties of $\mu_{\Omega,b}$ on the generalized weighted Morrey spaces, the general Orlicz–Morrey spaces, etc.*

The rest of this paper is organized as follows: In Section 2, we will recall and establish some auxiliary lemmas. Section 3 will establish the pointwise estimate for the sharp maximal operator of $\mu_{\Omega,b}$, and the proof of Theorem 1 will be given in Section 4.

Finally, we make some conventions on notation. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p, are dependent on the subscripts. We denote $f \lesssim g$ if $f \le Cg$, and $f \sim g$ if $f \lesssim g \lesssim f$. For $1 \le p \le \infty$, p' is the conjugate index of p, and $1/p + 1/p' = 1$.

2. Preliminaries

For a function $\rho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, the generalized Hardy–Littlewood maximal operator is defined by

$$\mathcal{M}_\rho(f)(x) = \sup_{B \ni x} \rho(B) \fint_B |f(y)|dy.$$

Clearly, if $\rho \equiv 1$, then $\mathcal{M}_\rho(f)(x)$ is the Hardy–Littlewood maximal operator M, and if $\rho(B) = |B|^{\alpha/n}$, then $\mathcal{M}_\rho(f)$ is the fraction maximal operator M_α defined by

$$M_\alpha(f)(x) = \sup_{B \ni x} |B|^{\alpha/n} \fint_B |f(y)|dy.$$

For the generalized Hardy–Littlewood maximal operator $\mathcal{M}_\rho$, we have the following lemma:

Lemma 1 ([19])**.** *Let $1 < p < q < \infty$ and $\rho,\ \varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that φ is in $\mathcal{G}^{dec}$ and satisfies (6). Assume also that there exists a positive constant C_0, such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$\rho(x,r)\varphi(x,r)^{1/p} \le C_0 \varphi(x,r)^{1/q}. \tag{11}$$

Then, $\mathcal{M}_\rho$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

Next, we recall John–Nirenberg inequality. Let $b \in BMO(\mathbb{R}^n)$, and there are constants C_1, $C_2 > 0$, such that for all $\beta > 0$, $B \subset \mathbb{R}^n$,

$$|\{x \in B : | b(x) - b_B | > \beta\}| \le C_1 \mid B \mid e^{-C_2\beta/\|b\|_{BMO}},$$

which yields that

$$\left(\sup_B \fint |b(x) - b_B|^p \right)^{1/p} \approx \|b\|_{BMO(\mathbb{R}^n)}.$$

The following lemma is a corollary of the John–Nirenberg inequality.

Lemma 2 ([19])**.** *Let $p \in (1,\infty)$ and $\psi \in \mathcal{G}^{inc}$. Assume that ψ satisfies (4). Then, $\mathcal{L}^{(p,\psi^p)}(\mathbb{R}^n) = \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ with equivalent norms.*

Lemma 3 ([19]). *Let $p \in (1, \infty)$ and $\psi \in \mathcal{G}^{inc}$. Assume that ψ satisfies (4). Then, there exists a positive constant C dependent only on n, p and ψ such that, for all $f \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$ and $r, s \in (0, \infty)$,*

$$\left(\fint_{B(x,s)} |f(y) - f_{B(x,r)}|^p dy \right)^{1/p} \leq C \int_r^s \frac{\psi(x,t)}{t} dt \|f\|_{\mathcal{L}^{(1,\psi)}}, \text{ if } 2r < s, \tag{12}$$

and

$$\left(\fint_{B(x,s)} |f(y) - f_{B(x,r)}|^p dy \right)^{1/p} \leq C \Big(\log_2 \frac{s}{r} \Big) \psi(x,s) \|f\|_{\mathcal{L}^{(1,\psi)}}, \text{ if } 2r < s. \tag{13}$$

Lemma 4 ([19]). *Let φ satisfy the doubling condition (3) and (9), that is,*

$$\int_r^\infty \frac{\varphi(x,t)}{t} dt \leq C \varphi(x,r).$$

Then, for all $p \in (0, \infty)$, there exists a positive constant C_p such that, for all $x \in \mathbb{R}^n$ and $r > 0$,

$$\int_r^\infty \frac{\varphi(x,t)^{1/p}}{t} dt \leq C_p \varphi(x,r)^{1/p}.$$

Proposition 1. *Let $1 < p < \infty$, $\varphi \in \mathcal{G}^{\rm dec}$ and satisfy (9). Suppose that $\Omega \in L^\infty(S^{n-1})$. Then, μ_Ω defined in (7) is bounded on $L^{(p,\varphi)}(\mathbb{R}^n)$. That is, there exists a positive constant C such that, for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$,*

$$\|\mu_\Omega(f)\|_{L^{(p,\varphi)}} \leq C \|f\|_{L^{(p,\varphi)}}.$$

Proof. For $x \in \mathbb{R}^n$, we take any ball $B = B(z,r) \ni x$. Set $B^* = 2B$. Then, we have

$$\mu_\Omega(f)(x) \leq \mu_\Omega(f\chi_{B^*})(x) + \mu_\Omega(f\chi_{(B^*)^\complement})(x).$$

By the boundedness of μ_Ω on $L^p(\mathbb{R}^n)$ and the doubling condition of φ, we have

$$\begin{aligned}
\Big(\frac{1}{\varphi(B)} \fint_B |\mu_\Omega(f\chi_{B^*})(x)|^p dx \Big)^{1/p} &\lesssim \Big(\frac{1}{\varphi(B)|B|} \int_{B^*} |f(x)|^p dx \Big)^{1/p} \\
&\lesssim \Big(\frac{1}{\varphi(B^*)|B^*|} \int_{B^*} |f(x)|^p dx \Big)^{1/p} \leq \|f\|_{L^{(p,\varphi)}}.
\end{aligned}$$

Hence, $\|\mu_\Omega(f\chi_{B^*})\|_{L^{(p,\varphi)}} \lesssim \|f\|_{L^{(p,\varphi)}}$.

For $\mu_\Omega(f\chi_{(B^*)^\complement})(x)$, note that, if $x \in B$ and $y \in (B^*)^\complement$, then $|y-z|/2 \leq |x-y| \leq 3|y-z|/2$. By the generalized Minkowski inequality and the doubling condition of φ, we have

$$\begin{aligned}
\mu_\Omega(f\chi_{(B^*)^\complement})(x) &\lesssim \int_{(B^*)^\complement} \frac{|f(y)|}{|z-y|^n} dy = \sum_{j=1}^\infty \int_{2^{j+1}B \backslash 2^j B} \frac{|f(y)|}{|z-y|^n} dy \\
&\lesssim \sum_{j=1}^\infty \fint_{2^{j+1}B} |f(y)| dy \leq \sum_{j=1}^\infty \Big(\fint_{2^{j+1}B} |f(y)|^p dy \Big)^{1/p} \\
&\lesssim \sum_{j=1}^\infty \int_{2^j r}^{2^{j+1} r} \frac{\varphi(z,t)^{1/p}}{t} dt \|f\|_{L^{(p,\varphi)}} \lesssim \int_{2r}^\infty \frac{\varphi(z,r)^{1/p}}{t} dt \|f\|_{L^{(p,\varphi)}} \\
&\lesssim \varphi(z,r)^{1/p} \|f\|_{L^{(p,\varphi)}}, \quad x \in B,
\end{aligned}$$

which leads to $\|\mu_\Omega(f\chi_{(B^*)^\complement})\|_{L^{(p,\varphi)}} \lesssim \|f\|_{L^{(p,\varphi)}}$ and completes the proof of Proposition 1. □

Lemma 5. *Under the assumption of Theorem 1, there exists a positive constant C such that, for all $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and all balls $B = B(z,r)$,*

$$\fint_B \left(\int_0^\infty \left| [b, F_{\Omega,t}](f\chi_{(2B)^\complement})(x) \right|^2 \frac{dt}{t^3} \right)^{1/2} dx \leq C\varphi(z,r)^{1/q} \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}}.$$

Proof. For $x \in B$, we have

$$\begin{aligned}\left(\int_0^\infty |[b, F_{\Omega,t}](f\chi_{(2B)^\complement})(x)|^2 \frac{dt}{t^3} \right)^{1/2} &\lesssim \int_{(2B)^\complement} \frac{|\Omega(x-y)|}{|x-y|^n} |b(x) - b_B + b_B - b(y)||f(y)|dy \\ &\leq |b(x) - b_B| \int_{(2B)^\complement} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)|dy \\ &\quad + \int_{(2B)^\complement} \frac{|\Omega(x-y)|}{|x-y|^n} |b(y) - b_B||f(y)|dy \\ &=: G_1(x) + G_2(x).\end{aligned}$$

Note that $x \in B$ and $y \notin 2B$, and we have $|y-z|/2 \leq |x-y| \leq 3|y-z|/2$. By Hölder's inequality and the doubling condition of φ, we obtain

$$\begin{aligned}G_1(x) &\lesssim |b(x) - b_B| \int_{(2B)^\complement} \frac{1}{|y-z|^n} |f(y)|dy| = |b(x) - b_B| \sum_{j=1}^\infty \int_{2^{j+1}B \setminus 2^j B} \frac{|f(y)|}{|x-y|^n} dy \\ &\lesssim |b(x) - b_B| \sum_{j=1}^\infty \left(\fint_{2^{j+1}B} |f(y)|^p dy \right)^{1/p} \lesssim |b(x) - b_B| \sum_{j=1}^\infty \int_{2^j r}^{2^{j+1} r} \frac{\varphi(z,t)^{1/p}}{t} dt \|f\|_{L^{(p,\varphi)}} \\ &\lesssim |b(x) - b_B| \int_{2r}^\infty \frac{\varphi(z,t)^{1/p}}{t} dt \|f\|_{L^{(p,\varphi)}} \lesssim |b(x) - b_B| \varphi(z,r)^{1/p} \|f\|_{L^{(p,\varphi)}}.\end{aligned}$$

Therefore, invoking Lemma 4 and (10) implies that

$$\begin{aligned}\fint_B G_1(x)dx &\lesssim \fint_B |b(x) - b_B| dx \varphi(z,r)^{1/p} \|f\|_{L^{(p,\varphi)}} \\ &\lesssim \psi(z,r)\varphi(z,r)^{1/p} \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}} \\ &\lesssim \varphi(z,r)^{1/q} \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}}.\end{aligned}$$

Similarly, by Hölder's inequality, Lemma 3 together with the doubling condition of ψ and φ, (2) and (10), we have

$$\begin{aligned}G_2(x) &\lesssim \sum_{j=1}^\infty \int_{2^{j+1}B \setminus 2^j B} |b(y) - b_B| \frac{|f(y)|}{|y-z|^n} dy \\ &\lesssim \sum_{j=1}^\infty \left(\fint_{2^{j+1}B} |b(y) - b_B|^{p'} dy \right)^{1/p'} \left(\fint_{2^{j+1}B} |f(y)|^p dy \right)^{1/p} \\ &\lesssim \varphi(z,r)^{1/q} \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}},\end{aligned}$$

which immediately includes that

$$\fint_B G_2(x)dx \lesssim \varphi(z,r)^{1/q} \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}}.$$

This leads to the desired conclusion and completes the proof of Lemma 5. □

Remark 2. *Under the assumptions in Theorem 1, let $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ and $f \in L^{(p,\varphi)}(\mathbb{R}^n)$. Then, $\mu_{\Omega,b}(f)$ in (8) is well defined.*

Indeed, it is obvious that $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and $bf \in L^{p_1}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $p_1 < p$ by Lemma 2. Hence, $\mu_\Omega(f\chi_{2B})$ and $\mu_\Omega(bf\chi_{2B})$ are well defined for any ball $B = B(z,r)$. That is, $\mu_{\Omega,b}(f\chi_{2B})$ is well defined for any ball $B = B(z,r)$.

On the other hand, it follows from the proof of Lemma 5 that $\mu_{\Omega,b}(f\chi_{(2B)^\complement})$ is well defined for any ball $B = B(z,r)$. In addition, by Minkowski's inequality, we have

$$\Big(\int_0^\infty \Big|[b,F_{\Omega,t}](f\chi_{2B})(x) + [b,F_{\Omega,b}](f\chi_{(2B)^\complement})(x)\Big|^2 \frac{dt}{t^3}\Big)^{1/2} \leq \mu_{\Omega,b}(f\chi_{2B})(x) + \mu_{\Omega,b}(f\chi_{(2B)^\complement})(x), \quad x \in B.$$

Therefore, we can write

$$\mu_{\Omega,b}(f)(x) = \Big(\int_0^\infty \Big|[b,F_{\Omega,t}](f\chi_{2B})(x) + [b,F_{\Omega,b}](f\chi_{(2B)^\complement})(x)\Big|^2 \frac{dt}{t^3}\Big)^{1/2}, \quad x \in B.$$

Moreover, if $x \in B_1 \cap B_2$, then, taking B_3 such that $B_1 \cup B_2 \subset B_3$, we have

$$\begin{aligned}&\big([b,F_{\Omega,t}](f\chi_{2B_i})(x) + [b,F_{\Omega,t}](f\chi_{(2B_i)^\complement})(x)\big)\\ &\quad - \big([b,F_{\Omega,t}](f\chi_{2B_3})(x) + [b,F_{\Omega,t}](f\chi_{(2B_3)^\complement})(x)\big)\\ &= -[b,F_{\Omega,t}](f\chi_{2B_3\setminus 2B_i})(x) + [b,F_{\Omega,t}](f\chi_{2B_3\setminus 2B_i})(x) = 0, \quad i = 1, 2,\end{aligned}$$

which implies that

$$\begin{aligned}&\big([b,F_{\Omega,t}](f\chi_{2B_1})(x) + [b,F_{\Omega,t}](f\chi_{(2B_1)^\complement})(x)\big)\\ &= \big([b,F_{\Omega,t}](f\chi_{2B_2})(x) + [b,F_{\Omega,t}](f\chi_{(2B_2)^\complement})(x)\big).\end{aligned}$$

Consequently,

$$\begin{aligned}\mu_{\Omega,b}(f)(x) &= \Big(\int_0^\infty \Big|[b,F_{\Omega,t}](f\chi_{2B_1})(x) + [b,F_{\Omega,t}](f\chi_{(2B_1)^\complement})(x)\Big|^2 \frac{dt}{t^3}\Big)^{1/2}\\ &= \Big(\int_0^\infty \Big|[b,F_{\Omega,t}](f\chi_{2B_2})(x) + [b,F_{\Omega,t}](f\chi_{(2B_2)^\complement})(x)\Big|^2 \frac{dt}{t^3}\Big)^{1/2}.\end{aligned}$$

This shows that $\mu_{\Omega,b}(f)(x)$ in (8) is independent of the choice of the ball B containing x.

3. Sharp Maximal Operator and Pointwise Estimate

In this section, we will establish a sharp maximal inequality on $\mu_{\Omega,b}$. For $f \in L^1_{loc}(\mathbb{R}^n)$, let

$$M^\sharp f(x) = \sup_{B \ni x} \fint_B |f(y) - f_B|dy, \quad x \in \mathbb{R}^n, \tag{14}$$

where the supremum is taken over all balls B containing x.

For sharp maximal operator, the following lemma is known.

Lemma 6 ([19]). *Let $p \in [1,\infty)$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $\varphi \in \mathcal{G}^{dec}$ and satisfies (9). For $f \in L^1_{loc}(\mathbb{R}^n)$, if $\lim_{r\to\infty} f_{B(0,r)} = 0$, then*

$$\|f\|_{L^{(p,\varphi)}} \leq C\|M^\sharp f\|_{L^{(p,\varphi)}},$$

where C is a positive constant independent of f.

Proposition 2. *Let $p,\eta \in (1,\infty)$ and $\varphi,\psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, Ω be as in Theorem 1. Assume that $\varphi \in \mathcal{G}^{dec}$ and $\psi \in \mathcal{G}^{inc}$. Assume that ψ satisfies (4) that φ satisfies (9), and that*

$\int_r^\infty \frac{\psi(x,t)\varphi(x,t)^{1/p}}{t}dt < \infty$, *for each* $x \in \mathbb{R}^n$ *and* $r > 0$. *Then, there exists a positive constant* C *such that, for all* $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ *and* $x \in \mathbb{R}^n$,

$$M^{\#}(\mu_{\Omega,b}(f))(x) \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\Big(\big(\mathcal{M}_{\psi^\eta}(|\mu_\Omega(f)|^\eta)(x)\big)^{1/\eta} + \big(\mathcal{M}_{\psi^\eta}(|f|^\eta)(x)\big)^{1/\eta}\Big), \tag{15}$$

where C *is a positive constant independent of* f.

Proof. Employing the vector-valued singular integral notation of Benedek et al. in [20], let $\mathcal{H}$ be the Hilbert space defined by

$$\mathcal{H} = \Big\{h : \|h\|_{\mathcal{H}} = \Big(\int_0^\infty \frac{|h(t)|^2}{t^3}dt\Big)^{1/2} < \infty\Big\},$$

and $F_{\Omega,t}(f)(x)$, $[b, F_{\Omega,t}](f)(x)$ be as before. Then, we can write

$$\mu_\Omega(f)(x) = \|F_{\Omega,t}(f)(x)\|_{\mathcal{H}}, \quad \mu_{\Omega,b}(f)(x) = \|[b, F_{\Omega,t}](f)(x)\|_{\mathcal{H}}.$$

For $x \in \mathbb{R}^n$, let B be a ball centered at x. Take $B^* = 2B$. We decompose $f = f\chi_{B^*} + f\chi_{(B^*)^\complement} =: f_1 + f_2$ and write

$$\begin{aligned}\mu_{\Omega,b}(f)(y) &= \mu_{\Omega,b-b_{B^*}}(f)(y) = \|[b - b_{B^*}, F_{\Omega,t}](f)(y)\|_{\mathcal{H}} := \|F_{\Omega,t}^{b-b_{B^*}}(f)(y)\|_{\mathcal{H}} \\ &= \|(b(y) - b_{B^*})F_{\Omega,t}(f)(y) - F_{\Omega,t}((b - b_{B^*})f_1)(y) - F_{\Omega,t}((b - b_{B^*})f_2(y)\|_{\mathcal{H}}.\end{aligned}$$

Let $C_B = \mu_\Omega((b - b_{B^*})f_2)(x) = \|F_{\Omega,t}((b - b_{B^*})f_2)(x)\|_{\mathcal{H}}$. Then, for $y \in B$,

$$\begin{aligned}|\mu_{\Omega,b}(f)(y) - C_B| &= \Big|\|F_{\Omega,t}^{b-b_{B^*}}(f)(y)\|_{\mathcal{H}} - \|F_{\Omega,t}((b - b_{B^*})f_2)(x)\|_{\mathcal{H}}\Big| \\ &\le |b(y) - b_{B^*}|\|F_{\Omega,t}(f)(y)\|_{\mathcal{H}} + \|F_{\Omega,t}((b - b_{B^*})f_1)(y)\|_{\mathcal{H}} \\ &\quad + \|F_{\Omega,t}((b - b_{B^*})f_2(y) - F_{\Omega,t}((b - b_{B^*})f_2)(x)\|_{\mathcal{H}} \\ &\le |b(y) - b_{B^*}|\mu_\Omega(f)(y) + \mu_\Omega((b(\cdot) - b_{B^*})f_1)(y) \\ &\quad + \int_{(B^*)^\complement} |b(z) - b_{B^*}|\Big|\frac{\Omega(x-z)}{|x-z|^{n-1}} - \frac{\Omega(y-z)}{|y-z|^{n-1}}\Big|\frac{1}{|y-z|}|f(z)|dz \\ &=: I_1(y) + I_2(y) + I_3(y).\end{aligned}$$

Next, we estimate each term separately. For $1 < \eta < \infty$, by Hölder's inequality and Lemma 2, we have

$$\begin{aligned}\fint_{B(x,r)} |I_1(y)|dy &= \fint_{B(x,r)} |b(y) - b_{B^*}|\mu_\Omega(f)(y)dy \\ &\le \frac{1}{\psi(B)}\Big(\fint_{B(x,r)} |b(y) - b_{B^*}|^{\frac{1}{\eta}}dy\Big)^{\eta'}\Big(\psi(B)^\eta \fint_{B(x,r)} |\mu_\Omega(f)(y)|^\eta\Big)^{\frac{1}{\eta}} \\ &\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}}\big(\mathcal{M}_{\psi^\eta}(|\mu_\Omega(f)|^\eta)(x)\big)^{1/\eta}.\end{aligned}$$

For the second term $I_2(y)$, choose $\nu \in (1, \eta)$ and let $1/\nu = 1/u + 1/\eta$. Then, by the boundedness of μ_Ω on $L^\nu(\mathbb{R}^n)$, together with Hölder's inequality and Lemma 2, we obtain

$$
\begin{aligned}
\fint_{B(x,r)} I_2(y)dy &= \fint_B \mu_\Omega((b-b_{B^*})f_1)(y)dy \\
&\leq \left(\fint_B \mu_\Omega((b-b_{B^*})f_1)(y)^\nu dy\right)^{1/\nu} \\
&\lesssim \left(\frac{1}{|B|}\int_{B^*} |(b(y)-b_{B^*})f(y)|^\nu dy\right)^{1/\nu} \\
&\lesssim \frac{1}{\psi(B^*)}\Big(\fint_{B^*} |b(y)-b_{B^*}|^u\Big)^{1/u}\left(\psi(B^*)^\eta \fint_{B^*} |f(y)|^\eta dy\right)^{1/\eta} \\
&\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \mathcal{M}_{\psi^\eta}(|f|^\eta)(x)^{1/\eta}.
\end{aligned}
$$

Finally, for $I_3(y)$, we write

$$
\begin{aligned}
I_3(y) \leq &\Big(\int_0^\infty \Big|\int_{|y-z|<t\leq|x-z|} (b(z)-b_{B^*})f_2(z)\frac{\Omega(y-z)}{|y-z|^{n-1}}dz\Big|^2\frac{dt}{t^3}\Big)^{1/2} \\
&+\Big(\int_0^\infty \Big|\int_{|x-z|<t\leq|y-z|} (b(z)-b_{B^*})f_2(z)\frac{\Omega(x-z)}{|x-z|^{n-1}}dz\Big|^2\frac{dt}{t^3}\Big)^{1/2} \\
&+\Big(\int_0^\infty \Big|\int_{|x-z|\leq t,\,|y-z|\leq t} (b(z)-b_{B^*})f_2(z)\Big[\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(x-z)}{|x-z|^{n-1}}\Big]dz\Big|^2\frac{dt}{t^3}\Big)^{1/2} \\
=:&\, A(y)+B(y)+C(y).
\end{aligned}
$$

In what follows, we estimate $A(y)$, $B(y)$ and $C(y)$, respectively. Note that, for $x, y \in B, z \in (B^*)^\complement$, we have $|x-z| \sim |y-z|$. By the Hölder inequality and Lemma 2,

$$
\begin{aligned}
A(y) &\leq \int_{(B^*)^\complement} |b(z)-b_{B^*}||f(z)|\frac{|\Omega(y-z)|}{|y-z|^{n-1}}\left|\frac{1}{|y-z|^2}-\frac{1}{|x-z|^2}\right|^{1/2}dz \\
&\leq \int_{(B^*)^\complement} |b(z)-b_{B^*}||f(z)|\frac{1}{|y-z|^{n-1}}\,\frac{|x-y|^{1/2}}{|x-z|^{3/2}}dz \\
&\leq \sum_{j=1}^\infty \int_{2^{j+1}B\setminus 2^jB} |b(z)-b_{B^*}||f(z)|\frac{|x-y|^{1/2}}{|x-z|^{n+1/2}}dz \\
&\lesssim \sum_{j=1}^\infty 2^{-j/2}\fint_{2^{j+1}B} |b(z)-b_{B^*}||f(z)|dz \\
&\lesssim \sum_{j=1}^\infty \frac{j}{2^{j/2}}\|b\|_{\mathcal{L}^{(1,\psi)}} M_{\psi^\eta}(|f|^\eta)(x)^{1/\eta} \\
&\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \mathcal{M}_{\psi^\eta}(|f|^\eta)(x)^{1/\eta}.
\end{aligned}
$$

By the same arguments as in estimating $A(y)$, we obtain

$$
B(y) \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \mathcal{M}_{\psi^\eta}(|f|^\eta)(x)^{1/\eta}.
$$

For $C(y)$, by the general Minkowski inequality, we have

$$
\begin{aligned}
C(y) &\lesssim \int_{(B^*)^\complement} |b(z) - b_{B^*}||f(z)|\left|\frac{\Omega(y-z)}{|y-z|^{n-1}} - \frac{\Omega(x-z)}{|x-z|^{n-1}}\right|\frac{1}{|x-z|}dz \\
&\leq \int_{(B^*)^\complement} |b(z) - b_{B^*}||f(z)|\frac{|\Omega(x-z)|}{|x-z|}\left|\frac{1}{|y-z|^{n-1}} - \frac{1}{|x-z|^{n-1}}\right|dz \\
&\quad + \int_{(B^*)^\complement} |b(z) - b_{B^*}||f(z)|\frac{|\Omega(y-z) - \Omega(x-z)|}{|x-z|^n} \\
&=: C_1(y) + C_2(y).
\end{aligned}
$$

As in estimating $A(y)$, we have

$$
\begin{aligned}
C_1(y) &\lesssim \int_{(B^*)^\complement} |b(z) - b_{B^*}||f(z)|\frac{|x-y|}{|x-z|^{n+1}}dz \\
&\lesssim \sum_{j=1}^{\infty} 2^{-j} \fint_{2^{j+1}B} |b(z) - b_{B^*}||f(z)|dz \\
&\lesssim \sum_{j=1}^{\infty} \frac{j}{2^j}\|b\|_{\mathcal{L}^{(1,\psi)}} M_{\psi^\eta}(|f|^\eta)(x)^{1/\eta} \\
&\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \mathcal{M}_{\psi^\eta}(|f|^\eta)(x)^{1/\eta}.
\end{aligned}
$$

For $C_2(y)$, invoking the condition (2), we obtain

$$
\begin{aligned}
C_2(y) &\lesssim \int_{(B^*)^\complement} |b(z) - b_{B^*}|\frac{|f(z)|}{|x-z|^n}\left(\log\frac{2|x-z|}{|x-y|}\right)^{-\gamma}dz \\
&\leq \sum_{j=1}^{\infty} \int_{2^{j+1}B\setminus 2^jB} |b(z) - b_{B^*}|\frac{|f(z)|}{|x-z|^n}\left(\log\frac{2|x-z|}{|x-y|}\right)^{-\gamma}dz \\
&\lesssim \sum_{j=1}^{\infty} \frac{j}{(j+1)^\gamma}\|b\|_{\mathcal{L}^{(1,\psi)}} \mathcal{M}_{\psi^\eta}(|f|^\eta)(x)^{1/\eta} \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \mathcal{M}_{\psi^\eta}(|f|^\eta)(x)^{1/\eta}.
\end{aligned}
$$

Summing up the estimates of $A(y)$, $B(y)$, $C_1(y)$ and $C_2(y)$, we obtain

$$
\fint_{B(x,r)} I_3(y)dy \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \mathcal{M}_{\psi^\eta}(|f|^\eta)(x)^{1/\eta}.
$$

This, together with the estimates for $I_1(y)$, $I_2(y)$, immediately yields that

$$
M^\#(\mu_{\Omega,b}(f))(x) \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}}\left((\mathcal{M}_{\psi^\eta}(|\mu_\Omega(f)|^\eta)(x))^{1/\eta} + (\mathcal{M}_{\psi^\eta}(|f|^\eta)(x))^{1/\eta}\right),
$$

which completes the proof of Proposition 2. □

4. Boundedness for $\mu_{\Omega,b}$ on the Generalized Morrey Spaces

This section is devoted to the proof of Theorem 1. At first, we note that, for $0 < \eta < \infty$,

$$
\||f|^\eta\|_{L^{(p,\varphi)}} = (\|f\|_{L^{(p\eta,\varphi)}})^\eta. \tag{16}
$$

Proof of Theorem 1. By Remark 2, we know that, for $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ and $f \in L^{(p,\varphi)}(\mathbb{R}^n)$, $\mu_{\Omega,b}(f)$ defined in (8) is well defined. Therefore, we need only to show

$$
\|\mu_{\Omega,b}(f)\|_{L^{(q,\varphi)}} \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.
$$

By the assumption of Theorem 1 and Proposition 1, we have

$$
\|\mu_\Omega(f)\|_{L^{(p,\varphi)}(\mathbb{R}^n)} \leq C\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)}.
$$

Let $1 < \eta < p$. It follows from (10) that

$$\psi(x,r)^{\eta}\varphi(x,r)^{\eta/p} \le C^{\eta}\varphi(x,r)^{\eta/q}.$$

Then, by Lemma 1, we know that

$$\|\mathcal{M}_{\psi^{\eta}}(f)\|_{L^{(q/\eta,\varphi)}(\mathbb{R}^n)} \lesssim \|f\|_{L^{(p/\eta,\varphi)}(\mathbb{R}^n)}.$$

This, together with the $L^{(p,\varphi)}(\mathbb{R}^n)$-boundedness of μ_Ω (see Proposition 1), leads to

$$\left\|(\mathcal{M}_{\psi^{\eta}}(|\mu_\Omega(f)|^{\eta}))^{1/\eta}\right\|_{L^{(q,\varphi)}} \lesssim (\||\mu_\Omega(f)|^{\eta}\|_{L^{(p/\eta,\varphi)}})^{1/\eta} \lesssim \|f\|_{L^{(p,\varphi)}},$$

and

$$\left\|(\mathcal{M}_{\psi^{\eta}}(|f|^{\eta}))^{1/\eta}\right\|_{L^{(q,\varphi)}} \lesssim (\||f|^{\eta}\|_{L^{(p/\eta,\varphi)}})^{1/\eta} = \|f\|_{L^{(p,\varphi)}}.$$

Therefore, if we can show that, for $B_r = B(0,r)$,

$$\fint_{B_r} \mu_{\Omega,b}(f)(x)dx \to 0, \text{ as } r \to \infty, \tag{17}$$

then, by Lemma 6 and Proposition 2, we have

$$\|\mu_{\Omega,b}(f)\|_{L^{(q,\varphi)}} \lesssim \left\|M^{\#}(\mu_{\Omega,b}(f))\right\|_{L^{(q,\varphi)}} \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

which is the desired conclusion.

It remains to show that (17) holds. Notice that

$$\mu_{\Omega,b}(f)(x) \le |b(x)|\mu_\Omega(f)(x) + \mu_\Omega(bf)(x) =: \mu_b^1(f)(x) + \mu_b^2(f)(x).$$

To prove (17), it suffices to show that

$$\fint_{B_r} \mu_b^1(f)(x)dx \to 0 \quad \text{and} \quad \fint_{B_r} \mu_b^2(f)(x)dr \to 0 \ \ as \ \ r \to \infty.$$

In what follows, we will prove the facts above in the following two cases.

Case 1. We first consider the case of that $f \in L^{(p,\varphi)}(\mathbb{R}^n)$, with compact support. Let $\operatorname{supp} f \subset B_s := B(0,s)$ with $s \ge 1$, $B_{2s} := 2B_s$. Then, $f \in L^p(\mathbb{R}^n)$ and $b \in L^{p_0}_{loc}(\mathbb{R}^n)$ for all $p_0 \in (1,\infty)$ since $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathcal{L}^{(p_0,\psi^{p_0})}(\mathbb{R}^n)$. By the L^p-boundednes of μ_Ω, it is easy to check that $\mu_b^1(f)(x)\chi_{B_{2s}}$ and $\mu_b^2(f)(x)\chi_{B_{2s}}$ are in $L^1(\mathbb{R}^n)$. Then,

$$\fint_{B_r} \mu_b^1(f)(x)\chi_{B_{2s}}(x)dx \to 0, \quad \text{and} \quad \fint_{B_r} \mu_b^2(f)(x)\chi_{B_{2s}}(x)dx \to 0 \text{ as } r \to \infty.$$

Next, we show that

$$\fint_{B_r} \mu_b^1(f)(x)\chi_{(B_{2s})^{\complement}}(x)dx \to 0, \quad \text{and} \quad \fint_{B_r} \mu_b^2(f)(x)\chi_{(B_{2s})^{\complement}}(x)dx \to 0 \text{ as } r \to \infty.$$

Note that, for $x \in (B_{2s})^{\complement}$ and $y \in B_s$, we have $|x|/2 \le |x-y| \le 3|x|/2$. Then, for $x \in (B_{2s})^{\complement}$,

$$\mu_\Omega(f)(x) \le \int_{B_s} \frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f(y)|\left(\int_{|x-y|}^{\infty}\frac{dt}{t^3}\right)^{1/2} \lesssim \int_{B_s}\frac{|\Omega(x-y)|}{|x-y|^{n}}|f(y)|dy \lesssim \frac{1}{|x|^n}\|f\|_{L^1(\mathbb{R}^n)},$$

and

$$\mu_\Omega(bf)(x) \le \int_{B_s} \frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(y)f(y)|\left(\int_{|x-y|}^{\infty}\frac{dt}{t^3}\right)^{1/2}$$
$$\lesssim \int_{B_s} \frac{|\Omega(x-y)|}{|x-y|^{n}}|b(y)f(y)|dy \lesssim \frac{1}{|x|^n}\|bf\|_{L^1(\mathbb{R}^n)},$$

which yields that

$$\fint_{B_r} \mu_b^2(f)(x)\chi_{(B_{2s})^\complement}(x)dx \lesssim \fint_{B_r}\frac{1}{|x|^n}\chi_{(B_{2s})^\complement}(x)dx\|bf\|_{L^1(\mathbb{R}^n)}$$
$$\lesssim \frac{1}{r^n}(\log\frac{r}{2s})\,\|bf\|_{L^1(\mathbb{R}^n)} \to 0 \text{ as } r\to\infty,$$

and

$$\fint_{B_r} \mu_b^1(f)(x)\chi_{(B_{2s})^\complement}(x)dx \lesssim \fint_{B_r}\frac{|b(x)-b_{B_{2s}}|}{|x|^n}\chi_{(B_{2s})^\complement}(x)dx\|f\|_{L^1(\mathbb{R}^n)}$$
$$+\fint_{B_r}\frac{|b_{B_{2s}}|}{|x|^n}\chi_{(B_{2s})^\complement}dx\|f\|_{L^1(\mathbb{R}^n)} =: F_1+F_2.$$

For F_2, we have

$$F_2 = |b_{B_{2s}}|\fint_{B_r}\frac{1}{|x|^n}\chi_{(B_{2s})^\complement}(x)dx\|f\|_{L^1(\mathbb{R}^n)} \lesssim |b_{B_{2s}}|\frac{1}{r^n}(\log\frac{r}{2s})\,\|f\|_{L^1(\mathbb{R}^n)} \to 0 \text{ as } r\to\infty.$$

To estimate F_1, we take $\varepsilon\in(0,1)$ such that $1+1/q-1/p>\varepsilon$ and let $v=1/(1-\varepsilon)$. Then, for $r>4s$, Hölder's inequality and Lemma 3 tell us that

$$F_1 \le \Big(\fint_{B_r}|b(x)-b_{B_{2s}}|^{v'}dx\Big)^{1/v'}\Big(\fint_{B_r}\frac{1}{|x|^{nv}}\chi_{(B_{2s})^\complement}(x)dx\Big)^{1/v}\|f\|_{L^1(\mathbb{R}^n)}$$
$$\lesssim (\log\frac{r}{2s})\psi(0,r)\|b\|_{\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)}\frac{1}{r^{n/v}}\|f\|_{L^1(\mathbb{R}^n)}$$
$$\lesssim \varphi(0,r)^{1/q-1/p}\frac{1}{r^{n/v}}(\log r)\|b\|_{\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)}\|f\|_{L^1(\mathbb{R}^n)}$$
$$\lesssim \frac{\log r}{r^{n(1+1/q-1/p-\varepsilon)}}\left(\frac{1}{r^n\varphi(0,r)}\right)^{1/p-1/q}\|b\|_{\mathcal{L}(1,\psi)(\mathbb{R}^n)}\|f\|_{L^1(\mathbb{R}^n)} \to 0 \text{ as } r\to\infty.$$

Summing up the estimates of F_1 and F_2, we obtain

$$\fint_{B_r}\mu_b^1(f)(x)dx \to 0 \text{ as } r\to\infty.$$

This completes the proof of Case 1.

Case 2. For general $f\in L^{(p,\varphi)}(\mathbb{R}^n)$, fix $r>0$, we write $f=f\chi_{B_{2r}}+f\chi_{(B_{2r})^\complement}$. For $f\chi_{B_{2r}}$, using Case 1, we have

$$\|\mu_{\Omega,b}(f\chi_{B_{2r}})\|_{L^{(p,\varphi)}(\mathbb{R}^n)} \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)}\|f\chi_{B_{2r}}\|_{L^{(p,\varphi)}(\mathbb{R}^n)} \le \|b\|_{\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)}\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)}.$$

Then,

$$\fint_{B_r}\mu_{\Omega,b}(f\chi_{B_{2r}})(x)dx \lesssim \varphi(0,r)^{1/q}\|\mu_{\Omega,b}(f\chi_{B_{2r}})\|_{L^{(p,\varphi)}(\mathbb{R}^n)} \le \varphi(0,r)^{1/q}\|b\|_{\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)}\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)}.$$

This, together with Lemma 5, implies that

$$\fint_{B_r}\mu_{\Omega,b}(f)(x)dx \lesssim \varphi(0,r)^{1/q}\|b\|_{\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)}\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)} \to 0 \text{ as } r\to\infty,$$

which completes the proof of Theorem 1. □

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References

1. Stein, E.M. On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz. *Trans. Amer. Math. Soc.* **1958**, *88*, 430–466. [CrossRef]
2. Al-Salman, A.; Al-Qassem, H.; Cheng, L.C.; Pan, Y. L^p bounds for the function of Marcinkiewicz. *Math. Res. Lett.* **2002**, *9*, 697–700.
3. Deringoz, F.; Hasanov, S.G. Parametric Marcinkiewicz integral operator on generalized Orlicz–Morrey spaces. *Trans. Natl. Acad. Sci. Azerb. Ser. Phys. Tech. Math. Sci.* **2016**, *36*, 70–76.
4. Scapellato, A. Riesz potential, Marcinkiewicz integral and their commutators on mixed Morrey spaces. *Filomat* **2020**, *34*, 931–944. [CrossRef]
5. Wu, H. On Marcinkiewicz integral operators with rough kernels. *Integral Equ. Oper. Theory* **2005**, *52*, 285–298. [CrossRef]
6. Wu, H. L^p bounds for Marcinkiewicz integrals associated with surfaces of revolution. *J. Math. Anal. Appl.* **2006**, *321*, 811–827. [CrossRef]
7. Walsh, T. On the function of Marcinkiewicz. *Stud. Math.* **1972**, *44*, 203–217. [CrossRef]
8. Torchinsky, A.; Wang, S. A note on the Marcinkiewicz integral. *Colloq. Math.* **1990**, *60/61*, 235–243. [CrossRef]
9. Aliev, S.S.; Guliyev, V.S. Boundedness of the parametric Marcinkiewicz integral operator and its commutators on generalized Morrey spaces. *Georgian Math. J.* **2012**, *19*, 195–208. [CrossRef]
10. Cui, R.; Li, Z. Boundedness of Marcinkiewicz integrals and their commutators on generalized weighted Morrey spaces. *J. Funct. Spaces* **2015**, *2015*, 450145. [CrossRef]
11. Deringoz, F. Parametric Marcinkiewicz integral operator and its higher order commutators on generalized weighted Morrey spaces. *Trans. Natl. Acad. Sci. Azerb. Ser. Phys. Tech. Math. Sci.* **2017**, *37*, 24–32.
12. Chen, Y.; Ding, Y. L^p boundedness of the commutators of Marcinkiewicz integrals with rough kernels. *Forum Math.* **2015**, *27*, 2087–2111. [CrossRef]
13. Ding, Y.; Lu, S.; Yabuta, K. On commutators of Marcinkieiwcz integrals with rough kernel. *J. Math. Anal. Appl.* **2002**, *275*, 60–68. [CrossRef]
14. Hu, G.; Yan, D. On the commutator of Marcinkiewicz integral. *J. Math. Anal. Appl.* **2003**, *283*, 351–361. [CrossRef]
15. Chen, Y.; Ding, Y. Commutators of Littlewood-Paley operators. *Sci. China Math.* **2009**, *52*, 2493–2505. [CrossRef]
16. Wen, Y.; Wu, H. On the commutators of Marcinkiewicz integrals with rough kernels in weighted Lebesgue spaces. *Anal. Math.* **2020**, *46*, 619–638. [CrossRef]
17. Deringoz, F. Commutators of parametric Marcinkiewicz integrals on generalized Orlicz–Morrey spaces. *Commun. Fac. Sci. Univ. Ank. Sér. Al Math. Stat.* **2017**, *66*, 115–123.
18. Lu, G. Parametric Marcinkiewicz integral and its commutator on generalized Orlicz–Morrey spaces. *J. Korea Math. J.* **2021**, *58*, 383–400.
19. Arai, R.; Nakai, E. Commutators of Calderón-Zygmund and generalized fractional integral operators on generalized Morrey spaces. *Rev. Mat. Complut.* **2018**, *31*, 287–331. [CrossRef]
20. Benedek, A.; Calderón, A.P.; Panzone, R. Convolution operators on Banach space valued function. *Proc. Nat. Acad. Sci. USA* **1962**, *48*, 356–365. [CrossRef] [PubMed]