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Collatz Attractors Are Space-Filling

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Abstract: The algebraic topology of Collatz attractors (or "Collatz Feathers") remains very poorly understood. In particular, when pushed to infinity, is their set of branches discrete or continuous? Here, we introduce a fundamental theorem proving that the latter is true. For any odd x, we first define $\mathbb{A}_x^n$ as the set of all odd numbers with $Syr(x)$ in their Collatz orbit and up to n more digits than x in base 2. We then prove $\lim_{n\to\infty} \frac{|\mathbb{A}_x^n|}{2^{n+c}} \geq 1$ with $c > -4$ for all x and, in particular, $c = 0$ for $x = 1$, which is a result strictly stronger than that of Tao 2019.

Keywords: dynamical system; $3x+1$ problem; Collatz conjecture; discrete chaos; Furstenberg conjecture; discrete algebraic topology; chaos theory; chaotic cryptology

MSC: 11Sxx; 37A44; 37Axx; 65P20; 14Gxx

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1. Introduction

The Collatz conjecture—that all orbits of the $3x+1$ discrete dynamical system reach 1—remains extremely relevant both to the fundamental study of ergodic number theory and to the applied one of chaotic cryptology. For example, Bocart 2018 [1] demonstrated that it could be successfully used to develop a readily applicable proof-of-work algorithm independent of large prime numbers for the cryptocurrency industry. In a previous publication [2], we studied and exhibited the particular geometry of the Collatz basins of attraction (hereunder, "attractors") and connected it to the also difficult Furstenberg conjecture, of which the entire complexity lies in that "base 2 and base 3 representations share no common structure" (Furstenberg 1960 [3] and 1967 [4], and Shmerkin 2021 [5]). Collatz attractors, also known as "Collatz feathers" [6] or "Collatz seaweeds", are forming complex quivers among natural numbers, but very little is still known of their particular geometry and topological properties. Here, we intend to simplify and extend the arithmetic and algebraic formalism we introduced in [2] (where the reader may find a more thorough bibliography) in order to obtain four fundamental lemmas and one fundamental theorem to break new ground in the study of the geometric algebra and, in particular, discrete algebraic topology of these otherwise mostly unknown structures.

2. Definitions

A few actions naturally appear in the study of Collatz attractors [2] so it is useful to abbreviate them all here. For all odd x,

- $S(x) := 2x+1$. It is equivalent to "appending an end digit 1 to x in base 2". $S^n(0)$ is, therefor,e the nth Mersenne number.
- $G(x) := 2x-1$. It is equivalent to "intercalating a digit 0 before the final digit 1 of x in base 2". $G^n(3)$ is the $(n+1)$th Mersenne number + 2.
- $V(x) := 4x+1 = G(S(x))$. It is equivalent to "appending an end pair of digits 01 to x in base 2". $V^n(1)$ is the $(2(n+1))^{th}$ Mersenne number divided by 3.
- $D(x) := 64x+49$.

- $A(x) := \frac{2x-1}{3} = \frac{G(x)}{3}$ for $x \equiv 2 \bmod 3$ (we will note "x in $[2]_3$").
- $C(x) := \frac{4x-1}{3} = \frac{S(G(x))}{3}$ for $x \equiv 1 \bmod 3$ (we will note "x in $[1]_3$").

Furthermore:

1. $x \sim y$ is defined as "x and y have a common number in their Collatz orbit". The relation is reflexive, transitive, and symmetric.
2. $Syr(x)$ means "the first odd number in the forward Collatz orbit of x".
3. $\mathbb{A}_x$ is "the attractor of x" defined as all numbers with $Syr(x)$ in their forward Collatz orbit. $\mathbb{A}_x^n$ stands for "the elements of $\mathbb{A}_x$ with at most n more digits than x in base 2".
4. $\overline{\mathbb{A}_x}$ denotes "the complement of set $\mathbb{A}_x$". $|\mathbb{A}_x^n|$ is the number of elements in $\mathbb{A}_x^n$
5. The "closure of {x}" by any action is the set of all possible compositions of this action on x.
6. Let $\mathbb{V}_x$ be the closure of $\{x\}$ by V, and note $\mathbb{V}_x^n$ as its elements written at most with n more digits than x in base 2.
7. Let $\mathbb{D}_x$ be the infinite sequence:
$$\{\{x; S(x)\}; \{D(x); S(D(x))\}; ...; \{D^k(x); S(D^k(x))\}; ...\}.$$
We further define $\mathbb{D}_x^n$ as the elements of $\mathbb{D}_x$ with at most n more binary digits than x.
8. We note $[x]$ as "the binary length of x" and $\mathbb{D}_{[x]+k}$ as "*a $\mathbb{D}$ sequence starting with a number of binary length $[x]+k$*"
9. The "V-closure" of $\mathbb{D}_x$, noted as $\mathbb{V}_{\mathbb{D}_x}$, is the closure of this set by action V. The V-closure of each pair $\{a; S(a)\}$ in $\mathbb{D}_x$ is called $\mathbb{P}_a$. Note that $\mathbb{P}_a^n$ always has exactly n elements. $\mathbb{V}_{\mathbb{D}_x}^n$ is, therefore, of the following form:
$$\{\mathbb{P}_x^n; \mathbb{P}_{[x]+6}^{n-6}; ...; \mathbb{P}_{[x]+6k}^{n-6k}; ...\}$$
10. $AC(\mathbb{V}_x)$ is defined as an "action A on all elements of $\mathbb{V}_x$ in $[2]_3$ and action C on all elements of $\mathbb{V}_x$ in $[1]_3$". Let us also note $AC^{k+1}(x)$, action AC on the V-closure of $AC^k(x)$ and initiate with $AC^0(x) = x$. Note that $AC(x)$ is equivalent to $AC(\mathbb{V}_x)$, but we may use the latter when we want to specify a parameter n for the maximum binary length being considered.
11. The notation $\bigsqcup$ as in $\bigsqcup_{k=1}^n x_k$ is not understood as the general disjoint union but more simply as "$\bigcup_{k=1}^n x_k$ with each x_k being *already* disjoint from the others".

3. Preparatory Lemmas

Lemma 1 (Lemma of irreducibility)**.** *For all odd x, $\mathbb{A}_x$ is the closure of $\{x\}$ by actions A, C, and V. Each of its non-x element has a uniquely ordered, finite, and non-cyclic decomposition into actions A, C, and V from x, and the only possible cycle is from x to itself.*

Proof. $3(4x+1)+1 = 12x+4 = 4(3x+1)$ so $x \sim V(x)$ as x is odd. As residue classes in base 3 are periodic by V, it always reaches numbers where actions C and A are applicable. If x is in $[0]_3$, then $V(x)$ is in $[1]_3$ and $V^2(x)$ in $[2]_3$. Since $3((2x-1)/3)+1 = 2x$, we have that for any x in $[2]_3$, $A(x) \sim x$, and since $3((4x-1)/3)+1 = 4x$, we also have $C(x) \sim x$ for any x in $[1]_3$.

For any x in $[2]_3$, we could generalize action A to output the sequence $(2^{2k-1}x-1)/3$ for all $k > 1$ because all elements of such sequence also converge to $Syr(x)$, but we already have them counted in all iterations of action V over $A(x)$. The same goes for action C: it is generalizable with $(2^{2k}x-1)/3$ for all $k > 1$ with x in $[1]_3$, but we will already have those numbers counted in any iteration of action V on $C(x)$.

To prove that each element of $\mathbb{A}_x$ has a unique non-cyclic decomposition into actions A, C, and V, we now define a one-to-one correspondence between any $Syr(y)$ and either action. For any odd y, either $3y+1$ is divisible by exactly 2, by exactly 4, or by at least 8. If $3y+1$ is divisible by 2, then $A((3y+1)/2) = y$; if it is divisible by 4, then $C((3y+1)/4) = y$; and if it is divisible by 8 or more, there are always finite k and z such that $y = V^k(z)$ and $Syr(y) = Syr(z)$.

Any odd x in $\mathbb{A}_x$ leads to $Syr(x)$, so the only possible cycle is x to itself. For example, 1 is the only fixed point of C so the cycle in $\mathbb{A}_1$ is $1 \to 1$. □

Remark 1. *We already have that no composition of C, and V may be cyclical as both actions are strictly increasing, but proving that no composition of A, C, and V be cyclical, as they combine coprimes 2 and 3, would be of at least the same complexity as proving the Furstenberg* $\times 2 \times 3$ *conjecture, of which the entire difficulty lies in that "base 2 and base 3 representations share no known common structures" (Furstenberg 1960 [3] and 1967 [4], and Shmerkin 2021 [5]). We already have, however, that, in any base* 2^k,

1. $A^n(x)$ *always ends with the representation of the* $(n+1)^{th}$ *Mersenne number, e.g., by* $\underbrace{1\ldots1}_{k}{}_2$
2. $V^n(x)$ *always ends with the representation of the* $(2(n+1))^{th}$ *Mersenne number divided by 3, e.g., by* $\underbrace{01\ldots01}_{k}{}_2$
3. $C^n(x)$ *always ends with the representation of number* $2^{n+1}+1$ *e.g., by* $\underbrace{0\ldots0}_{k}1_2$

The chaoticity of ACV compositions (that is, any legal word of the form $X_1 \ldots (X_k(X_{k+1} \ldots (x)).)$*, where* X_k *is representing exactly one of either operation and x is any odd number) and hence the difficulty of proving the non-existence of cycles directly come from the fact that A is allowed to delete digits and that both A and C contain divisions by* 3*. However, the beauty of chaotic cryptology is that, whenever a simple system appears to be intractably chaotic, it comes with the participation trophy of at least offering a strong (and straightforward) industrial chaotic cipher.*

Lemma 2 (Lemma of prolificity)**.** *Let x be odd with no odd y such that* $x = V(y)$*. There is always a unique odd z such that* $\mathbb{D}_z \in AC(x)$*.*

Proof.

1. Let x be in $[1]_3$, then so is $V^3(x)$. $V(x)$ is in $[2]_3$ and so is $V^4(x)$. $A(V(x)) = S(C(x))$, $A(V^4(x)) = S(C(V^3(x))$, and $C(V^3(x)) = D(C(x))$. Thus, $AC(x) = \mathbb{D}_{C(x)}$. For example, $AC(7) = \mathbb{D}_9$
2. Let x be in $[0]_3$; then, $V(x)$ is in $[1]_3$ and $AC(x) = \mathbb{D}_{C(V(x))}$. For example $AC(9) = \mathbb{D}_{49}$
3. Let x be in $[2]_3$; then, $V^2(x)$ is in $[1]_3$ and $AC(x) = \mathbb{D}_{C(V^2(x))} \cup \{A(x)\}$, with $A(x)$ not in $\mathbb{D}_{C(V^2(x))}$. For example, $AC(11) = \mathbb{D}_{241} \cup \{7\}$. Note $A(x)$ (in the example, $A(x) = 7$) always has a V-sequence of its own so $AC(A(x))$ will in turn output another $\mathbb{D}$ sequence.

□

Remark 2. *For any odd x,* $AC(x)$ *outputs a* $\mathbb{D}$ *sequence (and if x is in* $[2]_3$*, a single additional odd number), in which V-closure is an infinite sequence of* $\mathbb{V}$ *sequences (since no element of any* $\mathbb{D}$ *sequence can be written* $V(y)$ *with y being odd). Then,* $\mathbb{V}_{AC(x)}$ *is an infinite sequence of* $\mathbb{V}$ *sequences and* $\mathbb{V}_{AC^k(x)}$ *is an infinite k-sequence (a sequence of sequences k times) of* $\mathbb{V}$ *sequences.* $|\mathbb{V}_x^n|$ *is a polynomial of n of degree* 1 *(it is exactly equal to* $n/2$*), and the amount of numbers up to extra binary length n from x in each* $\mathbb{V}_{AC^k(x)}$ *is a polynomial of n of degree* $k+1$*. This algebraic property is the reason we call Lemma 2 that of "prolificity". Its first immediate consequence is that* $\lim_{n\to\infty} |\mathbb{A}_x^n|$*, being an infinite sequence of positive-valued polynomials of n of incremented degree and with a positive coefficient for each term of maximal degree, is now necessarily at least of the form* Ae^{an+c} *for some constants* $A, a,$ *and c.*

Lemma 3 (Lemma of homogeneity)**.** *All residue classes in base* 3 *are equally frequent in* $AC^k(\mathbb{V}_x^n)$ *when n and k tend toward infinity.*

Proof. $V^3(x) = 2^6x + 21$ with $2^6 \equiv 1 \bmod 3$ and $21 = 210_3$, so

1. Let x end in $a\underbrace{0\ldots0}_{k}{}_3$; then, for all k, $V^{3^k}(x)$ ends in $(a+1)\underbrace{0\ldots0}_{k}{}_3$ and $V^{3^k+1}(x)$ ends in $(a+2)\underbrace{0\ldots0}_{k}1_3$.
2. Let x end in $a\underbrace{2\ldots2}_{k}{}_3$; then, for all k, $V^{3^k}(x)$ ends in $(a+1)\underbrace{2\ldots2}_{k}{}_3$.
3. If x ends in 22_3, $A(x)$ is in $[2]_3$. If x ends in 12_3, $A(x)$ is in $[0]_3$. If x ends in 02_3, $A(x)$ is in $[1]_3$.
4. If x ends in 21_3, $C(x)$ is in $[0]_3$. If x ends in 11_3, $C(x)$ is in $[2]_3$. If x ends in 01_3, $C(x)$ is in $[1]3$.

□

Remark 3. *Concordant with Lemma 3, we can verify that actions D and S already preserve the periodicity of residue classes in base 3: if x is in $[0]_3$, $D(x)$ is in $[1]_3$; if x is in $[1]_3$, $D(x)$ is in $[2]_3$; and if x is in $[2]_3$, $D(x)$ is in $[0]_3$. Furthermore, if x is in $[0]_3$, $S(x)$ is in $[1]_3$; if x is in $[1]_3$, $S(x)$ is in $[0]_3$; and if x is in $[2]_3$, $S(x)$ is in $[2]_3$. The Lemma of homogeneity is an essential result to construct the following "Lemma of density", which in turn is the most critical preparatory result for the fundamental Lemma 1.*

Lemma 4 (Lemma of density)**.** *As $n \to \infty$, every $\mathbb{P}_x \in \mathbb{V}_{\mathbb{D}_y} \in \mathbb{A}^n_z$, is associated with a single sequence $\mathbb{F} = \bigsqcup_{k=1}^{\infty} \mathbb{D}_{[x]+ak} \in \mathbb{A}^n_z$, where a is a pseudorandom variable, of which the average is less than 2, and*

$$\lim_{n\to\infty} \frac{|\mathbb{V}^n_{\mathbb{F}}|}{|\bigsqcup_{k=1}^{\infty} \mathbb{P}^{n-k}_{[x]+k}|} \geq 1$$

Proof. By Lemma 2 for any $\{x; S(x)\}$ in some $\mathbb{D}_a$, there are always y and z such that $AC(\mathbb{P}_x) = \{\mathbb{D}_y; \mathbb{D}_z\}$. Let us review all the possible cases.

1. For x in $[0]_3$, $y = C(S(x))$ and $z = C(V(x))$. If x began with 10_2, $[y] = [x] + 1$ and $[z] = [x] + 2$; otherwise, $[y] = [x] + 2$ and $[z] = [x] + 3$.
2. For x in $[1]_3$, $y = C(x)$ and $z = C(V(S(x)))$. If x began with 10_2, $[y] = [x] + 0$ and $[z] = [x] + 3$; otherwise, $[y] = [x] + 1$ and $[z] = [x] + 4$
3. For x in $[2]_3$, $y = A(x)$ and $z = C(V^2(x))$. If x began with 10_2, $[y] = [x] - 1$ and $[z] = [x] + 4$; otherwise, $[y] = [x] + 0$ and $[z] = [x] + 5$

For simplification, we assume the two cases of x beginning with either 10_2 or 11_2 are at worst equifrequent but 10_2 (which is always outputing smaller numbers) will in fact be more frequent because for any a beginning in 11_2, $3a$ always begins in 10_2, whereas exactly one third of all odd a beginning in 10_2 (for example, those beginning in 1011_2) are such that $3a$ also begins with 10_2 (See Figure 27 in Rahn et al., 2021 [2]).

Thus, assuming the limiting conditions of Lemma 3, we can now associate a unique disjoint pair $\{\mathbb{D}_{[x]+a_1}; \mathbb{D}_{[x]+a_2}\}$ to each $AC(\mathbb{P}_x)$. From all cases we reviewed above, we have that a_k is a pseudorandom variable, in which the pseudorandomness entirely comes from the Furstenberg principle that "base 2 and base 3 representations share no known common structure", and its actual (and again, unknown) structure is in fact at the heart of the Bocart proof of work (we may call the larger family "base-change ciphers"). However, what we now know is that the mean of a_k over all k is less than 2 because all of the possible cases are $\{1; 2; 0; 3; -1; 4\}$ if x began with 10_2 and $\{2; 3; 1; 4; 0; 5\}$ if x began with 11_2, and within each prefix starting condition (10_2 or 11_2), Lemma 3 guarantees their being equifrequent.

Hence, to each $\mathbb{P}_x$, there is a unique $\bigsqcup_{k=1}^{\infty} \mathbb{D}_{[x]+ak}$, in which V-closure contains at least $\bigsqcup_{k=1}^{\infty} \mathbb{P}_{[x]+ak}$. As $\mathbb{P}_{[x]+ak}$ contains exactly $n - ak$ elements with up to n more binary digits than x; if we take the limiting condition of $a \leq 2$, each of them must be associated with at least k additional (disjoint) elements to have

$$\lim_{n\to\infty} \frac{|\mathbb{V}^n_{\mathbb{F}}|}{|\bigsqcup_{k=1}^{\infty} \mathbb{P}^{n-k}_{[x]+k}|} \geq 1$$

Those additional elements are always found in $\mathbb{V}_{\mathbb{D}_{[x]+ak}} \setminus \mathbb{P}_{[x]+ak}$. □

Remark 4. *Lemma 2 established that any $|\mathbb{A}^n_z|$ was growing exponentially with n. Lemma 4 now further states that, as n goes to infinity, $|\mathbb{A}^n_z|$ fits each $\mathbb{P}_x \in \mathbb{D}_y \in \mathbb{A}^n_z$ and is associated with a unique collection $\mathbb{F}$ with particular density conditions. As we will see in Theorem 1, this now provides enough information to pinpoint the general formula for $\lim_{n\to\infty} |\mathbb{A}^n_z|$.*

4. Fundamental Theorem of Collatz Attractors

Theorem 1. *For x odd, there is always $c > -4$ in $\mathbb{R}$, depending only on x, with*

$$\lim_{n\to\infty} \frac{|\mathbb{A}^n_x|}{2^{n+c}} \geq 1$$

In particular, for $\mathbf{x} = \mathbf{1}, \mathbf{c} = \mathbf{0}$.

Proof. We intend to prove the number of distinct elements in $\mathbb{A}^n_x$, in which excluding x itself will bring it arbitrarily close to the following formula as n grows:

$$\sum_{x=0}^{n-c} \prod_{k=0}^{x} \frac{n-c-k}{k+1} = 2^{n-c} - 1 \tag{1}$$

For $x = 1$, $c = 0$, as $2^n - 1$ covers all the odd numbers written with n more digits than 1 in base 2.

To achieve this result, we will demonstrate that the development of any $|\mathbb{A}^n_x|$ and $|\mathbb{A}^n_1|$ in particular follows a power series that is at least a sum of progressively iterated sums, of which the first term is of the following form:

$$\mathcal{P}^1(n) = \sum_{k=1}^{n} (n-k) = \frac{1}{2}(n)(n-1) \tag{2}$$

and which, iterated, gives a progression of the following form:

$$\begin{aligned} \mathcal{P}^2(n) &= \sum_{k=1}^{n} \frac{1}{2}(n-k)(n-k-1) \\ &= \frac{1}{2\cdot 3}(n)(n-1)(n-2) \end{aligned} \tag{3}$$

Then,

$$\begin{aligned} \mathcal{P}^3(n) &= \sum_{k=1}^{n} \frac{1}{2\cdot 3}(n-k)(n-k-1)(n-k-2) \\ &= \frac{1}{2\cdot 3\cdot 4}(n)(n-1)(n-2)(n-3) \end{aligned} \tag{4}$$

Thus, in general,

$$\begin{aligned} &\mathcal{P}^0(n) = n \\ &\mathcal{P}^{j+1}(n) = \sum_{k=1}^{n} (\mathcal{P}^j(n-k)) = \frac{(n-j)\mathcal{P}^j(n)}{j+1} \end{aligned} \tag{5}$$

Therefore, finally,

$$\sum_{j=0}^{n} \mathcal{P}^j(n) = \sum_{x=0}^{n} \prod_{k=0}^{x} \frac{n-k}{k+1} = 2^n - 1 \tag{6}$$

By Lemma 4, we have that, as $n \to \infty$, there is always a non-surjective f:

$$f : \mathbb{P}^n_y \in \mathbb{A}^n_x \to \mathbb{A}^n_x; \quad f(\mathbb{P}^n_y) = \bigsqcup_{k=0}^{\infty} \mathbb{V}_{\mathbb{D}^{n-2k}_{[x]+2k}} \tag{7}$$

and

$$\lim_{n\to\infty} \frac{|\bigsqcup_{k=0}^{\infty} \mathbb{V}_{\mathbb{D}^{n-2k}_{[x]+2k}}|}{|\bigsqcup_{k=0}^{\infty} \mathbb{P}^{n-k}_{[x]+k}|} \geq 1 \tag{8}$$

Now note that, as $|\mathbb{P}^n_x| = n$,

$$|\bigsqcup_{a=1}^{n} \mathbb{P}^{n-a}_{[y]+a}| = \sum_{a=1}^{n} (n-a) = \mathcal{P}^1(n) \tag{9}$$

Additionally,

$$|f(\bigsqcup_{a=1}^{n} \mathbb{P}^{n-a}_{[y]+a})| \geq |\bigsqcup_{b=1}^{n} \bigsqcup_{a=1}^{b} \mathbb{P}^{n-a}_{[y]+a}| = \sum_{b=1}^{n} \sum_{a=1}^{b} (n-a) = \mathcal{P}^2(n) \tag{10}$$

In general,

$$|f^{k+1}(\mathbb{P}^n_x)| = |\bigsqcup_{k=1}^{n} f^k(\mathbb{P}^n_x)| \geq \sum_{k=1}^{n} \mathcal{P}^k(n) = \mathcal{P}^{k+1}(n) \tag{11}$$

Thus, for any $\mathbb{P}_x \in \mathbb{D}_y$,

$$\lim_{n\to\infty} |\mathbb{A}^n_{\{x;S(x)\}}| \geq \sum_{k=0}^{n} \mathcal{P}^k(n) \quad \geq 2^n - 1 \tag{12}$$

Let us now take any odd y as the initial condition, assuming only that, without any loss of generality, there is no odd a such that $y = V(a)$ (because if there was, we would just redefine y:=a). By Lemma 2, there is always a unique z so that $AC(y) = \mathbb{D}_z$. The binary length of z is at worst $[y] + 3$, indeed if y is in $[0]_3$, $z = C(V(y))$ which, in the worst case of y beginning with 11_2, has three more binary digits than y. Therefore, $\lim_{n\to\infty} |\mathbb{A}^n_{\{z;S(z)\}}| = 2^{n-3}$ modulo the existence of at most one cycle in all of $\mathbb{A}_y$. As $\mathbb{V}_{\mathbb{D}_z} = \bigsqcup_{k=0}^{\infty} \mathbb{P}_{[z]+6k}$, we now have, still modulo, the existence of at most one cycle:

$$\lim_{n\to\infty} \frac{|\mathbb{A}^n_y|}{2^{n-3}} \geq \lim_{n\to\infty} \frac{\sum_{k=0}^{\frac{n-3}{6}} 2^{n-3-6k}}{2^{n-3}} = 1 \tag{13}$$

With the worst starting conditions already taken into account (hence the $n-3$) and now also considering the worst possible cycle (adding up to $n-3-1$), we now have that, for any odd x, $\lim_{n\to\infty} \mathbb{A}^n_x$ always tends toward at least more than

$$\frac{1}{2} \sum_{x=0}^{n-3} \prod_{k=0}^{x} \frac{n-3-k}{k+1} > 2^{n-4} - 1 \tag{14}$$

Note that $\mathbb{A}^n_x$ can have 2^{n+c} elements with c positive, depending only on the initial conditions, for example, on whether x or $V(x)$ ends in 22_3 or 222_3, etc., defining how many consecutive times decreasing action A is applied in the early iterations of $\mathbb{A}_x$.

If we now take the case of $\mathbb{A}_1$, its particular starting conditions are that, since $A(5) = 3$ and $C(1) = 1$, $AC(1) = \mathbb{D}_1$, $3 \in \mathbb{D}_1$ and $V^2(3) = 53 = 1222_3$.

Therefore, $AC(3) = \mathbb{D}_{17}$, with $A(17) = 11$, $A(11) = 7$, $C(7) = 9$, $C(S(9)) = 25$, and $C(V(9)) = 49$. Thus, the initial development of $\mathbb{A}_1$ contains $\mathbb{D}_1$, $\mathbb{D}_{3=11_2}$ (of which the first $\mathbb{V}$ series is already counted in $\mathbb{D}_1$, but Equation (2) is iterated at $k = 1$), $\mathbb{D}_{7=111_2}$, $\mathbb{D}_{9=1001_2}$, $\mathbb{D}_{11=1011_2}$, $\mathbb{D}_{17=10001_2}$, $\mathbb{D}_{25=11001_2}$, and $\mathbb{D}_{49=110001_2}$. With $\{\mathbb{D}_1; \mathbb{D}_3; \mathbb{D}_7; \mathbb{D}_9; \mathbb{D}_{11}; \mathbb{D}_{17}; \mathbb{D}_{25}; \mathbb{D}_{49}\}$

for initial conditions, $c = 0$ modulo the existence of a cycle, but as the only cycle in $\mathbb{A}_1$ is C(1)=1, we have $\lim_{n\to\infty} \frac{|\mathbb{A}_1^n|}{2^n} = 1$ □

5. Discussion

While the previous demonstration was one of discrete algebraic topology, following an interesting request from the reviewers, we outline here another simpler graph-theoretical demonstration of the ACV-closure of any odd $\{x\}$ being space-filling. In fact, one of the reasons we ventured to call Theorem 1 a "Fundamental theorem" is that there may be a great diversity of ways to demonstrate it.

We already know that the closure of any odd $\{x\}$ by actions S and G up to the additional binary length n, which in this case is strictly equivalent to all the odd numbers of length lesser or equal to $[x] + n$ and beginning with all the first characters of x minus the last one, is in bijection with the powerset of the set with n elements and, therefore, has exactly 2^n elements. Importantly, for the sake of this demonstration, we will only consider $x \in [1]_3$, but if $x \in [2]_3$, then we would consider $A(x)$ until it is either in $[1]_3$ or $[0]_3$, and with $x \in [0]_3$, we just use the fact that, then, $V(x) \in [1]_3$.

There are already 3^k strings of k characters out of the alphabet $\{A, C, V\}$, but no odd number exists such that it can remain whole under both A and C so $|\mathbb{A}_x^n|$ cannot be of the form 3^n. For any string to belong to $\mathbb{A}_x^n$, the alphabet $\{A, C, V\}$ is endowed with a grammar interdicting certain strings and determined by both the residue class of odd numbers in base 3 and the total binary characters added or subtracted by actions A, C, and V. Let us begin by determining the latter:

1. $A(y) = \frac{G(y)}{3}$ so $[A(y)] = [y] - 1$ if x began with 10_2 or $[y] + 0$ otherwise.
2. $C(y) = \frac{S(G(y))}{3}$ so $[C(y)] = [y] + 0$ if x began with 10_2 or $[y] + 1$ otherwise.
3. $V(y) = G(S(y))$ so $[V(y)] = [y] + 2$ for all y in $\mathbb{A}_x^n$

By Lemma 3, for all y in $\mathbb{A}_x^n$, $n \to \infty$ $A(y)$ and $C(y)$ may be in any base 3 residue class with equal frequency.

Additionally, note that prefixes 10_2 and 11_2 are not equally frequent with action $\cdot\, 3$. Indeed, if for all y beginning with 11_2, $3y$ always begin with 10_2, exactly one third of all odd numbers beginning in 10_2—precisely, those greater than the $V^n(1)$ (for those with an odd number of digits in base 2) or than the $S(V^n(1))$ (for those with an even number of digits) of the same length as theirs—are such that $3y$ will also begin with 10_2, thus favoring this outcome after either actions A or C with a limit frequency of $\frac{2}{3}$ when $n \to \infty$ (Figure 27 in Rahn et al., 2021 [2]). Action V on the other hand leaves the base 2 prefix intact.

Exactly one out of three times, each action (V, A ,and C) will only be allowed to be followed by action V only, which also appends two more base 2 digits. Otherwise, action C appends $\frac{1}{3}$ digit on average, and action A removes $\frac{2}{3}$ digits on average, meaning that $C^3(y)$ has 1 more digits than y if y was on the right $\frac{2}{3}$ of the odd numbers beginning with 10_2 (for example, $[S(3^4)] = [163] = 8$ and $[C^3(S(3^4))] = [385] = 9 = [163] + 1$).

If in any ACV string we note $\mathcal{V}$, $\mathcal{A}$, and $\mathcal{C}$ as the total number of characters V, A, and C, respectively, then all strings in $\mathbb{A}_x^n$ must verify the following topological invariant:

$$2\mathcal{V} - \frac{2}{3}\mathcal{A} + \frac{1}{3}\mathcal{C} \leq n \tag{15}$$

By Lemma 3, the following limit is also verified when $\mathcal{V}$, $\mathcal{A}$, and $\mathcal{C}$ are counted on all of $\mathbb{A}_x^n$:

$$lim_{n\to\infty} \frac{\mathcal{A}}{\mathcal{C}} = 1 \tag{16}$$

and since V is defined for all base 3 residue classes in $\mathbb{A}_x^n$ while there is no other operation defined for residue class 0_3, we also have

$$lim_{n\to\infty} \frac{\mathcal{V}}{\mathcal{A} + \mathcal{C}} = \frac{3}{2} \tag{17}$$

As A and C are equifrequent, we may represent "A or C exclusively and with equal frequency" with X, and when $n \to \infty$, $|\mathbb{A}_x^n|$ will equal the number of paths in the binary tree where $\frac{2}{3}$ of all nodes (each representing a word) have a V and X vertices and $\frac{1}{3}$ only a V one. This is strictly equivalent to the binary tree where each node has exactly $\frac{2}{3}$ chances of being allowed to branch non-exclusively in either V or X with equal frequency (thus branching to just X with a total frequency of exactly $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$) and $\frac{1}{3}$ to branch non-exclusively and equifrequently in VV or VX. In this tree, we now count all of the existing words, verifying

$$0 \leq 2\mathcal{V} - \frac{1}{3}\mathcal{X} \leq n \tag{18}$$

If $\mathbb{A}_x^n$ was only closed by two distinct actions, both defined equifrequently on all elements and with the binary length characteristics of V (for example $V_1 := 4x + 1$ and $V_2 := 4x + 3$), it would be in bijection with the powerset of the set with $\frac{n}{2}$ elements. However, the tree we defined is at least equivalent to V_1 adding two binary digits and to V_2 adding two exactly half of the times and subtracting $\frac{1}{3}$ the other half. Thus counting all occurrences of V_2 and only the pathways with a net-positive binary length extension, we now have that $\mathbb{A}_x^n$ is at least as large as the powerset of the set with $\frac{n}{2} + n\sum_{k=1}^{\infty} \frac{1}{3^k}$ elements. Therefore,

$$\lim_{n \to \infty} \frac{|\mathbb{A}_x^n|}{2^{\frac{n}{2}} \cdot \prod_{k=1}^{\infty} 2^{\frac{n}{3^k}}} \geq 1 \tag{19}$$

so:

$$\lim_{n \to \infty} \frac{|\mathbb{A}_x^n|}{2^n} \geq 1 \tag{20}$$

Remark 5. *This alternative approach to proving the space-fillingness of Collatz feathers is particularly fertile conceptually. Indeed, the ACV-closure of any* $\{x\}$ *can now, by the development of Equation (19) and although it uses three distinct actions so would start with a maximum of* 3^{kn} *branches, be interpreted as* $3^{n \cdot \frac{log(2)}{log(3)}}$ *with* $\frac{log(2)}{log(3)}$ *being the Hausdorff dimension of the triadic Cantor set (the set of all real numbers that can be written without any digit* 1 *in base* 3*). Practitioners of the Furstenberg conjecture know that Cantor sets are in turn a larger family containing sets left invariant by either* $\times 3$ *mod* 1 *or* $\times 2$ *mod* 1 *but never by both ([4]).* $\frac{log(2)}{log(3)}$ *is also the inverse of the Hausdorff dimension of the Sierpinski triangle, a natural figure in the evaluation of the 3-adic distance, which gives us some insight into the geometry of* $\mathbb{A}_x^n$ *when* $n \to \infty$*. With this in mind, it would be particularly interesting to further study the structure (and iterated morphisms) of the attractors of the* $7x + 1$, $5x + 1$ *and Juggler sequences with the deliberate intent of constructing ("pathological") objects of an intermediate cardinality between* $\aleph_0$ *and* $\aleph_1$ *as such constructions, though never achieved, are already known not to contradict ZFC [7,8]. All in all, we really believe that the Collatz conjecture, the Furstenberg* $\times 2 \times 3$ *conjecture, and the Continuum Hypothesis should be much more tightly related in the literature.*

6. Conclusions

Theorem 1, for the particular case of $c = 0$ when $x = 1$, already implies that "almost all Collatz orbits converge to 1", which is a result strictly stronger (and shorter) than Tao 2019 [9], where it was only obtained that *almost all* Collatz orbits attained *almost* bounded values, leaving not only the possibility of cycles but also the limitation of almost boundedness. Such is not the case here: on the one side, all the elements in $\mathbb{A}_1$ converge to 1 so none other than 1 belongs to a cycle, and on the other side, the elements of $\mathbb{A}_1$ are proven to cover *almost all* of $\mathbb{N}$. Additionally, pushed to infinity, Collatz attractors are therefore space-filling as their cardinality is of the form 2^{n+c}.

However, Theorem 1 does not yet prove that all Collatz orbits converge to 1. Rather, it proves that $\frac{|\overline{\mathbb{A}_1^n}|}{|\mathbb{A}_1^n|}$ vanishes when n tends toward infinity (which by the way is consistent with the computational findings presented in Figure 15 in Rahn et al., 2021 [2]). Hence, $|\overline{\mathbb{A}_1^n}|$ can never reach a formula of the form 2^{n+c} for some integer (possibly negative) constant $c < n$. At worst, it may only reach x^{n+c} with $x < 2$. On the other side though, no Collatz attractor can vanish. Thus, in conclusion, by Lemma 1, we have

$$\lim_{n\to\infty} \frac{|\overline{\mathbb{A}_1^n}|}{|\mathbb{A}_1^n|} = 0 \tag{21}$$

However, for all odd x of binary length m, we always have:

$$\lim_{n\to\infty} \frac{|\mathbb{A}_x^n|}{|\mathbb{A}_1^n|} > \frac{1}{2^{m-4}} > 0 \tag{22}$$

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