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Finite-Time Stability Analysis of Fractional Delay Systems

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Abstract: Nonhomogeneous systems of fractional differential equations with pure delay are considered. As an application, the representation of solutions of these systems and their delayed Mittag-Leffler matrix functions are used to obtain the finite time stability results. Our results improve and extend the previous related results. Finally, to illustrate our theoretical results, we give an example.

Keywords: finite time stability; fractional delay systems; delayed Mittag-Leffler matrix function; fractional derivative

MSC: 34K20; 34K37; 34A08

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1. Introduction

Numerous processes in mechanical and technological systems were described using fractional delay differential equations (DDE). These systems are frequently utilized in the modelling of phenomena in technological and scientific problems. These models have applications in diffusion processes [1], viscoelastic systems [2,3], modeling disease [4], forced oscillations, signal analysis, control theory, biology, computer engineering, finance, and population dynamics; see for instance [5–7]. Khusainov and Shuklin [8] constructed a new idea of a delayed exponential matrix function, in 2003, to express the solutions of linear DDEs. By generating a delayed matrix sine and a delayed matrix cosine, Khusainov et al. [9] used this approach to describe the solutions of an oscillating system with pure delay, in 2008. These pioneering research yielded plenty of novel results on the representation of solutions that are employed in the stability analysis and control problems of time-delay systems; see, for example [10–20] and the references therein.

Finite-time stability is a novel definition that involves a fixed finite-time interval and a prescribed constraint for the system, as opposed to the exponential/asymptotic stability definition, which is exposed to an infinite-time interval. In recent decades, there has been a growing interest in fractional delay system finite time stability (FTS) analysis, and several methods for studying FTS of fractional delay systems have been developed; see for example [21–32].

To our knowledge, there is no study dealing with the fractional system's finite temporal stability analysis with a single delay

$$\begin{aligned} \left({}^{C}D^{\alpha}_{0+}y\right)(u) &= -Ay(u-\kappa) + f(u), \quad \text{for } \kappa > 0,\ u \in W := [0, L], \\ y(u) &\equiv \psi(u),\ y'(u) \equiv \psi'(u) \quad \text{for } -\kappa \le u \le 0, \end{aligned} \tag{1}$$

where ${}^{C}D^{\alpha}_{0^+}$ is said to be the Caputo fractional derivative of order $\alpha \in (1,2]$ with the lower index zero, κ is a delay, L is a pre-fixed positive number, $y(u) \in \mathbb{R}^n$, $\psi \in C^2([-\kappa,0],\mathbb{R}^n)$, $A \in \mathbb{R}^{n\times n}$ is a constant nonzero matrix and $f \in C([0,\infty),\mathbb{R}^n)$ is a given function.

Recently, Elshenhab and Wang [14] gave a new representation of solutions of (1) of the form

$$\begin{aligned} y(u) = {} & \mathcal{H}_{\kappa,\alpha}\big(A(u-\kappa)^{\alpha}\big)\psi(0) + \mathcal{M}_{\kappa,\alpha}\big(A(u-\kappa)^{\alpha}\big)\psi'(0) \\ & - A\int_{-\kappa}^{0} \mathcal{S}_{\kappa,\alpha}\big(A(u-2\kappa-\xi)^{\alpha}\big)\psi(\xi)d\xi \\ & + \int_{0}^{u} \mathcal{S}_{\kappa,\alpha}\big(A(u-\kappa-\xi)^{\alpha}\big)f(\xi)d\xi, \end{aligned} \tag{2}$$

where $\mathcal{H}_{\kappa,\alpha}(Au^{\alpha})$, $\mathcal{M}_{\kappa,\alpha}(Au^{\alpha})$ and $\mathcal{S}_{\kappa,\alpha}(Au^{\alpha})$ are called the delayed Mittag-Leffler type matrix functions formulated by

$$\mathcal{H}_{\kappa,\alpha}(Au^{\alpha}) := \begin{cases} \Theta, & -\infty < u < -\kappa, \\ I, & -\kappa \le u < 0, \\ I - A\frac{u^{\alpha}}{\Gamma(1+\alpha)}, & 0 \le u < \kappa, \\ \quad\vdots & \quad\vdots \\ I - A\frac{u^{\alpha}}{\Gamma(1+\alpha)} + A^2\frac{(u-\kappa)^{2\alpha}}{\Gamma(1+2\alpha)} & \\ + \cdots + (-1)^m A^m \frac{(u-(m-1)\kappa)^{m\alpha}}{\Gamma(1+m\alpha)}, & (m-1)\kappa \le u < m\kappa, \end{cases} \tag{3}$$

$$\mathcal{M}_{\kappa,\alpha}(Au^{\alpha}) := \begin{cases} \Theta, & -\infty < u < -\kappa, \\ I(u+\kappa), & -\kappa \le u < 0, \\ I(u+\kappa) - A\frac{u^{\alpha+1}}{\Gamma(2+\alpha)}, & 0 \le u < \kappa, \\ \quad\vdots & \quad\vdots \\ I(u+\kappa) - A\frac{u^{\alpha+1}}{\Gamma(2+\alpha)} + A^2\frac{(u-\kappa)^{2\alpha+1}}{\Gamma(2+2\alpha)} & \\ + \cdots + (-1)^m A^m \frac{(u-(m-1)\kappa)^{m\alpha+1}}{\Gamma(2+m\alpha)}, & (m-1)\kappa \le u < m\kappa, \end{cases} \tag{4}$$

and

$$\mathcal{S}_{\kappa,\alpha}(Au^{\alpha}) := \begin{cases} \Theta, & -\infty < u < -\kappa, \\ I\frac{(u+\kappa)^{\alpha-1}}{\Gamma(\alpha)}, & -\kappa \le u < 0, \\ I\frac{(u+\kappa)^{\alpha-1}}{\Gamma(\alpha)} - A\frac{u^{2\alpha-1}}{\Gamma(2\alpha)}, & 0 \le u < \kappa, \\ \quad\vdots & \quad\vdots \\ I\frac{(u+\kappa)^{\alpha-1}}{\Gamma(\alpha)} - A\frac{u^{2\alpha-1}}{\Gamma(2\alpha)} + A^2\frac{(u-\kappa)^{3\alpha-1}}{\Gamma(3\alpha)} & \\ + \cdots + (-1)^m A^m \frac{(u-(m-1)\kappa)^{\alpha(m+1)-1}}{\Gamma(\alpha(m+1))}, & (m-1)\kappa \le u < m\kappa, \end{cases} \tag{5}$$

respectively, where $m = 0,1,2,\ldots$, the notations I is the $n\times n$ identity matrix, Θ is the $n \times n$ null matrix and Γ is a gamma function.

Motivated by [15], as an application, the explicit formulas of solutions of (1) and the delayed Mittag-Leffler matrix functions are used to get FTS results on $W = [0, L]$.

The rest of this paper is organized as follows: In Section 2, we give preliminaries on fractional calculus theory and FTS. Moreover, we give alternative formulas of solutions of (1) and estimations of norms for the delayed Mittag-Leffler matrix functions, which are used while discussing FTS. In Section 3, as an application, the representation of solutions of (1) is used to obtain FTS results. Finally, to illustrate our theoretical results, we give an example.

2. Preliminaries

Throughout the paper, we denote the vector norm as $\|y\| = \sum_{i=1}^{n} |y_i|$ and the matrix norm as $\|A\| = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$; y_i and a_{ij} are the elements of the vector y and the matrix A, respectively. Denote $C(W, \mathbb{R}^n)$ the Banach space of vector-value continuous function from $W \to \mathbb{R}^n$ endowed with the norm $\|y\|_C = \max_{u\in W} \|y(u)\|$ for a norm $\|\cdot\|$ on $\mathbb{R}^n$. We introduce a space $C^1(W, \mathbb{R}^n) = \{y \in C(W, \mathbb{R}^n) : y' \in C(W, \mathbb{R}^n)\}$. Furthermore, we see $\|\psi\|_C = \max_{v\in[-\kappa,0]} \|\psi(v)\|$.

We recall some basic definitions of fractional calculus theory and FTS.

Definition 1 ([6]). *The two-parameter Mittag-Leffler function is given by*

$$\mathbb{E}_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)}, \quad \alpha,\ \gamma > 0,\ z \in \mathbb{C}.$$

Especially, if $\gamma = 1$, then

$$\mathbb{E}_{\alpha,1}(z) = \mathbb{E}_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0.$$

Definition 2 ([6]). *The Caputo fractional derivative with lower index 0 of a function $y : [-\kappa, \infty) \to \mathbb{R}^n$ is defined as*

$$\left({}^{C}D_{0^+}^{\alpha} y\right)(u) = \frac{1}{\Gamma(2-\alpha)} \int_0^u \frac{y''(\xi)}{(u-\xi)^{\alpha-1}} d\xi, \quad u > 0.$$

Definition 3 ([23]). *The system (1) is finite time stable with respect to $\{0, W, \kappa, \delta, \epsilon\}$, $\delta < \epsilon$ if and only if $\eta < \delta$ implies $\|y(u)\| < \epsilon$ for all $u \in W$, where $\eta = \max\{\|\psi\|_C, \|\psi'\|_C, \|\psi''\|_C\}$ and δ, ϵ are real positive numbers.*

Next, we obtain some alternative formulas of solutions of (2) used in analyzing the FTS.

Lemma 1 (Theorem 2.6 [18]). *The system (1) has a unique solution $y(u)$, and*

$$\begin{aligned} y(u) &= \mathcal{H}_{\kappa,\alpha}(Au^{\alpha})\psi(-\kappa) + \mathcal{M}_{\kappa,\alpha}(Au^{\alpha})\psi'(-\kappa) \\ &+ \int_{-\kappa}^{0} \mathcal{M}_{\kappa,\alpha}\big(A(u-\kappa-\xi)^{\alpha}\big)\psi''(\xi)d\xi \\ &+ \left({}^{C}D_{0^+}^{2-\alpha} y\right)(u-\kappa) * f(u). \end{aligned}$$

Remark 1. *We can obtain some alternative formulas of solutions of (2) by applying integration by parts and simplification of the conclusion of Lemma 1 to derive that*

$$\begin{aligned} y(u) &= \mathcal{H}_{\kappa,\alpha}(Au^{\alpha})\psi(-\kappa) + \mathcal{M}_{\kappa,\alpha}(Au^{\alpha})\psi'(-\kappa) \\ &+ \int_{-\kappa}^{0} \mathcal{M}_{\kappa,\alpha}\big(A(u-\kappa-v)^{\alpha}\big)\psi''(v)dv \\ &+ \int_{0}^{u} \mathcal{S}_{\kappa,\alpha}\big(A(u-\kappa-v)^{\alpha}\big)f(v)dv, \end{aligned} \tag{6}$$

or

$$\begin{aligned} y(u) &= \mathcal{H}_{\kappa,\alpha}(Au^{\alpha})\psi(-\kappa) + \mathcal{M}_{\kappa,\alpha}\big(A(u-\kappa)^{\alpha}\big)\psi'(0) \\ &+ \int_{-\kappa}^{0} \mathcal{H}_{\kappa,\alpha}\big(A(u-\kappa-v)^{\alpha}\big)\psi'(v)dv \\ &+ \int_{0}^{u} \mathcal{S}_{\kappa,\alpha}\big(A(u-\kappa-v)^{\alpha}\big)f(v)dv. \end{aligned} \tag{7}$$

To conclude this section, we provide estimations of norms for the delayed Mittag-Leffler matrix functions, which are used in discussing FTS.

Lemma 2. *For any $u \in [(m-1)\kappa, m\kappa]$, $m = 1, 2, \ldots$, we have*

$$\|\mathcal{H}_{\kappa,\alpha}(Au^\alpha)\| \le \mathbb{E}_\alpha(\|A\|u^\alpha).$$

Proof. Using (3), we get

$$\begin{aligned}
\|\mathcal{H}_{\kappa,\alpha}(Au^\alpha)\| &\le 1 + \|A\|\frac{u^\alpha}{\Gamma(1+\alpha)} + \|A\|^2\frac{(u-\kappa)^{2\alpha}}{\Gamma(1+2\alpha)} \\
&\quad + \cdots + \|A\|^m\frac{(u-(m-1)\kappa)^{m\alpha}}{\Gamma(1+m\alpha)} \\
&\le 1 + \|A\|\frac{u^\alpha}{\Gamma(1+\alpha)} + \|A\|^2\frac{u^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots + \|A\|^m\frac{u^{m\alpha}}{\Gamma(1+m\alpha)} \\
&\le \sum_{k=0}^{\infty}\frac{(\|A\|u^\alpha)^k}{\Gamma(1+k\alpha)} = \mathbb{E}_\alpha(\|A\|u^\alpha).
\end{aligned}$$

Hence, the proof is complete. □

Lemma 3. *For any $u \in [(m-1)\kappa, m\kappa]$, $m = 1, 2, \ldots$, we have*

$$\|\mathcal{M}_{\kappa,\alpha}(Au^\alpha)\| \le (u+\kappa)\mathbb{E}_{\alpha,2}\big(\|A\|(u+\kappa)^\alpha\big).$$

Proof. Using (4), we get

$$\begin{aligned}
\|\mathcal{M}_{\kappa,\alpha}(Au^\alpha)\| &\le (u+\kappa) + \|A\|\frac{u^{\alpha+1}}{\Gamma(2+\alpha)} + \|A\|^2\frac{(u-\kappa)^{2\alpha+1}}{\Gamma(2+2\alpha)} \\
&\quad + \cdots + \|A\|^m\frac{(u-(m-1)\kappa)^{m\alpha+1}}{\Gamma(2+m\alpha)} \\
&\le (u+\kappa) + \|A\|\frac{(u+\kappa)^{\alpha+1}}{\Gamma(2+\alpha)} + \|A\|^2\frac{(u+\kappa)^{2\alpha+1}}{\Gamma(2+2\alpha)} \\
&\quad + \cdots + \|A\|^m\frac{(u+\kappa)^{m\alpha+1}}{\Gamma(2+m\alpha)} \\
&\le \sum_{k=0}^{\infty}\frac{\big[\|A\|(u+\kappa)^\alpha\big]^k(u+\kappa)}{\Gamma(2+k\alpha)} = (u+\kappa)\mathbb{E}_{\alpha,2}\big(\|A\|(u+\kappa)^\alpha\big).
\end{aligned}$$

Hence, the proof is complete. □

Lemma 4. *For any $u \in [(m-1)\kappa, m\kappa]$, $m = 1, 2, \ldots$, we have*

$$\|\mathcal{S}_{\kappa,\alpha}(Au^\alpha)\| \le (u+\kappa)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\big(\|A\|(u+\kappa)^\alpha\big).$$

Proof. Using (5), we get

$$\begin{aligned}\|\mathcal{S}_{\kappa,\alpha}(Au^{\alpha})\| &\leq \frac{(u+\kappa)^{\alpha-1}}{\Gamma(\alpha)} + \|A\|\frac{u^{2\alpha-1}}{\Gamma(2\alpha)} + \|A\|^{2}\frac{(u-\kappa)^{3\alpha-1}}{\Gamma(3\alpha)} \\ &+ \cdots + \|A\|^{m}\frac{(u-(m-1)\kappa)^{\alpha(m+1)-1}}{\Gamma(\alpha(m+1))} \\ &\leq \frac{(u+\kappa)^{\alpha-1}}{\Gamma(\alpha)} + \|A\|\frac{(u+\kappa)^{2\alpha-1}}{\Gamma(2\alpha)} + \|A\|^{2}\frac{(u+\kappa)^{3\alpha-1}}{\Gamma(3\alpha)} \\ &+ \cdots + \|A\|^{m}\frac{(u+\kappa)^{\alpha(m+1)-1}}{\Gamma(\alpha(m+1))} \\ &\leq \sum_{k=0}^{\infty}\frac{[\|A\|(u+\kappa)^{\alpha}]^{k}(u+\kappa)^{\alpha-1}}{\Gamma(\alpha k+\alpha)} \\ &= (u+\kappa)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\|A\|(u+\kappa)^{\alpha}).\end{aligned}$$

Hence, the proof is complete. □

3. Main Results

In this section, we establish some sufficient conditions for FTS results of (1) by making use of the three possible formulas of solutions (2), (6) and (7), respectively.

Theorem 1. *The system (1) is finite time stable with respect to* $\{0, W, \kappa, \delta, \epsilon\}$, $\delta < \epsilon$ *if*

$$\mathbb{E}_{\alpha}(\|A\|(L-\kappa)^{\alpha}) < \frac{\alpha\epsilon - \delta\alpha L\mathbb{E}_{\alpha,2}(\|A\|L^{\alpha}) - L^{\alpha}(\delta\|A\| + \|f\|_{C})\mathbb{E}_{\alpha,\alpha}(\|A\|L^{\alpha})}{\alpha\delta}. \tag{8}$$

Proof. By using Definition 3 and (2), we have $\eta < \delta$ and

$$\begin{aligned}\|y(u)\| &\leq \|\mathcal{H}_{\kappa,\alpha}(A(u-\kappa)^{\alpha})\|\|\psi(0)\| + \|\mathcal{M}_{\kappa,\alpha}(A(u-\kappa)^{\alpha})\|\|\psi'(0)\| \\ &+ \|A\|\left\|\int_{-\kappa}^{0}\mathcal{S}_{\kappa,\alpha}(A(u-2\kappa-\xi)^{\alpha})\psi(\xi)d\xi\right\| \\ &+ \left\|\int_{0}^{u}\mathcal{S}_{\kappa,\alpha}(A(u-\kappa-\xi)^{\alpha})f(\xi)d\xi\right\| \\ &\leq \|\mathcal{H}_{\kappa,\alpha}(A(u-\kappa)^{\alpha})\|\|\psi(0)\| + \|\mathcal{M}_{\kappa,\alpha}(A(u-\kappa)^{\alpha})\|\|\psi'(0)\| \\ &+ \|A\|\int_{-\kappa}^{0}\|\mathcal{S}_{\kappa,\alpha}(A(u-2\kappa-\xi)^{\alpha})\|\|\psi(\xi)\|d\xi \\ &+ \int_{0}^{u}\|\mathcal{S}_{\kappa,\alpha}(A(u-\kappa-\xi)^{\alpha})\|\|f(\xi)\|d\xi \\ &\leq \delta\|\mathcal{H}_{\kappa,\alpha}(A(u-\kappa)^{\alpha})\| + \delta\|\mathcal{M}_{\kappa,\alpha}(A(u-\kappa)^{\alpha})\| \\ &+ \delta\|A\|\int_{-\kappa}^{0}\|\mathcal{S}_{\kappa,\alpha}(A(u-2\kappa-\xi)^{\alpha})\|d\xi \\ &+ \|f\|_{C}\int_{0}^{u}\|\mathcal{S}_{\kappa,\alpha}(A(u-\kappa-\xi)^{\alpha})\|d\xi.\end{aligned} \tag{9}$$

Note that $\mathcal{S}_{\kappa,\alpha}(Au^{\alpha}) = \Theta$ if $u \in (-\infty, -\kappa)$. For $-\kappa \leq \xi \leq 0$, we get

$$\mathcal{S}_{\kappa,\alpha}(A(u-2\kappa-\xi)^{\alpha}) = \begin{cases} \mathcal{S}_{\kappa,\alpha}(A(u-2\kappa-\xi)^{\alpha}), & \xi \in [-\kappa, u-\kappa], \\ \Theta, & \xi \in (u-\kappa, 0]. \end{cases}$$

Thus

$$\|\mathcal{S}_{\kappa,\alpha}(A(u-2\kappa-\xi)^{\alpha})\| = \begin{cases} \|\mathcal{S}_{\kappa,\alpha}(A(u-2\kappa-\xi)^{\alpha})\|, & \xi \in [-\kappa, u-\kappa], \\ 0, & \xi \in (u-\kappa, 0]. \end{cases}$$

Therefore, from Lemma 4, we have

$$\begin{aligned}\|\mathcal{S}_{\kappa,\alpha}(A(u-2\kappa-\xi)^{\alpha})\| &\le (u-\kappa-\xi)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\|A\|(u-\kappa-\xi)^{\alpha})\\ &\le (u-\kappa-\xi)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\|A\|u^{\alpha}),\end{aligned} \tag{10}$$

for $-\kappa \le \xi \le 0$, $u \in W$, and since $\mathbb{E}_{\alpha,\alpha}(\|A\|u^{\alpha})$ is increasing function when $u \ge 0$. From (10), we get

$$\int_{-\kappa}^{0}\|\mathcal{S}_{\kappa,\alpha}(A(u-2\kappa-\xi)^{\alpha})\|d\xi \le \frac{u^{\alpha}}{\alpha}\mathbb{E}_{\alpha,\alpha}(\|A\|u^{\alpha}), \tag{11}$$

and

$$\begin{aligned}\int_{0}^{u}\|\mathcal{S}_{\kappa,\alpha}(A(u-\kappa-\xi)^{\alpha})\|d\xi &\le \mathbb{E}_{\alpha,\alpha}(\|A\|u^{\alpha})\int_{0}^{u}(u-\xi)^{\alpha-1}d\xi\\ &= \frac{u^{\alpha}}{\alpha}\mathbb{E}_{\alpha,\alpha}(\|A\|u^{\alpha}).\end{aligned} \tag{12}$$

From (9), (11) and (12), we have

$$\begin{aligned}\|y(u)\| \le{}& \delta\mathbb{E}_{\alpha}(\|A\|(u-\kappa)^{\alpha}) + \delta u\mathbb{E}_{\alpha,2}(\|A\|u^{\alpha})\\ &+ \left(\frac{\delta}{\alpha}\|A\| + \frac{\|f\|_C}{\alpha}\right)u^{\alpha}\mathbb{E}_{\alpha,\alpha}(\|A\|u^{\alpha}),\end{aligned} \tag{13}$$

for all $u \in W$. Combining (8) with (13), we get $\|y(u)\| < \epsilon$ for all $u \in W$. Hence, the proof is complete. □

Theorem 2. *The system (1) is finite time stable with respect to* $\{0, W, \kappa, \delta, \epsilon\}$, $\delta < \epsilon$ *if*

$$\mathbb{E}_{\alpha}(\|A\|L^{\alpha}) < \frac{\epsilon - \frac{\delta(L+\kappa)(L+\kappa+2)}{2}\mathbb{E}_{\alpha,2}(\|A\|(L+\kappa)^{\alpha}) - \frac{L^{\alpha}\|f\|_C}{\alpha}\mathbb{E}_{\alpha,\alpha}(\|A\|L^{\alpha})}{\delta}. \tag{14}$$

Proof. By using Definition 3 and (6), we have $\eta < \delta$ and

$$\begin{aligned}\|y(u)\| \le{}& \|\mathcal{H}_{\kappa,\alpha}(Au^{\alpha})\|\|\psi(-\kappa)\| + \|\mathcal{M}_{\kappa,\alpha}(Au^{\alpha})\|\|\psi'(-\kappa)\|\\ &+ \left\|\int_{-\kappa}^{0}\mathcal{M}_{\kappa,\alpha}(A(u-\kappa-\xi)^{\alpha})\psi''(\xi)d\xi\right\|\\ &+ \left\|\int_{0}^{u}\mathcal{S}_{\kappa,\alpha}(A(u-\kappa-\xi)^{\alpha})f(\xi)d\xi\right\|\\ \le{}& \delta\|\mathcal{H}_{\kappa,\alpha}(Au^{\alpha})\| + \delta\|\mathcal{M}_{\kappa,\alpha}(Au^{\alpha})\|\\ &+ \delta\int_{-\kappa}^{0}\|\mathcal{M}_{\kappa,\alpha}(A(u-\kappa-\xi)^{\alpha})\|d\xi\\ &+ \|f\|_C\int_{0}^{u}\|\mathcal{S}_{\kappa,\alpha}(A(u-\kappa-\xi)^{\alpha})\|d\xi.\end{aligned} \tag{15}$$

From Lemma 3, we have

$$\begin{aligned}\int_{-\kappa}^{0}\|\mathcal{M}_{\kappa,\alpha}(A(u-\kappa-\xi)^{\alpha})\|d\xi &\le \mathbb{E}_{\alpha,2}(\|A\|(u+\kappa)^{\alpha})\int_{-\kappa}^{0}(u-\xi)d\xi\\ &\le \frac{(u+\kappa)^2}{2}\mathbb{E}_{\alpha,2}(\|A\|(u+\kappa)^{\alpha}).\end{aligned} \tag{16}$$

From (12), (15) and (16), we get

$$\begin{aligned}\|y(u)\| \le{}& \delta\mathbb{E}_{\alpha}(\|A\|u^{\alpha}) + \delta(u+\kappa)\mathbb{E}_{\alpha,2}(\|A\|(u+\kappa)^{\alpha})\\ &+ \frac{\delta(u+\kappa)^2}{2}\mathbb{E}_{\alpha,2}(\|A\|(u+\kappa)^{\alpha}) + \frac{\|f\|_C}{\alpha}u^{\alpha}\mathbb{E}_{\alpha,\alpha}(\|A\|u^{\alpha}).\end{aligned} \tag{17}$$

for all $u \in W$. Combining (14) with (17), we have $\|y(u)\| < \epsilon$ for all $u \in W$. Hence, the proof is complete. □

Theorem 3. *The system (1) is finite time stable with respect to* $\{0, W, \kappa, \delta, \epsilon\}$, $\delta < \epsilon$ *if*

$$\mathbb{E}_{\alpha}(\|A\|L^{\alpha}) < \frac{\alpha\epsilon - \delta\alpha L\mathbb{E}_{\alpha,2}(\|A\|L^{\alpha}) - L^{\alpha}\|f\|_{C}\mathbb{E}_{\alpha,\alpha}(\|A\|L^{\alpha})}{\alpha\delta(1+\kappa)}. \tag{18}$$

Proof. By using Definition 3 and (7), we have $\eta < \delta$ and

$$\begin{aligned}\|y(u)\| \le{}& \|\mathcal{H}_{\kappa,\alpha}(Au^{\alpha})\|\|\psi(-\kappa)\| + \|\mathcal{M}_{\kappa,\alpha}(A(u-\kappa)^{\alpha})\|\|\psi'(0)\| \\ &+ \left\|\int_{-\kappa}^{0}\mathcal{H}_{\kappa,\alpha}(A(u-\kappa-\xi)^{\alpha})\psi'(\xi)d\xi\right\| \\ &+ \left\|\int_{0}^{u}\mathcal{S}_{\kappa,\alpha}(A(u-\kappa-\xi)^{\alpha})f(\xi)d\xi\right\| \\ \le{}& \delta\|\mathcal{H}_{\kappa,\alpha}(Au^{\alpha})\| + \delta\|\mathcal{M}_{\kappa,\alpha}(A(u-\kappa)^{\alpha})\| \\ &+ \delta\int_{-\kappa}^{0}\|\mathcal{H}_{\kappa,\alpha}(A(u-\kappa-\xi)^{\alpha})\|d\xi \\ &+ \|f\|_{C}\int_{0}^{u}\|\mathcal{S}_{\kappa,\alpha}(A(u-\kappa-\xi)^{\alpha})\|d\xi.\end{aligned} \tag{19}$$

From Lemma 2, we have

$$\int_{-\kappa}^{0}\|\mathcal{H}_{\kappa,\alpha}(A(u-\kappa-\xi)^{\alpha})\|d\xi \le \kappa\mathbb{E}_{\alpha}(\|A\|u^{\alpha}). \tag{20}$$

From (12), (19) and (20), we get

$$\begin{aligned}\|y(u)\| \le{}& \delta\mathbb{E}_{\alpha}(\|A\|u^{\alpha}) + \delta u\mathbb{E}_{\alpha,2}(\|A\|u^{\alpha}) \\ &+ \delta\kappa\mathbb{E}_{\alpha}(\|A\|u^{\alpha}) + \frac{\|f\|_{C}}{\alpha}u^{\alpha}\mathbb{E}_{\alpha,\alpha}(\|A\|u^{\alpha}).\end{aligned} \tag{21}$$

for all $u \in W$. Combining (18) with (21), we have $\|y(u)\| < \epsilon$ for all $u \in W$. Hence, the proof is complete. □

Remark 2. *Let* $\alpha = 2$ *in (1). Then Theorems 1–3 coincide with the conclusion of Theorems 1–3 in [32].*

Remark 3. *Let* $\alpha = 2$, $A = A^2$ *in (1) such that the matrix* A *is a nonsingular* $n \times n$ *matrix. Then*

$$\mathcal{H}_{\kappa,2}\left(A^2u^2\right) = \cos_{\kappa}(Au), \quad \mathcal{M}_{\kappa,2}\left(A^2u^2\right) = A^{-1}\sin_{\kappa}(Au).$$

where $\cos_{\kappa}(Au)$ *and* $\sin_{\kappa}(Au)$ *are called the delayed matrix of cosine and sine type, respectively, defined in [9]. Thus, Theorems 1–3 coincide with the conclusion of Theorems 3.1–3.3 in [30]. Therefore, by dropping the nonsingularity criterion on a matrix coefficient* A *and making the matrix* A *an arbitrary, not necessarily squared matrix* A^2*, our results improve and extend the corresponding results in [30,32].*

4. An Example

Consider the fractional DDEs

$$\begin{aligned}&\left({}^{C}D_{0^+}^{1.8}y\right)(u) = -Ay(u-1/2) + f(u), \ \ u \in [0,1], \\ &\psi(u) = \left(0.1u^2, 0.2u\right)^T, \ \psi'(u) = (0.2u, 0.2)^T, \ \psi''(u) = (0.2, 0)^T, \ -1/2 \le u \le 0,\end{aligned} \tag{22}$$

where

$$\alpha = 1.8, \ \ \kappa = 1/2, \ \ A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \ \ f(u) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

From (6), for all $0 \le u \le 1$, and through a basic calculation, we can obtain

$$\begin{aligned} y(u) = & \begin{pmatrix} 0.025\mathcal{H}_{0.5,1.8}(2u^{1.8}) \\ -0.1\mathcal{H}_{0.5,1.8}(2u^{1.8}) \end{pmatrix} + \begin{pmatrix} -0.1\mathcal{M}_{0.5,1.8}(2u^{1.8}) \\ 0.2\mathcal{M}_{0.5,1.8}(2u^{1.8}) \end{pmatrix} \\ & + \begin{pmatrix} 0.2\int_{-0.5}^{0} \mathcal{M}_{0.5,1.8}\Big(2(u-1/2-\xi)^{1.8}\Big)d\xi \\ 0 \end{pmatrix} \\ & + \begin{pmatrix} \int_0^u \mathcal{S}_{0.5,1.8}\Big(2(u-0.5-\xi)^{1.8}\Big)d\xi \\ 2\int_0^u \mathcal{S}_{0.5,1.8}\Big(2(u-0.5-\xi)^{1.8}\Big)d\xi \end{pmatrix} = \begin{pmatrix} y_1(u) \\ y_2(u) \end{pmatrix}, \end{aligned}$$

which implies that

$$\begin{aligned} y_1(u) = & \ 0.025\mathcal{H}_{0.5,1.8}\Big(2u^{1.8}\Big) - 0.1\mathcal{M}_{0.5,1.8}\Big(2u^{1.8}\Big) \\ & + 0.2\int_{-0.5}^{0} \mathcal{M}_{0.5,1.8}\Big(2(u-1/2-\xi)^{1.8}\Big)d\xi \\ & + \int_0^u \mathcal{S}_{0.5,1.8}\Big(2(u-1/2-\xi)^{1.8}\Big)d\xi, \end{aligned}$$

and

$$\begin{aligned} y_2(u) = & -0.1\mathcal{H}_{0.5,1.8}\Big(2u^{1.8}\Big) + 0.2\mathcal{M}_{0.5,1.8}\Big(2u^{1.8}\Big) \\ & + 2\int_0^u \mathcal{S}_{0.5,1.8}\Big(2(u-1/2-\xi)^{1.8}\Big)d\xi, \end{aligned}$$

where

$$\mathcal{H}_{0.5,1.8}\Big(2u^{1.8}\Big) = \begin{cases} 1, & -1/2 \le u < 0, \\ 1 - \frac{4000}{3353}u^{1.8}, & 0 \le u < 1/2, \\ 1 - \frac{4000}{3353}u^{1.8} + \frac{4000}{13381}(u-1/2)^{3.6}, & 1/2 \le u < 1, \end{cases}$$

$$\mathcal{M}_{0.5,1.8}\Big(2u^{1.8}\Big) = \begin{cases} (u+1/2), & -1/2 \le u < 0, \\ (u+1/2) - \frac{10000}{23471}u^{2.8}, & 0 \le u < 1/2, \\ (u+1/2) - \frac{10000}{23471}u^{2.8} + \frac{2000}{30777}(u-1/2)^{4.6}, & 1/2 \le u < 1, \end{cases}$$

and

$$\mathcal{S}_{0.5,1.8}\Big(2u^{1.8}\Big) = \begin{cases} \frac{1}{0.93138}(u+1/2)^{0.8}, & -1/2 \le u < 0, \\ \frac{1}{0.93138}(u+1/2)^{0.8} - \frac{2000}{3717}u^{2.6}, & 0 \le u < 1/2, \\ \frac{1}{0.93138}(u+1/2)^{0.8} - \frac{2000}{3717}u^{2.6} \\ \quad + \frac{4000}{44599}(u-1/2)^{4.4}, & 1/2 \le u < 1. \end{cases}$$

Thus the explicit solutions of (22) are

$$\begin{aligned} y_1(u) = & \ 0.025\mathcal{H}_{0.5,1.8}\Big(2u^{1.8}\Big) - 0.1\mathcal{M}_{0.5,1.8}\Big(2u^{1.8}\Big) \\ & + 0.2\int_{-0.5}^{u-0.5} \mathcal{M}_{0.5,1.8}\Big(2(u-1/2-\xi)^{1.8}\Big)d\xi \\ & + 0.2\int_{u-0.5}^{0} \mathcal{M}_{0.5,1.8}\Big(2(u-1/2-\xi)^{1.8}\Big)d\xi \\ & + \int_0^u \mathcal{S}_{0.5,1.8}\Big(2(u-1/2-\xi)^{1.8}\Big)d\xi, \end{aligned}$$

$$y_2(u) = -0.1\mathcal{H}_{0.5,1.8}\left(2u^{1.8}\right) + 0.2\mathcal{M}_{0.5,1.8}\left(2u^{1.8}\right) + 2\int_0^u \mathcal{S}_{0.5,1.8}\left(2(u-1/2-\xi)^{1.8}\right)d\xi,$$

where $0 \leq u \leq 1/2$, which implies that

$$y_1(u) = -0.022424u^{3.8} + 0.04261u^{2.8} + 0.56666u^{1.8},$$

$$y_2(u) = -0.0852u^{2.8} + 1.3123u^{1.8} + 0.2u,$$

and

$$\begin{aligned} y_1(u) = {} & 0.025\mathcal{H}_{0.5,1.8}\left(2u^{1.8}\right) - 0.1\mathcal{M}_{0.5,1.8}\left(2u^{1.8}\right) \\ & + 0.2\int_{-0.5}^{u-1} \mathcal{M}_{0.5,1.8}\left(2(u-1/2-\xi)^{1.8}\right)d\xi \\ & + 0.2\int_{u-1}^{0} \mathcal{M}_{0.5,1.8}\left(2(u-1/2-\xi)^{1.8}\right)d\xi \\ & + \int_0^{u-1/2} \mathcal{S}_{0.5,1.8}\left(2(u-1/2-\xi)^{1.8}\right)d\xi \\ & + \int_{u-1/2}^{u} \mathcal{S}_{0.5,1.8}\left(2(u-1/2-\xi)^{1.8}\right)d\xi, \end{aligned}$$

$$\begin{aligned} y_2(u) = {} & -0.1\mathcal{H}_{0.5,1.8}\left(2u^{1.8}\right) + 0.2\mathcal{M}_{0.5,1.8}\left(2u^{1.8}\right) \\ & + 2\int_0^{u-1/2} \mathcal{S}_{0.5,1.8}\left(2(u-1/2-\xi)^{1.8}\right)d\xi \\ & + 2\int_{u-1/2}^{u} \mathcal{S}_{0.5,1.8}\left(2(u-1/2-\xi)^{1.8}\right)d\xi, \end{aligned}$$

where $1/2 \leq u \leq 1$, which implies that

$$\begin{aligned} y_1(u) = {} & 0.0023(u-1/2)^{5.6} - 0.0065(u-1/2)^{4.6} + 0.0224(u-1/2)^{3.8} \\ & - 0.14199(u-1/2)^{3.6} - 0.0224u^{3.8} + 0.0426u^{2.8} + 0.56666u^{1.8}, \end{aligned}$$

$$\begin{aligned} y_2(u) = {} & 0.012997(u-1/2)^{4.6} - 0.3288(u-1/2)^{3.6} \\ & - 0.0852u^{2.8} + 1.3123u^{1.8} + 0.2u. \end{aligned}$$

By calculating we obtain $\eta = \max\{\|\psi\|_C, \|\psi'\|_C, \|\psi''\|_C\} = 0.3$, $\|A\| = 2$, $\|f\|_C = 3$, $\mathbb{E}_{1.8}(2L^{1.8}) = 3.351$, $\mathbb{E}_{1.8,2}\left(2(L+1/2)^{1.8}\right) = 2.2152$, $\mathbb{E}_{1.8,1.8}(2L^{1.8}) = 1.7095$, then we set $\delta = 0.31 > 0.3 = \eta$. Figure 1 demonstrates $y(u)$ and the norm $\|y(u)\|$ of (22). Now Theorems 1–3 implies that $\|y(u)\| \leq 4.32597$, $\|y(u)\| \leq 5.690596$ and $\|y(u)\| \leq 4.871297$, respectively, we just take $\epsilon = 4.326, 5.691, 4.872$, respectively. Table 1 shows the data.

We can see $\|y(u)\| < \epsilon$ for all $u \in W$ and (22) is finite time stable under Theorems 1–3. Concerning on the definition of finite time stable, we need to determine a specific threshold ϵ. By checking the value of ϵ in Theorems 1–3, we find that in this example the result of Theorem 1 is the optimal.

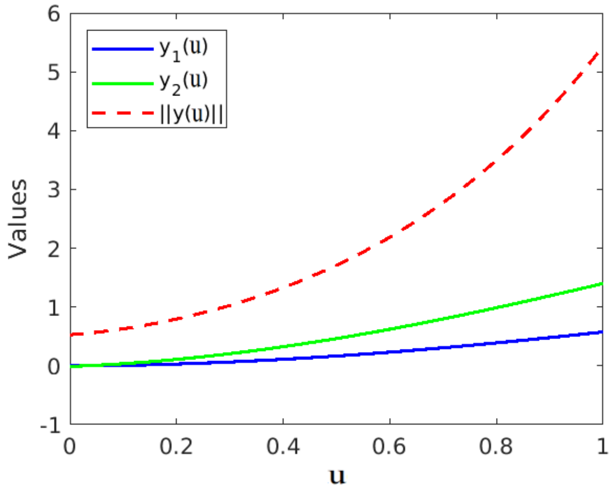

Figure 1. The state $y(u)$ and $||y(u)||$ of (22)

Table 1. Finite-time stability results of (22) and fixed the time $L = 1$.

Theorem	L	$\|\|A\|\|$	δ	$\|\|y(u)\|\|$	ϵ	κ	FTS
1	1	2	0.31	$\leq$ 4.32597	4.326 (optimal)	1/2	Yes
2	1	2	0.31	$\leq$ 5.690596	5.691	1/2	Yes
3	1	2	0.31	$\leq$ 4.871297	4.872	1/2	Yes

5. Conclusions

In this work, by making use of three possible formulas of solutions of nonhomogeneous systems governed by linear fractional differential equations with pure delay, and estimations of norms for the delayed Mittag-Leffler matrix functions, we derived finite-time stability results of these systems. Finally, we provided an example to demonstrate the effectiveness of the obtained results. The results are applicable to all singular, non-singular and arbitrary matrices, not necessarily squared. Consequently, our results improve and extend upon the existing results in [30,32].

One possible direction in which to extend the results of this paper is toward that of stochastic cases with various behaviours like impulses, delays in multistates, and neutral fractional differential and conformable fractional order time delay systems of order $\alpha \in (1,2]$.

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