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Solution Spaces Associated to Continuous or Numerical Models for Which Integrable Functions Are Bounded

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Abstract: Boundedness is an essential feature of the solutions for various mathematical and numerical models in the natural sciences, especially those systems in which linear or nonlinear preservation or stability features are fundamental. In those cases, the boundedness of the solutions outside a set of zero measures is not enough to guarantee that the solutions are physically relevant. In this note, we will establish a criterion for the boundedness of integrable solutions of general continuous and numerical systems. More precisely, we establish a characterization of those measures over arbitrary spaces for which real-valued integrable functions are necessarily bounded at every point of the domain. The main result states that the collection of measures for which all integrable functions are everywhere bounded are exactly all of those measures for which the infimum of the measures for nonempty sets is a positive extended real number.

Keywords: bounded solutions; integrable functions; real function spaces; complete characterization

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1. Introduction

Many models from mathematical biology, physics, and chemistry are used to quantify real-valued functions measured in absolute scales. That is the case with mathematical and numerical models which describe the distribution of temperatures in Kelvin on a region of space [1–3], solid fuel combustion in a chamber [4,5], and population densities or sizes [6,7], among other scientifically relevant problems [8]. In those cases, solutions which take on negative values are considered physically irrelevant and, thus, are considered as unrealistic. In some of those problems, the boundedness of solutions is also an important feature. As examples, we can mention that boundedness is an important characteristic in some chemotaxis systems with logistic sources [9], heat equations describing combustion phenomena [4,10], population problems in which limitation of resources or carrying capacities are considered [11,12], and mathematical models which describe the dynamics of fluids [13], among other physical examples. As the condition on the positivity of solutions, boundedness (not essential boundedness) is also a crucial feature to be observed in these models in order to assure that realistic solutions are obtained.

It is important to recall that, in general, a function is essentially bounded if it is bounded outside of a set with a zero measure [14]. Obviously, every bounded function is essentially bounded independently of the function space, but the converse is not true in general. Moreover, essential boundedness of a function is sometimes established as a consequence of the regularity of the function. As an example, every Lebesgue-integrable function is essentially bounded under suitable analytic conditions [15]. Unfortunately, essential boundedness is irrelevant in problems like those mentioned in the previous paragraph. In such cases, the full boundedness of solutions is required. More precisely, as in the case of essentially bounded functions, it is necessary to possess criteria that relate the regularity of the members in a function space to their boundedness considering the most general setting possible.

In the present letter, we will provide a complete characterization of those function spaces for which integrable functions are everywhere bounded. To that end, we will suppose that all the functions are real-valued, and we will consider only general assumptions on the domain of the functions and the definition of integrability. In view of that, all our functions will be defined on an arbitrary measurable space, and the integrals will be understood in the Lebesgue's sense. Our main result will propose a complete characterization of those functions spaces for which integrable functions are everywhere bounded. Precisely, we will show that the set of measures for which all integrable functions are everywhere bounded are exactly all of those measures for which the infimum of the measures for nonempty sets is a positive extended real number.

2. Preliminaries

Throughout this work, $(X, \mathcal{A}, \mu)$ will represent a measure space, that is, an ordered triplet where X is a nonempty set, $\mathcal{A}$ is a σ-algebra of subsets of X, and μ is an extended real-valued measure on $\mathcal{A}$. All functions in this manuscript will be extended real-valued functions, and they will be measurable with respect to $(X, \mathcal{A})$. Throughout, we will observe the conventions and nomenclature used in [16].

Definition 1. *The notation $L(X, \mathcal{A}, \mu)$ will represent the set of all functions which are integrable with respect to μ (see [15]).*

Definition 2. *We will use $\mathcal{P}(X, \mathcal{A})$ to represent the collection of all measures μ such that every function in $L(X, \mathcal{A}, \mu)$ is everywhere bounded. For each $0 \leq \alpha \leq \infty$, the collection $\mathcal{M}_\alpha(X, \mathcal{A})$ denotes the set of all measures μ such that:*

$$\alpha = \inf\{\mu(E) : E \in \mathcal{A} \text{ and } E \neq \emptyset\}. \tag{1}$$

For the remainder, we will consider a fixed—though arbitrary—measurable space $(X, \mathcal{A})$, and all measures will be defined on it. Using that nomenclature, the following two propositions are standard.

Lemma 1 (Yeh [17])**.** *If $f \in L(X, \mathcal{A}, \mu)$, then $\lim_{n\to\infty} \mu(E_n) = 0$, where:*

$$E_n = \{x \in X : |f(x)| \geq n\}, \quad \forall n \in \mathbb{N}.$$

Lemma 2 (Yeh [17])**.** *If $(f_n)_{n=1}^{\infty}$ is a sequence in $L(X, \mathcal{A}, \mu)$ such that:*

$$\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty,$$

then the series $\sum_{n=1}^{\infty} f_n$ converges μ almost everywhere to a function $f \in L(X, \mathcal{A}, \mu)$.

Lemma 3. *If $\mu \in \mathcal{P}(X, \mathcal{A})$, then $\mu(E) > 0$, for each $E \in \mathcal{A}$ with $E \neq \emptyset$.*

Proof. Assume that $\mu \in \mathcal{P}(X, \mathcal{A})$, and that $E \in \mathcal{A}$ is a nonempty set such that $\mu(E) = 0$. Define the function $f : X \to \overline{\mathbb{R}}$ for each $x \in X$, through

$$f(x) = \begin{cases} \infty, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

It is clear that f is measurable. Moreover, $f = 0$ μ almost everywhere, so it is integrable. However, $\mu \in \mathcal{P}(X, \mathcal{A})$, so $E = \emptyset$. The result readily follows. □

3. Main Result

We provide now the characterization of the measures in $\mathcal{P}(X, \mathcal{A})$. To that end, define the function $\mu_* : \mathcal{A} \to \overline{\mathbb{R}}$ as follows: for each $E \in \mathcal{A}$, we let

$$\mu_*(E) = \begin{cases} 0, & \text{if } E = \emptyset, \\ \infty, & \text{if } E \neq \emptyset. \end{cases}$$

It is readily shown that μ_* is a measure and that $\mathcal{M}_\infty(X, \mathcal{A}) = \{\mu_*\}$. For convenience, we will observe the following definition in our main result:

$$\mathcal{M}(X, \mathcal{A}) = \bigcup_{0 < \alpha \leq \infty} \mathcal{M}_\alpha(X, \mathcal{A})$$

The following is the main result of this manuscript.

Theorem 1. $\mathcal{P}(X, \mathcal{A}) = \mathcal{M}(X, \mathcal{A})$.

The proof of this result will be a consequence of the following lemmas.

Lemma 4. $\mathcal{M}_\infty(X, \mathcal{A}) \subseteq \mathcal{P}(X, \mathcal{A})$.

Proof. Let $f \in L(X, \mathcal{A}, \mu_*)$, and without loss of generality, suppose that $f \geq 0$. Let $\varphi = \sum_{k=1}^{n} a_k \chi_{E_k}$ be any simple function expressed in the *standard representation*, such that $0 \leq \varphi \leq f$. More precisely, suppose that the constants a_k are non-negative real numbers for each $k \in \{1, \ldots, n\}$ and that the finite sequence $(E_k)_{k=1}^{n}$ is a collection of measurable sets with respect to $\mathcal{A}$, which forms a partition of X (see [15]). By definition:

$$\sum_{k=1}^{n} a_k \mu_*(E_k) = \int_X \varphi d\mu_* \leq \int_X f d\mu_* < \infty.$$

Notice that the assumption that φ is given in the standard representation guarantees that the sets $E_1, \ldots, E_n$ form a partition of X with nonempty sets. Now, if one of the coefficients $a_k > 0$, then $a_k \mu_*(E_k) = \infty$, which is a contradiction. This implies that $\varphi = 0$, which means that $f = 0$. We conclude that f is bounded, as desired. □

Lemma 5. *If* $0 < \alpha < \infty$, *then* $\mathcal{M}_\alpha(X, \mathcal{A}) \subseteq \mathcal{P}(X, \mathcal{A})$.

Proof. Let $\mu \in \mathcal{M}_\alpha(X, \mathcal{A})$, for some $0 < \alpha < \infty$. Let $f \in L(X, \mathcal{A}, \mu)$, and define the sequence of sets $(E_n)_{n=1}^{\infty}$ as in Lemma 1. The conclusion of that lemma assures that $\lim_{n \to \infty} \mu(E_n) = 0$. As a consequence, there exists $N \in \mathbb{N}$ with the property that $\mu(E_N) < \alpha$. By the definition of $\mathcal{M}_\alpha(X, \mathcal{A})$, it follows that E_N must be empty, which means that $|f(x)| < N$, for each $x \in X$. In other words, $\mu \in \mathcal{P}(X, \mathcal{A})$. □

Lemma 6. $\mathcal{P}(X, \mathcal{A}) \subseteq \mathcal{M}(X, \mathcal{A})$.

Proof. Let $\mu \in \mathcal{P}(X, \mathcal{A})$, and define α by (1). If $\alpha = \infty$, then $\mu \in \mathcal{M}(X, \mathcal{A})$, so let us suppose that $0 \leq \alpha < \infty$. Note that Lemma 3 implies that $\mu(E) > 0$, for each $E \in \mathcal{A}$ with $E \neq \emptyset$. On the other hand, construct a sequence $(E_n)_{n=1}^{\infty}$ of measurable sets, such that $\mu(E_n) > 0$ and

$$\alpha \leq \mu(E_n) < \alpha + \frac{1}{n^3}, \quad \forall n \in \mathbb{N}. \tag{2}$$

For each $n \in \mathbb{N}$, define the function $f_n : X \to \mathbb{R}$ by:

$$f_n(x) = \begin{cases} n - \dfrac{n\alpha}{\mu(E_n)}, & \text{if } x \in E_n, \\ 0, & \text{if } x \notin E_n, \end{cases}$$

for each $x \in X$. The functions f_n are obviously measurable and non-negative, for each $n \in \mathbb{N}$. Moreover, notice that the inequality (2) yields:

$$\int_X f_n d\mu = n\mu(E_n) - n\alpha < \frac{1}{n^2}, \quad \forall n \in \mathbb{N}.$$

It follows that the functions f_n are integrable for each $n \in \mathbb{N}$. Moreover, the hypotheses of Lemma 2 are satisfied. This and Lemma 3 imply now that the series $\sum_{n=1}^{\infty} f_n$ converges everywhere to some $f \in L(X, \mathcal{A}, \mu)$. However, $\mu \in \mathcal{P}(X, \mathcal{A})$, so there exists $M \geq 0$, such that $0 \leq f(x) \leq M$, for each $x \in X$. In particular, this means that for each $n \in \mathbb{N}$ and $x \in E_n$, the following hold:

$$0 \leq 1 - \frac{\alpha}{\mu(E_n)} \leq \frac{f(x)}{n} \leq \frac{M}{n}.$$

Taking the limit when $n \to \infty$ on each term of these inequalities, we obtain that

$$\lim_{n\to\infty} \frac{\alpha}{\mu(E_n)} = 1.$$

This shows that $0 < \alpha < \infty$. So, in any case, $\mu \in \mathcal{M}(X, \mathcal{A})$, as desired. □

Proof of Theorem 1. The fact that $\mathcal{M}(X, \mathcal{A}) \subseteq \mathcal{P}(X, \mathcal{A})$ follows from Lemmas 4 and 5. Meanwhile, the converse was proved in Lemma 6. □

The following is a consequence from the main result.

Corollary 1. *Let μ be a measure on $(X, \mathcal{A})$, for which all integrable functions are bounded. Then, there exists $\alpha > 0$, such that $\mu(E) > \alpha$, for each $E \in \mathcal{A}$ which is nonempty.*

Before closing this stage of our work, we wish to establish that the set $M_\alpha(X, \mathcal{A})$ is nonempty for each $\alpha \in [0, \infty]$ and each measurable space $(X, \mathcal{A})$. To that end, notice that if α and β are positive real numbers and $\mu \in M_\alpha(X, \mathcal{A})$, then $\beta\mu \in M_{\alpha\beta}(X, \mathcal{A})$. In particular, it readily follows that $M_\alpha(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ is nonempty, for all $\alpha \in [0, \infty]$.

Definition 3. *Let $I(X, \mathcal{A})$ be the collection of all nonempty measurable sets $A \in \mathcal{A}$, with the property that $A' \subsetneq A$ for all $A' \in \mathcal{A}/\{\emptyset\}$. Let $E \in \mathcal{A}$ and define*

$$[E] := \{E^* \in I(X, \mathcal{A}) : E^* \subset E\}.$$

Lemma 7. *Let $\nu : \mathcal{A} \to \overline{\mathbb{R}}$ be defined by*

$$\nu(E) = |[E]|.$$

Then, ν is a measure. Moreover, $\nu \in M_1(X, \mathcal{A})$.

Proof. Let $(E_n)_{n=1}^{\infty}$ be pairwise disjoint measurable sets. Note that

$$\left[\bigcup_{n=1}^{\infty} E_n\right] = \bigcup_{n=1}^{\infty} [E_n].$$

Clearly, $[E_j] \subset [\bigcup E_n]$ for all $j \in \mathbb{N}$. Let $A^* \in [\bigcup E_n]$. If $A^* \cap E_{n_1} \neq \emptyset$ and $A^* \cap E_{n_2} \neq \emptyset$ with $n_1 \neq n_2$, then $A^* \cap E_{n_1} \in \mathcal{A}/\{\emptyset\}$ and $A^* \cap E_{n_1} \subset A^*$, but this is a contradiction. Hence, there is a unique $j \in \mathbb{N}$ such that $A^* \subset E_j$. Moreover, $[E_j] \cap [E_k] = \emptyset$ for all $j \neq k$. The result readily follows. □

Theorem 2. *$M_\alpha(X, \mathcal{A}) \neq \emptyset$ for all $\alpha \in [0, \infty]$.*

Proof. The proof of this result is a consequence of the above lemmas. □

4. Discussion and Conclusions

In the theory of integration by Riemann–Darboux, the condition of boundedness on an extended real-valued function $f : [a, b] \to \overline{\mathbb{R}}$ defined on a closed and bounded interval $[a, b] \subset \mathbb{R}$ is an indispensable requirement to guarantee the existence of upper and lower integrals [14]. If the upper and lower integrals of f coincide, then we say that the function f is a Riemann integrable over $[a, b]$, and various properties are derived then within that theory of integration. In Lebesgue's integration theory, a measurable, extended, real-valued function defined on a measure space need not be bounded in order to be integrable. In fact, Lebesgue integrable functions are real almost everywhere (for instance, in modeling and computations of highly oscillatory waves [18]). However, the current literature still lacks a characterization of those measures for which integrable functions are everywhere bounded. The main results demonstrates that the set of measures for which all integrable functions are everywhere bounded are exactly all of those measures for which the infimum of the measures for nonempty sets is a positive extended real number.

We would like to emphasize the fact that the results presented in this work are theoretical in nature. We are planning to employ them in the analysis of complex numerical models for systems of partial differential equations, in which the boundedness is also essential to prove properties of stability and convergence. Such studies require an extensive study; needless to mention that pertinent discrete operators and nomenclature will be required to that effect. We are definitely planning to carry it out in the future and show fruitful applications of the main result derived in this work. Moreover, since the most important result of this manuscript is also valid for continuous system, we expect to propose applications to the everywhere boundedness of the solutions of continuous systems governed by ordinary or partial differential equations.

Before closing this manuscript, we would like to point out that we focused on arbitrary function spaces and provide necessary and sufficient conditions for integrable functions to be everywhere bounded. This criterion can be an important tool to establish the boundedness, positivity, and numerical stabilities of the solutions of many analytical and numerical models in mathematical physics, biology, and chemistry. Details of our application results will be reported in forthcoming papers. In particular, the results derived in this work may be applied to problems in which the boundedness of solutions is an essential feature to be rigorously established. Moreover, this work provides only a characterization for the measure for which integrable functions are everywhere bounded. It is necessary to provide more practical characterizations following other perspectives and criteria. In particular, it is important to consider the relation between this problem and atomic measures on arbitrary measurable spaces.

An example of such applications is the nonlinear Kawarada equation:

$$u_t = \nabla(a\nabla u) + f(u), \quad x \in \Omega,\ t > t_0, \tag{3}$$

together with suitable initial and boundary conditions, where ∇ is the N-dimensional gradient vector, $a = a(x)$ is positive and bounded, and Ω is a convex, bounded, and connected domain in $\mathbb{R}^N$. The differential equation has been used to model a broad range of internal combustion engines. It is found that the solution u quenches in finite time only when certain Ω-dependent criteria are satisfied. It has also been observed that quenching solutions must preserve their positivity, monotonicity, and boundedness throughout their existences. The mathematical study of general Kawarada equation is still in its infancy. The results developed in this study may provide highly effective tools in the quantitative investigations of the Kawarada problems such as the above.

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