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Article

A Bivariate Post-Warranty Maintenance Model for the Product under a 2D Warranty

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Abstract: In this study, by integrating preventive maintenance (PM) into a two-dimensional warranty region, a two-dimensional warranty with customized PM (2D warranty with customized PM) is proposed from the manufacturer's perspective to reduce the warranty cost. The warranty cost of a 2D warranty with customized PM is derived. The manufacturer's tradeoff between PM cost and minimal repair cost saving is obtained by choosing the proper reliability threshold and the number of customized PMs, and the advantage of a 2D warranty with customized PM is illustrated. Second, by integrating PM into the post-warranty period, a bivariate post-warranty maintenance (BPWM) policy is proposed from the consumer's perspective to ensure the reliability of the product through the 2D warranty with customized PM. The expected cost rate model of BPWM is derived. Optimal BPWM is obtained in the numerical experiments. It is shown that a 2D warranty with customized PM is beneficial for both the manufacturer and the consumer, since both the manufacturer's warranty cost and the consumer's total cost are reduced.

Keywords: 2D warranty; customized PM; bivariate post-warranty maintenance; reliability; cost rate model

MSC: 93E20

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1. Introduction

Recently, warranties have been frequently used as a quality guarantee. Warranties require that manufacturers accept claims from consumers (users) and that they are responsible for ensuring product reliability. According to Warranty Week (see Ref. [1]), the base automotive companies in the US paid 15.6 billion USD on worldwide warranty claims in 2014, hitting a new record high with a 20% increase over 2013 spending. Similarly, US-based auto part suppliers' expenditure on claims is around 2.1 billion USD annually, corresponding to 0.55–0.8% of the annual revenue in 2015 (see Ref. [2]).

From the viewpoint of the reliability theory, the warranty cost is intensively influenced by maintenance policies in the warranty period (or region). In view of this, for the purpose of warranty cost reduction, some warranties integrated with preventive maintenance (PM) have been developed by academic researchers and industry practitioners. This type of warranty mainly concentrates upon the two-dimensional warranty (2D warranty), which includes two warranty limits, i.e., time limit (warranty period) and usage limit. For example, Ref. [3] investigated a periodic and imperfect PM strategy for a product covered by a fixed and combined base warranty and an extended warranty region from the manufacturer's perspective; Ref. [4] studied a 2D warranty policy with periodic PM under the assumption that the manufacturer and consumers can make flexible decisions in both basic and extended periods; Ref. [5] studied a servicing strategy that involves performing imperfect repairs in place of some of the minimal repairs of an 'all minimal repair' strategy; and Ref. [6] proposed a new warranty maintenance strategy for a 2D extended warranty based on dynamic usage rate.

In engineering practice, according to repair records in the base two-dimensional warranty region, the manufacturer can usually determine the usage rate category after the base 2D warranty expires. Based on this, Refs. [7,8] (other research on warranties can be found in Refs. [9–19]) classified the usage rate of the product into categories and developed customized PM to reduce warranty cost in the extended 2D warranty region. Without exception, if customized PM is introduced to the 2D warranty region of the product with an unknown usage rate category, the warranty cost of such a product can be reduced. However, in the existing literature, customized PM is rarely used to reduce the warranty cost of the product with an unknown usage rate category.

According to the warranty contract, the manufacturer, acting as a leader, ensures the product's reliability in the warranty region or period, but the consumer, acting as a follower, still confronts a problem of how to ensure the reliability of a product through warranties. The increase in the maintenance cost of a product through warranties prompts post-warranty maintenance policies to ensure the reliability of the product through warranties to be studied widely. Ref. [20] conducted early research on this type of maintenance policy by means of a stochastic degradation process, which was considered in Refs. [21–27]. Ref. [28] studied post-warranty condition-based maintenance policies to reduce the maintenance cost of a product through warranties; some subsequent developments can be found in Refs. [29–31]. It is worth emphasizing that the post-warranty maintenance policies in the above works have the following features: (1) the post-warranty maintenance policies are confined to the product through a 1D warranty, and there rarely exist post-warranty maintenance policies of the product through a 2D warranty; and (2) the post-warranty maintenance policies are time-based maintenance policies rather than reliability-centered (or based) maintenance policies, at which the product reliability is improved when the reliability function declines to a reliability threshold.

In this paper, by dividing the two-dimensional warranty region into two sub-regions and introducing PM to the second sub-region, a two-dimensional warranty with customized PM (2D warranty with customized PM) is proposed from the manufacturer's perspective to reduce the warranty cost of the product with an unknown usage rate category. In such a 2D warranty with customized PM, customized PM is modeled according to usage rate categories, which are used to improve the product's reliability when the reliability function declines to a reliability threshold. By integrating the reliability threshold and PM into the post-warranty period, a bivariate post-warranty maintenance (BPWM) policy to ensure the reliability of the product through a 2D warranty with PM is proposed from the consumer's perspective to reduce the maintenance cost of the product through warranties. The warranty cost of a 2D warranty with PM is derived, and the selection method of 2D warranties is presented from the viewpoint of the manufacturer. The expected cost rate of BPWM is obtained from the consumer's viewpoint. In numerical experiments, optimal maintenance plans (i.e., the optimal reliability threshold and the optimal number of PMs) for BPWM and 2D warranty with customized PM are obtained, and the advantages of 2D warranties with customized PM are explored.

The structure of this paper is listed as follows. Section 2 presents the model's formulation, where the intensity function influenced by both internal aging and external environmental conditions is modeled. Moreover, the 2D warranty with customized PM is described, and BPWM is defined. Section 3 derives the warranty cost of a 2D warranty with customized PM and the expected cost rate of BPWM. In Section 4, numerical experiments are used to illustrate the investigated approaches and to perform sensitivity analysis. Finally, Section 5 concludes this paper.

2. Model Formulation

2.1. Assumptions

The following assumptions are used in this paper.

- The warranty considered in this paper is a 2D warranty consisting of a time limit and a usage limit;
- Each failure in the 2D warranty region triggers a claim, and manufacturers accept all claims;
- Minimal repair time, PM time and replacement time are negligible;
- All products have the same shock process;
- The usage rate of each product is a constant in the whole life cycle;
- All usage rates can be obtained from the first sub-region and are classified into m ($m > 1$) usage rate categories;
- All information is symmetric, i.e., all parameters and cost items are known for both the manufacturer and the consumer.

2.2. Deterioration Process Modeling

In engineering practice, the deterioration of some products is simultaneously influenced by internal aging and external environmental conditions. Internal aging is strongly dependent on age (or operation time) and/or usage. In this paper, it is assumed that internal aging is characterized by a baseline intensity function given by $\gamma(x|r)$, where r is a random variable that represents the usage rate and satisfies a distribution function $G(r)$. We can use a shock process to characterize the influence of external environmental conditions on product deterioration (some recent developments about the shock process can be found in Refs. [32–37]). Let $\nu(x)$ and η be the arrival rate and increment/jump of the shock process, respectively. Therefore, by Ref. [38], the intensity function $\lambda(x|r)$ can be modeled as

$$\lambda(x|r) = \gamma(x|r) + \varphi(x) \tag{1}$$

where $\varphi(x) = \eta \int_0^x \exp\{-\eta(x-u)\}\nu(u)\mathrm{d}u$.

Proposition 1. *For a given x, the inequality $\lambda(x;r_i) \leq \lambda(x|r) \leq \lambda(x;r_{i+1})$ holds where the usage rate is $r \in [r_i, r_{i+1}]$ with $0 < r_i < r_{i+1}$.*

Since $\lambda(x|r)$ increases with respect to $r \in [r_i, r_{i+1}]$ for a given x, the result in Proposition 1 can be obtained easily.

This proposition implies that, if the usage rate category of the product is specified, then the intensity function is bounded.

2.3. Warranty Models

2.3.1. Benchmark Warranty

Minimal repair is frequently used to ensure product reliability in the warranty region (period). In this paper, we call the 2D warranty that satisfies this characteristic a benchmark warranty.

2.3.2. Proposed Warranty

Let w be the maximal time limit, u be the maximal usage limit and r^* be the average usage rate. Then, $r^* = u/w$, and the 2D warranty region can be defined as a set $\Omega = (w, u)$. Let $r_{\min}$ be the minimal usage rate and $r_{\max}$ be the maximal usage rate. For the sake of analytical tractability, we can divide the 2D warranty region Ω into two sub-regions, which are depicted in Figure 1.

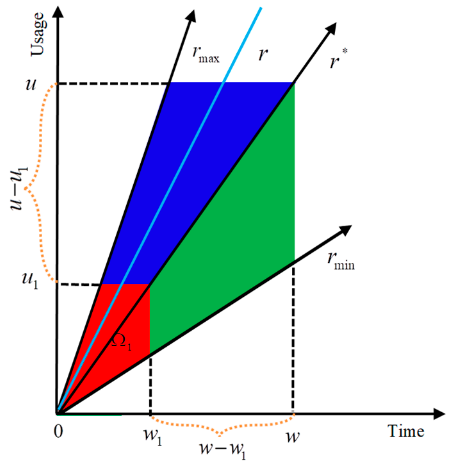

Figure 1. The warranty region description.

The first sub-region is a 2D warranty sub-region Ω_1 which is represented as $\Omega_1 = \{(w_1, u_1)|0 \leq w_1 < u/r_{\max}$ and $0 \leq u_1 < wr_{\min}\}$, where $u_1 = r^* w_1$. The second sub-region is a 2D warranty sub-region that is composed of the remaining time limit $w - w_1 > 0$ and the remaining usage limit $u - u_1 > 0$. The 2D warranty sub-region Ω_1 is a part of the 2D warranty region Ω.

As shown in Figure 1, the 2D warranty region Ω is classified into two sub-regions. According to all repair records in the 2D warranty sub-region Ω_1, the manufacturer can conclude the usage rates of all products and can classify these usage rates. Therefore, when a product with an unknown usage rate category goes through the 2D warranty sub-region Ω_1, the manufacturer can obtain the following information: (1) the usage rate category can be obtained; and (2) the remaining warranty limits in the 2D warranty region Ω, i.e., the remaining time limit $w - w_1 > 0$ and the remaining usage limit $u - u_1 > 0$, can be obtained.

Reliability-centered PM policies have been frequently applied in reliability management (see Refs. [39–41]). Such policies require that PM be performed to improve product reliability when the reliability function declines to a reliability threshold. In this paper, by integrating the reliability-centered PM policy into the second sub-region, we can design a 2D warranty where the minimal repair is used to ensure product reliability in the first sub-region, and the reliability-centered PM is used to ensure product reliability in the second sub-region, as shown below.

In this paper, the usage rates of the consumers are classified into two categories. The first category is the light usage category l, where the related consumers are defined as light consumers; the second category is the heavy usage category ϖ, at which the related consumers are defined as heavy consumers. Let ω_1 and ω_2 be the product age at which the product goes through the 2D warranty sub-region Ω_1 boundary and the product age at which the product goes through the 2D warranty region Ω boundary, respectively. Then,

$$\omega_1 = \begin{cases} w_1, & if\ r \leq r^* \\ u_1/r, & if\ r > r^* \end{cases} \text{ and } \omega_2 = \begin{cases} w, & if\ r \leq r^* \\ u/r, & if\ r > r^* \end{cases}.$$

Moreover, a 2D warranty can be designed as:

(A) *Before the remaining warranty limit ω_1 is reached, each failure is removed by minimal repair;*
(B) *Before the remaining warranty limit ω_2 is reached, whenever the reliability function of the product through 2D warranty sub-region Ω_1 declines to a reliability threshold, PM is performed to improve product reliability;*
(C) *Each failure between two successive PMs is removed by minimal repair.*

When the usage rate categories differ, the manufacturer's PM plan differs, i.e., according to the difference of usage rate categories, the manufacturer can customize its PM plan to be appropriate for each usage rate category. In view of this, the related PM is called a customized PM, and such a 2D warranty is regarded as a 2D warranty with customized PM. If $w_1 = 0$ and $u_1 = 0$, then customized PM is used to ensure product reliability in the 2D warranty region Ω. If $w_1 = w$ and $u_1 = u$, then customized PM is ignored, and minimal repair is used to ensure product reliability in the 2D warranty region Ω.

2.4. Maintenance Model after the Expiry of the 2D Warranty with Customized PM

With respect to warranties, the role between the manufacturer and the consumer is a leader–follower relationship. The manufacturer acts as a leader and provides warranty service for the consumer, and the consumer acts as a follower and accepts warranty service. From the viewpoint of product life cycle, the product life cycle is composed of two sub-periods, which are the warranty service period and the post-warranty period (see Figure 2). The manufacturer ensures product reliability in the warranty service period and does not ensure product reliability in the post-warranty period, i.e., the reliability of the product through the warranty. How to ensure the reliability of the product through a warranty is the problem of consumers.

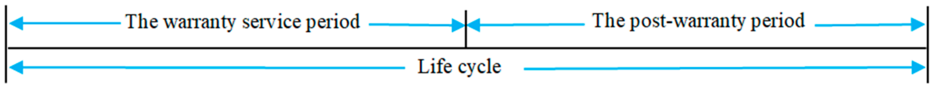

Figure 2. Life cycle description.

In view of this, a post-warranty maintenance policy is proposed from the consumer's perspective to ensure the reliability of the product through a 2D warranty with customized PM, as shown below:

(a) *Whenever product reliability in the post-warranty period declines to a reliability threshold $R_{i,j}$, wherej $= 1, 2, \cdots, N$, PM is performed to improve product reliability;*
(b) *Minimal repair removes all failures between two successive PMs;*
(c) *When product reliability after the* $(N-1)$th *PM declines to the reliability threshold $R_{i,j}$, the product is replaced at the expense of the consumer;*
(d) *Minimal repair removes all failures before replacement.*

Two decision variables, i.e., the reliability threshold $R_{i,j}$ and the number N of PMs, are considered in this maintenance policy. Thus, such a maintenance policy is called a bivariate post-warranty maintenance (BPWM) policy.

3. Model Analysis

3.1. Manufacturer Warranty Cost Model

3.1.1. Warranty Cost of the Benchmark Warranty

Under a benchmark warranty, usage rates cannot be classified, i.e., all products have the same usage rate category $r \in [r_{\min}, r_{\max}]$. We call this type of usage rate category a unified usage rate category in this paper. According to Ref. [42], furthermore, the warranty cost under a benchmark warranty can be obtained as

$$W_b^s = c_m\left(\int_{r^*}^{r_{\max}}\int_0^{u/r}\lambda(x|r)\mathrm{d}x\mathrm{d}G(r) + \int_{r_{\min}}^{r^*}\int_0^{w}\gamma(x|r)\mathrm{d}x\mathrm{d}G(r) + \int_0^{w}\varphi(x)\mathrm{d}x\right) \quad (2)$$

3.1.2. Warranty Cost of the Proposed Warranty

In this subsection, the warranty cost related to light consumers is derived, and the warranty cost related to heavy consumers is presented in Appendix A.

We use the reduction in the intensity function in Refs. [43,44] as a PM effect. As implied in Proposition 1, the intensity function $\lambda(x|r)$ $(r \in [r_i, r_{i+1}])$ is bounded for the same x, and its lower bound is $\lambda(x; r_i)$. In this setting, the reduction in the intensity function $\lambda(x|r)$ $(r \in (r_i, r_{i+1}])$ after the kth $(k = 1, 2, \cdots)$ PM cannot be less than the lower bound $\lambda(x; r_i)$, i.e., $\lambda(x|r) \geq \lambda(x; r_i)$. Let δ_k^i be a reduction in the intensity function $\lambda(x; r_i)$ between the

$(k-1)$th PM and the kth PM; let T_i^k be the time interval between the $(k-1)$th PM and the kth PM; and let S_k^i and S_{k-1}^i ($S_0^i = 0$) be the $(k-1)$th PM time and the kth PM time, respectively. Then, $\delta_k^i = \lambda(S_k^i; r_i) - \lambda(S_{k-1}^i; r_i)$, where $\delta_0^i = 0$. Based on these discussions, the intensity function $h_i^k(x\big|r)$ after the kth PM can be modeled as

$$h_i^k(x|r) = \lambda(w_1 + x|r) - \sum_{l=1}^{k} \alpha_l^i \delta_l^i = \gamma(w_1 + x|r) + \varphi(w_1 + x) - \sum_{l=1}^{k} \alpha_l^i \delta_l^i \tag{3}$$

where $t_k \leq x \leq t_{k+1}$, $h_i^0(x\big|r) = \lambda(w_1 + x\big|r)$ and $1 \geq \alpha_1^i \geq \alpha_2^i \geq \cdots \geq 0$.

From the viewpoint of reliability theory, the relationship between the reliability function and the intensity function is a one-to-one mapping. Let R_i be the reliability threshold related to the ith usage rate category $r \in (r_i, r_{i+1}]$. Then, the reliability threshold R_i can be expressed as

$$R_i = \exp\left\{-\int_{r_i}^{r_{i+1}} \left(\int_{S_{k-1}}^{S_{k-1}+T_k^i} \gamma(w_1 + x|r)\,\mathrm{d}x\right)\mathrm{d}G(r) - \int_{S_{k-1}}^{S_{k-1}+T_k^i} \varphi(w_1 + x)\mathrm{d}x + T_k^i \sum_{l=0}^{k-1} \alpha_l^i \delta_l^i\right\} \tag{4}$$

subject to

$$\begin{cases} T_0^i = S_0^i = \Delta_0^i = 0 \\ \mathbf{H}_k^i(T_k^i) = \int_{r_i}^{r_{i+1}} \left(\int_{S_{k-1}^i}^{S_{k-1}^i+T_k^i} \gamma(w_1 + x|r)\,\mathrm{d}x\right)\mathrm{d}G(r) + \int_{S_{k-1}^i}^{S_{k-1}^i+T_k^i} \varphi(w_1 + x)\mathrm{d}x - T_k^i \sum_{l=0}^{k-1} \alpha_l^i \delta_l^i \\ T_k^i = H_{k-inv}^i(-\ln R_i) \\ S_k^i = \sum_{l=0}^{k} T_l^i \\ \delta_k^i = \lambda(w_1 + S_k^i; r_i) - \lambda(w_1 + S_{k-1}^i; r_i) \end{cases}$$

where $H_{k-inv}^i(\bullet)$ is an inverse function of $\mathbf{H}_k^i(\bullet)$.

For light consumers, when the product age reaches the time limit w, the warranty expires. Therefore, the warranty cost related to light consumers includes two portions, which are minimal repair cost in the 2D warranty sub-region Ω_1 and minimal repair cost as well as PM cost in the remaining time limit $w - w_1$. Since each failure between two successive PMs is removed by minimal repair with the unit cost c_m,, the warranty cost $C_k^i(T_k^i)$ in the kth PM interval $(0, T_k^i]$ is equal to the minimal repair costs, i.e.,

$$C_k^i(T_k^i) = c_m \int_{r_i}^{r_{i+1}} \left(\int_{S_{k-1}^i}^{S_{k-1}^i+T_k^i} \gamma(w_1 + x|r)\,\mathrm{d}x\right)\mathrm{d}G(r) + c_m \int_{S_{k-1}^i}^{S_{k-1}^i+T_k^i} \varphi(w_1 + x)\mathrm{d}x - c_m T_k^i \sum_{l=0}^{k-1} \alpha_l^i \delta_l^i \tag{5}$$

The warranty cost $C_{n_i}^i$ after the last PM (i.e., the n_ith PM) can be given by

$$C_{n_i}^i = c_m \int_{r_i}^{r_{i+1}} \left(\int_{S_{n_i}^i}^{w} \gamma(w_1 + x|r)\,\mathrm{d}x\right)\mathrm{d}G(r) + c_m \int_{S_{n_i}^i}^{w} \varphi(w_1 + x)\mathrm{d}x - c_m(w - w_1 - S_{n_i}^i) \sum_{k=1}^{n_i} \alpha_k^i \delta_k^i \tag{6}$$

where $n_i = \max\left\{n_i > 0 \middle| S_{n_i}^i \leq w - w_1\right\}$.

Let $C_{p_i}^k(\bullet)$ be the PM cost resulting from the reduction $\bullet$, which increases with respect to $\bullet$. The warranty cost $W_i(R_i, n_i)$ in the remaining time limit $w - w_1$ is equal to the sum of the minimal repair cost $\sum_{k=1}^{n_i-1} C_k^i(T_k^i)$ produced by the first $n_i - 1$ PM intervals, the minimal repair cost $C_{n_i}^i$ after the last PM and the PM cost $\sum_{k=1}^{n_i} C_{p_i}^k(\alpha_k^i \delta_k^i)$ resulting from n_i PMs. By summing, the warranty cost $W_i(R_i, n_i)$ can be expressed as

$$\begin{aligned} W_i(R_i, n_i) &= \sum_{k=1}^{n_i-1} C_k^i(T_k^i) + C_{n_i}^i + \sum_{k=1}^{n_i} C_{p_i}^k(\alpha_k^i \delta_k^i) \\ &= c_m \int_{r_i}^{r_{i+1}} \left(\int_0^w \gamma(w_1 + x|r)\,\mathrm{d}x\right)\mathrm{d}G(r) + c_m \int_0^w \varphi(w_1 + x)\mathrm{d}x + \sum_{k=1}^{n_i} \left(C_{p_i}^k(\alpha_k^i \delta_k^i) - c_m \alpha_k^i \delta_k^i (w - w_1 - S_k)\right) \end{aligned} \tag{7}$$

Our target is to seek the optimal reliability threshold R_i^* and the optimal number n_i^* of PMs by minimizing the warranty cost $W_i(R_i, n_i)$, as in (7). For a given usage rate category, a given intensity function and a given remaining time limit $w - w_1$, if the reliability threshold is greater, then the number of PMs is larger, whereas the number of minimal repairs is smaller. If the reliability threshold is lower, then the number of PMs is lower, whereas the number of minimal repairs is larger. A larger number of PMs must lead to a greater PM cost and a smaller minimal repair cost; inversely, a smaller number of PMs must lead to a lower PM cost and a greater minimal repair cost. These monotonous regularities mean that the optimal reliability threshold can be determined by balancing PM cost and minimal repair cost saving. Once the optimal reliability threshold is determined, then the optimal number of PMs can be uniquely determined. The analytical solution is difficult to obtain; thus, the optimal maintenance plan (R_i^*, n_i^*) is searched numerically hereinafter.

3.1.3. Improvement Evaluation and Warranty Selection

How to select warranties is an important problem for the manufacturer. In practice, a key method to select warranties is to compare the warranty costs of all warranties. In this subsection, we consider a qualitative analysis method to select warranties.

After the optimal customized PM related to each usage rate category is obtained, the manufacturer uses a 2D warranty with optimal customized PM as a quality guarantee. As mentioned in the Assumption section, usage rates can be classified into m usage rate categories. In the remaining warranty limits (including the remaining time limit and the remaining usage limit), the warranty cost related to m usage rate categories can be obtained as

$$W_2^T(\mathbf{R}^*, \mathbf{N}^*) = \sum_{i=1}^{m} W_i(R_i^*, n_i^*) \tag{8}$$

where $\mathbf{R}^*$ is a vector composed of all optimal reliability thresholds and satisfies $\mathbf{R}^* = \{R_1^*, R_2^*, \cdots, R_m^*\}$; and $\mathbf{N}^*$ is a vector composed of all optimal number of PMs and satisfies $\mathbf{N}^* = \{n_1^*, n_2^*, \cdots, n_m^*\}$.

The warranty cost in (2) is a warranty cost related to the unified usage rate category, whereas the warranty cost in (8) is a total warranty cost related to m usage rate categories. This means that both are not equivalent, so both cannot be compared directly. In order to compare both, we must transform the warranty cost related to m usage rate categories into the warranty cost related to the unified usage rate category. Ref. [45] used root mean square (RMS) to transform the data. Similarly, RMS is used to make the transformations in this paper. By means of RMS, the warranty cost related to m usage rate categories can be approximately transformed as

$$W_2^u(\mathbf{R}^*, \mathbf{N}^*) = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(W_i(R_i^*, n_i^*)\right)^2} \tag{9}$$

Since minimal repair can remove all failures in the 2D warranty sub-region Ω_1, the warranty cost W_1^s in the 2D warranty sub-region Ω_1 can be given by

$$W_1^s = c_m\left(\int_{r^*}^{r_{\max}}\int_0^{u_1/r}\lambda(x|r)\mathrm{d}x\mathrm{d}G(r) + \int_{r_{\min}}^{r^*}\int_0^{w_1}\gamma(x|r)\mathrm{d}x\mathrm{d}G(r) + \int_0^{w}\varphi(x)\mathrm{d}x\right) \tag{10}$$

By summing (10) and (9), the warranty cost $W_p^s(\mathbf{R}^*, \mathbf{N}^*)$ of the proposed warranty can be calculated as

$$\begin{aligned} W_p^s(\mathbf{R}^*, \mathbf{N}^*) &= W_1^s + W_2^s(\mathbf{R}^*, \mathbf{N}^*) \\ &= c_m\left(\int_{r^*}^{r_{\max}}\int_0^{u_1/r}\lambda(x|r)\mathrm{d}x\mathrm{d}G(r) + \int_{r_{\min}}^{r^*}\int_0^{w_1}\gamma(x|r)\mathrm{d}x\mathrm{d}G(r) + \int_0^{w}\varphi(x)\mathrm{d}x\right) + \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(W_i(R_i^*, n_i^*)\right)^2} \end{aligned} \tag{11}$$

The warranty cost in (11) is a warranty cost related to the unified usage rate category, which is equivalent to the warranty cost in (3). This equivalency is facilitated to select a warranty by comparing warranty costs. In order to compare and select, the error produced by RMS is negligible. Then,

i. *If* $W_b^s > W_p^s(\mathbf{R}^*, \mathbf{N}^*)$*, then the manufacturer should select the proposed warranty as a quality guarantee of the product. Under this situation, the relative improvement* ξ_p *of the warranty cost can be measured as* $\xi_p = \left(\left(W_p^s(\mathbf{R}^*, \mathbf{N}^*) - W_b^s\right)/W_p^s(\mathbf{R}^*, \mathbf{N}^*)\right) \times 100\%$;

ii. *If* $W_b^s < W_p^s(\mathbf{R}^*, \mathbf{N}^*)$*, then the manufacturer should select benchmark warranty as a quality guarantee of the product. Under this situation, the relative improvement* ξ_b *of the warranty cost can be measured as* $\xi_b = \left(\left(W_b^s - W_p^s(\mathbf{R}^*, \mathbf{N}^*)\right)/W_b^s\right) \times 100\%$.

3.2. Consumer Cost Rate Model

Similar to Refs. [29–31,46], we can use the cost rate model as the consumer objective function. On basis of the renewal theorem in Ref. [47], we need to derive the total cost in the life cycle and the length of the life cycle and then derive cost rate model. The proposed warranty is flexible, whereas the benchmark warranty is a special case of the proposed warranty, when $w_1 = w$ and $u_1 = u$. Taking this case into consideration, here, we only derive the cost rate model associated with the proposed warranty, whereas the cost rate model associated with benchmark warranty can be obtained by adjusting parameters in the cost rate model associated with the proposed warranty.

It has been assumed that the usage rate of each product is a constant in the whole life cycle. For the jth product with the usage rate falling the ith interval $(r_i, r_{i+1}]$,, let $r_{i,j}$, $\omega_1^{i,j}$ and $\omega_2^{i,j}$ be, respectively, the usage rate in the life cycle, its age through the 2D warranty sub-region Ω_1 and its age through the 2D warranty region Ω. Then,

$$\omega_1^{i,j} = \begin{cases} w_1, & if\ r_{i,j} \le r^* \\ u_1/r_{i,j}, & if\ r_{i,j} > r^* \end{cases} \text{ and } \omega_2^{i,j} = \begin{cases} w, & if\ r_{i,j} \le r^* \\ u/r_{i,j}, & if\ r_{i,j} > r^* \end{cases}.$$

3.2.1. PM Model

In the remaining time limit $w - w_1$, the accumulative reduction in the intensity function is strongly influenced by maintenance history (i.e., the manufacturer's optimal customized PM). For a product with a usage rate $r_{i,j}$, its accumulative reduction in the intensity function can be given by $\sum_{k=1}^{n_i^*} \alpha_k^i \delta_k^i$. The intensity function $h_w(x; r_{i,j})$ in the warranty period w can be expressed as

$$h_w(x; r_{i,j}) = \lambda(w + x; r_{i,j}) - \sum_{k=1}^{n_i^*} \alpha_k^i \delta_k^i \tag{12}$$

Similar to PM modeling in Section 3, we can assume that the PM effect arithmetically reduces the intensity function. Under this assumption, the intensity function $h_w^k(x; r_{i,j})$ after the kth PM can be expressed as

$$h_w^k(x; r_{i,j}) = h_w(x; r_{i,j}) - \sum_{l=1}^{k} \alpha_l^{i,j} \delta_l^{i,j} \tag{13}$$

where $h_w^0(x; r_{i,j}) = h_w(x; r_{i,j})$ and $1 \ge \alpha_1^{i,j} \ge \alpha_2^{i,j} \ge \cdots \ge 0$. Note that $\delta_l^{i,j} = h_w(t_l; r_{i,j}) - h_w(t_{l-1}; r_{i,j})$ with $\delta_0^{i,j} = 0$, where t_{l-1} and t_l are the PM times of the $(l-1)$th PM and the lth PM, respectively.

As mentioned above, the relationship between the reliability function and the intensity function is a one-to-one mapping. Using this relationship, the reliability threshold $R_{i,j}$ can be expressed as

$$R_{i,j} = \exp\{-\int_{S_{k-1}^{i,j}}^{S_{k-1}^{i,j}+T_k^{i,j}} h_w^{k-1}(x; r_{i,j})\mathrm{d}x\} = \exp\{-\int_{S_{k-1}^{i,j}}^{S_{k-1}^{i,j}+T_k^{i,j}} h_w(x; r_{i,j})\mathrm{d}x + T_k^{i,j} \sum_{l=0}^{k-1} \alpha_l^{i,j} \delta_l^{i,j}\} \tag{14}$$

$$\text{subject to}\begin{cases} T_0^{i,j} = S_0^{i,j} = \delta_0^{i,j} = 0 \\ \mathbf{H}_k^{i,j}(T_k^{i,j}) = \int_{S_{k-1}^{i,j}}^{S_{k-1}^{i,j}+T_k^{i,j}} h_w(x;r_{i,j})\mathrm{d}x - T_k^{i,j}\sum_{l=0}^{k-1}\alpha_l^{i,j}\delta_l^{i,j} \\ T_k^{i,j} = H_{k-inv}^{i,j}(-\ln R_{i,j}) \\ S_k^{i,j} = \sum_{l=0}^{k} T_l^{i,j} \\ \delta_l^{i,j} = h_w(S_l^{i,j};r_{i,j}) - h_w(S_{l-1}^{i,j};r_{i,j}) \end{cases}$$

where $T_k^{i,j}$ is a time interval between the $(k-1)$th PM and the kth PM; $S_k^{i,j}$ is the PM time of the kth PM; and $H_{k-inv}^{i,j}(\bullet)$ is an inverse function of $\mathbf{H}_k^{i,j}(T_k^{i,j})$.

3.2.2. The Length of the Life Cycle

As depicted in Figure 2, the life cycle includes two sub-periods, which are the warranty service period produced by the 2D warranty with customized PM and the post-warranty period produced by BPWM. The warranty service period is equal to the warranty period w, and the post-warranty period is equal to the replacement time $S_N^{i,j}$ after the warranty expiration. Since minimal repair time, PM time and replacement time are negligible in this paper, the length $E[L_{i,j}]$ of the life cycle is equal to the sum of the warranty period w and the replacement time $S_N^{i,j}$, i.e.,

$$E[L_{i,j}] = w + S_N^{i,j} \tag{15}$$

3.2.3. The Total Cost in the Life Cycle

By the definition of the life cycle, the total cost $TC_{i,j}(R_{i,j},N)$ in the life cycle is equal to the sum of the total cost $C_w^{i,j}$ in the warranty period w, the total cost $C_P^{i,j}$ resulting from PM (including replacement cost) in the post-warranty period and the total cost $C_{pw}^{i,j}$ resulting from all failures in the post-warranty period, i.e.,

$$TC_{i,j}(R_{i,j},N) = C_w^{i,j} + C_p^{i,j} + C_{pw}^{i,j} \tag{16}$$

Next, we can derive them.

Since the usage rate of each product is a constant in the life cycle, the total cost in the 2D warranty region Ω is equal to the total cost in the warranty period w. The total cost in the warranty period w strongly depends on the length w of the warranty period and the manufacturer's PM in the remaining time limit $w - w_1$. Therefore, the total cost in the warranty period w can be obtained as

$$C_w^{i,j} = c_f\int_0^w \lambda(x;r_{i,j})\mathrm{d}x + \sum_{k=1}^{n_i^*}\Big(c_p - c_f\alpha_k^i\delta_k^i(w - w_1 - S_k^i)\Big) \tag{17}$$

where c_p represents the unit loss resulting from each PM and c_f is the unit failure cost resulting from each failure.

According to the definition of BPWM, we can draw a conclusion that the proposed BPWM includes $N-1$ PMs and a replacement action. Therefore, the total cost $C_P^{i,j}$ in the post-warranty period is equal to the sum of the PM cost produced by $N-1$ PMs and the replacement cost. Let the increasing function $C_{p_{i,j}}^k(\bullet)$ be a PM cost resulting from the reduction $\bullet$, where $C_{p_{i,j}}^0(\bullet) = 0$ and c_r is a replacement cost. Then, the total cost $C_P^{i,j}$ in the post-warranty period can be expressed as

$$C_P^{i,j} = \sum_{l=0}^{N-1}\Big(C_{p_{i,j}}^l(\alpha_l^{i,j}\delta_l^{i,j}) + c_p\Big) + c_r \tag{18}$$

where $\sum_{0}^{0} \bullet = 0$.

As mentioned earlier, all failures between two successive PMs are removed by minimal repair. Each failure makes the consumer incur a failure cost c_f and a minimal repair cost c_m. Therefore, the total cost $C_{pw}^{i,j}$ resulting from all failures in the post-warranty period can be obtained as

$$C_{pw}^{i,j} = (c_f + c_m)\sum_{k=1}^{N}\left(\int_{S_{k-1}^{i,j}}^{S_{k-1}^{i,j}+T_k^{i,j}} h_w(x; r_{i,j})\mathrm{d}x - T_k^{i,j}\sum_{l=0}^{k-1}\alpha_l^{i,j}\delta_l^{i,j}\right) = (c_f + c_m)\left(\int_0^{S_N^{i,j}} h_w(x; r_{i,j})\mathrm{d}x - \sum_{l=0}^{N-1}\alpha_l^{i,j}\delta_l^{i,j}(S_N^{i,j} - S_l^{i,j})\right) \tag{19}$$

By (16), the total cost $TC_{i,j}(R_{i,j}, N)$ in the life cycle is calculated as

$$\begin{aligned} TC_{i,j}(R_{i,j}, N) = \; & c_f\int_0^{w+S_N^{i,j}} \lambda(x; r_{i,j})\mathrm{d}x + c_m\int_w^{S_N^{i,j}} \lambda(x; r_{i,j})\mathrm{d}x + \sum_{k=1}^{n_i^*}\left(c_p - c_f\alpha_k^i\delta_k^i(w - w_1 - S_k^i)\right) + \\ & \sum_{l=0}^{N-1}\left(\left(C_{p_{i,j}}^l(\alpha_l^{i,j}\delta_l^{i,j}) + c_p\right) - \alpha_l^{i,j}\delta_l^{i,j}(S_N^{i,j} - S_l^{i,j})(c_f + c_m)\right) + c_r \end{aligned} \tag{20}$$

3.2.4. Cost Rate Model

The total cost $TC_{i,j}(R_{i,j}, N)$ in the life cycle and the length $E[L_{i,j}]$ of the life cycle are derived in (20) and (15), respectively. By the renewal reward theorem in Ref. [47], the expected cost rate $C_{i,j}(R_{i,j}, N)$ can be calculated as

$$\begin{aligned} C_{i,j}(R_{i,j}, N) &= \frac{TC_{i,j}(R_{i,j},N)}{E[L_{i,j}]} \\ &= \frac{A + c_f\int_0^{w+S_N^{i,j}} \lambda(x;r_{i,j})\mathrm{d}x + c_m\int_w^{S_N^{i,j}} \lambda(x;r_{i,j})\mathrm{d}x + \sum_{l=0}^{N-1}\left(\left(C_{p_{i,j}}^l(\alpha_l^{i,j}\delta_l^{i,j}) + c_p\right) - \alpha_l^{i,j}\delta_l^{i,j}(S_N^{i,j} - S_l^{i,j})(c_f + c_m)\right) + c_r}{w + S_N^{i,j}} \end{aligned} \tag{21}$$

where $A = \sum_{k=1}^{n_i^*}\left(c_p - c_f\alpha_k^i\delta_k^i(w - w_1 - S_k^i)\right)$.

Our objective is to seek the optimal maintenance plan $(R_{i,j}^*, N^*)$ by minimizing the objection function $C_{i,j}(R_{i,j}, N)$. The analytical solution is very difficult to obtain; thus, the numerical solution is sought.

By replacing w, w_1 and S_k^i in (21) with u, $u_1/r_{i,j}$ and $U_k^i/r_{i,j}$, (21) can be rewritten as an expected cost rate related to a heavy consumer.

Remark 1.

(1) *Let $\alpha_k^i\delta_k^i$ in (14) be zero, i.e., $\alpha_k^i\delta_k^i = 0$, and A in (21) be zero, i.e., $A = 0$. Then, (21) can be rewritten as*

$$C_{i,j}(R_{i,j}, N) = \frac{c_f\int_0^{w+S_N^{i,j}} \lambda(x; r_{i,j})\mathrm{d}x + c_m\int_w^{S_N^{i,j}} \lambda(x; r_{i,j})\mathrm{d}x + \sum_{l=0}^{N-1}\left(\left(C_{p_{i,j}}^l(\alpha_l^{i,j}\delta_l^{i,j}) + c_p\right) - \alpha_l^{i,j}\delta_l^{i,j}(S_N^{i,j} - S_l^{i,j})(c_f + c_m)\right) + c_r}{w + S_N^{i,j}} \tag{22}$$

This model is an expected cost rate associated with the benchmark warranty, where the proposed BPWM is used to ensure the reliability of the product through the benchmark warranty.

(2) *Let N be one, i.e., $N = 1$. Then, (21) can be rewritten as*

$$C_{i,j}(R_{i,j}, 1) = \frac{A + c_f\int_0^{w+T_1^{i,j}} \lambda(x; r_{i,j})\mathrm{d}x + c_m\int_w^{T_1^{i,j}} \lambda(x; r_{i,j})\mathrm{d}x + c_r}{w + T_1^{i,j}} \tag{23}$$

where $T_1^{i,j} = H_1^{i,j}(-\ln R_{i,j})$.

This model is an expected cost rate in which the proposed warranty is used to warrant products, and the proposed BPWM is transformed into a periodic replacement with minimal repair, as in Ref. [48].

4. Numerical Experiments

In real life, 2D warranties are frequently used as quality guarantees of vehicles. Recently, Ref. [8] utilized the data collected from a car dealer to demonstrate the effectiveness of the warranty. Similar to this reference, the intensity function is used as a baseline intensity function $\gamma(x|r)$ that reflects the interdependence between age and usage, and it is given by $\gamma(x|r) = 0.1 + 0.4r + (1.5x + 1.5r)x$. In addition, it is assumed that the random usage rate r is subject to the uniform distribution $G(r)$ with the maximum $r_{\max} = 10$ and the minimum $r_{\min} = 0.3$. Similar to Ref. [49], we used the linearly increasing arrival rate to characterize the shock process, and the corresponding expression is given by $\nu(x) = Vx$ where $V > 0$. The parameters of the shock process were set as $V = 0.5, 1, 1.5$ and $\eta = 0.5,1, 1.5$. By (1), the intensity function of vehicles is given by $\lambda(x|r) = 0.1 + 0.4r + (1.5x + 1.5r)x + \eta V \int_0^x \exp\{-\eta(x-u)\}u du$.

Ref. [50] recently used a power function to model PM cost, which is a non-negative and non-decreased function with respect to the maintenance level (quality). Similarly, this paper models the PM cost as $C^k_{p_{i,j}}(y) = c(y)^\rho$, where $c, \rho > 0$, and y is an independent variable representing a reduction in the intensity function. We can specify that $w = 5$ (years) and $u = 10$ (10^4 km). Then, the average usage rate $r^* = 2$ and the 2D warranty region can be given by $\Omega = (5, 10)$. The 2D warranty sub-regions include three cases, which are $\Omega_1 = (0.25, 0.5)$, $(0.5, 1)$ and $(0.75, 1.5)$. We classified all consumers into two categories, which are light consumers and heavy consumers. Under this classification, the usage rates for light consumers are confined to the range $[r_{\min}, r^*)$ $(= [0.3, 2))$, and the usage rates for heavy consumers are confined to the range $[r^*, r_{\max}] (= [2, 10])$. Some parameters are constant throughout the whole numerical experiment, and they are listed in Table 1. Since anther parameters may be non-constant, they are not presented in Table 1 and are assigned when used.

Table 1. Parameter settings.

α^i_k	$\alpha^{i,j}_k$	c	ρ	c_m	c_f	c_p	c_r
1	1	0.5	0.5	8	10	1	150

Although the intensity function and some parameters are presented above, the analytical solution is still difficult to obtain. Therefore, we sought the optimal maintenance plan by means of a search algorithm with a given step length. Since the reliability threshold is one of decision variables, we set the step length as 0.05 if it did not announce specially. In order to analyze sensitivity, we present some figures, which are used to describe the variation tendency related to parametric variation.

4.1. Sensitivity Analysis of the Proposed Warranty

In order to explore the existence and uniqueness of the optimal maintenance plan and to illustrate the effects of some key parameters on the optimal maintenance plan, we used the MATLAB software to make three figures, as follows.

By letting the step length be 0.1, we used the 2D warranty sub-region $\Omega_1 = (0.5, 1)$, the arrival rate $V = 0.5$ and the increment $\eta = 0.5$ to obtain Figure 3. As indicated in Figure 3, the optimal reliability threshold and the optimal number of PMs exist for light consumers and heavy consumers. This signals that the optimal maintenance plan exists uniquely.

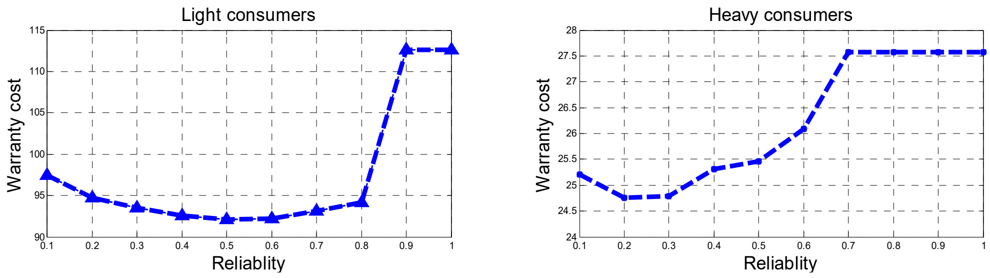

Figure 3. Optimal PM.

In order to illustrate the performance of the proposed warranty (PW), we plotted Figure 4, where the step length equates to 0.05 and $V = \eta = 0.5$.

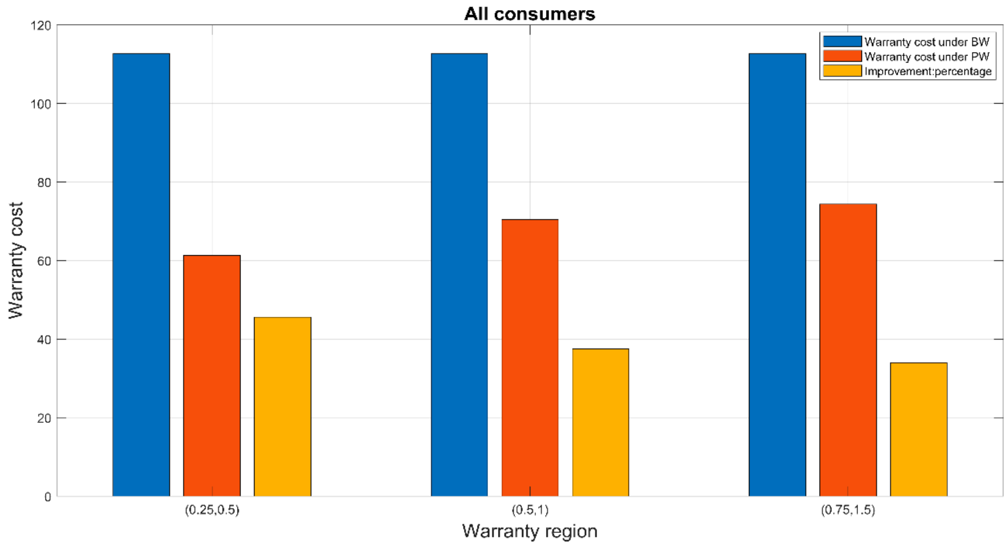

Figure 4. The effect of warranty region on the warranty cost.

From Figure 4, we obtained the following results.

(a) The warranty cost under PW increases and tends to the warranty cost under the benchmark warranty (BW) when the area of the 2D warranty sub-region Ω_1 becomes bigger;
(b) The relative improvement of the warranty cost under PW decreases when the area of the 2D warranty sub-region Ω_1 becomes bigger.

A bigger area of the 2D warranty sub-region Ω_1 produces a smaller remaining time limit. The smaller remaining time limit leads to a larger minimal repair cost and a lower number of PMs. Furthermore, a lower number of PMs leads to a lower reduction in the minimal repair cost. Thus, the results presented in (a) are apparent. Since the warranty costs for the two types of warranties tend to be the same when the area of the 2D warranty sub-region Ω_1 approaches the area of the 2D warranty region Ω, the relative improvement of the warranty cost under PW decreases. Therefore, the results presented in (b) are obtained. These changes illustrate that the performance of PW is superior to that of BW. Compared with BW, conclusively, it is more beneficial that the manufacturer uses PW as a quality guarantee. This is because the manufacturer incurs a lower warranty cost when PW is used as a quality guarantee. Compared with BW, additionally, a consumer may be inclined to the PW, since PM actions in the PW keep higher reliability, which slows down product deterioration in the post-warranty period and indirectly prolongs the product life cycle.

Compared with traditional 2D warranties, we modeled the intensity function as an additive function by introducing a shock process, which was used to model external environmental conditions. In order to show the effect of the shock process on warranty cost, we plotted Figure 5, where the 2D warranty sub-region Ω_1 satisfies $\Omega_1 = (0.5, 1)$.

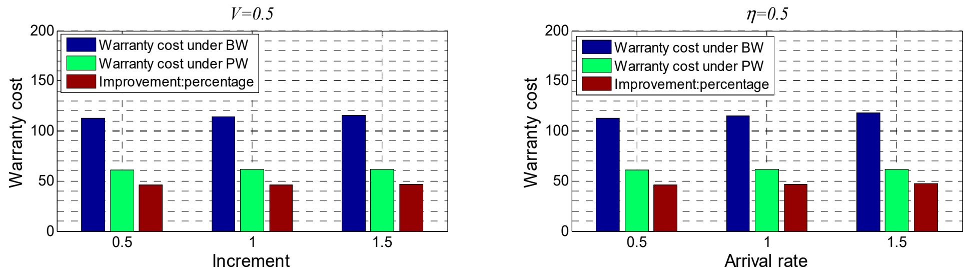

Figure 5. The effect of the shock process on the warranty cost.

As depicted in Figure 5, the warranty costs for two types of warranties increase for a given arrival rate (or increment) as the increment (or arrival rate) increases. Moreover, the relative improvement of the warranty cost under PW increases for a given arrival rate (or increment) as the increment (or arrival rate) increases. These changes are apparent because the intensity function must become bigger if the shock process is added to the baseline intensity function. Furthermore, a bigger intensity function necessarily results in increases in warranty costs. From the viewpoint of management, the consideration of the shock process contributes to scheduling the warranty cost budget and reducing budget risk.

4.2. Sensitivity Analysis of BPWM

In this subsection, from the consumer's perspective, we investigate the existence and uniqueness of the optimal maintenance plan and the effects of some key parameters on the optimal expected cost rate. Note that, as indicated in Section 4.2, the warranty cost under PW is less than the warranty cost under BW. This relationship means that the manufacturer is more inclined to use PW as a quality guarantee. Therefore, all discussion here is presented based on the expected cost rate associated with PW. In addition, the situation for light consumers is only investigated in this subsection, and the situation for light consumers is similar from the viewpoint of management.

By differing from the post-warranty maintenance policies in the traditional literature, in this paper, we propose a BPWM. In order to validate the feasibility of a BPWM, we plotted Figure 6, where the 2D warranty sub-region Ω_1 satisfies $\Omega_1 = (0.75, 1.5)$, the usage rate of the consumer is $r_{1,j} = 0.5$.

As shown in Figure 6, the optimal reliability threshold and the optimal number of PMs exist uniquely, i.e., $\left(R^*_{1,j}, N^*\right) = (0.0768, 9)$. Since the error exists in the search algorithm, a discrepancy exists between the optimal maintenance plan $\left(R^*_{1,j}, N^*\right)$ and its real optimal maintenance plan. This discrepancy can be lesser if and only if the step length used in the search algorithm closely approaches zero.

Figure 7 indicates that the optimal expected cost rate increases with respect to the usage rate, whereas the optimal reliability threshold and the optimal number of PMs decrease. These results show that the usage rate of a product has a significant effect on the optimal maintenance plan in the post-warranty period and the optimal expected cost rate. From the viewpoint of reliability theory, a larger usage rate can accelerate product deterioration, which can elevate failure frequency. Therefore, the optimal expected cost rate increases, whereas the life cycle decreases.

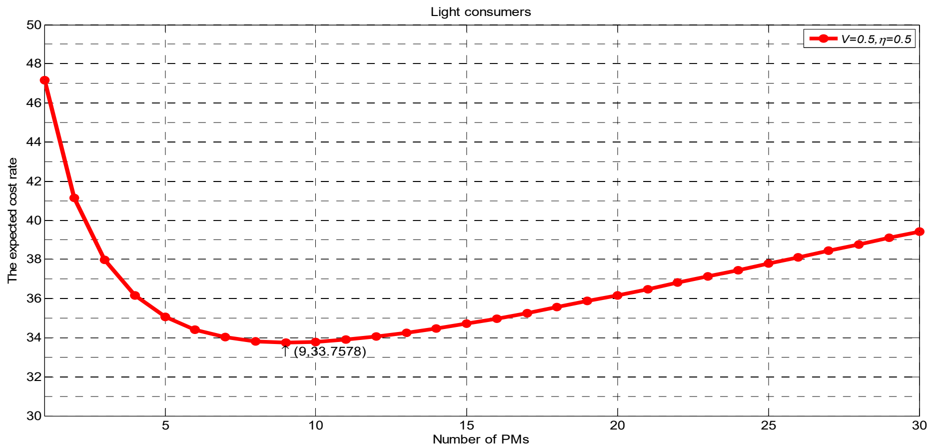

Figure 6. The feasibility of BPWM.

As mentioned above, the usage rate of the consumer is a constant in the whole life cycle. In order to analyze the effect of the usage rate on the optimal BPWM, we made Figure 7, where the 2D warranty sub-region Ω_1 satisfies $\Omega_1 = (0.75, 1.5)$ and $\eta = v = 0.5$.

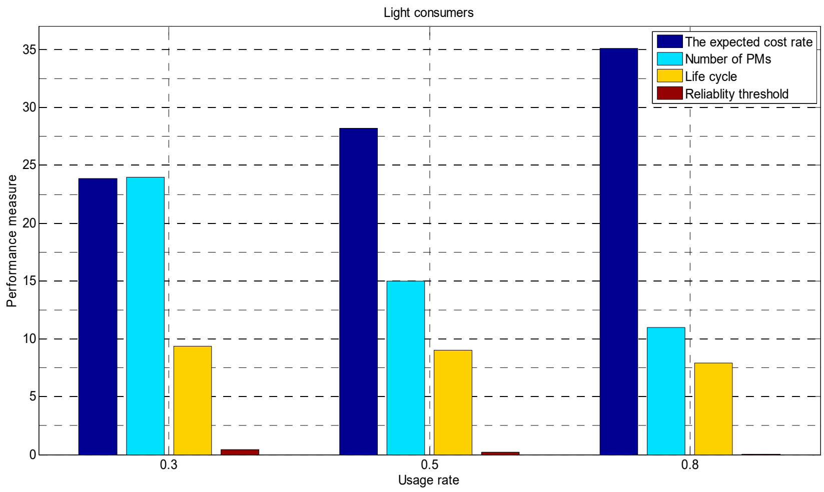

Figure 7. Performance measure versus usage rate.

By the definition of the life cycle, the warranty service period produced by the 2D warranty region is a portion of the life cycle. The total cost in the life cycle is indirectly dependent on the manufacturer's optimal maintenance plan, which is strongly influenced by the area of the 2D warranty region Ω_1. In order to illustrate this type of effect, we plotted Figure 8, where the usage rate $r_{1,j} = 0.5$. As indicated in Figure 8, the optimal reliability threshold and the optimal number of PMs decrease when the area of the 2D warranty region Ω_1 becomes bigger, whereas the expected cost rate increases. A bigger area of the 2D warranty region Ω_1 must make the remaining time limit smaller when the area of the 2D warranty region Ω is fixed. Furthermore, a smaller remaining time limit can reduce the number of PMs. A lower PM count necessarily results in accelerated deterioration of the product in the post-warranty period, which can increase minimal repair costs in a smaller remaining time limit. Therefore, the above results hold.

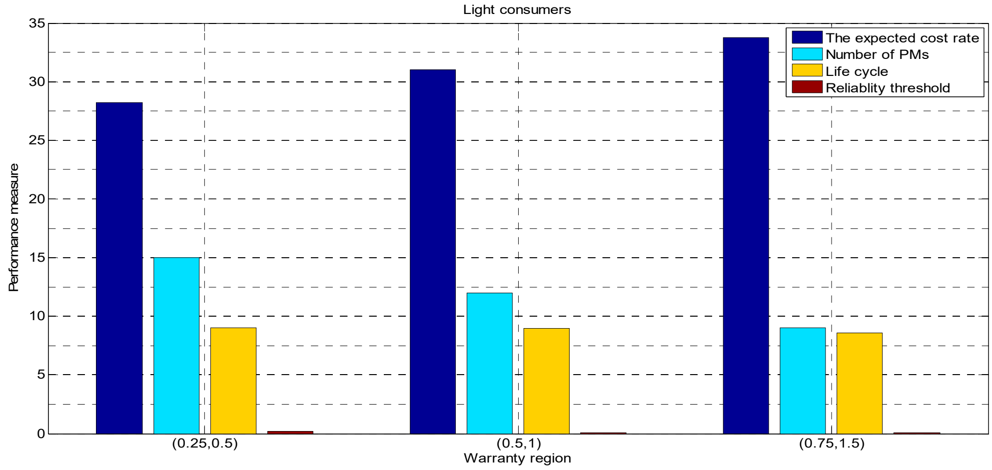

Figure 8. Performance measure versus warranty region.

Figure 8 indicates that the life cycle increases when the area of the 2D warranty sub-region Ω_1 becomes smaller. A smaller area of the 2D warranty sub-region Ω_1 leads to a greater number of PMs in a longer remaining time limit. Furthermore, a greater number of PMs results in a lower intensity function after the warranty expiry. A lower intensity function after the warranty expiry can decelerate product deterioration speed in the post-warranty period. Therefore, the life cycle increases when the area of the 2D warranty sub-region Ω_1 becomes smaller. Inversely, if the area of the 2D warranty sub-region Ω_1 becomes bigger, then the life cycle reduces. This implies indirectly that PW can prolong the life cycle.

Finally, it is worth mentioning that the shock process also has significant effects on the optimal BPWM. Concretely, the shock process can increase the optimal expected cost rate and shorten the life cycle. Here, we do not present a visible figure to exhibit these changes.

5. Conclusions

In this paper, by dividing the 2D warranty region into two sub-regions, a 2D warranty with customized preventive maintenance (2D warranty with customized PM), including two sub-regions, is proposed to ensure product reliability and to reduce the warranty cost. Such a 2D warranty with customized PM requires that minimal repair removes all failures in the first sub-region, and PM improves product reliability in the second sub-region, which is composed of the remaining warranty limits. Subsequently, from the consumer's perspective, a bivariate post-warranty maintenance (BPWM) policy is proposed to ensure the reliability of the product through the warranty and to reduce the maintenance cost of the product through the warranty. The numerical experiments shows that the performance of a 2D warranty with customized PM is superior to that of traditional 2D warranties, where minimal repair removes all failures in the whole warranty region. When the first sub-region becomes bigger, the cost rate increases, and the life cycle shortens.

In this paper, we used conditional probability to characterize the effect of the usage rate on the reliability of the product. In most cases, such a method can solve the problem, allowing warranty costs to be estimated and PM plans in the 2D warranty region to be optimized, but the obtained results may have a large deviation compared with the truth. In addition, the job cycle and usage in each job project are neglected in this paper. From the perspective of reliability theory, to neglect them means that the reliability of the product is difficult to ensure in real time. In view of these, some methods to reduce deviation and to ensure reliability in real time can be devised, as shown below.

(1) By characterizing the reliability of the product as 2D probability, some 2D warranties and the post-warranty maintenance policies can be designed and modeled;

(2) By introducing job cycle and usage into the product life cycle, some 4D warranties and some multivariable post-warranty maintenance policies can be devised and optimized, which can be used to ensure reliability in real time.

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Appendix A

For heavy consumers, the warranty expires only when the product usage reaches the usage limit u. Furthermore, a default assumption is that all symbols used in this section are similar to the case associated with light consumers; thus, we do not differentiate them from light consumers.

Appendix A.1. PM Model

The reliability threshold R_i associated with the ith usage rate category $r \in (r_i, r_{i+1}]$ can be expressed as

$$R_i = \exp\{-\int_{r_i}^{r_{i+1}} \left(\int_{(u_1+U_{k-1}^i)/r}^{(u_1+U_{k-1}^i+u_k^i)/r} \lambda(x|r)\,\mathrm{d}x - \left(u_k^i \sum_{l=0}^{k-1} \alpha_l \Delta_l^i \right)/r \right) \mathrm{d}G(r)\}$$
$$\text{subject to} \begin{cases} u_0^i = U_0^i = \delta_0^i = 0 \\ \pi_k^i(u_k^i) = \int_{r_i}^{r_{i+1}} \left(\int_{(u_1+U_{k-1}^i)/r}^{(u_1+U_{k-1}^i+u_k^i)/r} \lambda(x|r)\,\mathrm{d}x - \left(u_k^i \sum_{l=0}^{k-1} \alpha_l^i \delta_l^i \right)/r \right) \mathrm{d}G(r) \\ u_k^i = J_k^i(-lnR_i) \\ U_k^i = \sum_{l=0}^{k} u_l^i \\ \delta_k^i = \lambda((u_1+U_k^i)/r_i; r_i) - \lambda((u_1+U_{k-1}^i)/r_i; r_i) \end{cases}, \tag{A1}$$

where u_k^i is a usage interval between the $(k-1)$th PM and the kth PM, and $J_k^i(\bullet)$ is an inverse function of $\pi_k^i(u_k^i)$.

Appendix A.2. Warranty Cost Model in the Remaining Usage Limit

Since each failure between two successive PMs is removed by minimal repair, the warranty cost $C_i^k(u_i^k/r)$ between the $(k-1)$th PM and the kth PM can be calculated as

$$C_i^k(u_i^k/r) = c_m \int_{r_i}^{r_{i+1}} \left(\int_{(u_1+U_{k-1}^i)/r}^{(u_1+U_{k-1}^i+u_k^i)/r} \lambda(x|r)\,\mathrm{d}x - \left(u_k^i \sum_{l=0}^{k-1} \alpha_l^i \delta_l^i \right)/r \right) \mathrm{d}G(r). \tag{A2}$$

After the last PM, i.e., n_i, the warranty cost $C_i^{n_i}$ can be given by

$$C_i^{n_i} = c_m \int_{r_i}^{r_{i+1}} \left(\int_{(u_1+U_{n_i}^i)/r}^{u} \lambda(x|r)\,\mathrm{d}x - \left((ur - u_1 - U_{n_i}^i) \sum_{k=1}^{n_i^*} \alpha_k^i \delta_k^i \right)/r \right) \mathrm{d}G(r), \tag{A3}$$

where $n_i = \max\left\{ n_i > 0 \middle| U_{n_i}^i \le u \right\}$.

The warranty cost $W_i(R_i)$ in the remaining usage limit $u - u_1$ is equal to the sum of the total cost $\sum_{k=1}^{n_i-1} C_i^k(u_i^k/r)$ resulting from the first $n_i - 1$ PM intervals, the minimal repair cost $C_i^{n_i}$ after the last PM and the PM cost $\sum_{k=1}^{n_i} C_{p_i}^k(\alpha_k \Delta_k^i)$ resulting from n_i PMs. Mathematically, the warranty cost $W_i(R_i)$ can be expressed as

$$W_i(R_i) = c_m \int_{r_i}^{r_{i+1}} \left(\int_{u_1/r}^{u/r} \lambda(x|r) \mathrm{d}x + \sum_{k=1}^{n_i} \left(C_{p_i}^k(\alpha_k^i \delta_k^i) - c_m \alpha_k^i \delta_k^i (ur - u_1 - U_k)/r \right) \right) \mathrm{d}G(r). \tag{A4}$$

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