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Median Bernoulli Numbers and Ramanujan's Harmonic Number Expansion

Kwang-Wu Chen

Department of Mathematics, University of Taipei, No. 1, Ai-Guo West Road, Taipei 100234, Taiwan; kwchen@utaipei.edu.tw

Abstract: Ramanujan-type harmonic number expansion was given by many authors. Some of the most well-known are: $H_n \sim \gamma + \log n - \sum_{k=1}^{\infty} \frac{B_k}{k \cdot n^k}$, where B_k is the Bernoulli numbers. In this paper, we rewrite Ramanujan's harmonic number expansion into a similar form of Euler's asymptotic expansion as n approaches infinity: $H_n \sim \gamma + c_0(h) \log(q+h) - \sum_{k=1}^{\infty} \frac{c_k(h)}{k \cdot (q+h)^k}$, where $q = n(n+1)$ is the nth pronic number, twice the nth triangular number, γ is the Euler–Mascheroni constant, and $c_k(x) = \sum_{j=0}^{k} \binom{k}{j} c_j x^{k-j}$, with c_k is the negative of the median Bernoulli numbers. Then, $2c_n = \sum_{k=0}^{n} \binom{n}{k} B_{n+k}$, where B_n is the Bernoulli number. By using the result obtained, we present two general Ramanujan's asymptotic expansions for the nth harmonic number. For example, $H_n \sim \gamma + \frac{1}{2}\log(q+\frac{1}{3}) - \frac{1}{180(q+\frac{1}{3})^2}\left(\sum_{j=0}^{\infty} \frac{b_j(r)}{(q+\frac{1}{3})^j}\right)^{1/r}$ as n approaches infinity, where $b_j(r)$ can be determined.

Keywords: harmonic numbers; asymptotic expansion; median Bernoulli numbers

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1. Introduction

Leonhard Euler in 1755 applied the Euler–Maclaurin sum formula to find the famous standard Euler asymptotic expansion for H_n as $n \to \infty$:

$$H_n \sim \gamma + \log n - \sum_{k=1}^{\infty} \frac{B_k}{k \cdot n^k}, \tag{1}$$

where B_k is the Bernoulli number defined by $\frac{t}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k$, and $\gamma = 0.57721\cdots$ is the Euler–Mascheroni constant.

Ramanujan [1] proposed the following asymptotic expansion for H_n:

$$\begin{aligned} H_n \sim & \gamma + \frac{1}{2}\log(2m) + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} \\ & - \frac{191}{360360m^6} + \frac{29}{30030m^7} - \frac{2833}{1166880m^8} + \frac{140051}{17459442m^9} - \cdots, \end{aligned} \tag{2}$$

where $m = n(n+1)/2$ is the n-th triangular number. However, Ramanujan did not give any formulas for the general terms and also without any proof. Rewrite the above formula as the following notation:

$$H_n \sim \gamma + \frac{1}{2}\log(2m) + \sum_{k=1}^{\infty} \frac{R_k}{m^k}. \tag{3}$$

In 2008, Villarino [2] established an explicit expression for the coefficient sequence (R_k):

$$R_k = \frac{(-1)^{k-1}}{2k \cdot 8^k} \sum_{j=0}^{k} \binom{k}{j} (-4)^j B_{2j}(1/2), \tag{4}$$

where $B_k(x)$ are the Bernoulli polynomials defined by $\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!}$. In 2015, Chen and Cheng [3] reconsidered Ramanujan's formula and gave the following recurrence relation for (R_k):

$$R_1 = \frac{1}{12}, R_k = \frac{1}{2^k} \left\{ \frac{1}{4k} - \frac{B_{2k}}{2k} - \sum_{j=1}^{k-1} 2^j R_j \binom{2k-j-1}{j-1} \right\}, k \geq 2. \tag{5}$$

In 2019, Chen [4] improved the recurrence relation as

$$R_k = \frac{1}{2^{k+1}k} \left\{ \frac{1}{2k+1} - \sum_{j=1}^{k-1} 2^{j+1} R_j \binom{2k-j}{2k-2j+1} \right\}, \quad \text{for } k \geq 2. \tag{6}$$

Another Ramanujan-type harmonic number expansion was given by Wang [5] in 2018,

$$H_n \sim \gamma + \frac{1}{2}\log(2m+h) + \sum_{k=1}^{\infty} \frac{\alpha_k(h)}{(2m+h)^k}, \tag{7}$$

where h is a parameter and $(\alpha_k(h))$ is a coefficient sequence

$$\alpha_k(h) = -\frac{h^k}{2k} + \sum_{j=1}^{k} \binom{k-1}{j-1} R_j 2^j h^{k-j}. \tag{8}$$

In this paper, we rewrite Ramanujan's harmonic number expansion into a similar form of Euler's asymptotic expansion:

$$H_n \sim \gamma + c_0 \log q - \sum_{k=1}^{\infty} \frac{c_k}{k \cdot q^k}, \tag{9}$$

where $q = n(n+1) = 2m$ is the nth pronic number, twice the nth triangular number. In fact, we prove that the number c_k is the negative of the median Bernoulli number. The median Bernoulli number is studied by the author [6] in 2005. Then, we have for $n \geq 0$,

$$c_n = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} B_{n+k}. \tag{10}$$

Moreover, let

$$c_k(x) = \sum_{j=0}^{k} \binom{k}{j} c_j x^{k-j}. \tag{11}$$

Then, we could rewrite Wang's expansion Equation (7) as follows:

$$H_n \sim \gamma + c_0(h)\log(q+h) - \sum_{k=1}^{\infty} \frac{c_k(h)}{k \cdot (q+h)^k}. \tag{12}$$

We give simpler asymptotic expansion representations for H_n using Equations (10) and (11), which effectively integrate the results of Villarino, Chen and Cheng, Chen, and Wang (see Equations (4)–(7)) and make their representations more meaningful. We discuss some properties of the numbers c_n and the polynomials $c_n(x)$ in Sections 3 and 4, respectively. For example, the Hankel determinant of $c_n(x)$ for any x can be evaluated as

$$2^{n+1} \det_{0 \leq i,j \leq n} (c_{i+j}(x)) = \det_{0 \leq i,j \leq n} (B_{2i+2j}(1/2)). \tag{13}$$

Furthermore, Chen [7] gave a new asymptotic expansion. For any nonzero real number r, the n-th harmonic number H_n may have an asymptotic expansion as n approaches infinity:

$$\frac{1}{2}\log(2m)+\gamma+\frac{1}{12m}\left(\sum_{j=0}^{\infty}\frac{a_j(r)}{m^j}\right)^{1/r}, \tag{14}$$

where the parameters $a_j(r)$ satisfy the following recurrence relation

$$a_0(r)=1, a_j(r)=\frac{1}{j}\sum_{k=1}^{j}[k(1+r)-j](12R_{k+1})a_{j-k}(r), j\in\mathbb{N}.$$

Inspired by this, we give a more general asymptotic expansion in Section 5 using Equation (12). Given r,h real numbers with $r\neq 0$, $h\neq 1/3$, we get

$$H_n\sim\gamma+c_0(h)\log(q+h)-\frac{3h-1}{6(q+h)}\left(1+\sum_{j=1}^{\infty}\frac{a_j(r,h)}{(q+h)^j}\right)^{1/r}, \quad n\to\infty, \tag{15}$$

We know that the formula with $h=0$ is Equation (14) (see ([7], Theorem 2.3)). Since $c_1(h)=\frac{3h-1}{6}$, $h=1/3$ will remove the $(q+h)^{-1}$ term. This will improve the approximation. Thus, it can be seen that there are a lot of investigations for the $h=1/3$ case, see [4,8–10].

If $h=1/3$, then the asymptotic expansion will become

$$H_n\sim\gamma+\frac{1}{2}\log\left(q+\frac{1}{3}\right)-\frac{1}{180\left(q+\frac{1}{3}\right)^2}\left(1+\sum_{j=1}^{\infty}\frac{b_j(r)}{\left(q+\frac{1}{3}\right)^j}\right)^{1/r}. \tag{16}$$

The parameters $a_j(r,h)$ and $b_j(r)$ in Equations (15) and (16) are determined by some recurrence relations, which will be illustrated in Theorems 2 and 3, respectively. At the end of this paper, we will compare how close these asymptotic formulas are to H_n.

2. Median Bernoulli Numbers and R_k

Set $a_{0,n}=B_n$, for $n\geq 0$. And for $n\geq 1, k\geq 0$,

$$a_{n,k}=a_{n-1,k}+a_{n-1,k+1},$$

or equivalently,

$$a_{n,k}=\sum_{j=0}^{n}\binom{n}{j}a_{0,k+j}.$$

The corresponding matrix is represented as follows.

1	−1/2	1/6	0	−1/30	0	1/42	0
1/2	−1/3	1/6	−1/30	−1/30	1/42	1/42	…
1/6	−1/6	2/15	−1/15	−1/105	1/21	…	
0	−1/30	1/15	−8/105	4/105	…		
−1/30	1/30	−1/105	−4/105	…			
0	1/42	−1/21	…				
1/42	−1/42	…					
0	…						
…							

This matrix is called the "BS-matrix" in [6], which is a special Euler–Seidel matrix. Let

$$c_n=a_{n+1,n}$$

be the lower diagonal sequence of the BS-matrix. The number c_n is the negative of the median Bernoulli number K_n, which is the upper diagonal sequence of the BS-matrix, i.e., $K_n = a_{n,n+1} = -c_n$ (ref. [6]). Therefore, by [6], and Equations (8), (15) and (16), we have

$$c_n = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}B_{n+k} = \sum_{k=0}^{n+1}\binom{n+1}{k}B_{n+k} = -\sum_{k=0}^{n}\binom{n}{k}B_{n+1+k}. \tag{17}$$

Let the ordinary generating function of c_n as follows.

$$m(x) = \sum_{n=0}^{\infty} c_n x^{n+1}.$$

Let $\psi(x)$ be the formal Laplace transform of $t/\sinh t$. Then, the following relation was obtained ([6], Theorem 4.2, Equation (29))

$$2x \cdot \psi(x) = m\left(\frac{4x^2}{1-x^2}\right). \tag{18}$$

Since $t/\sinh t = \sum_{n=0}^{\infty} 4^n B_{2n}(1/2)t^{2n}/(2n)!$, we have that for $n \geq 0$ ([6], Equation (32)),

$$2^{2n+1}(-1)^n c_n = \sum_{j=0}^{n}\binom{n}{j}(-1)^j 2^{2j} B_{2j}(1/2). \tag{19}$$

Using Villarino's explicit formula for R_k, Equation (4), we have for $k \geq 1$,

$$-2^k \cdot k \cdot R_k = \sum_{j=0}^{k}\binom{k}{j}(-1)^{k+j}2^{2j-2k-1}B_{2j}(1/2) = c_k. \tag{20}$$

This implies that

$$H_n \sim \gamma + c_0 \log q - \sum_{k=1}^{\infty}\frac{c_k}{kq^k}, \quad n \to \infty. \tag{21}$$

On the other hand, we substitute R_k as Equation (20) in Wang's formula for $\alpha_k(h)$, (see Equation (8)), we have for $k \geq 1$,

$$\begin{aligned} -k \cdot \alpha_k(h) &= \frac{h^k}{2} - \sum_{j=1}^{k}\binom{k-1}{j-1}R_j 2^k h^{k-j} k \\ &= \frac{h^k}{2} + \sum_{j=1}^{k}\binom{k-1}{j-1}h^{k-j}\frac{c_j}{j}\cdot k \;=\; \sum_{j=0}^{k}\binom{k}{j}h^{k-j}c_j \;=\; c_k(h). \end{aligned} \tag{22}$$

Therefore, we conclude our result in the following.

Theorem 1. *For $n \to \infty$, we have*

$$H_n \sim \gamma + c_0(h)\log(q+h) - \sum_{k=1}^{\infty}\frac{c_k(h)}{k\cdot(q+h)^k}, \tag{23}$$

where $q = n(n+1)$ is the nth pronic number,

$$c_k(x) = \sum_{j=0}^{k}\binom{k}{j}c_j x^{k-j}, \quad \textit{and} \quad c_n = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}B_{n+k}. \tag{24}$$

3. Some Properties of c_n

Let the ordinary generating function of B_n as follows.

$$b(x) = \sum_{n=0}^{\infty} B_n x^{n+1}.$$

Using the relation between the ordinary generating functions of $a_{0,n}$, $a_{n,n}$, and $a_{n,n+1}$ of the BS-matrix, we have the following relation ([6], Theorem 4.2, Equation (29))

$$b(x) = \left(1 + \frac{2}{x}\right) m\left(\frac{x^2}{1+x}\right). \tag{25}$$

Then, the following identity is obtained ([6], Equation (27)).

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} \frac{n}{n-j} c_j = \begin{cases} -B_1, & \text{if } n = 1, \\ B_n, & \text{if } n \geq 2. \end{cases} \tag{26}$$

In the above formula, the formula obtained by substituting $n = 2k$ appears in the recurrence relation of R_k given by Chen and Cheng [3] in 2015 (see Equation (5)). Furthermore, if we substitute $n = 2k + 1$ into the above identity, we obtain Equation (6) given by Chen [4] in 2019.

There are a lot of properties of c_n obtained from [6]. For example, let the denominators and the numerators of the rational number c_n be D_n, N_n, respectively. We have the following properties ([6], Theorem 1.1):

- The denominator D_n is a square-free integer.
- The set of the all odd prime divisors of D_n is $\left\{p : \text{odd prime} \mid \frac{n}{m} \leq p - 1 \leq \frac{2n}{2m-1}, m \in \mathbb{N}\right\}$.
- The denominator D_n is an odd integer, for $n \geq 2$.
- The largest power of 2 that divides the numerator N_n is $2\lfloor \frac{n-1}{2} \rfloor$.

The ordinary generating function $m(x)$ has the following continued fraction representation ([6], Theorem 5.5)

$$m(x) = \frac{c_0 x}{1 + a_0 x} - \frac{b_1 x^2}{1 + a_1 x} - \frac{b_2 x^2}{1 + a_2 x} - \cdots, \tag{27}$$

where for $n \geq 0$,

$$a_n = \frac{8n^4 + 8n^3 + 6n^2 + 2n - 1}{(4n+3)(4n-1)},$$
$$b_{n+1} = \frac{(2n+1)^4 (n+1)^4}{(4n+1)(4n+3)^2(4n+5)}.$$

Using this representation, we have the Hankel determinant of c_n (ref. [6], Theorem 5.5)

$$\det_{0 \leq i,j \leq n} (c_{i+j}) = \left(\frac{1}{2}\right)^{n+1} \prod_{j=1}^{n} \left(\frac{(2j-1)^4 j^4}{(4j-3)(4j-1)^2(4j+1)}\right)^{n-j+1}. \tag{28}$$

Since the finite product in Equation (28) is the Hankel determinant of $B_{2n}(1/2)$ (see [6], Equation (41)), we have

$$\det_{0 \leq i,j \leq n} \left(B_{2i+2j}(1/2)\right) = 2^{n+1} \det_{0 \leq i,j \leq n} (c_{i+j}). \tag{29}$$

By Equation (19) and an integral representation of $B_{2n}(1/2)$ ([11], Equation (28))

$$B_{2n}(1/2) = (-1)^n \pi \int_0^\infty t^{2n} \operatorname{sech}^2(\pi t)\, dt,$$

we have an integral representation of c_n, for $n \geq 0$,

$$c_n = \frac{(-1)^n \pi}{2^{2n+1}} \int_0^\infty (4t^2+1)^n \operatorname{sech}^2(\pi t)\, dt. \tag{30}$$

4. Some Properties of $c_n(x)$

We first list $c_n(x)$ for $n = 0, 1, 2, \ldots, 5$.

$$\begin{aligned}
c_0(x) &= \frac{1}{2}, \\
c_1(x) &= -\frac{1}{6} + \frac{x}{2}, \\
c_2(x) &= \frac{1}{15} - \frac{x}{3} + \frac{x^2}{2}, \\
c_3(x) &= -\frac{4}{105} + \frac{x}{5} - \frac{x^2}{2} + \frac{x^3}{2}, \\
c_4(x) &= \frac{4}{105} - \frac{16x}{105} + \frac{2x^2}{5} - \frac{2x^3}{3} + \frac{x^4}{2}, \\
c_5(x) &= -\frac{16}{231} + \frac{4x}{21} - \frac{8x^2}{21} + \frac{2x^3}{3} - \frac{5x^4}{6} + \frac{x^5}{2}.
\end{aligned}$$

Differentiating Equation (11) with respect to x we obtain

$$\begin{aligned}
\frac{d}{dx} c_n(x) &= \sum_{k=0}^{n} \binom{n}{k} c_{n-k} k x^{k-1} \\
&= n \sum_{k=0}^{n-1} \binom{n-1}{k} c_{n-1-k} x^k = n c_{n-1}(x).
\end{aligned} \tag{31}$$

Therefore,

$$\int_x^y c_n(t)\, dt = \frac{c_{n+1}(y) - c_{n+1}(x)}{n+1}. \tag{32}$$

On the other hand, we use Equation (30) to get an integral representation of $c_n(x)$:

$$c_n(x) = \frac{(-1)^n \pi}{2^{2n+1}} \int_0^\infty (4t^2 - 4x + 1)^n \operatorname{sech}^2(\pi t)\, dt. \tag{33}$$

Let us consider the function $c_n(x+y)$. We express $(x+y)^k$ as its binomial expansion.

$$c_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} c_{n-k} (x+y)^k = \sum_{k=0}^{n} c_{n-k} \sum_{\ell=0}^{k} \binom{k}{\ell} x^\ell y^{k-\ell}.$$

We interchange the order of summation and the inner sum becomes $c_{n-\ell}(y)$:

$$c_n(x+y) = \sum_{\ell=0}^{n} \sum_{k=\ell}^{n} \binom{n}{k} \binom{k}{\ell} c_{n-k} x^\ell y^{k-\ell} = \sum_{\ell=0}^{n} \binom{n}{\ell} x^\ell c_{n-\ell}(y).$$

Thus, we have

$$c_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} c_k(x) y^{n-k}. \tag{34}$$

Using the inversion binomial theorem to Equation (19) we have

$$4^n B_{2n}(1/2) = \sum_{k=0}^{n} \binom{n}{k} 2^{2k+1} c_k = 2^{2n+1} c_n(1/4).$$

This implies that

$$2\, c_n(1/4) = B_{2n}(1/2). \tag{35}$$

Applying the above identity and Equation (29), we know that the Hankel determinant of $c_n(1/4)$ is the same as the Hankel determinant of c_n. However, we use ([12], Proposition 1), indeed that for any value of x,

$$\det_{0\le i,j\le n}(c_{i+j}) = \det_{0\le i,j\le n}(c_{i+j}(x)). \tag{36}$$

5. New Asymptotic Expansions

To derive our new asymptotic expansions are inspired by ([7], Theorem 2.3). We need the following lemma.

Lemma 1 ([7], Lemma 1). *If $\sum_{j=0}^{\infty} q_j x^{-j}$ is an asymptotic expansion for $g(x)$ as x approaches infinity. Given any real number r, the parameters $Q_j(r)$ are defined by $Q_0(r) = 1$ and for $j \in \mathbb{N}$,*

$$Q_j(r) = \frac{1}{j} \sum_{k=1}^{j} (k(1+r) - j) q_k Q_{j-k}(r).$$

Then $\sum_{j=0}^{\infty} Q_j(r) x^{-j}$ is an asymptotic expansion for $g(x)^r$.

Our new asymptotic expansions are derived from Equation (23). It is note that $c_1(h) = \frac{3h-1}{6}$. Therefore, we divide into two cases depending on whether h is $1/3$ or not.

Theorem 2. *Let r and h be any given real numbers with $r \neq 0$ and $h \neq 1/3$. The n-th harmonic number H_n has the following asymptotic expansion as n approaches infinity:*

$$\gamma + c_0(h)\log(q+h) - \frac{3h-1}{6(q+h)} \left(1 + \sum_{j=1}^{\infty} \frac{a_j(r,h)}{(q+h)^j}\right)^{1/r} \tag{37}$$

where the parameters $a_j(r,h)$ given by the recurrence relation

$$\begin{aligned} a_0(r,h) &= 1, \\ a_j(r,h) &= \frac{1}{j} \sum_{k=1}^{j} (k(1+r) - j) \frac{6c_{k+1}(h)}{(3h-1)(k+1)} a_{j-k}(r,h), j \geq 1. \end{aligned} \tag{38}$$

Proof. Rewrite Equation (37) as the following representation:

$$\frac{6(q+h)}{3h-1}(H_n - \gamma - c_0(h)\log(q+h)) \sim \left(1 + \sum_{j=1}^{\infty} \frac{a_j(r,h)}{(q+h)^j}\right)^{1/r}.$$

In view of Equation (23), we have

$$\frac{6(q+h)}{3h-1}(H_n - \gamma - c_0(h)\log(q+h)) \sim \left(1 + \frac{6}{3h-1} \sum_{k=1}^{\infty} \frac{c_{k+1}(h)}{(k+1)(q+h)^k}\right).$$

Comparing the above two expressions, we know that

$$\left(1 + \frac{6}{3h-1} \sum_{k=1}^{\infty} \frac{c_{k+1}(h)}{(k+1)(q+h)^k}\right)^r \sim 1 + \sum_{j=1}^{\infty} \frac{a_j(r,h)}{(q+h)^j}.$$

We apply Lemma 1 and get the result we want. □

Using a similar approach, we can easily derive the following theorem for the situation $h = 1/3$.

Theorem 3. *Given a real number r with $r \neq 0$. The n-th harmonic number H_n has the asymptotic expansion as n approaches infinity:*

$$\gamma + \frac{1}{2}\log(q + \frac{1}{3}) - \frac{1}{180(q+\frac{1}{3})^2}\left(1 + \sum_{j=1}^{\infty}\frac{b_j(r)}{(q+\frac{1}{3})^j}\right)^{1/r} \tag{39}$$

where the parameters $b_j(r)$ are defined by the following relation

$$b_0(r) = 1, \quad b_j(r) = \frac{1}{j}\sum_{k=1}^{j}(k(1+r) - j)\frac{180\, c_{k+2}(\frac{1}{3})}{k+2} b_{j-k}(r), \quad j \geq 1. \tag{40}$$

Chen [7] discussed many properties of the $h = 0$ case. Therefore, we mainly deal with the case of $h = 1/3$ here.

The first few parameters $b_j(r)$ are:

$$\begin{aligned}
b_0(r) &= 1,\\
b_1(r) &= -\frac{32}{63}r,\\
b_2(r) &= \frac{3701}{7938}r + \frac{512}{3969}r^2,\\
b_3(r) &= -\frac{7264240}{8251551}r - \frac{59216}{250047}r^2 - \frac{16384}{750141}r^3,\\
b_4(r) &= \frac{47882328785}{18021387384}r + \frac{2311659673}{4158781704}r^2 + \frac{947456}{15752961}r^3 + \frac{131072}{47258883}r^4,\\
b_5(r) &= -\frac{8014919889976}{709592128245}r - \frac{749340134980}{425755276947}r^2 - \frac{789621116}{4678629417}r^3\\
&\quad - \frac{30318592}{2977309629}r^4 - \frac{4194304}{14886548145}r^5.
\end{aligned}$$

For $r = 1$ in Equation (39), the resulting asymptotic expansion is as follows ([10], Equation (3.24)):

$$\begin{aligned}
H_n \sim & \gamma + \frac{1}{2}\log(q+\frac{1}{3}) - \frac{1}{180(q+\frac{1}{3})^2} + \frac{8}{2835(q+\frac{1}{3})^3} - \frac{5}{1512(q+\frac{1}{3})^4}\\
&+ \frac{592}{93555(q+\frac{1}{3})^5} - \frac{796801}{43783740(q+\frac{1}{3})^6} + \frac{268264}{3648645(q+\frac{1}{3})^7} - \cdots
\end{aligned} \tag{41}$$

as $n \to \infty$.

For $r = -1$ in Equation (39), we obtain a new asymptotic expansion:

$$\begin{aligned}
H_n \sim & \gamma + \frac{1}{2}\log(q+\frac{1}{3}) - \left[180(q+\frac{1}{3})^2 + \frac{640}{7}(q+\frac{1}{3}) - \frac{26770}{441} + \frac{36602240}{305613(q+\frac{1}{3})}\right.\\
&\left. - \frac{97247611025}{250297047(q+\frac{1}{3})^2} + \frac{27515011460000}{15768713961(q+\frac{1}{3})^3} - \cdots\right]^{-1}
\end{aligned} \tag{42}$$

as $n \to \infty$.

For $r = -23/2$ in Equation (39), we obtain a new asymptotic expansion:

$$H_n \sim \gamma + \frac{1}{2}\log(q+\frac{1}{3}) - \frac{1}{180(q+\frac{1}{3})^2}\Bigg[1 + \frac{368}{63(q+\frac{1}{3})} + \frac{185725}{15876(q+\frac{1}{3})^2} + \frac{3674204}{305613(q+\frac{1}{3})^3} - \frac{5793677}{728136864(q+\frac{1}{3})^4} + \frac{1021070020123}{31537427922(q+\frac{1}{3})^5} + \cdots\Bigg]^{-2/23} \tag{43}$$

as $n \to \infty$.

From a computational point of view, the formulas Equations (42) and (43) are better than Equation (41).

It follows from Equations (41)–(43) that for $n \to \infty$,

$$H_n \sim \gamma + \frac{1}{2}\log(q+\frac{1}{3}) - \frac{1}{180(q+\frac{1}{3})^2} + \frac{8}{2835(q+\frac{1}{3})^3} - \frac{5}{1512(q+\frac{1}{3})^4} + \frac{592}{93555(q+\frac{1}{3})^5} := u_n, \tag{44}$$

$$H_n \sim \gamma + \frac{1}{2}\log(q+\frac{1}{3}) - \frac{1}{180(q+\frac{1}{3})^2 + \frac{640}{7}(q+\frac{1}{3}) - \frac{26770}{441} + \frac{36602240}{305613(q+\frac{1}{3})}} := v_n, \tag{45}$$

$$H_n \sim \gamma + \frac{1}{2}\log(q+\frac{1}{3}) - \frac{1}{180(q+\frac{1}{3})^2\left[1 + \frac{368}{63(q+\frac{1}{3})} + \frac{185725}{15876(q+\frac{1}{3})^2} + \frac{3674204}{305613(q+\frac{1}{3})^3}\right]^{2/23}} := w_n. \tag{46}$$

From Table 1, we observe that, among approximation formulas Equations (44)–(46), for $n \geq 1$, the formula Equation (46) would be the best one. There seems to be an optimal real number r in Equation (39), and when we substitute it in this formula, the resulting approximation should be optimal. We guess that this real number r should be close to $-11.502534\ldots$

Table 1. Comparison of approximation Formulas (44)–(46).

n	$u_n - H_n$	$v_n - H_n$	$w_n - H_n$
1	4.625×10^{-5}	1.997×10^{-5}	-1.405×10^{-6}
10	9.735×10^{-15}	6.332×10^{-15}	-6.750×10^{-17}
10^2	1.713×10^{-26}	1.129×10^{-26}	2.162×10^{-30}
10^3	1.809×10^{-38}	1.192×10^{-38}	3.805×10^{-42}
10^4	1.819×10^{-50}	1.198×10^{-50}	3.841×10^{-54}

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