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Wiener Process Effects on the Solutions of the Fractional (2 + 1)-Dimensional Heisenberg Ferromagnetic Spin Chain Equation

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Abstract: The stochastic fractional (2 + 1)-dimensional Heisenberg ferromagnetic spin chain equation (SFHFSCE), which is driven in the Stratonovich sense by a multiplicative Wiener process, is considered here. The analytical solutions of the SFHFSCE are attained by utilizing the Jacobi elliptic function method. Various kinds of analytical fractional stochastic solutions, for instance, the elliptic functions, are obtained. Physicists can utilize these solutions to understand a variety of important physical phenomena because magnetic solitons have been categorized as one of the interesting groups of non-linear excitations representing spin dynamics in semi-classical continuum Heisenberg systems. To study the impact of the Wiener process on these solutions, the 3D and 2D surfaces of some achieved exact fractional stochastic solutions are plotted.

Keywords: fractional Heisenberg ferromagnetic equation; stochastic Heisenberg ferromagnetic equation; Wiener process; Jacobi elliptic function method

MSC: 35A20; 60H10; 83C15; 60H15

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1. Introduction

Over the last few decades, stochastic partial differential equations (SPDEs) have been intensively investigated as mathematical models for spatial-temporal chemical, biological, and physical equations that are sensitive to random perturbations. In complex system modeling, the necessity to include stochastic effects has been highlighted. For example, there is growing interest in the mathematical modeling of complex processes in climate systems, finance, biology, condensed matter physics, materials sciences, information systems, and mechanical and electrical engineering utilizing SPDEs.

Researchers and scientists, on the other hand, have concentrated their efforts on fractional differential equations (FDEs), which have been proved to be more accurate than classical differential equations in describing complicated physical events in the actual world. Various phenomena, such as nuclear physics, viscoelastic materials, signal processing, fluid dynamics porous medium, plasma physics, photonics, chaotic systems, electromagnetism, propagation of waves, optical fiber communication, ocean wave and many others, have been explained using the idea of fractional derivatives. Because FDEs are so important, several efficient and powerful approaches for determining the precise solutions to these equations have been developed. Some of these methods are the Riccati–Bernoulli sub-ODE [1], the bifurcation [2,3], the tanh-sech [4,5], the Jacobi elliptic function [6], the Hirota's [7], the $exp(-\phi(\varsigma))$-expansion [8], perturbation [9–11], the $(\frac{G'}{G})$-expansion [12–14], and the sine-cosine [15,16].

Models of fractional differential equations with random forces appear to be more important. As a result, one of the most significant equations in modern magnet theory is considered here. This equation is defined as the (2 + 1)-dimensional stochastic fractional Heisenberg ferromagnetic spin chain equation (SFHFSCE), and is written as follows:

$$idU + [k_1\mathbb{T}_x^{2\alpha}U + k_2\mathbb{T}_y^{2\alpha}U + k_3\mathbb{T}_{xy}^{2\alpha}U - k_4|U|^2U]dt + i\rho U \circ d\eta = 0, \tag{1}$$

where U is a complex stochastic function of the variable x, y and t,

$$k_1 = \sigma^4(J + J_2),\ k_2 = \sigma^4(J_1 + J_2),\ k_3 = 2\sigma^4 J_2,\ k_3 = 2\sigma^4 A,$$

J, J_1, J_2 are the constant coefficients of bilinear exchange interactions in two dimensions, σ is a lattice parameter and A represents the crystal field anisotropic interaction, ρ is the noise strength, η is the standard Wiener process (SWP) in one variable t and $U \circ d\eta$ is multiplicative noise in the Stratonovich sense.

To understand magnetic ordering in ferromagnetic materials, the deterministic Heisenberg ferromagnet equation (DHFE) was created. It is employed in optical fibers and plays a significant role in the modern theory of magnets, which models non-linear magnet dynamics. Due to the significance of DHFE, numerous authors have used a variety of approaches, such as generalized Riccati mapping and improved auxiliary equation [17], Sine–Gordon and modified exp-function expansion [18], the F-expansion method combined with Jacobi elliptic [19], the Darboux transformation [20–22], the Hirota bilinear [23,24], the auxiliary ordinary differential equation [25], a new extended FAN sub-equation [26] and Jacobi elliptic functions [27], to find the exact solution for this equation. Many authors have investigated the analytical solutions of fractional DHFE using various methods, including generalized Riccati equation mapping [28], the complete discrimination system [29], exponential methods and the new Kudryashov [30], the new extended generalized Kudryashov [31], the Jacobi elliptic function [32], the extended tanh-function and the $exp(-\phi(\varsigma))$-expansion [33]. However, to the best of our knowledge, the space-fractional stochastic solutions of Equation (1) have not been investigated until now.

Our objective for this paper was to obtain the exact stochastic fractional solutions of Equation (1) using the Jacobi elliptic function method (for more details about this method see, for instance, [6,27]). Because magnetic solitons are one of the interesting groups of non-linear excitations reflecting spin dynamics in semi-classical continuum Heisenberg systems, physicists might use the obtained solution to understand a range of fascinating physical phenomena. Also, we demonstrate the impact of the Wiener process on the behavior of these solutions by displaying different graphical representations using MATLAB tools.

The rest of this article is ordered as follows: We define and state some features of the SWP and conformable derivative (CD), In Section 2. In Section 3, we apply suitable wave transformation to derive the wave equation of the SFSHFSCE (1). In Section 4, we apply the Jacobi elliptic function method to attain the analytical solutions of the SFSHFSCE (1). In Section 5, we discuss how SWP influences the analytical solutions of the SFSHFSCE (1). Lastly, we provide the article's conclusions.

2. Preliminaries

We present here some definitions and properties of SWP and CD. First, let us define SWP $\eta(t)$ as follows:

Definition 1. *A stochastic process $\{\eta(t)\}_{t\geq 0}$ is called SWP if*

1. $\eta(0) = 0$,
2. $\eta(t)$ *is a continuous function for* $t \geq 0$,
3. *For* $t_1 < t_2$, $\eta(t_1) - \eta(t_2)$ *is independent,*
4. $\eta(t_2) - \eta(t_1)$ *has a normal distribution with variance* $t_2 - t_1$ *and mean* 0.

are satisfied

We note that the two most commonly utilized variants of the stochastic integral are the Stratonovich and Itô variants [34]. The modeling problem mainly determines what form is appropriate; even so, once it is chosen, an equivalent equation of the other kind can be created using the same solutions. So, the next relation can be used to swap between Itô (denoted by $\int_0^t \Phi d\eta$) and Stratonovich (denoted by $\int_0^t \Phi \circ d\eta$):

$$\int_0^t \Phi(\tau, Z_\tau) d\eta(\tau) = \int_0^t \Phi(\tau, Z_\tau) \circ d\eta(\tau) - \frac{1}{2}\int_0^t \Phi(\tau, Z_\tau)\frac{\partial \Phi(\tau, Z_\tau)}{\partial z} d\tau, \tag{2}$$

where Φ is assumed to be sufficiently regular and $\{Z_t, t \geq 0\}$ is a stochastic process.

Definition 2 ([35])**.** *The CD of order* $\alpha \in (0,1]$ *for* $\phi : \mathbb{R}^+ \to \mathbb{R}$ *is defined as*

$$\mathbb{T}_x^\alpha \phi(y) = \lim_{h\to 0} \frac{\phi(y + hy^{1-\alpha}) - \phi(y)}{h}.$$

Let us exhibit some features of the CD:

1. $\mathbb{T}_y^\alpha[a\phi(y) + b\Psi(y)] = a\mathbb{T}_y^\alpha \phi(y) + b\mathbb{T}_y^\alpha \Psi(y)$,
2. $\mathbb{T}_x^\alpha(\phi \circ \Psi)(y) = y^{1-\alpha}\Psi'(y)\phi(\Psi(y))$,
3. $\mathbb{T}_y^\alpha[b] = 0$,
4. $\mathbb{T}_y^\alpha[y^b] = by^{b-\alpha}$,
5. $\mathbb{T}_y^\alpha \Psi(y) = y^{1-\alpha\frac{d\Psi}{dy}}$,

for any real constants a, b.

3. The Wave Equation of the SFSHFSCE

To achieve the wave equation for the SFSHFSCE, we use the following transformation

$$U(x,y,t) = \varphi(\eta)e^{(i\theta - \rho\eta(t) - \rho^2 t)}, \quad \eta = \frac{\eta_1}{\alpha}x^\alpha + \frac{\eta_2}{\alpha}y^\alpha + \eta_3 t,\ \theta = \frac{\theta_1}{\alpha}x^\alpha + \frac{\theta_2}{\alpha}y^\alpha + \theta_3 t, \tag{3}$$

where φ is a real deterministic function, and η_k and θ_k for all $k = 1,2,3$ are constants. We see that

$$\begin{aligned}
\mathbb{T}_x^{2\alpha} U &= [\eta_1^2\varphi'' + 2i\eta_1\theta_1\varphi' - \theta_1^2\varphi]e^{(i\theta - \rho\eta(t) - \rho^2 t)}, \\
\mathbb{T}_y^{2\alpha} U &= [\eta_2^2\varphi'' + 2i\eta_2\theta_2\varphi' - \theta_2^2\varphi]e^{(i\theta - \rho\eta(t) - \rho^2 t)}, \\
\mathbb{T}_{xy}^{\alpha} U &= [\eta_1\eta_2\varphi'' + i(\eta_1\theta_2 + \eta_2\theta_1)\varphi' - \theta_1\theta_2\varphi]e^{(i\theta - \rho\eta(t) - \rho^2 t)},
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
dU &= [(\eta_3\varphi' + i\theta_3\varphi + \frac{1}{2}\rho^2\varphi - \rho^2\varphi)dt - \rho\varphi d\eta]e^{(i\theta - \rho\eta(t) - \rho^2 t)}dt \\
&= [(\eta_3\varphi' + i\theta_3\varphi)dt - (\frac{1}{2}\rho^2\varphi dt + \rho\varphi d\eta)]e^{(i\theta - \rho\eta(t) - \rho^2 t)}dt,
\end{aligned} \tag{5}$$

where the term $+\frac{1}{2}\rho^2\varphi$ is the Itô correction. Using Equation (2) in differential form, we have

$$dU = [(\eta_3\varphi' + i\theta_3\varphi)dt - \rho\varphi \circ d\eta]e^{(i\theta - \rho\eta(t) - \rho^2 t)}dt. \tag{6}$$

Substituting Equation (3) into Equation (1) and utilizing (4) and (5), we have, for the imaginary part:

$$(\eta_3 + 2k_1\eta_1\theta_1 + 2k_2\eta_2\theta_2 + k_3\eta_1\theta_2 + k_3\eta_2\theta_1)\varphi' = 0, \tag{7}$$

we assume

$$\eta_3 = -k_1\eta_1\theta_1 - 2k_2\eta_2\theta_2 - k_3\eta_1\theta_2 - k_3\eta_2\theta_1.$$

And we have, for the real part:

$$\varphi'' - H_1 e^{(2\rho\eta(t) - 2\rho^2 t)}\varphi^3 - H_2\varphi = 0, \tag{8}$$

where

$$H_1 = \frac{k_4}{k_1\eta_1^2 + k_2\eta_2^2 + k_3\eta_1\eta_2} \text{ and } H_2 = \frac{\theta_3 + k_1\theta_1^2 + k_2\theta_2^2 + k_3\theta_1\theta_2}{k_1\eta_1^2 + k_2\eta_2^2 + k_3\eta_1\eta_2}.$$

Considering the expectation on both sides of (8), yields

$$\varphi'' - H_1\varphi^3 e^{-2\rho^2 t}\mathbb{E}(e^{2\rho\eta(t)}) - H_2\varphi = 0, \tag{9}$$

where φ is a deterministic function. We note, for every Gaussian process Y and real number ϱ, that

$$\mathbb{E}(e^{\varrho Y}) = e^{\frac{\varrho^2}{2}t}. \tag{10}$$

The identity (10) relates to the fact that $\rho\eta(t)$ is distributed as $\rho\sqrt{t}Y$. Then, Equation (9) takes the form

$$\varphi'' - H_1\varphi^3 - H_2\varphi = 0. \tag{11}$$

4. Analytical Solutions of the SFSHFSCE

To acquire the wave solutions of (11), we utilize the Jacobi elliptic function method. Then we obtain the SFSHFSCE (1) solutions. Considering the solutions to Equation (11), they take the following form:

$$\varphi(\xi) = A + Bcn(\lambda\xi), \tag{12}$$

where $cn(\lambda\xi) = cn(\lambda\xi, m)$ is a Jacobi elliptic sine function for $0 < m < 1$ and $A,\ B,\ \lambda$ are undefined constants. By differentiating Equation (12) twice, we obtain

$$\varphi''(\xi) = -(2m^2 - 1)B\lambda^2 cn(\lambda\xi) - 2m^2 B\lambda^2 cn^3(\lambda\xi). \tag{13}$$

Inserting Equations (12) and (13) into Equation (11), we have

$$(2m^2 B\lambda^2 + H_1 B^3)cn^3(\lambda\xi) + 3H_1 AB^2 cn^2(\lambda\xi)$$

$$+[(2m^2 - 1)B\lambda^2 + 3H_1 A^2 B + H_2 B]cn(\lambda\xi) + (H_1 A^3 + AH_2) = 0.$$

Balancing the coefficient of $[cn(\lambda\xi)]^n$ to zero for $n = 0, 1, 2, 3$, we have

$$H_1 A^3 + AH_2 = 0,$$

$$(2m^2 - 1)B\lambda^2 + 3H_1 A^2 B + H_2 B = 0,$$

$$3H_1 AB^2 = 0,$$

and

$$2m^2 B\lambda^2 + H_1 B^3 = 0.$$

When we solve the previously mentioned equations, we get

$$A=0,\ B=\pm\sqrt{\frac{-2m^2H_2}{(2m^2-1)H_1}},\ \lambda^2=\frac{H_2}{(2m^2-1)}.$$

Thus, Equation (11), using (12), has the solution

$$\varphi(\xi)=\pm\sqrt{\frac{-2m^2H_2}{(2m^2-1)H_1}}cn(\sqrt{\frac{H_2}{(2m^2-1)}}\xi).$$

Therefore, the solution of SFSHFSCE (1) is

$$U(x,y,t)=\pm\sqrt{\frac{-2m^2H_2}{(2m^2-1)H_1}}cn\left(\sqrt{\frac{H_2}{(2m^2-1)}}\xi\right)e^{(i\theta-\rho\eta(t)-\rho^2t)}, \tag{14}$$

for $\frac{H_2}{(2m^2-1)}>0$ and $H_1<0$. If $m\to1$, then Equation (14) takes the form

$$U(x,y,t)=\pm\sqrt{\frac{-2H_2}{H_1}}\operatorname{sech}\left(\sqrt{H_2}\xi\right)e^{(i\theta-\rho\eta(t)-\rho^2t)}. \tag{15}$$

Similarly, we can change cn in (12) by dn and sn to acquire the solutions of (11), as follows:

$$\varphi(\xi)=\pm\sqrt{\frac{2m^2H_2}{(2-m^2)H_1}}dn(\frac{-H_2}{(2-m^2)}\xi),$$

and

$$\varphi(\xi)=\pm\sqrt{\frac{-2m^2H_2}{(m^2+1)H_1}}sn(\frac{-H_2}{(m^2+1)}\xi),$$

respectively. As a result, the solutions of SFSHFSCE (1) are as follows:

$$U(x,y,t)=\pm\sqrt{\frac{-2m^2H_2}{(2m^2-1)H_1}}dn\left(\sqrt{\frac{-H_2}{(2m^2-1)}}\xi\right)e^{(i\theta-\rho\eta(t)-\rho^2t)}, \tag{16}$$

for $\frac{H_2}{(2m^2-1)}<0$, $H_1>0$, and

$$U(x,y,t)=\pm\sqrt{\frac{-2m^2H_2}{(m^2+1)H_1}}sn\left(\sqrt{\frac{-H_2}{(m^2+1)}}\xi\right)e^{(i\theta-\rho\eta(t)-\rho^2t)}, \tag{17}$$

for $H_2<0$, $H_1>0$, respectively. If $m\to1$, then the solutions (16) and (17) take the form

$$U(x,y,t)=\pm\sqrt{\frac{-2H_2}{H_1}}\operatorname{csch}\left(\sqrt{-H_2}\xi\right)e^{(i\theta-\rho\eta(t)-\rho^2t)}, \tag{18}$$

and

$$U(x,y,t)=\pm\sqrt{\frac{-H_2}{H_1}}\tanh\left(\sqrt{\frac{-H_2}{2}}\xi\right)e^{(i\theta-\rho\eta(t)-\rho^2t)}, \tag{19}$$

for $H_2<0$, $H_1>0$.

Remark 1. *If we put $\alpha=1$ and $\rho=0$ in Equations (14), (16) and (17), then the identical solutions as in [27] are obtained.*

Remark 2. *If we put $\rho = 0$ and $\alpha = 1$ in Equations (14)–(19), then we have the same solutions stated in [19] .*

5. The Impact of Noise on the SFSHFSCE Solutions

We address the impact of the SWP on the solutions of the SFSHFSCE (1). We provide numerous graphical-representations to check the impact of the SWP on the behavior of these solutions. First, let us fix the parameters $k_1 = 2.5$, $k_2 = k_3 = 1.5$, $k_4 = 0.5$, $\eta_1 = \eta_2 = \theta_1 = \theta_2 = 1$. In the following, we utilize the MATLAB tools [36] to simulate the solutions (14) for $t \in [0, 5]$, $x \in [0, 6]$ and , $y = 1$ and for various ρ (noise intensity):

When we look at Figures 1–3 above, we can see that there is some variation and that the surface is not flat when we examine at $\rho = 0$. When noise is included and its strength is increased by $\rho = 0.5,\ 1,\ 2$, the surface becomes significantly more flat after minor transit patterns. This displays that the SWP has an effect on the solutions of the SFSHFSCE and stabilizes the solutions around zero.

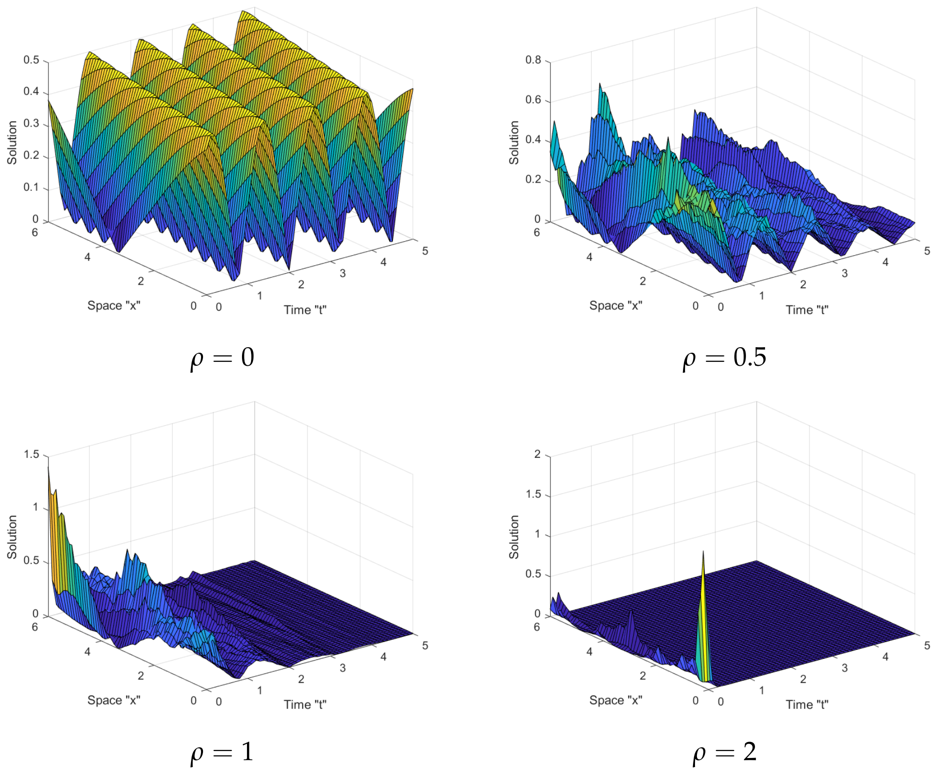

Figure 1. Profile picture of $|U|$ given in (14) in three dimension for $\alpha = 1$ and $\theta_3 = -5$.

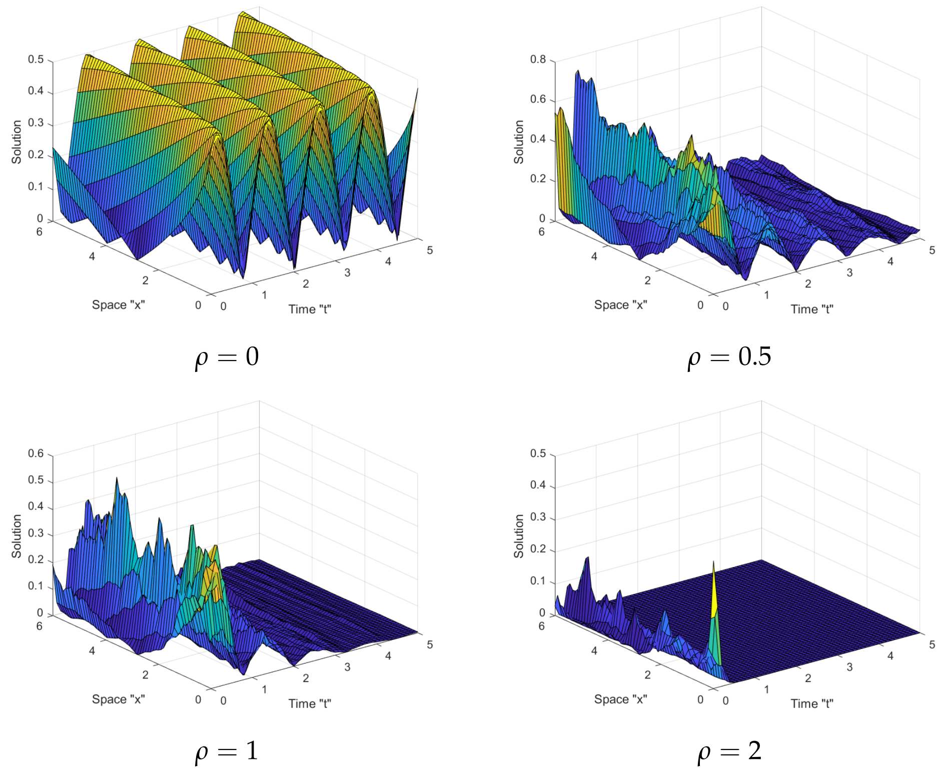

Figure 2. Profile picture of $|U|$ given in (14) in three dimensions for $\alpha = 0.5$ and $\theta_3 = -5$.

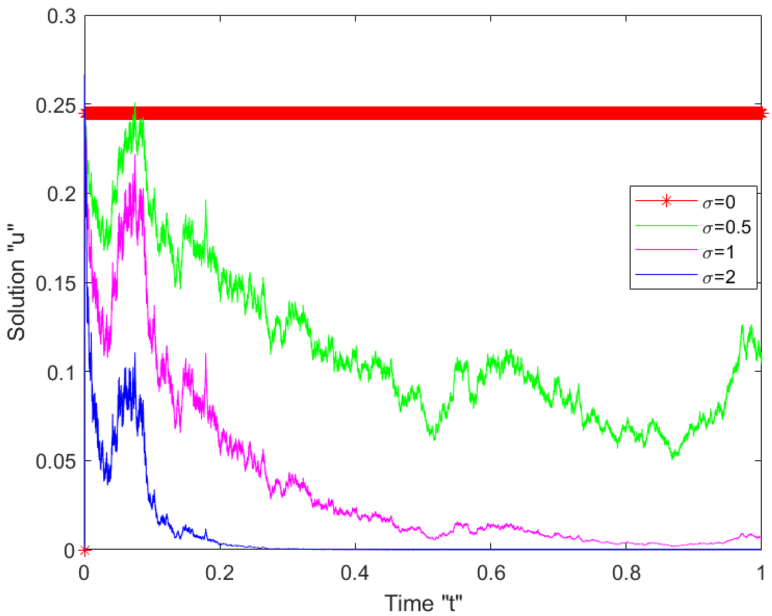

Figure 3. Profile picture of $|U|$ given in (14) in two dimensions for $\alpha = 1$ and different $\rho = 0, 0.5, 1, 2$.

6. Conclusions

We successfully acquired the analytical space-fractional stochastic solutions of the SFSHFSCE (1) forced by multiplicative SWP. This Equation (1) has never been studied before with a stochastic term. We used the Jacobi elliptic function method to obtain elliptic and hyperbolic stochastic solutions. These stochastic solutions are much more accurate and effective in understanding some important complex physical phenomena. In addition, we extended some previously obtained solutions, such as those reported in [19,27]. Finally, we implemented Matlab tools to demonstrate how the multiplicative Wiener process affected the SFSHFSCE solutions. In future work, we will consider the SFSHFSCE (1) with additive noise or with the infinite dimension Wiener process.

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